Tables for Digital Signal Processing

1 Mathematical formulae

Pythagoras's theorem: For a right-angle triangle with hypotenuse c and two other sides a, b,

$$c = \sqrt{a^2 + b^2}$$

Quadratic formula: If

$$ax^2 + bx + c = 0$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.1 Circular functions

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\tan(\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \\
\sin\theta + \sin\phi &= 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} \\
\sin\theta - \sin\phi &= 2\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2} \\
\cos\theta + \cos\phi &= 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} \\
\cos\theta - \cos\phi &= -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2} \\
2\sin\theta\cos\phi &= \sin(\theta+\phi) + \sin(\theta-\phi) \\
2\cos\theta\sin\phi &= \sin(\theta+\phi) - \sin(\theta-\phi) \\
2\cos\theta\cos\phi &= \cos(\theta-\phi) + \cos(\theta+\phi) \\
2\sin\theta\sin\phi &= \cos(\theta-\phi) - \cos(\theta+\phi) \\
\tan\theta\tan\phi &= \frac{\cos(\theta-\phi) - \cos(\theta+\phi)}{\cos(\theta-\phi) + \cos(\theta+\phi)}
\end{aligned}$$

1.2 Euler's formula

$$
\begin{aligned}
\mathrm{e}^{\mathrm{j}x} &= \cos x + \mathrm{j}\sin x \\
\cos x &= \frac{\mathrm{e}^{\mathrm{j}x} + \mathrm{e}^{-\mathrm{j}x}}{2} \\
\sin x &= \frac{\mathrm{e}^{\mathrm{j}x} - \mathrm{e}^{-\mathrm{j}x}}{2\mathrm{j}}
\end{aligned}
$$

1.3 Series

$$
\begin{aligned}
\sum_{k=0}^{n-1}(a + kd) &= \frac{n}{2}[2a + (n-1)d] \\
\sum_{k=0}^{n-1} ar^k &= \frac{a(1-r^n)}{1-r} \\
\sum_{k=0}^{\infty} ar^k &= \frac{a}{1-r} \qquad \text{if } |r| < 1
\end{aligned}
$$

2 Transforms

z transform:

$$
X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}
$$

Fourier transform:

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t \\
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega
\end{aligned}
$$

Discrete-time Fourier transform:

$$
\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)\,\mathrm{e}^{-\mathrm{j}\omega n} \\
x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\,\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega
\end{aligned}
$$

Discrete Fourier transform (DFT):

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)\,\mathrm{e}^{-\mathrm{j}2\pi kn/N}, && 0 \le k \le N-1 \\
x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\mathrm{e}^{\mathrm{j}2\pi kn/N}, && 0 \le n \le N-1
\end{aligned}
$$

2.1 Discrete-time convolution

Linear convolution:

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

N-point circular convolution:

$$x(n) \circledN h(n) = \sum_{k=0}^{N-1} x(n-k \mod N)h(k) = \sum_{k=0}^{N-1} h(n-k \mod N)x(k)$$

2.2 Discrete-time correlation

Auto-correlation:

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) = x(l) * x(-l)$$

Cross-correlation of energy signals:

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) = x(l) * y(-l)$$

Cross-correlation of periodic signals with period N:

$$r_{xy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)y(n-l)$$

2.3 Bounded-Input Bounded-Output (BIBO) stability

A system is stable if input $x(n)$ and output $y(n)$ satisfy

$$|x(n)| < \infty \implies |y(n)| < \infty$$

A linear time-invariant system with impulse response $h(n)$ is stable if and only if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

2.4 Common z transform pairs

Time domain $x(n)$	z domain $X(z)$	ROC
$\delta(n)$	1	All z
$\mathrm{U}(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$n\mathrm{U}(n)$	$\dfrac{z^{-1}}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n\mathrm{U}(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n\mathrm{U}(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n\mathrm{U}(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$-na^n\mathrm{U}(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\cos(\omega_0 n)\mathrm{U}(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)\mathrm{U}(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos(\omega_0 n)\mathrm{U}(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n\sin(\omega_0 n)\mathrm{U}(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

2.5 Properties of the z transform

Property	Time domain	z domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	$ROC,\ r_\mathrm{l} < \lvert z\rvert < r_\mathrm{u}$ $ROC_1,\ r_{1\mathrm{l}} < \lvert z\rvert < r_{1\mathrm{u}}$ $ROC_2,\ r_{2\mathrm{l}} < \lvert z\rvert < r_{2\mathrm{u}}$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	Includes the intersection of ROC_1 and ROC_2
Time shifting (two sided)	$x(n-k)$	$z^{-k}X(z)$	ROC, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Time delay (one sided)	$x(n-k), \quad k > 0$	$z^{-k}\left[X(z) + \sum_{n=1}^{k} x(-n)z^{n}\right]$	ROC, except $z = 0$
Time advance (one sided)	$x(n+k), \quad k > 0$	$z^{k}\left[X(z) - \sum_{n=0}^{k-1} x(n)z^{-n}\right]$	ROC, except $z = \infty$
Scaling in the z domain	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert r_\mathrm{l} < \lvert z\rvert < \lvert a\rvert r_\mathrm{u}$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_\mathrm{u}} < \lvert z\rvert < \frac{1}{r_\mathrm{l}}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\mathrm{Re}\, x(n)$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\mathrm{Im}\, x(n)$	$\frac{1}{2}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z domain	$nx(n)$	$-z\frac{\mathrm{d}X(z)}{\mathrm{d}z}$	$r_\mathrm{l} < \lvert z\rvert < r_\mathrm{u}$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	Includes the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) =$ $x_1(l) * x_2(-l)$	$R_{x_1x_2} =$ $X_1(z)X_2(z^{-1})$	Includes the intersection of ROC_1 and ROC of $X_2(z^{-1})$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi \mathrm{j}}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}\,\mathrm{d}v$	Includes $r_{1\mathrm{l}}r_{2\mathrm{l}} < \lvert z\rvert < r_{1\mathrm{u}}r_{2\mathrm{u}}$
Initial value theorem	If $x(n)$ is causal, then $x(0) = \lim_{z\to\infty} X(z)$		
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi \mathrm{j}}\oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}\,\mathrm{d}v$		

2.6 Common discrete-time Fourier transform pairs

Note: $X(\omega)$ is periodic with period 2π. The table below only shows $X(\omega)$ in the range $-\pi < \omega \leq \pi$.

Time domain $x(n)$	Frequency domain (one period) $X(\omega), \quad -\pi < \omega \leq \pi$
1	$2\pi\delta(\omega)$
$\delta(n)$	1
$\mathrm{U}(n)$	$\frac{1}{1-\mathrm{e}^{-\mathrm{j}\omega}} + \pi\delta(\omega)$
$a^n\mathrm{U}(n), \quad \lvert a\rvert < 1$	$\frac{1}{1-a\mathrm{e}^{-\mathrm{j}\omega}}$
$(n+1)a^n\mathrm{U}(n), \quad \lvert a\rvert < 1$	$\frac{1}{(1-a\mathrm{e}^{-\mathrm{j}\omega})^2}$
$\cos\omega_0 n, \quad -\pi < \omega_0 < \pi$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin\omega_0 n, \quad -\pi < \omega_0 < \pi$	$\frac{\pi}{\mathrm{j}}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$
$\cos\pi n = (-1)^n$	$2\pi\delta(\omega-\pi)$
$\sum_{m=-\infty}^{\infty}\delta(n-mN)$	$\frac{2\pi}{N}\sum_{m=\lfloor 1-N/2\rfloor}^{\lfloor N/2\rfloor}\delta\left(\omega-\frac{2\pi m}{N}\right)$
$\mathrm{U}(n+N)-\mathrm{U}(n-N-1)$	$\frac{\sin[\omega(N+1/2)]}{\sin(\omega/2)}$
$x(n) = \begin{cases} \frac{W}{\pi}, & n = 0 \\ \frac{\sin Wn}{\pi n}, & n \neq 0 \end{cases}$	$X(\omega) = \begin{cases} 1, & \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$

2.7 Properties of the discrete-time Fourier transform

Property	Time domain	Frequency domain
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(\omega)$ $X_1(\omega)$ $X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$\mathrm{e}^{-\mathrm{j}\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
Frequency shifting	$\mathrm{e}^{\mathrm{j}\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\dfrac{X(\omega-\omega_0) + X(\omega+\omega_0)}{2}$
	$x(n)\sin\omega_0 n$	$\dfrac{X(\omega-\omega_0) - X(\omega+\omega_0)}{2\mathrm{j}}$
Multiplication	$x_1(n)x_2(n)$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)\,\mathrm{d}\lambda$
Differentiation in the frequency domain	$nx(n)$	$\mathrm{j}\dfrac{\mathrm{d}X(\omega)}{\mathrm{d}\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\displaystyle\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)\,\mathrm{d}\omega$	

2.8 Properties of the discrete Fourier transform (DFT)

Property	Time domain	Frequency domain
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(k)$ $X_1(k)$ $X_2(k)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Circular time shifting	$x(n - l \mod N)$	$\mathrm{e}^{-\mathrm{j}2\pi kl/N}X(k)$
Time reversal	$x(N - n)$	$X(N - k)$
Circular convolution	$x_1(n) \circledN x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x_1(n) \circledN x_2^*(N - n)$	$X_1(k)X_2^*(k)$
Circular frequency shifting	$\mathrm{e}^{\mathrm{j}2\pi ln/N}x(n)$	$X(k - l \mod N)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \circledN X_2(k)$
Conjugation	$x^*(n)$	$X^*(N - k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x_1(n)x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2^*(k)$	
Symmetry	$x(n) = x_\mathrm{R}^\mathrm{e}(n) + x_\mathrm{R}^\mathrm{o}(n) + \mathrm{j}x_\mathrm{I}^\mathrm{e}(n) + \mathrm{j}x_\mathrm{I}^\mathrm{o}(n)$ $X(k) = X_\mathrm{R}^\mathrm{e}(k) + X_\mathrm{R}^\mathrm{o}(k) + \mathrm{j}X_\mathrm{I}^\mathrm{e}(k) + \mathrm{j}X_\mathrm{I}^\mathrm{o}(k)$ where $x_\mathrm{R}^\mathrm{e}(n), X_\mathrm{R}^\mathrm{e}(k)$ are real and even, $x_\mathrm{R}^\mathrm{o}(n), X_\mathrm{R}^\mathrm{o}(k)$ are real and odd, $x_\mathrm{I}^\mathrm{e}(n), X_\mathrm{I}^\mathrm{e}(k)$ are imaginary and even, $x_\mathrm{I}^\mathrm{o}(n), X_\mathrm{I}^\mathrm{o}(k)$ are imaginary and odd	

3 Filters

Kaiser formula to determine the equiripple filter order:

$$M \approx \frac{-10\log_{10}(\delta_1\delta_2) - 13}{14.6\Delta f} + 1$$

Order of Butterworth filter:

$$\frac{\log(\delta/\epsilon)}{\log(\Omega_\mathrm{s}/\Omega_\mathrm{p})}$$

Order of Chebyshev filter:

$$\frac{\operatorname{arcosh}(\delta/\epsilon)}{\operatorname{arcosh}(\Omega_\mathrm{s}/\Omega_\mathrm{p})}$$

Bilinear transform:

$$s = \frac{2}{T_\mathrm{s}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad z = \frac{\frac{2}{T_\mathrm{s}} + s}{\frac{2}{T_\mathrm{s}} - s}$$

$$\Omega = \frac{2}{T_\mathrm{s}}\tan\frac{\omega}{2} \qquad \omega = 2\arctan\frac{\Omega T_\mathrm{s}}{2}$$

where T_s is the sampling period.

3.1 Window functions for FIR filter design

Name	Time domain sequence
Bartlett	$1 - \frac{2\left\lvert n - \frac{M-1}{2}\right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\frac{2\pi n}{M-1}$
Hann	$\frac{1}{2}\left(1 - \cos\frac{2\pi n}{M-1}\right)$
Kaiser	$\frac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$

3.2 Frequency transformations for analogue filters

The prototype low-pass filter has band edge frequency Ω_p.

Type	Transformation	Parameters
Low pass	$s \to \dfrac{\Omega_\mathrm{p}}{\Omega'_\mathrm{p}} s$	Ω'_p = band edge frequency for new filter
High pass	$s \to \dfrac{\Omega_\mathrm{p}\Omega'_\mathrm{p}}{s}$	Ω'_p = band edge frequency for new filter
Band pass	$s \to \Omega_\mathrm{p} \dfrac{s^2 + \Omega_\mathrm{l}\Omega_\mathrm{u}}{s(\Omega_\mathrm{u} - \Omega_\mathrm{l})}$	Ω_l = lower band edge frequency Ω_u = upper band edge frequency
Band stop	$s \to \Omega_\mathrm{p} \dfrac{s(\Omega_\mathrm{u} - \Omega_\mathrm{l})}{s^2 + \Omega_\mathrm{l}\Omega_\mathrm{u}}$	Ω_l = lower band edge frequency Ω_u = upper band edge frequency

3.3 Frequency transformations for digital filters

The prototype low-pass filter has band edge frequency ω_p.

Type	Transformation	Parameters
Low pass	$z^{-1} \to \dfrac{z^{-1} - a}{1 - az^{-1}}$	ω'_p = band edge frequency for new filter $a = \dfrac{\sin[(\omega_\mathrm{p} - \omega'_\mathrm{p})/2]}{\sin[(\omega_\mathrm{p} + \omega'_\mathrm{p})/2]}$
High pass	$z^{-1} \to -\dfrac{z^{-1} + a}{1 + az^{-1}}$	ω'_p = band edge frequency for new filter $a = -\dfrac{\cos[(\omega_\mathrm{p} + \omega'_\mathrm{p})/2]}{\cos[(\omega_\mathrm{p} - \omega'_\mathrm{p})/2]}$
Band pass	$z^{-1} \to \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l = lower band edge frequency ω_u = upper band edge frequency $a_1 = 2\alpha K/(K+1)$ $a_2 = (K-1)/(K+1)$ $\alpha = \dfrac{\cos[(\omega_\mathrm{u} + \omega_\mathrm{l})/2]}{\cos[(\omega_\mathrm{u} - \omega_\mathrm{l})/2]}$ $K = \cot \dfrac{\omega_\mathrm{u} - \omega_\mathrm{l}}{2} \tan \dfrac{\omega_\mathrm{p}}{2}$
Band stop	$z^{-1} \to \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l = lower band edge frequency ω_u = upper band edge frequency $a_1 = 2\alpha/(K+1)$ $a_2 = (1-K)/(1+K)$ $\alpha = \dfrac{\cos[(\omega_\mathrm{u} + \omega_\mathrm{l})/2]}{\cos[(\omega_\mathrm{u} - \omega_\mathrm{l})/2]}$ $K = \tan \dfrac{\omega_\mathrm{u} - \omega_\mathrm{l}}{2} \tan \dfrac{\omega_\mathrm{p}}{2}$

3.4 Lattice filters

Conversion of lattice coefficients to direct-form filter coefficients:

$$A_0(z) = B_0(z) = 1$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z) \qquad m = 1, 2, \dots, M-1$$

$$B_m(z) = z^{-m} A_m(z^{-1}) \qquad m = 1, 2, \dots, M-1$$

Conversion of direct-form filter coefficients to lattice coefficients:

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2} \qquad m = M-1, M-2, \dots, 1$$

$$B_m(z) = z^{-m} A_m(z^{-1}) \qquad m = 1, 2, \dots, M-1$$

$$A_0(z) = 1$$

Conversion of direct-form IIR filter coefficients to lattice coefficients (in addition to FIR filter coefficient above):

$$C_M(z) = \sum_{m=0}^{M} v_m B_m(z)$$

$$C_{m-1}(z) = C_m(z) - v_m B_m(z)$$

4 Wavelets

Haar wavelet function:

$$\psi(t) = \begin{cases} 1, & \text{for } 0 \le t < 1/2 \\ -1, & \text{for } 1/2 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Haar scaling function:

$$\phi(t) = \begin{cases} 1, & \text{for } 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Wavelet dilation and translation:

$$\psi_{j,n}(t) = \frac{1}{\sqrt{2^j}}\, \psi\left(\frac{t - 2^j n}{2^j}\right)$$

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