L-Università ta' Malta

CCE3206—Digital Signal Processing

Practical 4

Trevor Spiteri
trevor.spiteri@um.edu.mt

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Poles and zeros

Objective

The objective of this practical is to determine the effects of the location of the poles and zeros of a system on its behaviour.

Background

The transfer function $H(z)$ of most practical systems may be expressed as a rational function in the form

$$H(z) = Gz^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

where $z_k, k = 1, 2, \dots, M$ are M zeros,
$p_k, k = 1, 2, \dots, N$ are N poles,
if $N > M$, z^{N-M} are $N - M$ zeros at $z = 0$,
if $M > N$, z^{N-M} are $M - N$ poles at $z = 0$, and
G is the gain.

The system's frequency response may be obtained by setting $z = \mathrm{e}^{\mathrm{j}\omega}$.

$$\begin{aligned} H(\omega) &= H(z)|_{z=\mathrm{e}^{\mathrm{j}\omega}} \\ &= G\mathrm{e}^{\mathrm{j}\omega(N-M)} \frac{\prod_{k=1}^{M}(\mathrm{e}^{\mathrm{j}\omega} - z_k)}{\prod_{k=1}^{N}(\mathrm{e}^{\mathrm{j}\omega} - p_k)} \end{aligned}$$

Note that for $H(z)$ to have real coefficients, all the zeros must be real or complex conjugate pairs. Similarly, all the poles must be real or complex conjugate pairs.

If $z_k = r\mathrm{e}^{\pm \mathrm{j}\omega_k}$, then there is increased *attenuation* near the frequency ω_k. The closer r is to 1, the greater the attenuation. If $r = 1$, the output will be zero at that frequency.

If $p_k = r\mathrm{e}^{\pm \mathrm{j}\omega_k}$, then there is increased *gain* near the frequency ω_k. The closer r is to 1, the greater the gain. If $r \geq 1$, the system will be unstable.

Tasks

1. A first-order system is described by the system transfer function

$$H(z) = \frac{z - z_1}{z - p_1}$$

where z_1 is a zero and p_1 is a pole.

Setting $z_1 = 0$, determine the impulse response of the system for

(a) $p_1 = 0.2$
(b) $p_1 = 0.8$
(c) $p_1 = 1$
(d) $p_1 = 1.1$
(e) $p_1 = -0.8$
(f) $p_1 = -1$

In each case observe both the time response and the magnitude frequency response.

This task can be divided into the following subtasks:

(a) Define the discrete linear time-invariant system using the SciPy `signal.dlti` function. For example, for $z_1 = 0$ and $p_1 = 0.2$,

```
from scipy import signal
z1 = 0
p1 = 0.2
G = 1
# parameters: array of zeros, array of poles, gain
system = signal.dlti([z1], [p1], G)
```

(b) The impulse response in time can be found using the `impulse` method.

```
import numpy as np
import matplotlib.pyplot as plt
# impulse response with 20 points
n, h = system.impulse(n=20)
h = np.squeeze(h)

# plot on matplotlib axis ax
ax.stem(n, h)
```

(c) The magnitude frequency response can be found using the `freqresp` method.

```
# 256 frequency points from -π to π
w = np.linspace(
    -np.pi, np.pi,
    num=256, endpoint=False
)
_, H = system.freqresp(w)

# plot on matplotlib axis ax
ax.plot(w, abs(H))
```

(d) Comment on the results obtained and compare these with theory. Do *not* comment on each and every individual plot, but rather on the general trends observed over all plots (eight time plots and eight frequency plots).

(20 marks)

2. Repeat Task 1 with $p_1 = 0$ and with

   (a) $z_1 = 0.2$
   (b) $z_1 = 0.8$
   (c) $z_1 = 1$
   (d) $z_1 = 1.1$
   (e) $z_1 = -0.8$
   (f) $z_1 = -1$

   Comment on the effect of the zero on the system response.

   (10 marks)

3. Repeat Tasks 1–2, but instead of the impulse response, obtain the time response for a 10-point pulse input $x(n) = \mathrm{U}(n) - \mathrm{U}(n - 10)$. Do *not* obtain the frequency response; the time response is enough for this task.

```
# step response with 20 points
t, y = system.step(n=20)
y = np.squeeze(y)
# Replace y(n) with y(n) - y(n - 10)
y[10:] -= y[:-10]
```

   (10 marks)

4. An all-pole second-order system is described by the system transfer function

$$H(z) = \frac{z^2}{(z - p_1)(z - p_2)}$$

   where p_1 and p_2 are two poles. We can also say that we have two trivial zeros $z_1 = z_2 = 0$.

   Using code similar to Task 1, determine the impulse response of the system, both in time and in frequency, for the following.

   (a) $p_1 = p_2 = -0.9$
   (b) $p_1 = -0.9, p_2 = 0.9$
   (c) $p_1 = p_2 = 1$
   (d) $p_1 = -1, p_2 = 1$
   (e) $p_1 = 0.9\mathrm{e}^{\mathrm{j}\pi/6}, p_2 = 0.9\mathrm{e}^{-\mathrm{j}\pi/6}$
   (f) $p_1 = \mathrm{e}^{\mathrm{j}\pi/6}, p_2 = \mathrm{e}^{-\mathrm{j}\pi/6}$
   (g) $p_1 = 1.1\mathrm{e}^{\mathrm{j}\pi/6}, p_2 = 1.1\mathrm{e}^{-\mathrm{j}\pi/6}$

   For this task, it is better to have a longer time response, using for example `system.impulse(n=100)`.

   (10 marks)

5. A second-order system is described by the system transfer function

$$H(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

where z_1 and z_2 are two zeros, and p_1 and p_2 are two poles.

Determine the impulse response, both in time and frequency, for $z_1 = \mathrm{e}^{\mathrm{j}\pi/2}, z_2 = \mathrm{e}^{-\mathrm{j}\pi/2}$, $p_1 = 0.9\mathrm{e}^{\mathrm{j}3\pi/4}$, $p_2 = 0.9\mathrm{e}^{-\mathrm{j}3\pi/4}$.

In particular, determine

- the null frequencies, where $|H(\omega)| = 0$, and
- the peak frequencies, where $|H(\omega)|$ is at a maximum.

Comment on the results.

(20 marks)

6. Using the insight from the previous tasks, design a second-order notch filter that removes the frequency 1 kHz while passing all other frequencies. You may assume that the input is an audio signal with sampling frequency $F_\mathrm{s} = 44.1$ kHz.

Test the designed filter and comment on its effectiveness.

(30 marks)

Report

- If there are answers that require some calculations on plot readings, the report must show both the actual readings and the calculated answers.
- Include any general observations and comments in your report.

Acknowledgements

This practical is based on a similar practical by Victor Buttigieg.