

An Introduction to Statistical Learning with Applications in R

Second Edition

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2 Statistical Learning

2.1

- (a) Given a very large sample size n and a small number of predictors p , an inflexible method would be better than a flexible one, since the risk of overfitting is less.
- (b) For the same reasons as (a), a flexible method would yield better results for small n and large p .
- (c) A flexible method would yield better results, since non-linear functions cannot be accurately modelled by linear functions.
- (d) If there is high variance in the error terms, an inflexible method would be better, since a flexible method would introduce even more variance in the values of \hat{f} .

2.2

- (a) This is a classification problem, since we are trying to identify a qualitative trend in the data. It is an inference problem, since we are not trying to estimate future values of f . In this case, we have $n = 500$ and $p = 4$.

- (b) This is a classification problem, since we are trying to classify the product as either a success or a failure. It is also a prediction problem, since we are looking to estimate a future output. We have $n = 20$ and $p = 14$.
- (c) This is a regression problem since we have quantitative data and assume that it fits some function f , which we are attempting to estimate. Since this is a future estimate, it is a prediction problem. We have $n = 52$ and $p = 4$.

2.3

- (a) We have an R plot of various error curves in Figure 2.1.

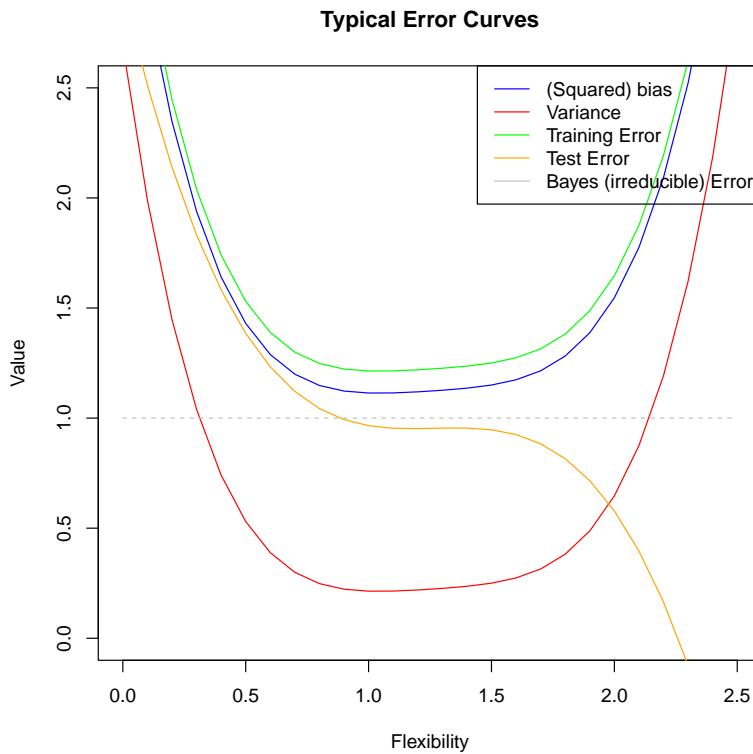


Figure 2.1: Typical bias, variance, training error, test error, and Bayes error

Arbitrary polynomials of the appropriate degree were used to generate these plots, since they are approximations, using the following script:

```
# Exercise 2.3(a) - Plots of error curves
squared_bias <- function(x) {
  return((x - 1.25)^4 + 0.01 * x^3 + 0.05 * x^2 - 0.1 * x + 1.15)
```

```

}

variance <- function(x) {
  return((x - 1.25)^4 + 0.01 * x^3 + 0.05 * x^2 - 0.1 * x + 0.25)
}

training_err <- function(x) {
  return((x - 1.25)^4 + 0.01 * x^3 + 0.05 * x^2 - 0.1 * x + 1.25)
}

test_err <- function(x) {
  return(-(x - 1.25)^3 + 0.05 * x^2 - 0.1 * x + 1.0)
}

bayes_err <- function(x) {
  return(rep(1.0, length(x)))
}

x <- seq(0, 2.5, by = 0.1)

y1 <- squared_bias(x)
y2 <- variance(x)
y3 <- training_err(x)
y4 <- test_err(x)
y5 <- bayes_err(x)

pdf("ex2_3_a.pdf")

plot(x, y1,
      type = "l",
      col = "blue",
      lwd = 1,
      ylim = c(0, 2.5),
      xlab = "Flexibility",
      ylab = "Value",
      main = "Typical Error Curves"
    )

lines(x, y2, col = "red", lwd = 1)
lines(x, y3, col = "green", lwd = 1)
lines(x, y4, col = "orange", lwd = 1)
lines(x, y5, col = "gray", lwd = 1, lty = 2)

legend("topright",
       legend = c(
         "(Squared) bias", "Variance", "Training Error", "Test Error",
         "Bayes (irreducible) Error"
       ),
       col = c("blue", "red", "green", "orange", "gray"), lwd = 1
     )

dev.off()

```

- (b) We are given that the training error will always be greater than the (squared) bias, and that both will be u-shaped. The variance will always be less than the (squared) bias, while the test error will decrease as the model becomes more flexible. The Bayes error is irreducible, so it remains constant.

2.4

- (a) Three real-life applications in which classification might be useful include:

- (i) Determining a voter's party preference based upon the results of a census. The response would be a classification of a voter as being likely to vote for one of the possible political parties and the predictors would be features such as age, demographic, or location, which could be quantitative or qualitative. The goal in this case would be prediction, since we would most likely want to determine their vote in the next election.
 - (ii) Classification could also be used to determine the medical condition causing certain symptoms via medical imaging analysis. The response would be any of a number of medical conditions and the predictors would be features of the image, like abnormal textures or shapes within images provided by medical scanning. The goal would be inference, given that the condition already exists and we would like to determine its nature.
 - (iii) Another application would be determining a credit score for an individual. The response would be a natural number within a pre-determined range and the predictors would be factors such as time taken to repay debts, number of credit lines awarded to the individual, and total accumulated debt. The goal would be prediction, since the credit score is used to determine the likelihood that the individual will repay their debts on time.
- (b) Three real-life applications in which regression might be useful include:
- (i) Attempting to predict the performance of a stock, given its past history would be a suitable application for regression. The response would be a numerical value indicating the expected price at a certain time and the predictors would be past values on a set of times. The goal would be prediction, given that we are estimating future performance.
 - (ii) Regression could also be used to determine the probability that a person will purchase a certain item from a retailer, given their past purchase history. The response would be a probability ($\text{Pr}(X) \in [0, 1]$) and the predictors would be factors such as number of past purchases from that same retailer, spending history, credit rating, etc. The goal would also be prediction.
 - (iii) Another application would be determining the levels of a contaminant in a water supply, based upon readings from sources which are not directly drawn from the water supply itself, e.g., taps and waste water. The response would be a numeric value (say in mg/L) and the predictors would be the equivalent values in the other sources. The goal would be inference, since we are attempting to determine a current value.
- (c) Three real-life applications in which cluster analysis might be useful include:

- (i) A useful application of cluster analysis would be attempting to determine the species of certain related organisms given the degree to which they exhibit certain features.
- (ii) Cluster analysis could also be used to determine whether certain geological samples fit within distinct groups based upon their chemical composition.
- (iii) Another application of cluster analysis could be using segmenting consumers into certain target markets based upon their response to marketing surveys.

2.5

The advantages of a very flexible approach for regression or classification include accuracy if over-fitting has not been exhibited, while the disadvantages include interpretability, large variance in errors, and tendency for over-fitting to occur. A more flexible approach might be preferred when a set of data contains a large variation in the response and a less flexible approach would be preferable if the data tends to contain few outliers.

2.6

When utilising a parametric statistical learning approach, we attempt to estimate a countable number of parameters β_i for $i \in \{1, \dots, p\}$ s.t. the observation (X, Y) has Y s.t.

$$Y \approx f(X) = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p.$$

This means we need to select the number of parameters used in order to minimize the error. In doing so, we risk overfitting the model to the data, so that the approximation does not closely match the form of f .

In contrast, a non-parametric method makes no assumption about the parameters used, with the advantage that we do not need to concern ourselves with the number of parameters used, but with the disadvantage that we must select a level of smoothness which makes the approximation to f easy to calculate without introducing unnecessary variation in the shape of the approximation relative to f .

2.7

- (a) If we denote the number of observations by n and the number of variables by p , then we let x_{ij} denote the i th observation of the j th variable. We calculate the Euclidian distances using the formula

$$\begin{aligned} d((X_1 = 0, X_2 = 0, X_3 = 0), (x_{i1}, x_{i2}, x_{i3})) \\ = \sqrt{(0 - x_{i1})^2 + (0 - x_{i2})^2 + (0 - x_{i3})^2} \\ = \sqrt{(-x_{i1})^2 + (-x_{i2})^2 + (-x_{i3})^2}, \end{aligned}$$

for $i \in \{1, \dots, 6\}$.

Thus we have

$$\begin{aligned}
& d((X_1 = 0, X_2 = 0, X_3 = 0), (x_{11}, x_{12}, x_{13})) \\
&= \sqrt{9^2} \\
&= 9, \\
& d((X_1 = 0, X_2 = 0, X_3 = 0), (x_{21}, x_{22}, x_{23})) \\
&= \sqrt{2^2} \\
&= 2, \\
& d((X_1 = 0, X_2 = 0, X_3 = 0), (x_{31}, x_{32}, x_{33})) \\
&= \sqrt{1^2 + 3^2} \\
&= \sqrt{10} \\
&\approx 3.162278, \\
& d((X_1 = 0, X_2 = 0, X_3 = 0), (x_{41}, x_{42}, x_{43})) \\
&= \sqrt{1^2 + 2^2} \\
&= \sqrt{5} \\
&\approx 2.236068, \\
& d((X_1 = 0, X_2 = 0, X_3 = 0), (x_{51}, x_{52}, x_{53})) \\
&= \sqrt{(-1)^2 + 1^2} \\
&= \sqrt{2} \\
&\approx 1.414214,
\end{aligned}$$

and

$$\begin{aligned}
& d((X_1 = 0, X_2 = 0, X_3 = 0), (x_{61}, x_{62}, x_{63})) \\
&= \sqrt{1^2 + 1^2 + 1^2} \\
&= \sqrt{3} \\
&\approx 1.732051.
\end{aligned}$$

- (b) Our prediction with $K = 1$ is that $Y = \text{Green}$ since the nearest neighbour to the point $(X_1 = 0, X_2 = 0, X_3 = 0)$ is the point given by (x_{51}, x_{52}, x_{53}) corresponding to observation 5, which yields the value $y = \text{Green}$.
- (c) With $K = 3$, we have $Y = \text{Red}$, since of the 3 nearest neighbours calculated in (a), we have two with value $y = \text{Red}$ and only one with value $y = \text{Green}$.
- (d) We would expect the best value to be small, since this would yield a decision boundary which is highly flexible and more easily approximates the non-linear nature of the Bayes decision boundary.

2.8

- (c) i. We have the following output from the `summary` function:

```
> summary(college)
Private          Apps           Accept        Enroll      Top10perc
No :212   Min.   : 81   Min.   : 72   Min.   : 35   Min.   : 1.00
Yes:565  1st Qu.: 776  1st Qu.: 604  1st Qu.: 242  1st Qu.:15.00
          Median :1558  Median :1110  Median : 434  Median :23.00
          Mean   :3002  Mean   :2019  Mean   : 780  Mean   :27.56
          3rd Qu.:3624  3rd Qu.:2424  3rd Qu.: 902  3rd Qu.:35.00
          Max.   :48094 Max.   :26330 Max.   :6392  Max.   :96.00
Top25perc     F.Undergrad    P.Undergrad    Outstate
Min.   : 9.0   Min.   : 139  Min.   : 1.0   Min.   : 2340
1st Qu.: 41.0  1st Qu.: 992  1st Qu.: 95.0  1st Qu.: 7320
Median : 54.0  Median :1707  Median : 353.0 Median : 9990
Mean   : 55.8  Mean   :3700  Mean   : 855.3 Mean   :10441
3rd Qu.: 69.0  3rd Qu.:4005  3rd Qu.: 967.0 3rd Qu.:12925
Max.   :100.0  Max.   :31643 Max.   :21836.0 Max.   :21700
Room.Board    Books          Personal       PhD
Min.   :1780  Min.   : 96.0  Min.   : 250  Min.   : 8.00
1st Qu.:3597  1st Qu.: 470.0 1st Qu.: 850  1st Qu.: 62.00
Median :4200  Median :500.0  Median :1200  Median : 75.00
Mean   :4358  Mean   :549.4  Mean   :1341  Mean   : 72.66
3rd Qu.:5050  3rd Qu.:600.0  3rd Qu.:1700 3rd Qu.: 85.00
Max.   :8124  Max.   :2340.0 Max.   :6800  Max.   :103.00
Terminal      S.F.Ratio    perc.alumni    Expend
Min.   : 24.0  Min.   : 2.50  Min.   : 0.00  Min.   : 3186
1st Qu.: 71.0  1st Qu.:11.50 1st Qu.:13.00  1st Qu.: 6751
Median : 82.0  Median :13.60  Median :21.00  Median : 8377
Mean   : 79.7  Mean   :14.09  Mean   :22.74  Mean   : 9660
3rd Qu.: 92.0  3rd Qu.:16.50  3rd Qu.:31.00 3rd Qu.:10830
Max.   :100.0  Max.   :39.80  Max.   :64.00  Max.   :56233
Grad.Rate
Min.   : 10.00
1st Qu.: 53.00
Median : 65.00
Mean   : 65.46
3rd Qu.: 78.00
Max.   :118.00
```

- ii. In Figure 2.2, we have the scatterplot matrix of the first ten columns of `College` data.
- iii. In Figure 2.3, we have the plots of the `Outstate` versus `Private` College data.
- iv. We have the following output from the `summary(Elite)` function call:

```
> summary(Elite)
No Yes
699 78
```

In Figure 2.4, we have the plots of the `Outstate` versus `Elite` College data.

- v. In Figure 2.5, we have the histogram plots for four of the variables.
- vi. We can say broadly that the elite colleges have higher out-of-state tuition costs and that the number of applications received is strongly correlated with the number of applicants accepted (obviously this is to be expected), but not as strongly correlated with the number of new students enrolled.

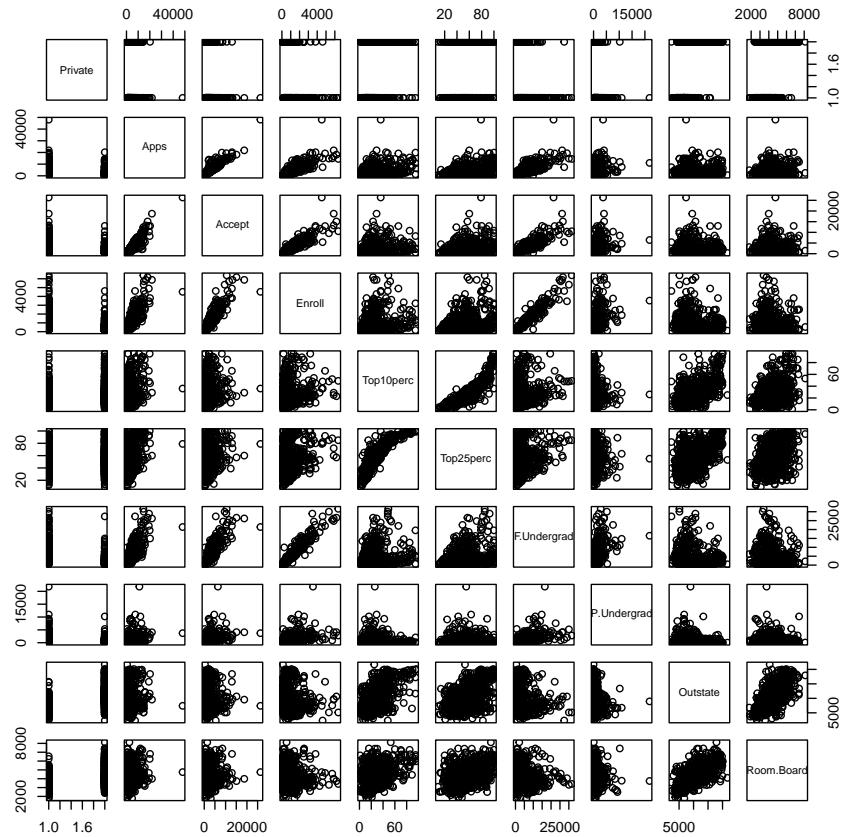


Figure 2.2: Scatterplot matrix of the first ten columns of College data

2.9

- (a) The quantitative predictors are `mpg`, `cylinders`, `displacement`, `horsepower`, `weight`, `acceleration`, `year`, and `origin`. The only qualitative predictor is `name`.
- (b) The range of each quantitative predictor is as follows:

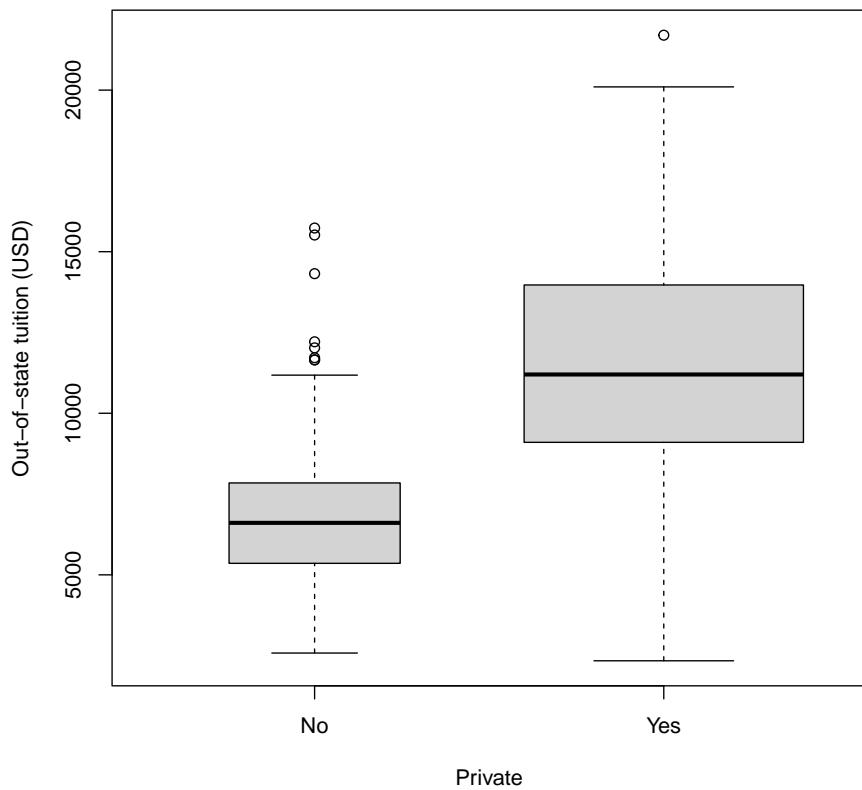


Figure 2.3: Outstate versus Private College data

Predictor	Minimum	Maximum
mpg	9.0	46.6
cylinders	3	8
displacement	68	455
horsepower	46	230
weight	1649	4997
acceleration	8.0	24.8
year	70	82
origin	1	3

- (c) We have the following values for the mean and standard deviation:

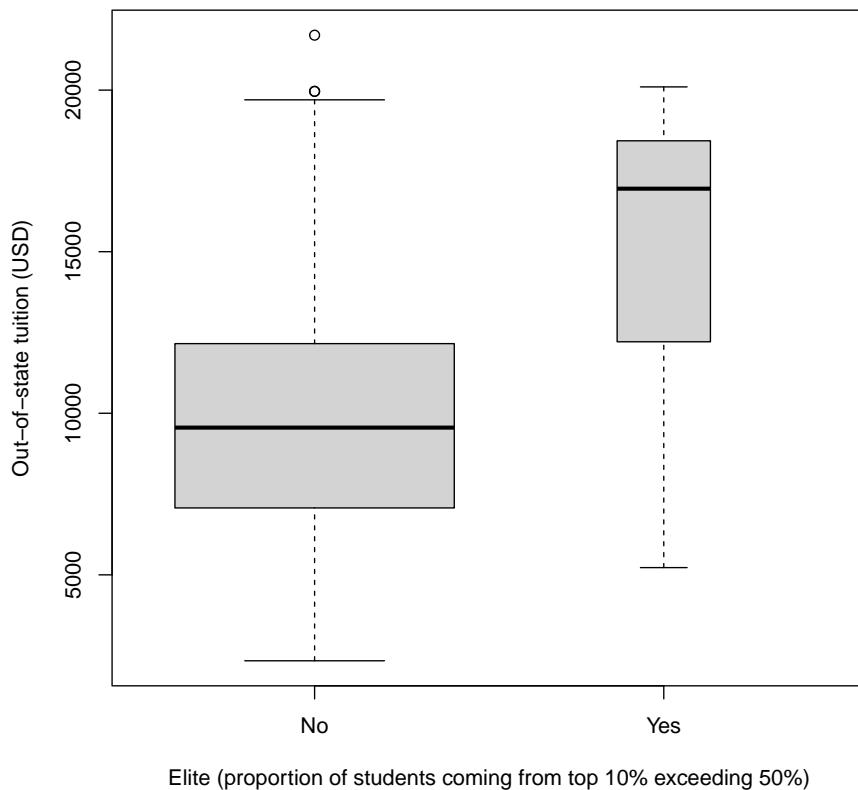


Figure 2.4: Outstate versus Elite College data

Predictor	Mean	Standard Deviation
mpg	23.44592	7.805007
cylinders	5.471939	1.705783
displacement	194.412	104.644
horsepower	104.4694	38.49116
weight	2977.584	849.4026
acceleration	15.54133	2.758864
year	75.97959	3.683737
origin	1.576531	0.8055182

- (d) We have the following adjusted values for the range, mean, and standard deviation:

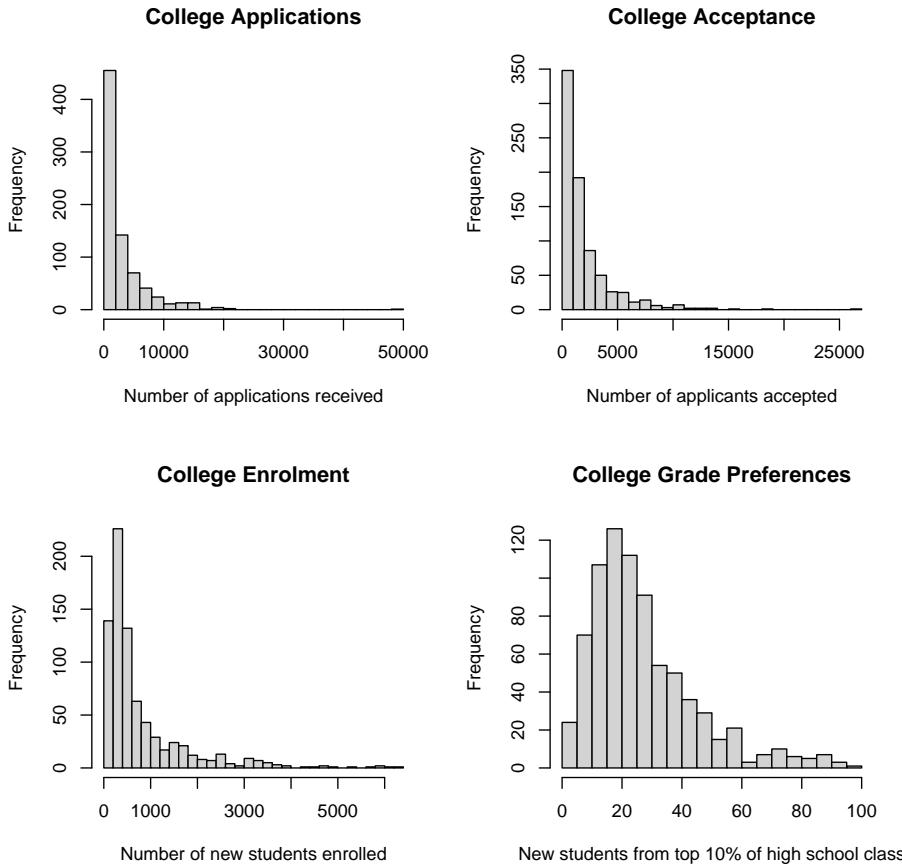


Figure 2.5: Plots of College data

Predictor	Min.	Max.	Mean	S.D.
mpg	11.0	46.6	24.40443	7.867283
cylinders	3	8	5.373418	1.654179
displacement	68	455	187.2405	99.67837
horsepower	46	230	100.7215	35.70885
weight	1649	4997	2935.972	811.3002
acceleration	8.5	24.8	15.7269	2.693721
year	70	82	77.14557	3.106217
origin	1	3	1.601266	0.81991

- (e) In Figure 2.6, we have plots of the `Auto` data highlighting the relationships between some of the variables. As can be seen by these plots, there is a strong correlation between the number of cylinders and the gas mileage (`mpg`) of the vehicle in question. This relationship is somewhat the inverse

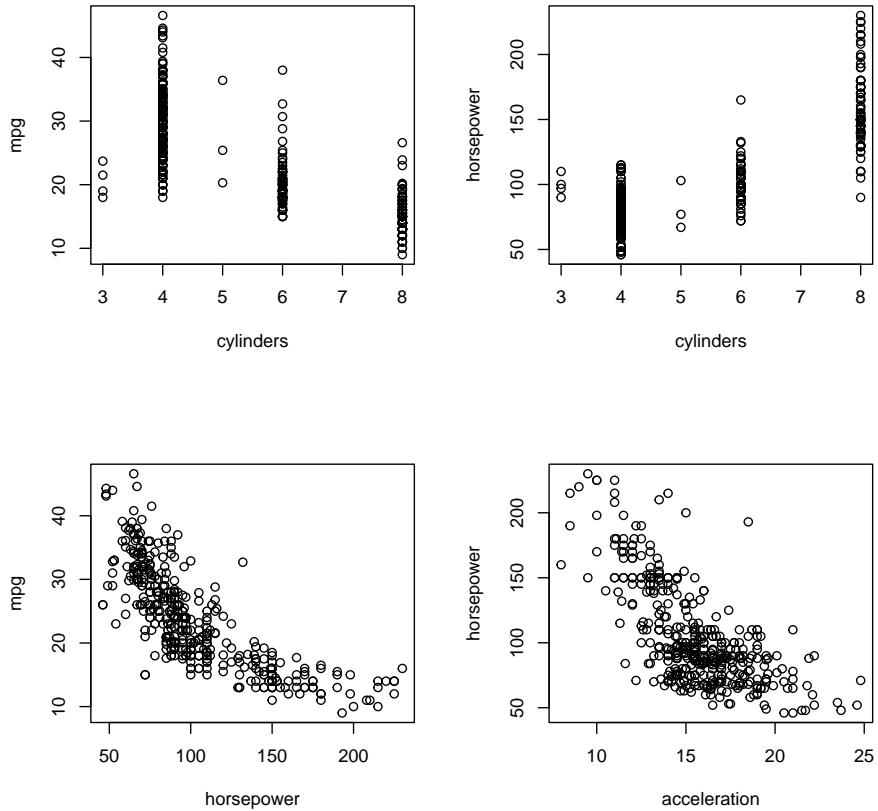


Figure 2.6: Scatterplots of Auto data

of that between cylinders and horsepower. We can also see that mpg and horsepower as well and, to a lesser extent, acceleration and horsepower are correlated.

- (f) Yes, we can use both the number of cylinders and the horsepower to estimate the gas mileage of the vehicle, since these two variables have a strongly correlated relationship to gas mileage.

2.10

- (a) We have the following output from calling ?Boston:

```
Boston          package:ISLR2          R Documentation
Boston Data
```

Description:

A data set containing housing values in 506 suburbs of Boston.

Usage:

Boston

Format:

A data frame with 506 rows and 13 variables.

'crim' per capita crime rate by town.

'zn' proportion of residential land zoned for lots over 25,000 sq.ft.

'indus' proportion of non-retail business acres per town.

'chas' Charles River dummy variable (= 1 if tract bounds river; 0 otherwise).

'nox' nitrogen oxides concentration (parts per 10 million).

'rm' average number of rooms per dwelling.

'age' proportion of owner-occupied units built prior to 1940.

'dis' weighted mean of distances to five Boston employment centres.

'rad' index of accessibility to radial highways.

'tax' full-value property-tax rate per \$10,000.

'ptratio' pupil-teacher ratio by town.

'lstat' lower status of the population (percent).

'medv' median value of owner-occupied homes in \$1000s.

- (b) In Figure 2.7, we have some of the data from the Boston data set displayed in scatterplots.

From the scatterplots in this figure, we may see that as the distance to employment centres increases, the nitrogen oxide concentration decreases, perhaps indicating higher levels of nitrogen oxide in the inner city due to vehicles and industrial pollution, which coincides with where employment centres are located. Likewise, we see that owner-occupied units built prior to the year 1940 are concentrated in the city centre.

We also see that a higher number of rooms and value of owner-occupied homes tend to be concentrated amongst the lower status of the population.

- (c) We can see that per capita crime rates are correlated with the lower status of the population by looking at Figure 2.8. It appears that high per capita crime rates are associated with a larger population in the lower status category.

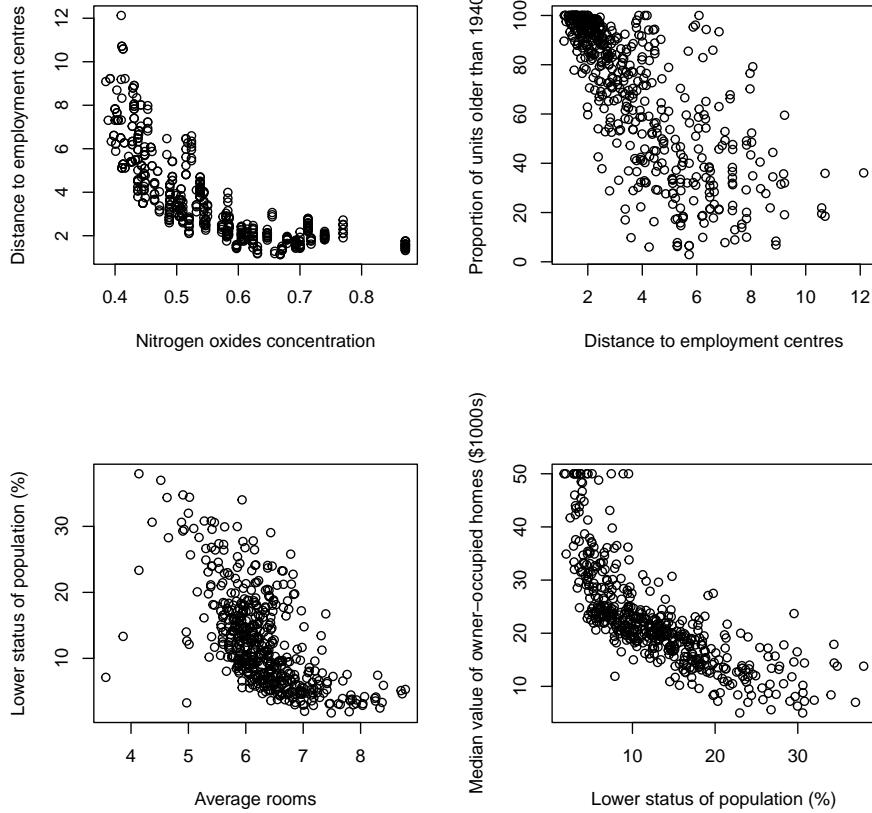


Figure 2.7: Scatterplots of Boston data

(d) Yes, we have the following summary:

```
> summary(Boston$crim)
   Min. 1st Qu. Median      Mean 3rd Qu.    Max.
0.00632  0.08204  0.25651  3.61352  3.67708 88.97620
```

While the median and mean for the `crim` data are 0.25651 and 3.61352, respectively, the range is [0.00632, 88.97620], so that some census tracts must have particularly high crime rates. Tax rates appear to be more evenly distributed, as can be seen from the following:

```
> summary(Boston$tax)
   Min. 1st Qu. Median      Mean 3rd Qu.    Max.
187.0  279.0  330.0  408.2  666.0  711.0
```

The same applies to pupil-teacher ratios:

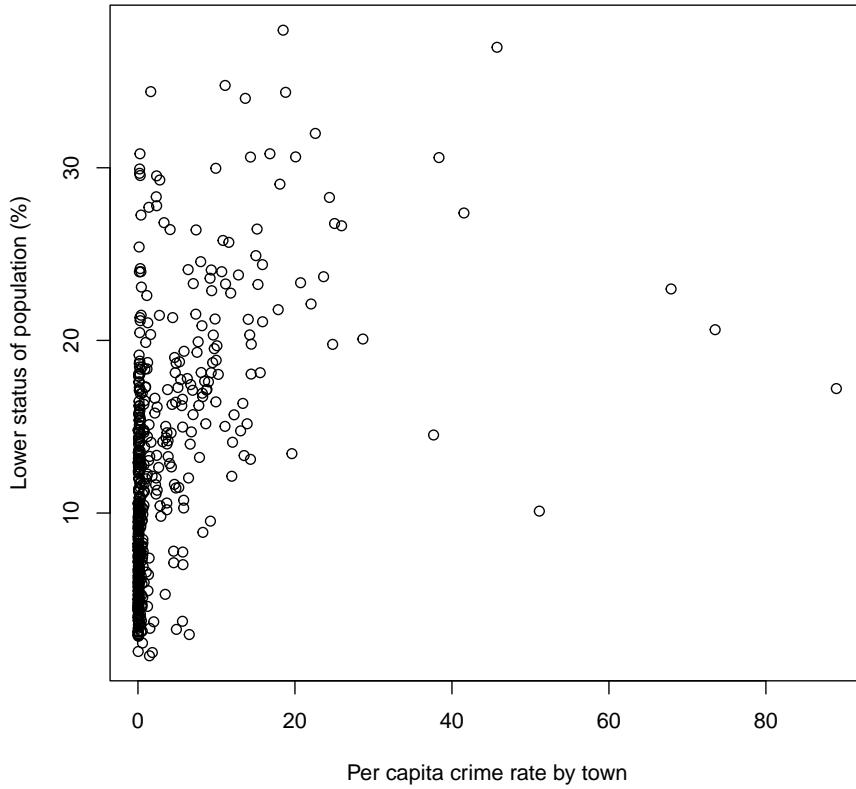


Figure 2.8: Scatterplot of crim vs lstat data

```
> summary(Boston$ptratio)
   Min. 1st Qu. Median     Mean 3rd Qu.    Max.
12.60   17.40  19.05  18.46  20.20  22.00
```

(e) We have the following:

```
> sum(Boston$chas)
[1] 35
```

(f) We have the following:

```
> median(Boston$ptratio)
[1] 19.05
```

(g) The tract corresponding to observation $i = 399$, for which we have

```
> median(Boston$ptratio)
[1] 19.05
```

For that census tract, we have the following data:

```
> Boston[399,]
  crim zn indus chas   nox    rm age    dis rad tax ptratio lstat medv
399 38.3518 0 18.1    0 0.693 5.453 100 1.4896 24 666    20.2 30.59     5
```

We can see that the **crim** variable is far higher than the median value of 3.61352, and is also in the higher range for the variables **tax** and **ptratio** for which the range was determined in (d). This would likely indicate a correlation between these variables.

- (h) We have the following code to find the number of census tracts which average more than 7 and 8 rooms per dwelling:

```
> sum(Boston$rm > 7)
[1] 64
> sum(Boston$rm > 8)
[1] 13
```

We have the following summary of all census tracts averaging more than 8 rooms:

```
> summary(Boston[Boston$rm > 8, ])
      crim          zn         indus        chas
Min. :0.02009  Min. : 0.00  Min. : 2.680  Min. :0.0000
1st Qu.:0.33147 1st Qu.: 0.00  1st Qu.: 3.970  1st Qu.:0.0000
Median :0.52014  Median : 0.00  Median : 6.200  Median :0.0000
Mean   :0.71879  Mean   :13.62  Mean   : 7.078  Mean   :0.1538
3rd Qu.:0.57834 3rd Qu.:20.00  3rd Qu.: 6.200  3rd Qu.:0.0000
Max.   :3.47428  Max.   :95.00   Max.   :19.580  Max.   :1.0000

      nox          rm         age         dis
Min. :0.4161  Min. :8.034  Min. : 8.40  Min. :1.801
1st Qu.:0.5040 1st Qu.:8.247  1st Qu.:70.40  1st Qu.:2.288
Median :0.5070  Median :8.297  Median :78.30  Median :2.894
Mean   :0.5392  Mean   :8.349  Mean   :71.54  Mean   :3.430
3rd Qu.:0.6050 3rd Qu.:8.398  3rd Qu.:86.50  3rd Qu.:3.652
Max.   :0.7180  Max.   :8.780  Max.   :93.90  Max.   :8.907

      rad          tax        ptratio       lstat        medv
Min. : 2.000  Min. :224.0  Min. :13.00  Min. :2.47  Min. :21.9
1st Qu.: 5.000 1st Qu.:264.0  1st Qu.:14.70  1st Qu.:3.32  1st Qu.:41.7
Median : 7.000  Median :307.0  Median :17.40  Median :4.14  Median :48.3
Mean   : 7.462  Mean   :325.1  Mean   :16.36  Mean   :4.31  Mean   :44.2
3rd Qu.: 8.000 3rd Qu.:307.0  3rd Qu.:17.40  3rd Qu.:5.12  3rd Qu.:50.0
Max.   :24.000  Max.   :666.0  Max.   :20.20  Max.   :7.44  Max.   :50.0
```

From the above summary, it becomes clear that census tracts with a large average number of rooms per dwelling have correspondingly low crime rates, relatively high tax rates, and relatively high pupil-teacher ratios.

3 Linear Regression

3.1

The four null hypotheses to which the p -values in Table 3.4 correspond are the following:

H_{0I} : There is no relationship between **sales** and β_0 .

H_{0T} : There is no relationship between **sales** and **TV**.

H_{0r} : There is no relationship between **sales** and **radio**.

H_{0n} : There is no relationship between **sales** and **newspaper**.

The p -values for H_{0I} , H_{0T} , and H_{0r} are all below 0.0001, so that we can reject the null hypothesis in these four cases, and we can conclude that there do exist relationships between **sales**, β_0 , **TV**, and **radio**. Since the p -value of a regression of **sales** on **newspaper** is relatively large, $p = 0.8599$, we can conclude that it is unlikely that **newspaper** alone will have an effect upon **sales**.

3.2

For the *K-nearest neighbours regression* (KNN regression), we are given a value for K and a prediction point x_0 and we first identify the K training observations that are closest to x_0 . Then we estimate $f(x_0)$ using the average of all training responses in this set (\mathcal{N}_0), using

$$\hat{f}(x_0) = \frac{1}{K} \sum_{x_i \in \mathcal{N}_0} y_i.$$

The *K-nearest neighbours* (KNN) classifier follows the same method, but instead of estimating $f(x_0)$ using an average, we estimate the conditional probability for class j as the fraction of points in \mathcal{N}_0 whose response values equal j :

$$\Pr(Y = j | X = x_0) = \frac{1}{K} \sum_{i \in \mathcal{N}_0} I(y_i = j).$$

Then we classify the test observation x_0 to the class with the largest such probability.

Thus KNN regression estimates a function for prediction, while the KNN classifier method classifies an observation.

3.3

- (a) Since $\hat{\beta}_1 = 20$, $\hat{\beta}_2 = 0.07$, and $\hat{\beta}_3 = 35$, we can conclude that IQ has little effect on starting salary by itself, while GPA and level have significant effects. The interaction term between GPA and level is given by $\hat{\beta}_5 = -10$, so that we can see a significant negative correlation between GPA and

level, while $\hat{\beta}_4 = 0.01$ shows that the interaction between IP and GPA does not have a significant effect. Thus answer iv. is correct, since level will be a strong predictor, but GPA interacting with level will bring down the response if GPA is not high enough.

- (b) For a college graduate with IQ of 110 and a GPA of 4.0, we have the following

$$\begin{aligned}\hat{y} &= \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3 + \hat{\beta}_4 x_4 + \hat{\beta}_5 x_5 \\ &= 50 + 20x_1 + 0.07x_2 + 35x_3 + 0.01x_4 - 10x_5 \\ &= 50 + 20(4.0) + 0.07(110) + 35(1) + 0.01(4.0 \cdot 110) - 10(4.0) \\ &\approx 137.1,\end{aligned}$$

so that the student is expected to earn a salary of \$137 100 per year.

- (c) False, we would need to perform statistical tests in order to evaluate that the interaction terms have little effect, since we would then be able to compare *p*-values and standard errors. Thus our answer in (a) is only a guess using relative sizes, but we cannot be entirely sure that this provides the full picture without the required data.

3.4

- (a) In both cases, the training RSS will be the same, since the least squares model uses the best linear approximation to the true relationship between X and Y .
- (b) The test RSS will be greater for the cubic regression, since it does not model the real relationship between X and Y as well as the linear regression.
- (c) The answer is the same as (a), since the training RSS will be minimised by the choice of any linear regression approach.
- (d) There is not enough information, since we do not know which model more closely matches the non-linear relationship between X and Y .

3.5

Proof. We have

$$\begin{aligned}
\hat{y}_i &= x_i \hat{\beta} \\
&= \frac{x_i \sum_{i'=1}^n x_{i'} y_{i'}}{\sum_{i'=1}^n x_{i'}^2} \\
&= \sum_{i'=1}^n \left(\frac{x_i x_{i'} y_{i'}}{\sum_{i'=1}^n x_{i'}^2} \right) \\
&= \sum_{i'=1}^n \left(\frac{x_i x_{i'}}{\sum_{i'=1}^n x_{i'}^2} y_{i'} \right) \\
&= \sum_{i'=1}^n a_{i'} y_{i'},
\end{aligned}$$

where

$$a_{i'} = \frac{x_i x_{i'}}{\sum_{i'=1}^n x_{i'}^2},$$

for fixed i and variable i' . \square

3.6

From (3.4), we have

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x},$$

so that we have

$$\begin{aligned}
\hat{y}_i(\bar{x}) &= \hat{\beta}_0 + \hat{\beta}_1 \bar{x} \\
&= \bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 \bar{x} \\
&= \bar{y},
\end{aligned}$$

and we conclude that the least squares line passes through the point (\bar{x}, \bar{y}) .

3.7

Proof. If we assume $\bar{x} = \bar{y} = 0$, then the simple linear regression of Y onto X is given by

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x.$$

Then we have

$$\begin{aligned}
R^2 &= \frac{\text{TSS} - \text{RSS}}{\text{TSS}} \\
&= \frac{\sum_{i=1}^n (y_i - \bar{y})^2 - \sum_{i=1}^n (y_i - \hat{y}_i)^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n [(y_i - \bar{y})^2 - (y_i - \hat{y}_i)^2]}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n [(y_i - \bar{y})^2 - (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2]}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n [(y_i - \bar{y})^2 - (y_i - \bar{y} + \hat{\beta}_1 \bar{x} - \hat{\beta}_1 x_i)^2]}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n [(y_i - \bar{y})^2 - (y_i - \bar{y} + \hat{\beta}_1 [\bar{x} - x_i])^2]}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n [-2\hat{\beta}_1(y_i - \bar{y})(\bar{x} - x_i) - \hat{\beta}_1^2(\bar{x} - x_i)^2]}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n \hat{\beta}_1(x_i - \bar{x})[2(y_i - \bar{y}) + \hat{\beta}_1(\bar{x} - x_i)]}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\hat{\beta}_1 \sum_{i=1}^n (x_i - \bar{x})[2(y_i - \bar{y}) + \hat{\beta}_1(\bar{x} - x_i)]}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \sum_{i=1}^n (x_i - \bar{x})[2(y_i - \bar{y}) + \hat{\beta}_1(\bar{x} - x_i)]}{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \sum_{i=1}^n (x_i - \bar{x})[2(y_i - \bar{y}) - \hat{\beta}_1(x_i - \bar{x})]}{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}.
\end{aligned}$$

Now,

$$\begin{aligned}
&2(y_i - \bar{y}) - \hat{\beta}_1(x_i - \bar{x})] \\
&= 2(y_i - \bar{y}) - (x_i - \bar{x}) \frac{\sum_{j=1}^n (x_j - \bar{x})(y_j - \bar{y})}{\sum_{j=1}^n (x_j - \bar{x})^2} \\
&= \frac{2(y_i - \bar{y}) \sum_{j=1}^n (x_j - \bar{x})^2 - (x_i - \bar{x}) \sum_{j=1}^n (x_j - \bar{x})(y_j - \bar{y})}{\sum_{j=1}^n (x_j - \bar{x})^2} \\
&= \frac{2(y_i - \bar{y}) \sum_{j=1}^n (x_j - \bar{x})^2 - (y_i - \bar{y}) \sum_{j=1}^n (x_j - \bar{x})^2}{\sum_{j=1}^n (x_j - \bar{x})^2} \\
&= \frac{(y_i - \bar{y}) \sum_{j=1}^n (x_j - \bar{x})^2}{\sum_{j=1}^n (x_j - \bar{x})^2} \\
&= (y_i - \bar{y}),
\end{aligned}$$

so that

$$\begin{aligned} [\text{Cor}(X, Y)]^2 &= \frac{[\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})]^2}{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2} \\ &= R^2. \end{aligned}$$

□

3.8

- (a) We have the following output:

```
> lm.fit <- lm(mpg ~ horsepower)
> summary(lm.fit)

Call:
lm(formula = mpg ~ horsepower)

Residuals:
    Min      1Q  Median      3Q     Max 
-13.5710 -3.2592 -0.3435  2.7630 16.9240 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 39.935861   0.717499   55.66   <2e-16 ***
horsepower -0.157845   0.006446  -24.49   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.906 on 390 degrees of freedom
Multiple R-squared:  0.6059,    Adjusted R-squared:  0.6049 
F-statistic: 599.7 on 1 and 390 DF,  p-value: < 2.2e-16
```

- i. Yes, it is safe to conclude that there is a relationship between the predictor and the response. We can see that from the p -values for `(Intercept)` and `horsepower`, which are very small, i.e., $p < 2.2 \times 10^{-16}$, we may reject the null hypothesis.
- ii. Since we have multiple R^2 and adjusted R^2 statistics of 0.6059 and 0.6049, respectively, we conclude that close to 2/3 of the variability in the response can be explained by the regression. From this, we conclude that the linear regression is a relatively good fit, and the relationship between the predictor and response is strong.
- iii. Since `mpg` decreases as `horsepower` increases along the regression line, we see that the relationship between the predictor and response is negative.
- iv. We have the following output:

```
> predict(lm.fit, data.frame(horsepower = 98), interval = "confidence")
      fit      lwr      upr
1 24.46708 23.97308 24.96108
```

from which we conclude that the associated 95% confidence interval is $[23.97308, 24.96108]$ for a predicted value of 24.46708. For the prediction interval, we have

```

> predict(lm.fit, data.frame(horsepower = 98), interval = "prediction")
      fit     lwr      upr
1 24.46708 14.8094 34.12476

```

so that the interval is $[14.8094, 34.12476]$ for the same predicted value.

- (b) In Figure 3.1, we have a plot of the response and predictor, as well as the regression line using the `abline()` function.

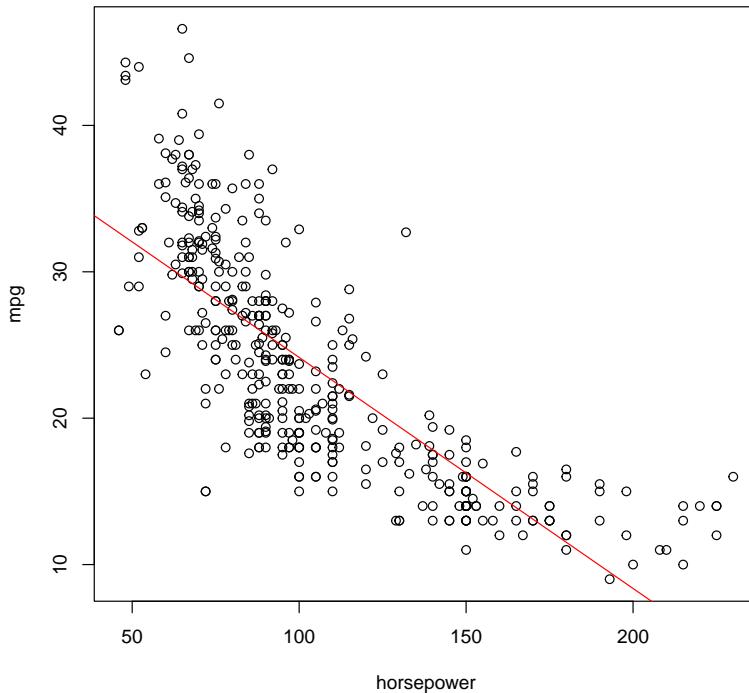


Figure 3.1: Response, predictor, and regression line for a linear regression of `mpg` on `horsepower`

- (c) In Figure 3.2, we have a plot of the default diagnostics for the regression of `mpg` on `horsepower`. We can see from these plots that there are 3 high leverage points corresponding to observations 323, 330, and 334. From the residual plots, we can also see the data follows a very clear pattern, bringing into question the conclusions of 3.8(a). From this we may conclude that there are some problems with applying a linear model to this data.

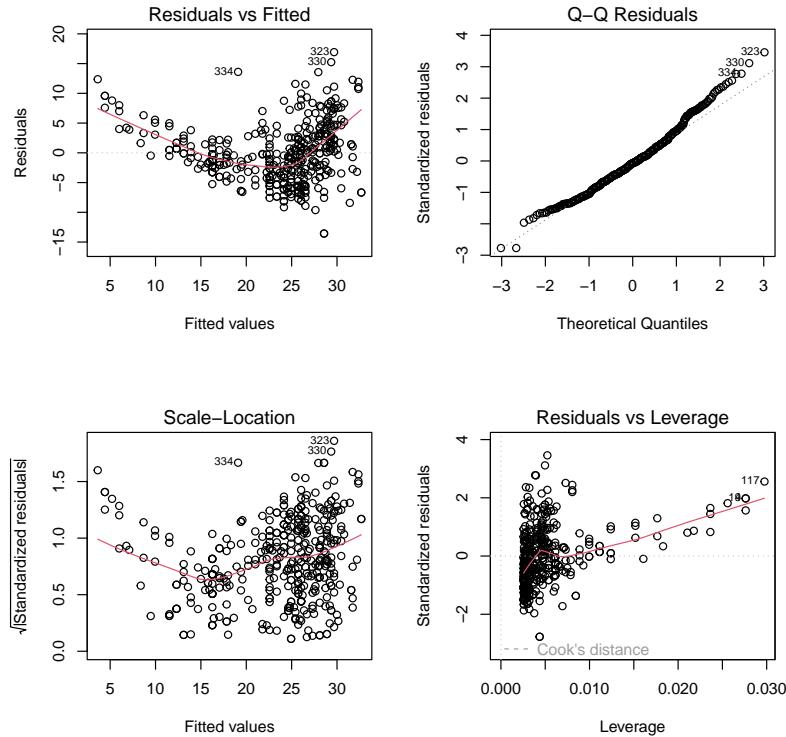


Figure 3.2: Diagnostic plots for a linear regression of `mpg` on `horsepower`

3.9

- (a) In Figure 3.3, we have a scatterplot matrix of all the `Auto` data.
- (b) We have the following output from using the `cor()` function:

```
> cor(Auto[,1:8])
      mpg cylinders displacement horsepower      weight
mpg       1.0000000 -0.7776175 -0.8051269 -0.7784268 -0.8322442
cylinders -0.7776175  1.0000000  0.9508233  0.8429834  0.8975273
displacement -0.8051269  0.9508233  1.0000000  0.8972570  0.9329944
horsepower  -0.7784268  0.8429834  0.8972570  1.0000000  0.8645377
weight     -0.8322442  0.8975273  0.9329944  0.8645377  1.0000000
acceleration 0.4233285 -0.5046834 -0.5438005 -0.6891955 -0.4168392
year        0.5805410 -0.3456474 -0.3698552 -0.4163615 -0.3091199
origin      0.5652088 -0.5689316 -0.6145351 -0.4551715 -0.5850054
               acceleration      year      origin
acceleration 0.4233285  0.5805410  0.5652088
cylinders    -0.5046834 -0.3456474 -0.5689316
displacement -0.5438005 -0.3698552 -0.6145351
horsepower   -0.6891955 -0.4163615 -0.4551715
weight       -0.4168392 -0.3091199 -0.5850054
acceleration  1.0000000  0.2903161  0.2127458
year         0.2903161  1.0000000  0.1815277
```

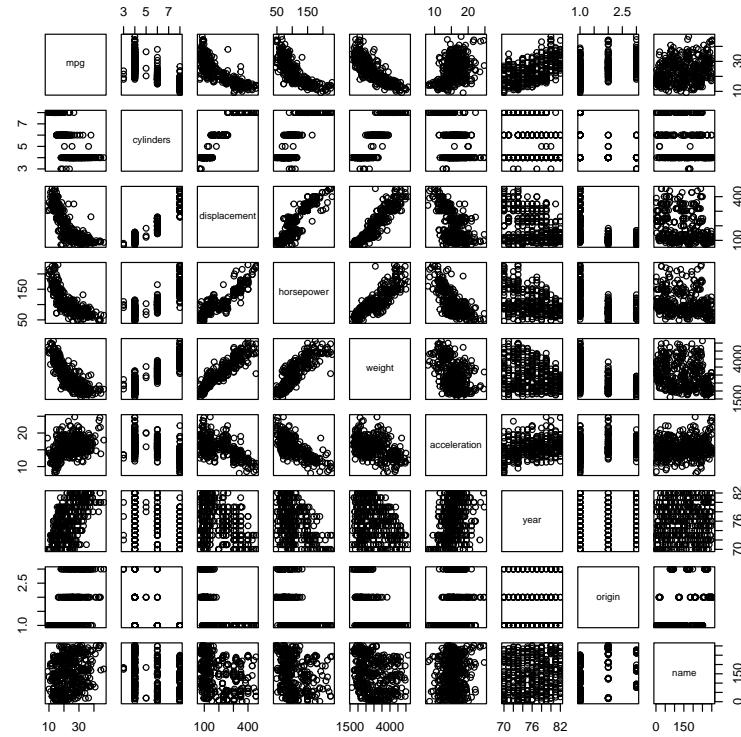


Figure 3.3: Scatterplot matrix of Auto data

```
origin          0.2127458  0.1815277  1.0000000
```

(c) We have the following output from using the `lm()` function:

```
> lm.fit <- lm(mpg ~ . - name, data = Auto)
> summary(lm.fit)

Call:
lm(formula = mpg ~ . - name, data = Auto)

Residuals:
    Min      1Q  Median      3Q     Max 
-9.5903 -2.1565 -0.1169  1.8690 13.0604 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -17.218435   4.644294 -3.707  0.00024 ***  
cylinders    -0.493376   0.323282 -1.526  0.12780    
displacement   0.019896   0.007515  2.647  0.00844 **  
horsepower    -0.016951   0.013787 -1.230  0.21963    
weight       -0.006474   0.000652 -9.929 < 2e-16 ***  
acceleration  0.080576   0.098845  0.815  0.41548    
year         0.750773   0.050973 14.729 < 2e-16 ***  

```

```

origin      1.426141   0.278136   5.127 4.67e-07 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 3.328 on 384 degrees of freedom
Multiple R-squared:  0.8215,    Adjusted R-squared:  0.8182
F-statistic: 252.4 on 7 and 384 DF,  p-value: < 2.2e-16

```

- (i) Yes, since we have several predictors with low p -values and/or low standard error relative to the estimate.
- (ii) The predictors with the most statistically significant relationship to the response appear to be `displacement`, `weight`, `year`, and `origin`.
- (iii) The coefficient of the `year` variable is given by 0.750773, suggesting that an increase of 1 year in the model's date of manufacture is associated with an increase of 0.750773 miles per gallon in fuel consumption by the vehicle.
- (d) In Figure 3.4, we have some diagnostic plots of the linear regression fit for the `Auto` data with `mpg` as response and all quantitative variables as the predictors. We can see that the residuals vs fitted plot shows a well-defined shape to the data that may suggest a non-linear relationship. We can see some high-leverage observations in the residuals vs leverage plot and the scale-location plot, with observations 323, 327, and 394 standing out most.
- (e) Yes, we can see the interaction between `horsepower` and `year` is statistically equivalent by analysing the following linear regression:

```

> summary(lm(mpg ~ horsepower * year))

Call:
lm(formula = mpg ~ horsepower * year)

Residuals:
    Min      1Q  Median      3Q     Max 
-12.3492 -2.4509 -0.4557  2.4056 14.4437 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -1.266e+02  1.212e+01 -10.449 <2e-16 ***
horsepower   1.046e+00  1.154e-01   9.063 <2e-16 ***
year         2.192e+00  1.613e-01  13.585 <2e-16 ***
horsepower:year -1.596e-02  1.562e-03 -10.217 <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 3.901 on 388 degrees of freedom
Multiple R-squared:  0.7522,    Adjusted R-squared:  0.7503 
F-statistic: 392.5 on 3 and 388 DF,  p-value: < 2.2e-16

```

All p -values are less than 2×10^{-16} and we have R^2 statistics both around 3/4 compared with the values around 2/3 for linear regression of `mpg` on `horsepower` and `age` without an interaction term, which can be seen in the following linear regression summary:

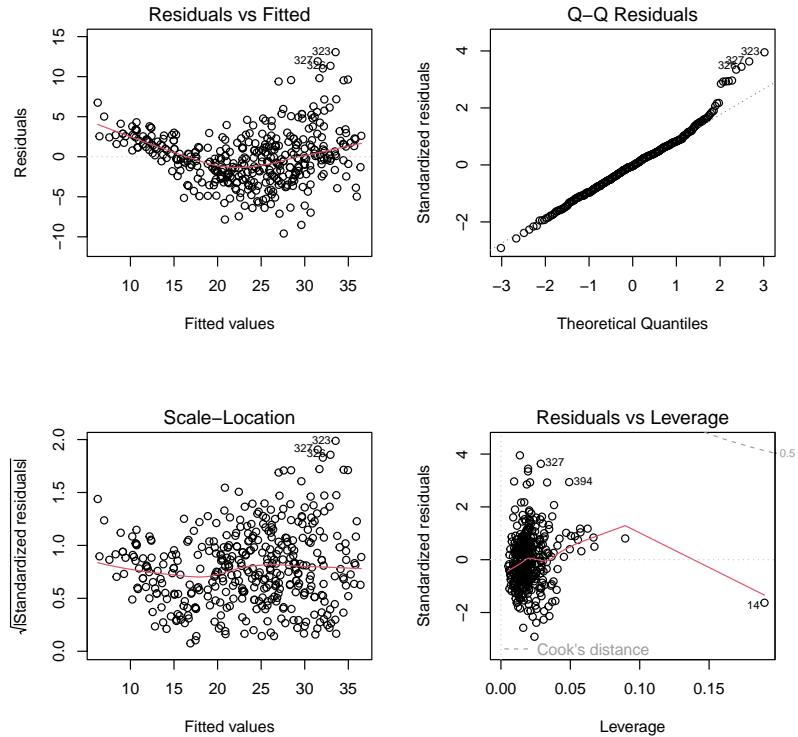


Figure 3.4: Diagnostic plots for regression of Auto data

```

> summary(lm(mpg ~ horsepower + year))

Call:
lm(formula = mpg ~ horsepower + year)

Residuals:
    Min      1Q  Median      3Q     Max 
-12.0768 -3.0783 -0.4308  2.5884 15.3153 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -12.739166   5.349027 -2.382   0.0177 *  
horsepower   -0.131654   0.006341 -20.761  <2e-16 *** 
year          0.657268   0.066262   9.919  <2e-16 *** 
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.388 on 389 degrees of freedom
Multiple R-squared:  0.6855,   Adjusted R-squared:  0.6839 
F-statistic: 423.9 on 2 and 389 DF,  p-value: < 2.2e-16

```

- (f) We can see from the following comparison of linear regression of `mpg` on `horsepower2`, $\sqrt{\text{horsepower}}$, and $\log(\text{horsepower})$ that the `horsepower2`

term provides the superior model, since it has the highest F -statistic (much greater than 1), indicating the lack of evidence for the null hypothesis.

```

> lm.fit <- lm(mpg ~ horsepower)
> lm.fit2 <- lm(mpg ~ horsepower + I(horsepower^2))
> anova(lm.fit, lm.fit2)
Analysis of Variance Table

Model 1: mpg ~ horsepower
Model 2: mpg ~ horsepower + I(horsepower^2)
  Res.Df   RSS Df Sum of Sq    F    Pr(>F)
1     390 9385.9
2     389 7442.0  1    1943.9 101.61 < 2.2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
> lm.fit2 <- lm(mpg ~ horsepower + I(sqrt(horsepower)))
> anova(lm.fit, lm.fit2)
Analysis of Variance Table

Model 1: mpg ~ horsepower
Model 2: mpg ~ horsepower + I(sqrt(horsepower))
  Res.Df   RSS Df Sum of Sq    F    Pr(>F)
1     390 9385.9
2     389 7502.2  1    1883.7 97.672 < 2.2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
> lm.fit2 <- lm(mpg ~ horsepower + I(log(horsepower)))
> anova(lm.fit, lm.fit2)
Analysis of Variance Table

Model 1: mpg ~ horsepower
Model 2: mpg ~ horsepower + I(log(horsepower))
  Res.Df   RSS Df Sum of Sq    F    Pr(>F)
1     390 9385.9
2     389 7581.2  1    1804.7 92.601 < 2.2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

```

3.10

- (a) We have the following output from a multiple regression of `Sales` on `Price`, `Urban`, and `US`:

```

> Carseats <- read.csv(
  "../datasets/Carseats.csv", header = T,
  na.strings = "?", stringsAsFactors = T
)
> lm.fit <- lm(Sales ~ Price + Urban + US, data = Carseats)
> summary(lm.fit)

Call:
lm(formula = Sales ~ Price + Urban + US, data = Carseats)

Residuals:
    Min      1Q  Median      3Q     Max 
-6.9206 -1.6220 -0.0564  1.5786  7.0581 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 13.043469  0.651012 20.036 < 2e-16 ***
Price       -0.054459  0.005242 -10.389 < 2e-16 ***
UrbanYes    -0.021916  0.271650 -0.081   0.936    
USYes       1.200573  0.259042  4.635 4.86e-06 *** 

```

```

---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.472 on 396 degrees of freedom
Multiple R-squared:  0.2393,    Adjusted R-squared:  0.2335
F-statistic: 41.52 on 3 and 396 DF,  p-value: < 2.2e-16

```

- (b) The **Price** coefficient indicates that a decrease of \$1 in price results in a sales increase of about 54 carseats. From the qualitative variables **Urban** and **US**, we can see that since **Urban** has a negative coefficient and **US** a positive coefficient, both corresponding to the dummy variable **Yes**, we can see that a store in an urban area is associated with a slight decrease in sales, while a store in the US is associated with an increase in sales.
- (c) We have the dummy variables

$$x_{i1} = \begin{cases} 1 & \text{if } i\text{th store is in an urban area} \\ 0 & \text{if } i\text{th store is not in an urban area} \end{cases}$$

and

$$x_{i2} = \begin{cases} 1 & \text{if } i\text{th store is in the US} \\ 0 & \text{if } i\text{th store is not in the US} \end{cases},$$

so that we have the regression model

$$\begin{aligned} y_i &= \beta_0 + \beta_1 x_i + \beta_2 x_{i1} + \beta_3 x_{i2} + \epsilon_i \\ &= \begin{cases} \beta_0 + \beta_1 x_i + \beta_2 + \beta_3 + \epsilon_i & \text{if } i\text{th store is urban in the US} \\ \beta_0 + \beta_1 x_i + \beta_2 + \epsilon_i & \text{if } i\text{th store is urban not in the US} \\ \beta_0 + \beta_1 x_i + \beta_3 + \epsilon_i & \text{if } i\text{th store is not urban in the US} \\ \beta_0 + \beta_1 x_i + \epsilon_i & \text{if } i\text{th store is neither urban nor US} \end{cases}. \end{aligned}$$

- (d) We can reject the null hypothesis for **Price** and **US**, since the associated *p*-values are less than 2^{-16} .
- (e) We have the following regression of **Sales** on **Price** and **US**:

```

> lm.fit <- lm(Sales ~ Price + US)
> summary(lm.fit)

Call:
lm(formula = Sales ~ Price + US)

Residuals:
    Min      1Q  Median      3Q     Max 
-6.9269 -1.6286 -0.0574  1.5766  7.0515 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 13.03079   0.63098 20.652 < 2e-16 ***
Price       -0.05448   0.00523 -10.416 < 2e-16 ***
USYes       1.19964   0.25846   4.641 4.71e-06 ***
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

```

Residual standard error: 2.469 on 397 degrees of freedom
Multiple R-squared:  0.2393,   Adjusted R-squared:  0.2354
F-statistic: 62.43 on 2 and 397 DF,  p-value: < 2.2e-16

```

- (f) The models fit the data reasonably well, since in both cases, we have a relatively high F -statistic and low p -values, but the model in (e) is improved by omitting `Urban` as a predictor.
- (g) We have the following:

```

> confint(lm.fit)
            2.5 %      97.5 %
(Intercept) 11.79032020 14.27126531
Price        -0.06475984 -0.04419543
USYes       0.69151957  1.70776632

```

This indicates that 95% confidence intervals for the intercept coefficient (β_0) are $[11.79032020, 14.27126531]$, $[-0.06475984, -0.04419543]$ for the `Price` (β_1) coefficient, and $[0.69151957, 1.70776632]$ for the `USYes` (β_2) coefficient, in the model

$$f(X) = \beta_0 + \beta_1 \text{Price} + \beta_2 \text{USYes}.$$

- (h) In Figure 3.5, we have a plot of some diagnostics for the model from (e). We can see some high-leverage observations in the residuals vs leverage plot and the scale-location plot, with observations 51, 69, and 377 standing out most.

3.11

- (a) We have the following values for the simple linear regression of `y` onto `x` without a coefficient:

```

> lm.fit <- lm(y ~ x + 0)
> summary(lm.fit)

Call:
lm(formula = y ~ x + 0)

Residuals:
    Min      1Q  Median      3Q     Max 
-1.9154 -0.6472 -0.1771  0.5056  2.3109 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
x          1.9939    0.1065  18.73   <2e-16 ***  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1 

Residual standard error: 0.9586 on 99 degrees of freedom
Multiple R-squared:  0.7798,   Adjusted R-squared:  0.7776 
F-statistic: 350.7 on 1 and 99 DF,  p-value: < 2.2e-16

```

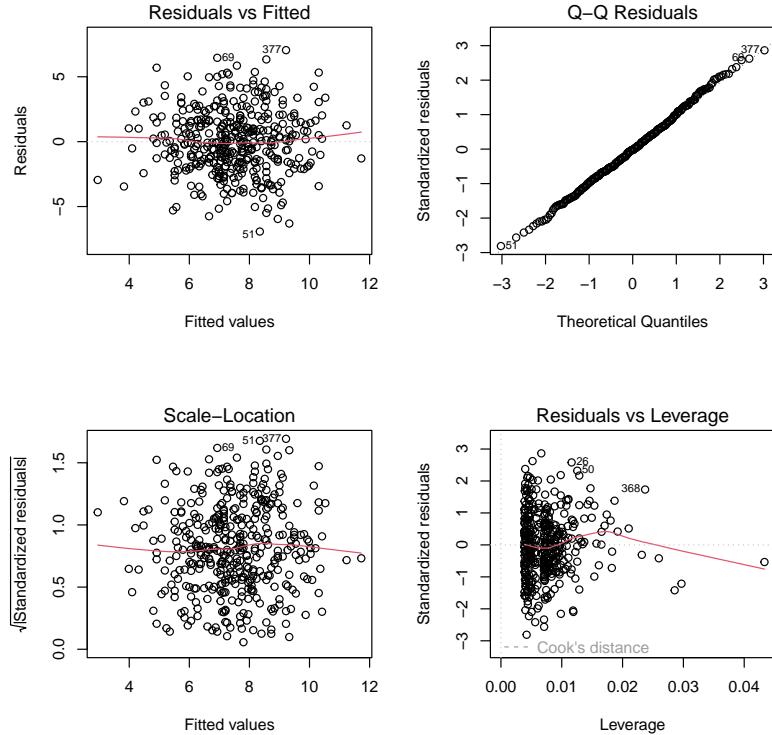


Figure 3.5: Diagnostic plots for regression of **Carseats** data

From the above, we can see that $\hat{\beta} = 1.9939$, the standard error of $\hat{\beta}$ is 0.1065, the t -statistic is 18.73, and the p -value associated with the null hypothesis $H_0 : \beta = 0$ is given by $p < 2.2 \times 10^{-16}$. It appears that a simple linear regression

$$\hat{Y} = \hat{\beta}X = 1.9939X$$

with an underlying linear relationship given by

$$Y = 2X + \varepsilon$$

produces a strong enough correlation that we may reject the null hypothesis, given the same criteria as for the previously-analysed data.

- (b) We have the following values for the simple linear regression of x onto y without a coefficient:

```
> lm.fit <- lm(x ~ y + 0)
> summary(lm.fit)
```

```

Call:
lm(formula = x ~ y + 0)

Residuals:
    Min      1Q  Median      3Q     Max 
-0.8699 -0.2368  0.1030  0.2858  0.8938 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
y          0.39111   0.02089 18.73   <2e-16 ***  
---      
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1 

Residual standard error: 0.4246 on 99 degrees of freedom
Multiple R-squared:  0.7798, Adjusted R-squared:  0.7776 
F-statistic: 350.7 on 1 and 99 DF, p-value: < 2.2e-16

```

From the above, we can see that $\hat{\beta} = 0.39111$, the standard error of $\hat{\beta}$ is 0.02089, the t -statistic is 18.73, and the p -value associated with the null hypothesis $H_0 : \beta = 0$ is given by $p < 2.2 \times 10^{-16}$. It appears that a simple linear regression

$$\hat{X} = \hat{\beta}Y = 0.39111Y$$

with an underlying linear relationship given by

$$X = 0.5Y + \varepsilon$$

produces a similar result, with the t -statistic and p values almost identical.

- (c) (a) is the inverse of the model in (b), ignoring the error term ε .
- (d) *Proof.* We have the t -statistic for $H_0 : \beta = 0$ given by

$$\begin{aligned}
\frac{\hat{\beta}}{\text{SE}(\hat{\beta})} &= \frac{\hat{\beta}}{\sqrt{\frac{\sum_{i=1}^n (y_i - x_i \hat{\beta})^2}{(n-1) \sum_{i'=1}^n x_{i'}^2}}} \\
&= \frac{\sum_{i=1}^n x_i y_i}{\sum_{i'=1}^n x_{i'}^2 \sqrt{\frac{\sum_{i=1}^n (y_i - x_i \hat{\beta})^2}{(n-1) \sum_{i'=1}^n x_{i'}^2}}} \\
&= \frac{\sum_{i=1}^n x_i y_i \sqrt{(n-1) \sum_{i'=1}^n x_{i'}^2}}{\sum_{i'=1}^n x_{i'}^2 \sqrt{\sum_{i=1}^n (y_i - x_i \hat{\beta})^2}} \\
&= \frac{\sum_{i=1}^n x_i y_i (\sqrt{n-1}) \sqrt{\sum_{i'=1}^n x_{i'}^2}}{\sum_{i'=1}^n x_{i'}^2 \sqrt{\sum_{i=1}^n (y_i - x_i \hat{\beta})^2}} \\
&= \frac{(\sqrt{n-1}) \sum_{i=1}^n x_i y_i}{\sqrt{\sum_{i'=1}^n x_{i'}^2} \sqrt{\sum_{i=1}^n (y_i - x_i \hat{\beta})^2}} \\
&= \frac{(\sqrt{n-1}) \sum_{i=1}^n x_i y_i}{\sqrt{(\sum_{i'=1}^n x_{i'}^2) [\sum_{i=1}^n (y_i - x_i \hat{\beta})^2]}}.
\end{aligned}$$

Now, we have

$$\begin{aligned} & \left(\sum_{i'=1}^n x_{i'}^2 \right) \left[\sum_{i=1}^n (y_i - x_i \hat{\beta})^2 \right] \\ &= \left(\sum_{i'=1}^n x_{i'}^2 \right) \left[\sum_{i=1}^n \left(y_i - x_i \left[\sum_{i'=1}^n x_{i'} y_{i'} \right] / \left[\sum_{i'=1}^n x_{i'}^2 \right] \right)^2 \right]. \end{aligned}$$

Then,

$$\begin{aligned} & \left(y_i - x_i \left[\sum_{i'=1}^n x_{i'} y_{i'} \right] / \left[\sum_{i'=1}^n x_{i'}^2 \right] \right)^2 \\ &= y_i^2 - 2y_i x_i \left[\sum_{i'=1}^n x_{i'} y_{i'} \right] / \left[\sum_{i'=1}^n x_{i'}^2 \right] + \left(x_i \left[\sum_{i'=1}^n x_{i'} y_{i'} \right] / \left[\sum_{i'=1}^n x_{i'}^2 \right] \right)^2 \\ &= y_i^2 - 2y_i x_i \left[\sum_{i'=1}^n x_{i'} y_{i'} \right] / \left[\sum_{i'=1}^n x_{i'}^2 \right] + x_i^2 \left[\sum_{i'=1}^n x_{i'} y_{i'} \right]^2 / \left[\sum_{i'=1}^n x_{i'}^2 \right]^2 \\ &= \left(y_i^2 \left[\sum_{i'=1}^n x_{i'}^2 \right] - 2y_i x_i \left[\sum_{i'=1}^n x_{i'}^2 \right] \left[\sum_{i'=1}^n x_{i'} y_{i'} \right] \right. \\ &\quad \left. + x_i^2 \left[\sum_{i'=1}^n x_{i'} y_{i'} \right]^2 \right) / \left[\sum_{i'=1}^n x_{i'}^2 \right]^2, \end{aligned}$$

so that

$$\begin{aligned} & \left(\sum_{i'=1}^n x_{i'}^2 \right) \left[\sum_{i=1}^n (y_i - x_i \hat{\beta})^2 \right] \\ &= \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{i'=1}^n y_{i'}^2 \right) - \left(\sum_{i'=1}^n x_{i'} y_{i'} \right)^2. \end{aligned}$$

Thus, we have

$$\frac{\hat{\beta}}{\text{SE}(\hat{\beta})} = \frac{(\sqrt{n-1}) \sum_{i=1}^n x_i y_i}{\sqrt{(\sum_{i=1}^n x_i^2)(\sum_{i'=1}^n y_{i'}^2) - (\sum_{i'=1}^n x_{i'} y_{i'})^2}}.$$

□

- (e) The t -statistic for the regression of \mathbf{y} onto \mathbf{x} is the same as the t -statistic for the regression of \mathbf{x} onto \mathbf{y} , since the expression in (d) is symmetric with respect to \mathbf{x} and \mathbf{y} .
- (f) We demonstrate using the following R code:

```

> set.seed(1)
> x <- rnorm(100)
> y <- 2 * x + rnorm(100)
> lm.fit <- lm(y ~ x)
> summary(lm.fit)

Call:
lm(formula = y ~ x)

Residuals:
    Min      1Q  Median      3Q     Max 
-1.8768 -0.6138 -0.1395  0.5394  2.3462 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -0.03769   0.09699 -0.389   0.698    
x            1.99894   0.10773 18.556  <2e-16 ***  
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.9628 on 98 degrees of freedom
Multiple R-squared:  0.7784,    Adjusted R-squared:  0.7762 
F-statistic: 344.3 on 1 and 98 DF,  p-value: < 2.2e-16

> lm.fit <- lm(x ~ y)
> summary(lm.fit)

Call:
lm(formula = x ~ y)

Residuals:
    Min      1Q  Median      3Q     Max 
-0.90848 -0.28101  0.06274  0.24570  0.85736 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.03880   0.04266   0.91   0.365    
y            0.38942   0.02099  18.56  <2e-16 ***  
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.4249 on 98 degrees of freedom
Multiple R-squared:  0.7784,    Adjusted R-squared:  0.7762 
F-statistic: 344.3 on 1 and 98 DF,  p-value: < 2.2e-16

```

We can see that the t -statistic given by 18.556 for the regression of y onto x is approximately the same as the t -statistic given by 18.56 of x onto y .