Learn Physics with Functional Programming - Scott N. Walck - Chapter 4: Exercises

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4.2 For $f(x) = x^3$, we have $f'(x) = 3x^2$, so that the relative error is defined by

$$
\begin{aligned}
\operatorname{err}(x,a) &= \left| \frac{\frac{f(x+a/2)-f(x-a/2)}{a} - f'(x)}{f'(x)} \right| \\
&= \left| \frac{\frac{(x+a/2)^3-(x-a/2)^3}{a} - 3x^2}{3x^2} \right| \\
&= \left| \frac{\frac{\begin{array}{c}[x^3 + (3x^2a)/2 + (3xa^2)/4 + a^3/8] \\ - [x^3 - (3x^2a)/2 + (3xa^2)/4 - a^3/8]\end{array}}{a} - 3x^2}{3x^2} \right| \\
&= \left| \frac{\frac{3x^2a+a^3/4-3x^2a}{a}}{3x^2} \right| \\
&= \left| \frac{\frac{a^2}{4}}{3x^2} \right| \\
&= \left| \frac{a^2}{12x^2} \right| \\
&= \frac{a^2}{12x^2},
\end{aligned}
$$

since $a^2 \geq 0$ and $x^2 \geq 0$.

Thus we have an error of 1 percent if

$$
\begin{aligned}
\operatorname{err}(x,a) &= 0.01 \\
\Leftrightarrow \frac{a^2}{12x^2} &= 0.01 \\
\Leftrightarrow a^2 &= 0.12x^2 \\
\Leftrightarrow a &= |x|\sqrt{0.12}
\end{aligned}
$$

Then, for $x = 4$, we have

$$\begin{aligned} a &= 4\sqrt{0.12} \\ &\approx 1.3856406460551018 \end{aligned}$$

and for $x = 0.1$, we have

$$\begin{aligned} a &= 0.1\sqrt{0.12} \\ &\approx 3.4641016151377546 \times 10^{-2}. \end{aligned}$$

4.3 Suppose we have a function f and independent variable, say x, such that `derivative 0.01 f x` produces at least a 10 percent error, $\mathrm{err}(x, \epsilon)$, compared to the exact derivative, $f'(x)$. Then, we have

$$\begin{aligned} \mathrm{err}(x, \epsilon) &= \mathrm{err}(x, 0.01) \\ &= \left| \frac{\frac{f(x+\epsilon/2)-f(x-\epsilon/2)}{\epsilon} - f'(x)}{f'(x)} \right| \\ &= \left| \frac{\frac{f(x+0.01/2)-f(x-0.01/2)}{0.01} - f'(x)}{f'(x)} \right| \\ &= \left| \frac{\frac{f(x+0.005)-f(x-0.005)}{0.01} - f'(x)}{f'(x)} \right| \\ &\geq 0.1. \end{aligned}$$

If we substitute $\epsilon = 0.01$, $x = \pi \approx 3.141592653589793$, and $f(x) = \cos x$ into the above, we have $f'(x) = -\sin x$, so that

$$\begin{aligned} \mathrm{err}(x, \epsilon) &= \mathrm{err}(\pi, 0.01) \\ &= \left| \frac{\frac{\cos(\pi+0.005)-\cos(\pi-0.005)}{0.01} + \sin\pi}{-\sin\pi} \right| \\ &\approx 1.0 \\ &\geq 0.1. \end{aligned}$$