Learn Physics with Functional Programming - Scott N. Walck - Chapter 4: Exercises

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4.2 For $f(x) = x^3$, we have $f'(x) = 3x^2$, so that the relative error is defined by

$$\begin{aligned}
\operatorname{err}(a) &= \left| \frac{f(x + a/2) - f(x - a/2)}{a} - 3x^2 \right| \\
&= \left| \frac{(x + a/2)^3 - (x - a/2)^3}{a} - 3x^2 \right| \\
&= \left| \frac{\begin{array}{l}[x^3 + (3x^2a)/2 + (3xa^2)/4 + a^3/8] \\ \quad - [x^3 - (3x^2a)/2 + (3xa^2)/4 - a^3/8]\end{array}}{a} - 3x^2 \right| \\
&= \left| \frac{3x^2a + a^3/4 - 3x^2a}{a} \right| \\
&= \left| \frac{a^2}{4} \right| .
\end{aligned}$$

Thus we have an error of 1 percent if

$$\begin{aligned}
\operatorname{err}(a) &= 0.01 \\
\Leftrightarrow \left| \frac{a^2}{4} \right| &= 0.01 \\
\Leftrightarrow \frac{a^2}{4} &= \pm 0.01 \\
\Leftrightarrow a^2 &= \pm 0.04 \\
\Leftrightarrow a &= \sqrt{0.04} \\
\Leftrightarrow a &= 0.2,
\end{aligned}$$

since $a \geq 0$, which is valid for $x = 4$ and $x = 0.1$, since $\operatorname{err}(a)$ does not depend on x.

4.3 Suppose we have a funtion f and independent variable, say x, such that `derivative 0.01 f x` produces at least a 10 percent error, $\mathrm{err}(\epsilon)$, compared to the exact derivative, $f'(x)$. Then, we have

$$\begin{aligned}
\mathrm{err}(\epsilon) = \left| \frac{f(x+\epsilon/2) - f(x-\epsilon/2)}{\epsilon} - f'(x) \right| &\leq 0.1 \\
\Leftrightarrow \left| \frac{f(x+0.01/2) - f(x-0.01/2)}{0.01} - f'(x) \right| &\leq 0.1 \\
\Leftrightarrow -0.1 \leq \frac{f(x+0.02) - f(x-0.02)}{0.01} - f'(x) &\leq 0.1 \\
\Leftrightarrow -0.001 \leq f(x+0.02) - f(x-0.02) - 0.01 f'(x) &\leq 0.001.
\end{aligned}$$

Now, if we let $f(x) = x^2$ and $x = 0$, we have $f'(x) = 2x$ and thus

$$\begin{aligned}
f(x+0.02) - f(x-0.02) - 0.01 f'(x) &= (0.02)^2 - (-0.02)^2 - 0.01 \cdot 2 \cdot 0 \\
&= 0.004 - 0.004 \\
&= 0.0,
\end{aligned}$$

so that we have

$$\Leftrightarrow -0.001 \leq f(x+0.02) - f(x-0.02) - 0.01 f'(x) \leq 0.001$$

and $f(x) = x^2$ for $x = 0$ is such that `derivative 0.01 f x` produces at least a 10 percent error.