Learn Physics with Functional Programming - Scott N. Walck - Chapter 4: Exercises

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4.2 For $f(x) = x^3$, we have $f'(x) = 3x^2$, so that the relative error is defined by

$$
\begin{aligned}
\text{err}(a) &= \left| \frac{f(x + a/2) - f(x - a/2)}{a} - 3x^2 \right| \\
&= \left| \frac{(x + a/2)^3 - (x - a/2)^3}{a} - 3x^2 \right| \\
&= \left| \frac{\begin{aligned}[x^3 + (3x^2 a)/2 + (3xa^2)/4 + a^3/8] \\ - [x^3 - (3x^2 a)/2 + (3xa^2)/4 - a^3/8]\end{aligned}}{a} - 3x^2 \right| \\
&= \left| \frac{3x^2 a + a^3/4 - 3x^2 a}{a} \right| \\
&= \left| \frac{a^3}{4} \right|.
\end{aligned}
$$

Thus we have an error of 1 percent if

$$
\begin{aligned}
\text{err}(a) &= 0.01 \\
\Leftrightarrow \left| \frac{a^3}{4} \right| &= 0.01 \\
\Leftrightarrow \frac{a^3}{4} &= \pm 0.01 \\
\Leftrightarrow a^3 &= \pm 0.04 \\
\Leftrightarrow a &= \pm 0.04^{(1/3)},
\end{aligned}
$$

which is valid for $x = 4$ and $x = 0.1$, since $\text{err}(a)$ does not depend on x.

4.3 Suppose we have a funtion f and independent variable, say x, such that `derivative 0.01 f x` produces at least a 10 percent error, $\text{err}(\epsilon)$, com-

pared to the exact derivative, $f'(x)$. Then, we have

$$\text{err}(\epsilon) = \left| \frac{f(x + \epsilon/2) - f(x - \epsilon/2)}{\epsilon} - f'(x) \right| \leq 0.1$$

$$\Leftrightarrow \left| \frac{f(x + 0.01/2) - f(x - 0.01/2)}{0.01} - f'(x) \right| \leq 0.1$$

$$\Leftrightarrow -0.1 \leq \frac{f(x + 0.02) - f(x - 0.02)}{0.01} - f'(x) \leq 0.1$$

$$\Leftrightarrow -0.001 \leq f(x + 0.02) - f(x - 0.02) - 0.01 f'(x) \leq 0.001.$$

Now, if we let $f(x) = x^2$ and $x = 0$, we have $f'(x) = 2x$ and thus

$$f(x + 0.02) - f(x - 0.02) - 0.01 f'(x) = (0.02)^2 - (-0.02)^2 - 0.01 \cdot 2 \cdot 0$$
$$= 0.004 - 0.004$$
$$= 0.0,$$

so that we have

$$\Leftrightarrow -0.001 \leq f(x + 0.02) - f(x - 0.02) - 0.01 f'(x) \leq 0.001$$

and $f(x) = x^2$ for $x = 0$ is such that `derivative 0.01 f x` produces at least a 10 percent error.