

Learn Physics with Functional Programming - Scott N. Walck - Chapter 4: Exercises

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4.2 For $f(x) = x^3$, we have $f'(x) = 3x^2$, so that the relative error is defined by

$$\begin{aligned}
 \text{err}(x, a) &= \left| \frac{\frac{f(x+a/2)-f(x-a/2)}{a} - f'(x)}{f'(x)} \right| \\
 &= \left| \frac{\frac{(x+a/2)^3-(x-a/2)^3}{a} - 3x^2}{3x^2} \right| \\
 &= \left| \frac{\frac{[x^3 + (3x^2a)/2 + (3xa^2)/4 + a^3/8] - [x^3 - (3x^2a)/2 + (3xa^2)/4 - a^3/8]}{a} - 3x^2}{3x^2} \right| \\
 &= \left| \frac{\frac{3x^2a + a^3/4 - 3x^2a}{a}}{3x^2} \right| \\
 &= \left| \frac{\frac{a^2}{4}}{3x^2} \right| \\
 &= \left| \frac{a^2}{12x^2} \right| \\
 &= \frac{a^2}{12x^2},
 \end{aligned}$$

since $a^2 \geq 0$ and $x^2 \geq 0$.

Thus we have an error of 1 percent if

$$\begin{aligned}
 \text{err}(x, a) &= 0.01 \\
 \Leftrightarrow \frac{a^2}{12x^2} &= 0.01 \\
 \Leftrightarrow a^2 &= 0.12x^2 \\
 \Leftrightarrow a &= |x|\sqrt{0.12}
 \end{aligned}$$

Then, for $x = 4$, we have

$$\begin{aligned} a &= 4\sqrt{0.12} \\ &\approx 1.3856406460551018 \end{aligned}$$

and for $x = 0.1$, we have

$$\begin{aligned} a &= 0.1\sqrt{0.12} \\ &\approx 3.4641016151377546 \times 10^{-2}. \end{aligned}$$

4.3 Suppose we have a function f and independent variable, say x , such that **derivative** 0.01 f x produces at least a 10 percent error, $\text{err}(x, \epsilon)$, compared to the exact derivative, $f'(x)$. Then, we have

$$\begin{aligned} &\text{err}(x, \epsilon) \\ &= \left| \frac{\frac{f(x+\epsilon/2)-f(x-\epsilon/2)}{\epsilon} - f'(x)}{f'(x)} \right| \\ &= \left| \frac{\frac{f(x+0.01/2)-f(x-0.01/2)}{0.01} - f'(x)}{f'(x)} \right| \\ &= \left| \frac{\frac{f(x+0.005)-f(x-0.005)}{0.01} - f'(x)}{f'(x)} \right| \\ &\geq 0.1, \end{aligned}$$

which holds if and only if

$$|f(x + 0.005) - f(x - 0.005) - 0.01f'(x)| \geq 0.0001|f'(x)|.$$