Learn Physics with Functional Programming - Scott N. Walck - Chapter 4: Exercises

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4.2 For $f(x) = x^3$, we have $f'(x) = 3x^2$, so that the relative error is defined by

$$\begin{aligned}
\mathrm{err}(a) &= \left| \frac{f(x + a/2) - f(x - a/2)}{a} - 3x^2 \right| \\
&= \left| \frac{(x + a/2)^3 - (x - a/2)^3}{a} - 3x^2 \right| \\
&= \left| \frac{\begin{array}{l}[x^3 + (3x^2 a)/2 + (3xa^2)/4 + a^3/8] \\ \quad - [x^3 - (3x^2 a)/2 + (3xa^2)/4 - a^3/8]\end{array}}{a} - 3x^2 \right| \\
&= \left| \frac{3x^2 a + a^3/4 - 3x^2 a}{a} \right| \\
&= \left| \frac{a^2}{4} \right|.
\end{aligned}$$

Thus we have an error of 1 percent if

$$\begin{aligned}
\mathrm{err}(a) &= 0.01 \\
\Leftrightarrow \left| \frac{a^2}{4} \right| &= 0.01 \\
\Leftrightarrow \frac{a^2}{4} &= \pm 0.01 \\
\Leftrightarrow a^2 &= \pm 0.04 \\
\Leftrightarrow a &= \sqrt{0.04} \\
\Leftrightarrow a &= 0.2,
\end{aligned}$$

since $a \geq 0$, which is valid for $x = 4$ and $x = 0.1$, since $\mathrm{err}(a)$ does not depend on x.

4.3 Suppose we have a funtion f and independent variable, say x, such that `derivative 0.01 f x` produces at least a 10 percent error, $\mathrm{err}(\epsilon)$, compared to the exact derivative, $f'(x)$. Then, we have

$$\begin{aligned}\mathrm{err}(\epsilon) &= \left|\frac{f(x+\epsilon/2)-f(x-\epsilon/2)}{\epsilon} - f'(x)\right| \leq 0.1\\ \Leftrightarrow &\left|\frac{f(x+0.01/2)-f(x-0.01/2)}{0.01} - f'(x)\right| \leq 0.1,\end{aligned}$$

and we have

$$\begin{aligned}&-\left[\frac{f(x+0.02)-f(x-0.02)}{0.01} - f'(x)\right]\\ &\leq 0.1\\ &\leq \frac{f(x+0.02)-f(x-0.02)}{0.01} - f'(x)],\end{aligned}$$

so that

$$\begin{aligned}&0.01f'(x) - f(x+0.02) + f(x-0.02)\\ &\leq 0.001\\ &\leq f(x+0.02) - f(x-0.02) - 0.01f'(x).\end{aligned}$$

Now, if we let $f(x) = \sin x$ and $x = 0$, we have $f'(x) = \cos x$ and thus

$$\begin{aligned}f(x+0.02) - f(x-0.02) - 0.01f'(x) &= \sin 0.02 - \sin(-0.02) - 0.01\cos 0\\ &\approx 2.999733338666616 \times 10^{-2}\end{aligned}$$

so that we have

$$|\sin 0.02 - \sin(-0.02) - 0.01\cos 0| \approx 2.999733338666616 \times 10^{-2} \leq 0.001$$

and $f(x) = \sin x$ for $x = 0$ is such that `derivative 0.01 f x` produces at least a 10 percent error.