

Nonlinear Dynamics and Chaos with Applications  
to Physics, Biology, Chemistry, and Engineering  
2nd Edition

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## 2 Flows on the Line

### 2.1 A Geometric Way of Thinking

#### 2.1.1

For a fixed point of the flow  $\dot{x} = \sin x$  on the line, we have

$$\dot{x} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \quad n \in \mathbb{Z}.$$

Thus all fixed points are given by  $x = n\pi, n \in \mathbb{Z}$ .

#### 2.1.2

The points  $x$  for which the flow has the greatest velocity to the right are those for which  $\dot{x} > 0$  and  $x$  is a local maximum. These are given by

$$x = \frac{(4n+1)\pi}{2}, \quad n \in \mathbb{Z}.$$

#### 2.1.3

(a) We have

$$\dot{x} = \sin x,$$

so that

$$\begin{aligned}\ddot{x} &= \frac{d}{dt}(\sin x) \\ &= \dot{x} \cos x \\ &= \sin x \cos x \\ &= \frac{1}{2} \sin(2x).\end{aligned}$$

- (b) The maximum positive acceleration is given by the local maxima of  $\ddot{x} = (1/2) \sin(2x)$ , i.e., where

$$2x = \frac{(4n+1)\pi}{2} \Rightarrow x = \frac{(4n+1)\pi}{4}, \quad n \in \mathbb{Z}.$$

#### 2.1.4

- (a) We begin by evaluating

$$t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$$

with  $x_0 = \pi/4$  to find

$$\begin{aligned}t &= \ln \left| \frac{\sqrt{2} + 1}{\csc x + \cot x} \right| \\ &= \ln \left( \frac{1 + \sqrt{2}}{\csc x + \cot x} \right).\end{aligned}$$

Then we apply the function  $e^z$  to both sides of to find

$$\begin{aligned}e^t &= \frac{1 + \sqrt{2}}{\csc x + \cot x} \\ &= \frac{1 + \sqrt{2}}{\cot(x/2)} \\ \Leftrightarrow \cot(x/2) &= \frac{1 + \sqrt{2}}{e^t} \\ \Leftrightarrow \frac{1}{\tan(x/2)} &= \frac{1 + \sqrt{2}}{e^t} \\ \Leftrightarrow \tan(x/2) &= \frac{e^t}{1 + \sqrt{2}} \\ \Leftrightarrow \frac{x}{2} &= \tan^{-1} \left( \frac{e^t}{1 + \sqrt{2}} \right) \\ \Leftrightarrow x &= 2 \tan^{-1} \left( \frac{e^t}{1 + \sqrt{2}} \right).\end{aligned}$$

Now, we have

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \left[ 2 \tan^{-1} \left( \frac{e^t}{1 + \sqrt{2}} \right) \right] = \pi,$$

since  $\lim_{t \rightarrow \infty} e^t = \infty$  and  $\lim_{t \rightarrow \infty} \tan^{-1} t = \pi$ .

- (b) An analytic solution is given by performing the steps in (a) without assuming  $x_0 = \pi/4$ , so that we have

$$\begin{aligned} x(t) &= 2 \tan^{-1} \left( \frac{e^t}{\csc x_0 + \cot x_0} \right) \\ &= 2 \tan^{-1} \left( \frac{e^t}{\cot(x_0/2)} \right). \end{aligned}$$

### 2.1.5

- (a) Since the time derivative oscillates with period  $2\pi$ , a mechanical system which is approximately governed by  $\dot{x} = \sin x$  is a pendulum, with angle of relative to the horizontal axis given by  $x$  and  $x = \pi$  the lowest point in the pendulum's trajectory, i.e., an upside-down Cartesian coordinate system.
- (b) Since it will always fall back towards the vertical axis when it is suspended perpendicular to the horizontal axis,  $x^* = 0$  is an unstable fixed point, while  $x^* = \pi$  corresponds to the lowest point in the trajectory, so that the pendulum will not move if it starts at this point.

## 2.2 Fixed Points and Stability

### 2.2.1

In Figure 2.1, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = 4x^2 - 16$$

An explicit solution for  $\dot{x} = 4x^2 - 16$  can be found by re-writing the equation as

$$\frac{dx}{dt} = 4x^2 - 16,$$

so that we have separable equations. Then, we may write

$$\frac{dx}{4x^2 - 16} = dt,$$

so that by using partial fraction decomposition, we have

$$\left[ \frac{\frac{1}{16}}{x-2} - \frac{\frac{1}{16}}{x+2} \right] dx = dt,$$

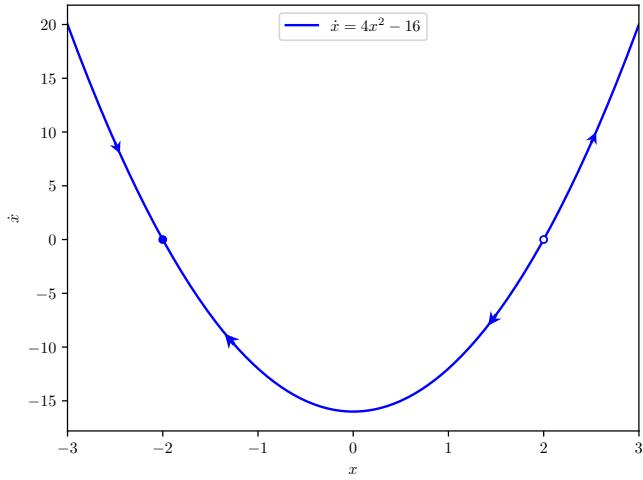


Figure 2.1: Vector field and fixed points for  $\dot{x} = 4x^2 - 16$

and integrate either sides with respect to the  $x$  and  $t$ , respectively, so that we have

$$\begin{aligned} \int \left[ \frac{\frac{1}{16}}{x-2} - \frac{\frac{1}{16}}{x+2} \right] dx &= \int dt \\ \Rightarrow \frac{1}{16} \ln|x-2| - \frac{1}{16} \ln|x+2| &= t + c_1 \\ \Rightarrow \ln \left| \frac{x-2}{x+2} \right| &= 16t + c_2 \\ \Rightarrow \frac{x-2}{x+2} &= \pm e^{16t+c_2}. \end{aligned}$$

After replacing  $\pm e^{c_2}$  by  $c$  and solving the last equation for  $x$ , we have the one-parameter family of solutions

$$x = 2 \frac{1 + ce^{16t}}{1 - ce^{16t}}.$$

If we let  $x(0) = 0$ , we have

$$\begin{aligned} 2 \frac{1 + c}{1 - c} &= 0 \\ \Rightarrow 1 + c &= 0 \\ \Rightarrow c &= -1, \end{aligned}$$

so that

$$x_1(t) = 2 \frac{1 - e^{16t}}{1 + e^{16t}}$$

is a solution corresponding to the initial condition  $x(0) = 0$ .

Likewise, if we let  $x(0) = 4$ , we have

$$\begin{aligned} 2\frac{1+c}{1-c} &= 4 \\ \Rightarrow 1+c &= 2(1-c) \\ \Rightarrow c &= \frac{1}{3}, \end{aligned}$$

so that

$$x_2(t) = 2\frac{1 + \frac{1}{3}e^{16t}}{1 - \frac{1}{3}e^{16t}}$$

is a solution corresponding to the initial condition  $x(0) = 4$ .

Finally, if we let  $x(0) = -4$ , we have

$$\begin{aligned} 2\frac{1+c}{1-c} &= -4 \\ \Rightarrow 1+c &= -2(1-c) \\ \Rightarrow c &= 3, \end{aligned}$$

so that

$$x_3(t) = 2\frac{1 + 3e^{16t}}{1 - 3e^{16t}}$$

is a solution corresponding to the initial condition  $x(0) = 4$ .

In Figure 2.2, we have a plot of these solutions for the given initial conditions.

## 2.2.2

In Figure 2.3, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = 1 - x^{14}$$

Again, we separate the equation to find

$$\frac{dx}{1 - x^{14}} = dt.$$

Integrating either side with respect to  $x$  and  $t$ , respectively, we have

$$\begin{aligned} \int \frac{dx}{1 - x^{14}} &= \int dt \\ \Rightarrow \ln|1 - x^{14}| &= t + c_1 \\ \Rightarrow 1 - x^{14} &= \pm e^{t+c_1} \\ \Rightarrow x^{14} &= \pm e^{t+c_1} + 1 \\ \Rightarrow x^{14} &= ce^t + 1 \\ \Rightarrow x &= \pm \sqrt[14]{ce^t + 1}, \end{aligned}$$

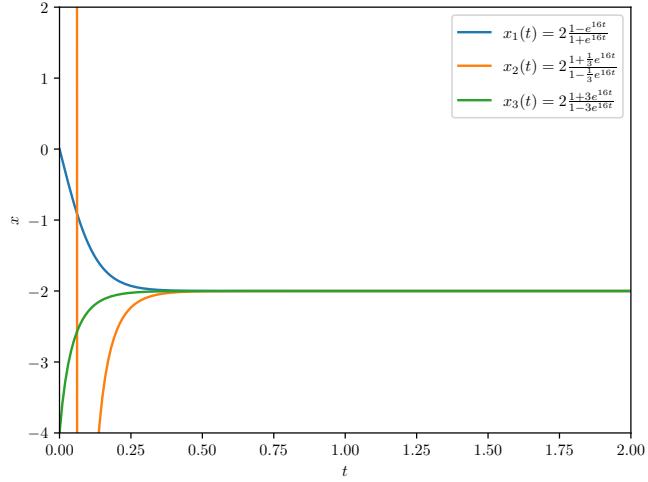


Figure 2.2: Graph of solutions of  $\dot{x} = 4x^2 - 16$

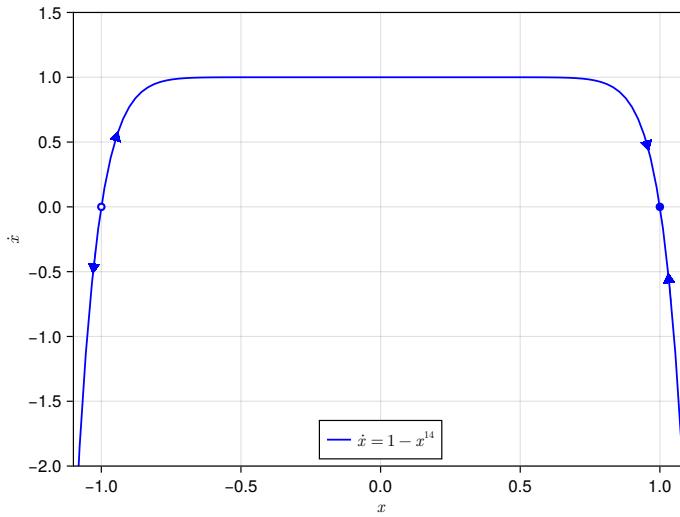


Figure 2.3: Vector field and fixed points for  $\dot{x} = 1 - x^{14}$

for  $ce^t + 1 \geq 0$ .

It does not seem possible to find a simple closed-form solution to this ODE.

### 2.2.3

In Figure 2.4, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = x - x^3$$

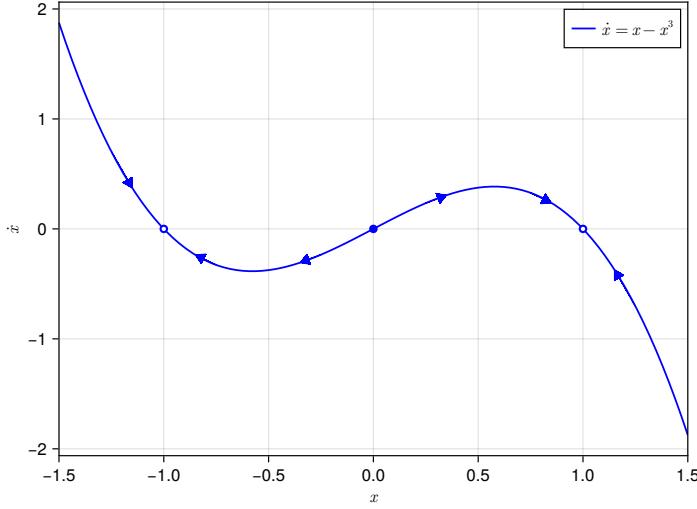


Figure 2.4: Vector field and fixed points for  $\dot{x} = x - x^3$

We use the method of separable equations again to find

$$\frac{dx}{x - x^3} = dt,$$

and using partial fraction decomposition, we have

$$\left[ \frac{1}{x} + \frac{\frac{1}{2}}{1-x} - \frac{\frac{1}{2}}{1+x} \right] dx = dt.$$

Integrating both sides yields

$$\begin{aligned} \int \frac{dx}{x} + \frac{1}{2} \int \frac{dx}{1-x} - \frac{1}{2} \int \frac{dx}{1+x} &= \int dt \\ \Leftrightarrow \ln|x| + \frac{1}{2} \ln|1-x| - \frac{1}{2} \ln|1+x| &= t + c_1 \\ \Leftrightarrow \ln \left| \frac{x^2(1-x)}{1+x} \right| &= 2t + c_2 \\ \Leftrightarrow \frac{x^2(1-x)}{1+x} &= \pm e^{2t+c_2} \\ &= ce^{2t}. \end{aligned}$$

Again, it does not seem possible to find a simple closed-form solution to this ODE.

#### 2.2.4

In Figure 2.5, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = e^{-x} \sin x$$

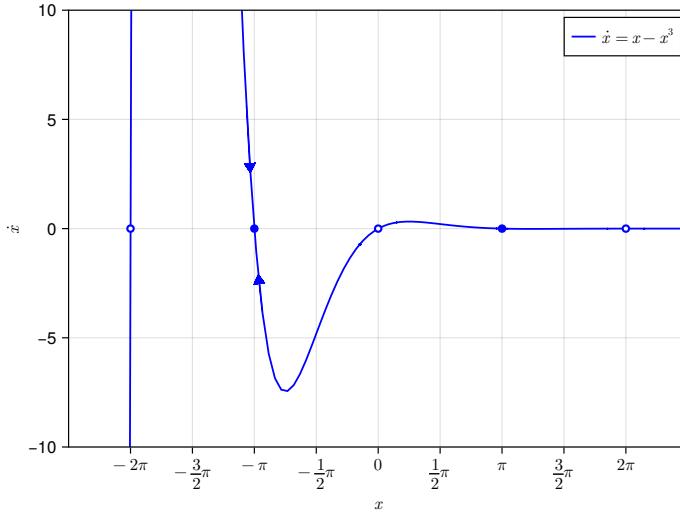


Figure 2.5: Vector field and fixed points for  $\dot{x} = e^{-x} \sin x$

We separate equations to find

$$\int e^x \csc x \, dx = \int dt,$$

but since  $e^x \csc x$  is not an elementary function, we cannot find a simple, closed-form solution.

#### 2.2.5

In Figure 2.6, we have a plot of the vector field and various graphs with different initial conditions for the equation

$$\dot{x} = 1 + \frac{1}{2} \cos x. \quad (2.1)$$

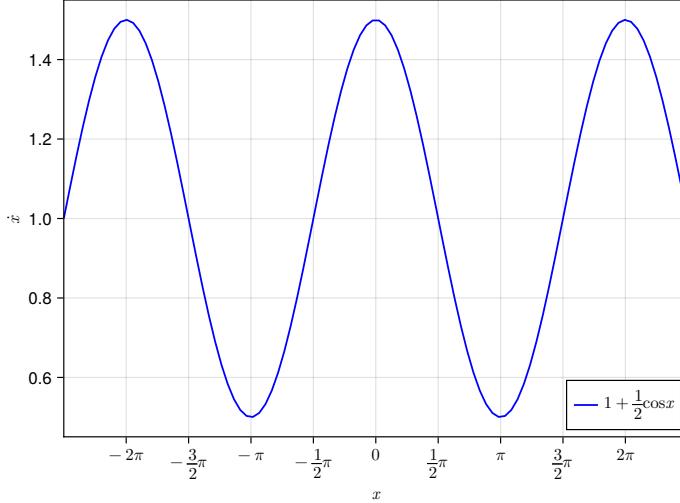


Figure 2.6: Vector field for  $\dot{x} = 1 + \frac{1}{2} \cos x$

Note that there are no fixed points for Equation (2.1), so that these are not included in Figure 2.6.

We separate equations to find

$$\begin{aligned} \frac{dx}{dt} &= 1 + \frac{1}{2} \cos x \\ \Leftrightarrow \int \frac{dx}{1 + \frac{1}{2} \cos x} &= \int dt. \end{aligned}$$

We introduce a new variable,  $u = \tan \frac{x}{2}$ , so that we have

$$\cos x = \frac{1 - u^2}{1 + u^2},$$

and

$$dx = \frac{2}{1 + u^2} dt.$$

Then we have

$$\begin{aligned} \int \frac{dx}{1 + \frac{1}{2} \cos x} &= 4 \int \frac{du}{3 + u^2} \\ &= \frac{4}{\sqrt{3}} \tan^{-1} \left( \frac{u}{\sqrt{3}} \right) + c_1 \\ &= \frac{4}{\sqrt{3}} \tan^{-1} \left( \frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + c_1. \end{aligned}$$

Thus our ordinary differential equation becomes

$$\frac{4}{\sqrt{3}} \tan^{-1} \left( \frac{\tan \frac{x}{2}}{\sqrt{3}} \right) = t + c_2,$$

so that we have

$$\begin{aligned} \tan^{-1} \left( \frac{\tan \frac{x}{2}}{\sqrt{3}} \right) &= \frac{\sqrt{3}}{4}(t + c_2) \\ \Leftrightarrow \frac{\tan \frac{x}{2}}{\sqrt{3}} &= \tan \left( \frac{\sqrt{3}}{4}t + c_3 \right) \\ \Leftrightarrow \tan \frac{x}{2} &= \sqrt{3} \tan \left( \frac{\sqrt{3}}{4}t + c_3 \right) \\ \Leftrightarrow x &= 2 \tan^{-1} \left[ \sqrt{3} \tan \left( \frac{\sqrt{3}}{4}t + c_3 \right) \right], \end{aligned}$$

where  $c_3 = \frac{\sqrt{3}}{4}c_2$ .

Substituting the initial condition  $x(0) = 0$  yields

$$\begin{aligned} \frac{4}{\sqrt{3}} \tan^{-1} \left( \frac{\tan 0}{\sqrt{3}} \right) &= c_2 \\ \Leftrightarrow \frac{4}{\sqrt{3}} \tan^{-1} 0 &= c_2 \\ \Leftrightarrow 0 &= c_2, \end{aligned}$$

so that  $c_3 = 0$ , and we have a solution corresponding to the initial condition  $x(0) = 0$  given by

$$x(t) = 2 \tan^{-1} \left[ \sqrt{3} \tan \left( \frac{\sqrt{3}}{4}t \right) \right].$$

In Figure 2.7, we have a plot of this solution.

### 2.2.6

In Figure 2.8, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = 1 - 2 \cos x$$

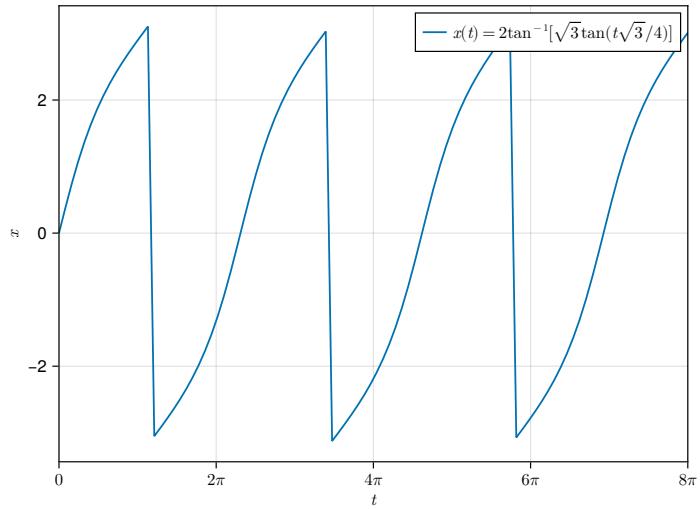


Figure 2.7: Graph of a solution of  $\dot{x} = 1 + \frac{1}{2} \cos x$

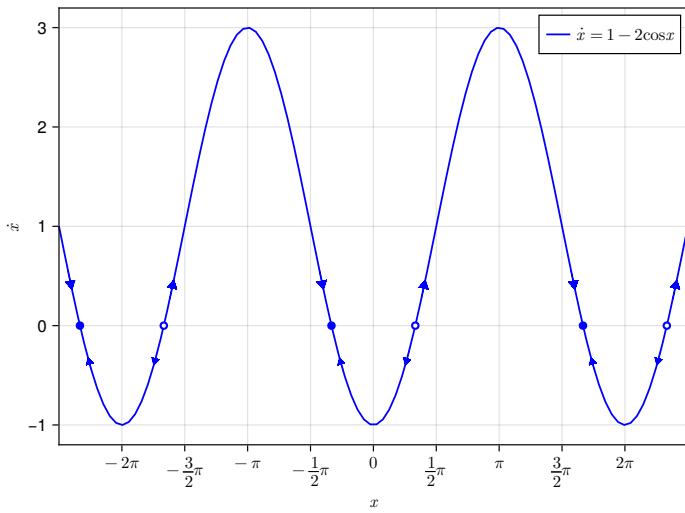


Figure 2.8: Vector field and fixed points for  $\dot{x} = 1 - 2 \cos x$