# Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering 2nd Edition

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# 2 Flows on the Line

# 2.1 A Geometric Way of Thinking

### 2.1.1

For a fixed point of the flow  $\dot{x} = \sin x$  on the line, we have

$$\dot{x} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \quad n \in \mathbb{Z}.$$

Thus all fixed points are given by  $x = n\pi$ ,  $n \in \mathbb{Z}$ .

#### 2.1.2

The points x for which the flow has the greatest velocity to the right are those for which  $\dot{x} > 0$  and x is a local maximum. These are given by

$$x = \frac{(4n+1)\pi}{2}, \quad n \in \mathbb{Z}.$$

## 2.1.3

(a) We have

$$\dot{x} = \sin x$$
,

so that

$$\ddot{x} = \frac{d}{dt}(\sin x)$$

$$= \dot{x}\cos x$$

$$= \sin x \cos x$$

$$= \frac{1}{2}\sin(2x).$$

(b) The maximum positive acceleration is given by the local maxima of  $\ddot{x} = (1/2)\sin(2x)$ , i.e., where

$$2x = \frac{(4n+1)\pi}{2} \Rightarrow x = \frac{(4n+1)\pi}{4}, \quad n \in \mathbb{Z}.$$

## 2.1.4

(a) We begin by evaluating

$$t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$$

with  $x_0 = \pi/4$  to find

$$t = \ln \left| \frac{\sqrt{2} + 1}{\csc x + \cot x} \right|$$
$$= \ln \left( \frac{1 + \sqrt{2}}{\csc x + \cot x} \right).$$

Then we apply the function  $e^z$  to both sides of to find

$$e^{t} = \frac{1 + \sqrt{2}}{\csc x + \cot x}$$

$$= \frac{1 + \sqrt{2}}{\cot (x/2)}$$

$$\Leftrightarrow \cot (x/2) = \frac{1 + \sqrt{2}}{e^{t}}$$

$$\Leftrightarrow \frac{1}{\tan (x/2)} = \frac{1 + \sqrt{2}}{e^{t}}$$

$$\Leftrightarrow \tan (x/2) = \frac{e^{t}}{1 + \sqrt{2}}$$

$$\Leftrightarrow \frac{x}{2} = \tan^{-1} \left(\frac{e^{t}}{1 + \sqrt{2}}\right)$$

$$\Leftrightarrow x = 2 \tan^{-1} \left(\frac{e^{t}}{1 + \sqrt{2}}\right).$$

Now, we have

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \left[ 2 \tan^{-1} \left( \frac{e^t}{1 + \sqrt{2}} \right) \right] = \pi,$$

since  $\lim_{t\to\infty} e^t = \infty$  and  $\lim_{t\to\infty} \tan^{-1} t = \pi$ .

(b) An analytic solution is given by performing the steps in (a) without assuming  $x_0 = \pi/4$ , so that we have

$$x(t) = 2 \tan^{-1} \left( \frac{e^t}{\csc x_0 + \cot x_0} \right)$$
$$= 2 \tan^{-1} \left( \frac{e^t}{\cot (x_0/2)} \right).$$

#### 2.1.5

- (a) Since the time derivative oscillates with period  $2\pi$ , a mechanical system which is approximately governed by  $\dot{x}=\sin x$  is a pendulum, with angle of relative to the horizontal axis given by x and  $x=\pi$  the lowest point in the pendulum's trajectory, i.e., an upside-down Cartesian coordinate system.
- (b) Since it will always fall back towards the vertical axis when it is suspended perpendiculat to the horizontal axis,  $x^*=0$  is an unstable fixed point, while  $x^*=\pi$  corresponds to the lowest point in the trajectory, so that the pendulum will not move if it starts at this point.

## 2.2 Fixed Points and Stability

#### 2.2.1

In Figure 1, we a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = 4x^2 - 16$$

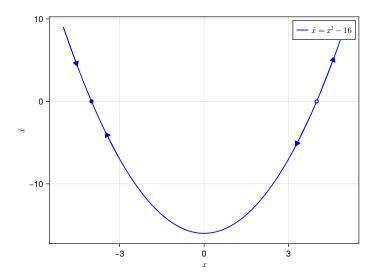


Figure 1: Vector field, fixed points, and graphs for  $\boldsymbol{x}(t)$