Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering
2nd Edition

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2 Flows on the Line

2.1 A Geometric Way of Thinking

2.1.1

For a fixed point of the flow $\dot{x} = \sin x$ on the line, we have

$$\dot{x} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \quad n \in \mathbb{Z}.$$

Thus all fixed points are given by $x = n\pi$, $n \in \mathbb{Z}$.

2.1.2

The points x for which the flow has the greatest velocity to the right are those for which $\dot{x} > 0$ and x is a local maximum. These are given by

$$x = \frac{(4n+1)\pi}{2}, \quad n \in \mathbb{Z}.$$

2.1.3

(a) We have

$$\dot{x} = \sin x,$$

so that

$$\begin{aligned}
\ddot{x} &= \frac{d}{dt}(\sin x) \\
&= \dot{x}\cos x \\
&= \sin x \cos x \\
&= \frac{1}{2}\sin(2x).
\end{aligned}$$

(b) The maximum positive acceleration is given by the local maxima of $\ddot{x} = (1/2)\sin(2x)$, i.e., where

$$2x = \frac{(4n+1)\pi}{2} \Rightarrow x = \frac{(4n+1)\pi}{4}, \quad n \in \mathbb{Z}.$$

2.1.4

(a) We begin by evaluating

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$$

with $x_0 = \pi/4$ to find

$$\begin{aligned}
t &= \ln\left|\frac{\sqrt{2}+1}{\csc x + \cot x}\right| \\
&= \ln\left(\frac{1+\sqrt{2}}{\csc x + \cot x}\right).
\end{aligned}$$

Then we apply the function e^z to both sides of to find

$$\begin{aligned}
e^t &= \frac{1+\sqrt{2}}{\csc x + \cot x} \\
&= \frac{1+\sqrt{2}}{\cot(x/2)} \\
\Leftrightarrow \cot(x/2) &= \frac{1+\sqrt{2}}{e^t} \\
\Leftrightarrow \frac{1}{\tan(x/2)} &= \frac{1+\sqrt{2}}{e^t} \\
\Leftrightarrow \tan(x/2) &= \frac{e^t}{1+\sqrt{2}} \\
\Leftrightarrow \frac{x}{2} &= \tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right) \\
\Leftrightarrow x &= 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right).
\end{aligned}$$

Now, we have

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right)\right] = \pi,$$

since $\lim_{t\to\infty} e^t = \infty$ and $\lim_{t\to\infty}\tan^{-1} t = \pi$.

(b) An analytic solution is given by performing the steps in (a) without assuming $x_0 = \pi/4$, so that we have

$$\begin{aligned} x(t) &= 2\tan^{-1}\left(\frac{e^t}{\csc x_0 + \cot x_0}\right) \\ &= 2\tan^{-1}\left(\frac{e^t}{\cot\left(x_0/2\right)}\right). \end{aligned}$$

2.1.5

(a) Since the time derivative oscillates with period 2π, a mechanical system which is approximately governed by $\dot{x} = \sin x$ is a pendulum, with angle of relative to the horizontal axis given by x and $x = \pi$ the lowest point in the pendulum's trajectory, i.e., an upside-down Cartesian coordinate system.

(b) Since it will always fall back towards the vertical axis when it is suspended perpendiculat to the horizontal axis, $x^* = 0$ is an unstable fixed point, while $x^* = \pi$ corresponds to the lowest point in the trajectory, so that the pendulum will not move if it starts at this point.

2.2 Fixed Points and Stability

2.2.1

In Figure 2.1, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = 4x^2 - 16$$

2.2.2

In Figure 2.2, we a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = 1 - x^{14}$$

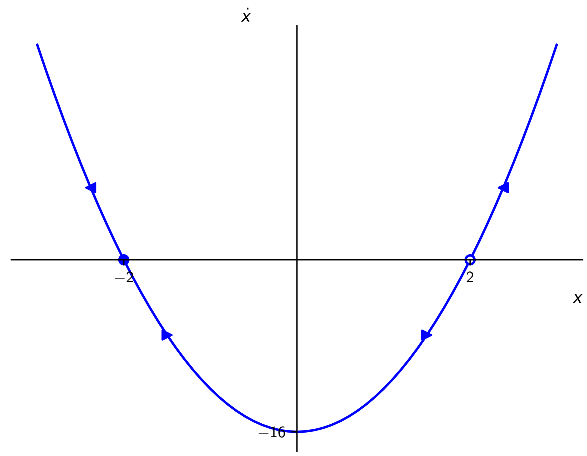

Figure 2.1: Vector field and fixed points for $\dot{x} = 4x^2 - 16$

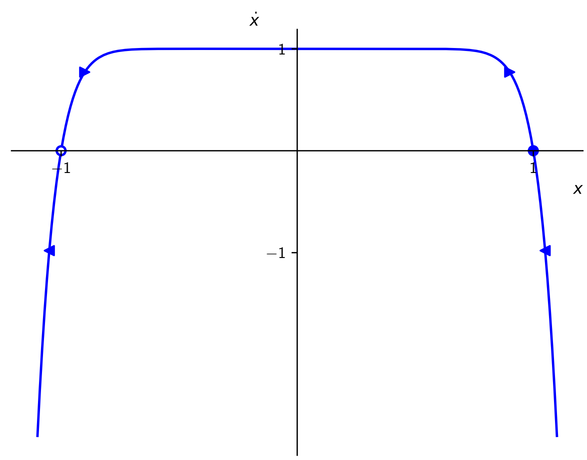

Figure 2.2: Vector field and fixed points for $\dot{x} = 1 - x^{14}$

2.2.3

In Figure 2.3, we a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = x - x^3$$

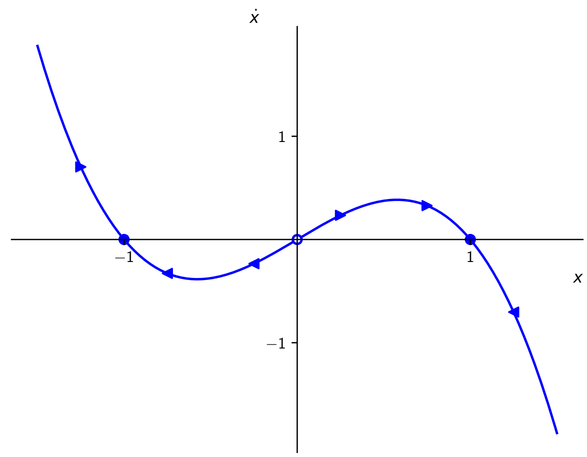

Figure 2.3: Vector field and fixed points for $\dot{x} = x - x^3$