Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering
2nd Edition

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2 Flows on the Line

2.1 A Geometric Way of Thinking

2.1.1

For a fixed point of the flow $\dot{x} = \sin x$ on the line, we have

$$\dot{x} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \quad n \in \mathbb{Z}.$$

Thus all fixed points are given by $x = n\pi$, $n \in \mathbb{Z}$.

2.1.2

The points x for which the flow has the greatest velocity to the right are those for which $\dot{x} > 0$ and x is a local maximum. These are given by

$$x = \frac{(4n+1)\pi}{2}, \quad n \in \mathbb{Z}.$$

2.1.3

(a) We have

$$\dot{x} = \sin x,$$

so that

$$\begin{aligned}\ddot{x} &= \frac{d}{dt}(\sin x)\\ &= \dot{x}\cos x\\ &= \sin x \cos x\\ &= \frac{1}{2}\sin(2x).\end{aligned}$$

(b) The maximum positive acceleration is given by the local maxima of $\ddot{x} = (1/2)\sin(2x)$, i.e., where

$$2x = \frac{(4n+1)\pi}{2} \Rightarrow x = \frac{(4n+1)\pi}{4}, \quad n \in \mathbb{Z}.$$

2.1.4

(a) We begin by evaluating

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$$

with $x_0 = \pi/4$ to find

$$\begin{aligned}t &= \ln\left|\frac{\sqrt{2}+1}{\csc x + \cot x}\right|\\ &= \ln\left(\frac{1+\sqrt{2}}{\csc x + \cot x}\right).\end{aligned}$$

Then we apply the function e^z to both sides of to find

$$\begin{aligned}e^t &= \frac{1+\sqrt{2}}{\csc x + \cot x}\\ &= \frac{1+\sqrt{2}}{\cot(x/2)}\\ \Leftrightarrow \cot(x/2) &= \frac{1+\sqrt{2}}{e^t}\\ \Leftrightarrow \frac{1}{\tan(x/2)} &= \frac{1+\sqrt{2}}{e^t}\\ \Leftrightarrow \tan(x/2) &= \frac{e^t}{1+\sqrt{2}}\\ \Leftrightarrow \frac{x}{2} &= \tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right)\\ \Leftrightarrow x &= 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right).\end{aligned}$$

Now, we have

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \left[2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right) \right] = \pi,$$

since $\lim_{t\to\infty} e^t = \infty$ and $\lim_{t\to\infty} \tan^{-1} t = \pi$.

(b) An analytic solution is given by performing the steps in (a) without assuming $x_0 = \pi/4$, so that we have

$$\begin{aligned} x(t) &= 2\tan^{-1}\left(\frac{e^t}{\csc x_0 + \cot x_0}\right) \\ &= 2\tan^{-1}\left(\frac{e^t}{\cot(x_0/2)}\right). \end{aligned}$$

2.1.5

(a)

(b)