

# Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering 2nd Edition

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## 2 Flows on the Line

### 2.1 A Geometric Way of Thinking

#### 2.1.1

For a fixed point of the flow  $\dot{x} = \sin x$  on the line, we have

$$\dot{x} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \quad n \in \mathbb{Z}.$$

Thus all fixed points are given by  $x = n\pi$ ,  $n \in \mathbb{Z}$ .

#### 2.1.2

The points  $x$  for which the flow has the greatest velocity to the right are those for which  $\dot{x} > 0$  and  $x$  is a local maximum. These are given by

$$x = \frac{(4n+1)\pi}{2}, \quad n \in \mathbb{Z}.$$

#### 2.1.3

(a) We have

$$\dot{x} = \sin x,$$

so that

$$\begin{aligned}
 \ddot{x} &= \frac{d}{dt}(\sin x) \\
 &= \dot{x} \cos x \\
 &= \sin x \cos x \\
 &= \frac{1}{2} \sin(2x).
 \end{aligned}$$

- (b) The maximum positive acceleration is given by the local maxima of  $\ddot{x} = (1/2) \sin(2x)$ , i.e., where

$$2x = \frac{(4n+1)\pi}{2} \Rightarrow x = \frac{(4n+1)\pi}{4}, \quad n \in \mathbb{Z}.$$

#### 2.1.4

- (a) We begin by evaluating

$$t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$$

with  $x_0 = \pi/4$  to find

$$\begin{aligned}
 t &= \ln \left| \frac{\sqrt{2} + 1}{\csc x + \cot x} \right| \\
 &= \ln \left( \frac{1 + \sqrt{2}}{\csc x + \cot x} \right).
 \end{aligned}$$

Then we apply the function  $e^z$  to both sides of to find

$$\begin{aligned}
 e^t &= \frac{1 + \sqrt{2}}{\csc x + \cot x} \\
 &= \frac{1 + \sqrt{2}}{\cot(x/2)} \\
 \Leftrightarrow \cot(x/2) &= \frac{1 + \sqrt{2}}{e^t} \\
 \Leftrightarrow \frac{1}{\tan(x/2)} &= \frac{1 + \sqrt{2}}{e^t} \\
 \Leftrightarrow \tan(x/2) &= \frac{e^t}{1 + \sqrt{2}} \\
 \Leftrightarrow \frac{x}{2} &= \tan^{-1} \left( \frac{e^t}{1 + \sqrt{2}} \right) \\
 \Leftrightarrow x &= 2 \tan^{-1} \left( \frac{e^t}{1 + \sqrt{2}} \right).
 \end{aligned}$$

Now, we have

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \left[ 2 \tan^{-1} \left( \frac{e^t}{1 + \sqrt{2}} \right) \right] = \pi,$$

since  $\lim_{t \rightarrow \infty} e^t = \infty$  and  $\lim_{t \rightarrow \infty} \tan^{-1} t = \pi$ .

- (b) An analytic solution is given by performing the steps in (a) without assuming  $x_0 = \pi/4$ , so that we have

$$\begin{aligned} x(t) &= 2 \tan^{-1} \left( \frac{e^t}{\csc x_0 + \cot x_0} \right) \\ &= 2 \tan^{-1} \left( \frac{e^t}{\cot(x_0/2)} \right). \end{aligned}$$

### 2.1.5

- (a) Since the time derivative oscillates with period  $2\pi$ , a mechanical system which is approximately governed by  $\dot{x} = \sin x$  is a pendulum, with angle of relative to the horizontal axis given by  $x$  and  $x = \pi$  the lowest point in the pendulum's trajectory, i.e., an upside-down Cartesian coordinate system.
- (b) Since it will always fall back towards the vertical axis when it is suspended perpendicular to the horizontal axis,  $x^* = 0$  is an unstable fixed point, while  $x^* = \pi$  corresponds to the lowest point in the trajectory, so that the pendulum will not move if it starts at this point.

## 2.2 Fixed Points and Stability

### 2.2.1

In Figure 2.1, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = 4x^2 - 16$$

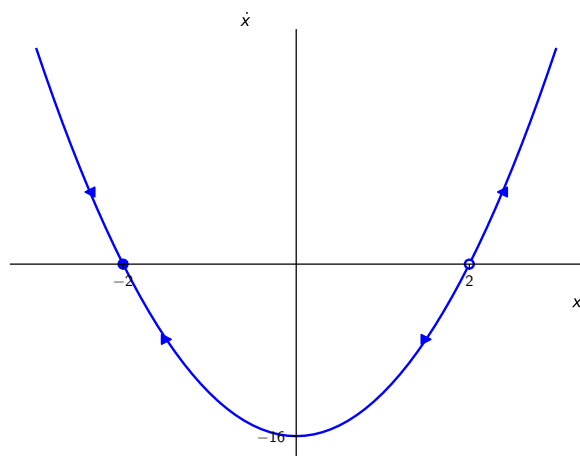


Figure 2.1: Vector field and fixed points for  $x(t) = 4x^2 - 16$