Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering 2nd Edition

—

Steven H. Strogatz

Graham Strickland

December 13, 2025

2 Flows on the Line

2.1 A Geometric Way of Thinking

2.1.1

For a fixed point of the flow $\dot{x} = \sin x$ on the line, we have

$$\dot{x} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \quad n \in \mathbb{Z}.$$

Thus all fixed points are given by $x = n\pi$, $n \in \mathbb{Z}$.

2.1.2

The points x for which the flow has the greatest velocity to the right are those for which $\dot{x} > 0$ and x is a local maximum. These are given by

$$x = \frac{(4n+1)\pi}{2}, \quad n \in \mathbb{Z}.$$

2.1.3

(a) We have

$$\dot{x} = \sin x,$$

so that

$$\begin{aligned}
\ddot{x} &= \frac{d}{dt}(\sin x) \\
&= \dot{x}\cos x \\
&= \sin x \cos x \\
&= \frac{1}{2}\sin(2x).
\end{aligned}$$

(b) The maximum positive acceleration is given by the local maxima of $\ddot{x} = (1/2)\sin(2x)$, i.e., where

$$2x = \frac{(4n+1)\pi}{2} \Rightarrow x = \frac{(4n+1)\pi}{4}, \quad n \in \mathbb{Z}.$$

2.1.4

(a) We begin by evaluating

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$$

with $x_0 = \pi/4$ to find

$$\begin{aligned}
t &= \ln\left|\frac{\sqrt{2}+1}{\csc x + \cot x}\right| \\
&= \ln\left(\frac{1+\sqrt{2}}{\csc x + \cot x}\right).
\end{aligned}$$

Then we apply the function e^z to both sides of to find

$$\begin{aligned}
e^t &= \frac{1+\sqrt{2}}{\csc x + \cot x} \\
&= \frac{1+\sqrt{2}}{\cot(x/2)} \\
\Leftrightarrow \cot(x/2) &= \frac{1+\sqrt{2}}{e^t} \\
\Leftrightarrow \frac{1}{\tan(x/2)} &= \frac{1+\sqrt{2}}{e^t} \\
\Leftrightarrow \tan(x/2) &= \frac{e^t}{1+\sqrt{2}} \\
\Leftrightarrow \frac{x}{2} &= \tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right) \\
\Leftrightarrow x &= 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right).
\end{aligned}$$

Now, we have

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right)\right] = \pi,$$

since $\lim_{t\to\infty} e^t = \infty$ and $\lim_{t\to\infty} \tan^{-1} t = \pi$.

(b) An analytic solution is given by performing the steps in (a) without assuming $x_0 = \pi/4$, so that we have

$$\begin{aligned} x(t) &= 2\tan^{-1}\left(\frac{e^t}{\csc x_0 + \cot x_0}\right) \\ &= 2\tan^{-1}\left(\frac{e^t}{\cot(x_0/2)}\right). \end{aligned}$$

2.1.5

(a) Since the time derivative oscillates with period 2π, a mechanical system which is approximately governed by $\dot{x} = \sin x$ is a pendulum, with angle of relative to the horizontal axis given by x and $x = \pi$ the lowest point in the pendulum's trajectory, i.e., an upside-down Cartesian coordinate system.

(b) Since it will always fall back towards the vertical axis when it is suspended perpendiculat to the horizontal axis, $x^* = 0$ is an unstable fixed point, while $x^* = \pi$ corresponds to the lowest point in the trajectory, so that the pendulum will not move if it starts at this point.

2.2 Fixed Points and Stability

2.2.1

In Figure 2.1, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = 4x^2 - 16$$

An explicit solution for $\dot{x} = 4x^2 - 16$ can be found by re-writing the equation as

$$\frac{dx}{dt} = 4x^2 - 16,$$

so that we have separable equations. Then, we may write

$$\frac{dx}{4x^2 - 16} = dt,$$

so that by using partial fraction decomposition, we have

$$\left[\frac{\frac{1}{16}}{x-2} - \frac{\frac{1}{16}}{x+2}\right]dx = dt,$$

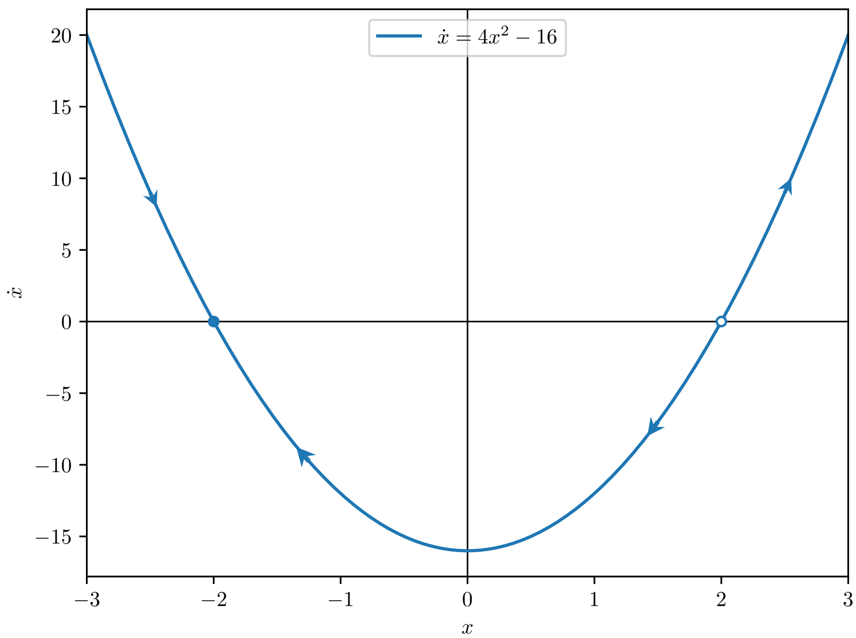

Figure 2.1: Vector field and fixed points for $\dot{x} = 4x^2 - 16$

and integrate either sides with respect to the x and t, respectively, so that we have

$$\begin{aligned} \int \left[\frac{\frac{1}{16}}{x-2} - \frac{\frac{1}{16}}{x+2} \right] dx &= \int dt \\ \Rightarrow \frac{1}{16} \ln|x-2| - \frac{1}{16} \ln|x+2| &= t + c_1 \\ \Rightarrow \ln\left|\frac{x-2}{x+2}\right| &= 16t + c_2 \\ \Rightarrow \frac{x-2}{x+2} &= \pm e^{16t+c_2}. \end{aligned}$$

After replacing $\pm e^{c_2}$ by c and solving the last equation for x, we have the one-parameter family of solutions

$$x = 2\frac{1+ce^{16t}}{1-ce^{16t}}.$$

If we let $x(0) = 0$, we have

$$\begin{aligned} 2\frac{1+c}{1-c} &= 0 \\ \Rightarrow 1+c &= 0 \\ \Rightarrow c &= -1, \end{aligned}$$

so that

$$x_1(t) = 2\frac{1-e^{16t}}{1+e^{16t}}$$

is a solution corresponding to the initial condition $x(0) = 0$.

Likewise, if we let $x(0) = 4$, we have

$$\begin{aligned} 2\frac{1+c}{1-c} &= 4 \\ \Rightarrow 1+c &= 2(1-c) \\ \Rightarrow c &= \frac{1}{3}, \end{aligned}$$

so that

$$x_2(t) = 2\frac{1+\frac{1}{3}e^{16t}}{1-\frac{1}{3}e^{16t}}$$

is a solution corresponding to the initial condition $x(0) = 4$.

Finally, if we let $x(0) = -4$, we have

$$\begin{aligned} 2\frac{1+c}{1-c} &= -4 \\ \Rightarrow 1+c &= -2(1-c) \\ \Rightarrow c &= 3, \end{aligned}$$

so that

$$x_3(t) = 2\frac{1+3e^{16t}}{1-3e^{16t}}$$

is a solution corresponding to the initial condition $x(0) = 4$.

In Figure 2.2, we have a plot of these solutions for the given initial conditions.

2.2.2

In Figure 2.3, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = 1 - x^{14}$$

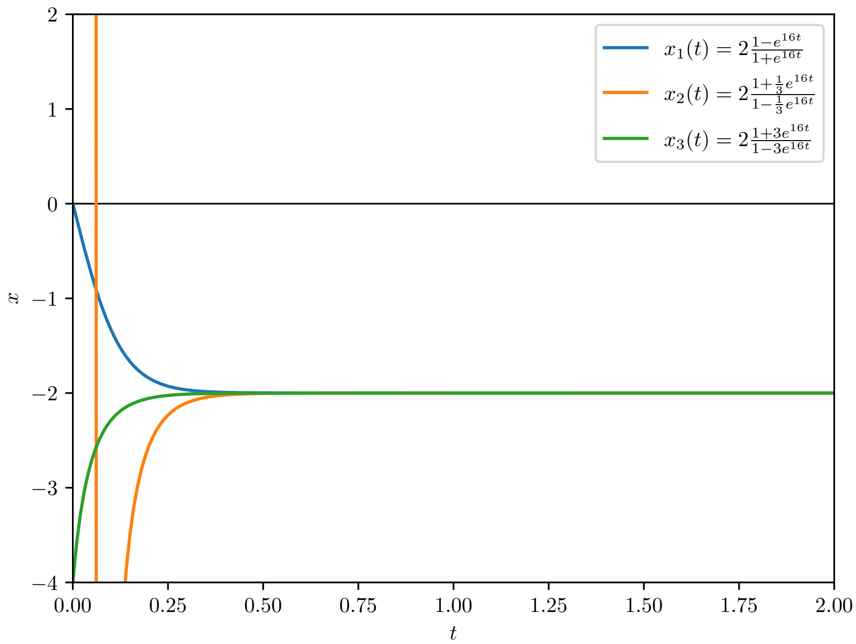

Figure 2.2: Graph of solutions of $\dot{x} = 4x^2 - 16$

Again, we separate the equation to find

$$\frac{dx}{1 - x^{14}} = dt.$$

Integrating either side with respect to x and t, respectively, we have

$$\begin{aligned}
\int \frac{dx}{1 - x^{14}} &= \int dt \\
\Rightarrow \ln |1 - x^{14}| &= t + c_1 \\
\Rightarrow 1 - x^{14} &= \pm e^{t+c_1} \\
\Rightarrow x^{14} &= \pm e^{t+c_1} + 1 \\
\Rightarrow x^{14} &= ce^t + 1 \\
\Rightarrow x &= \pm \sqrt[14]{ce^t + 1},
\end{aligned}$$

for $ce^t + 1 \geq 0$.

It does not seem possible to find a simple closed-form solution to this ODE.

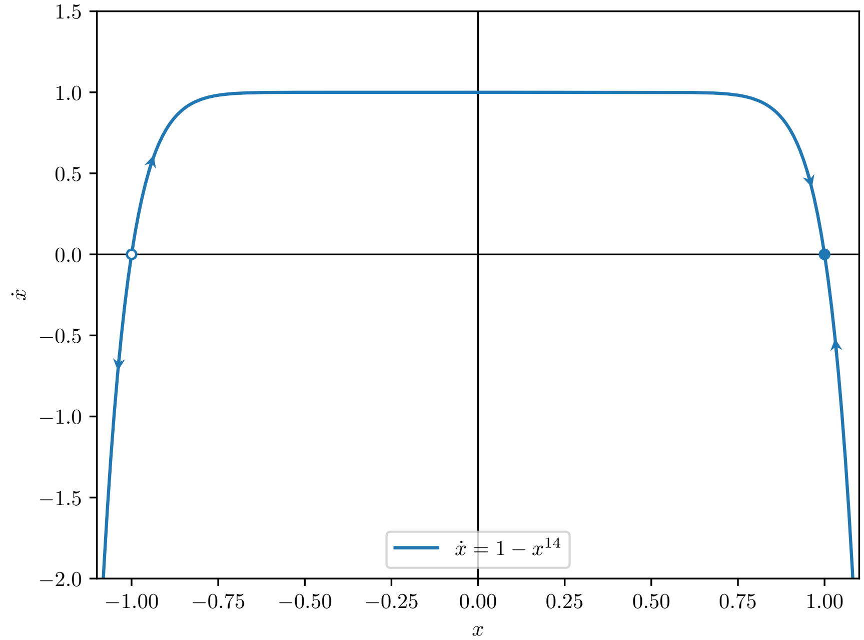

Figure 2.3: Vector field and fixed points for $\dot{x} = 1 - x^{14}$

2.2.3

In Figure 2.4, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = x - x^3$$

We use the method of separable equations again to find

$$\frac{dx}{x - x^3} = dt,$$

and using partial fraction decomposition, we have

$$\left[\frac{1}{x} + \frac{\frac{1}{2}}{1-x} - \frac{\frac{1}{2}}{1+x}\right]dx = dt.$$

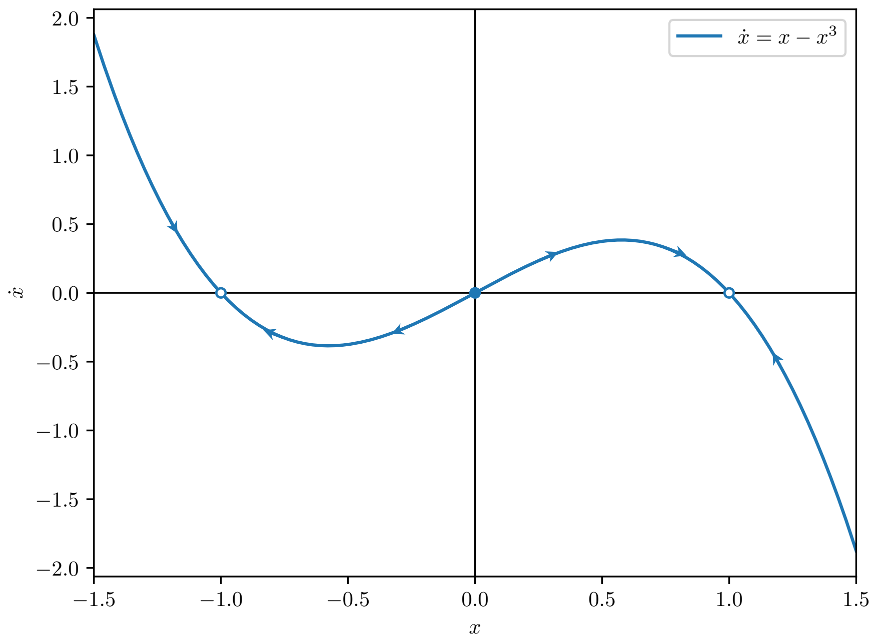

Figure 2.4: Vector field and fixed points for $\dot{x} = x - x^3$

Integrating both sides yields

$$\begin{aligned}
\int \frac{dx}{x} + \frac{1}{2}\int \frac{dx}{1-x} - \frac{1}{2}\int \frac{dx}{1+x} &= \int dt \\
\Leftrightarrow \ln|x| + \frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x| &= t + c_1 \\
\Leftrightarrow \ln\left|\frac{x^2(1-x)}{1+x}\right| &= 2t + c_2 \\
\Leftrightarrow \frac{x^2(1-x)}{1+x} &= \pm e^{2t+c_2} \\
&= ce^{2t}.
\end{aligned}$$

Again, it does not seem possible to find a simple closed-form solution to this ODE.

2.2.4

In Figure 2.5, we have a plot of the vector field, fixed points, and various graphs with different initial conditions for the equation

$$\dot{x} = e^{-x} \sin x$$

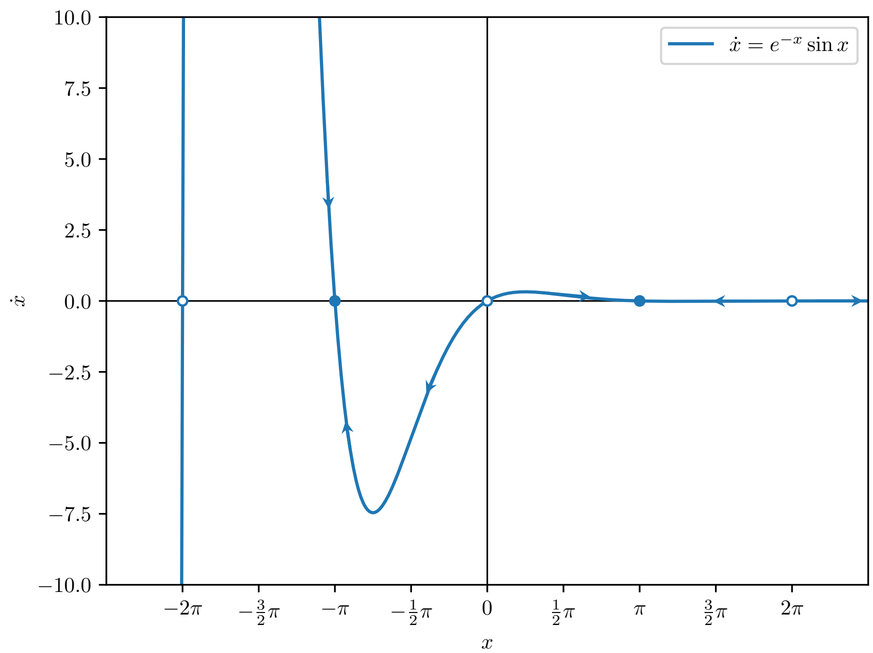

Figure 2.5: Vector field and fixed points for $\dot{x} = e^{-x} \sin x$

We separate equations to find

$$\int e^x \csc x \, dx = \int dt,$$

but since $e^x \csc x$ is not an elementary function, we cannot find a simple, closed-form solution.

2.2.5

In Figure 2.6, we have a plot of the vector field and various graphs with different initial conditions for the equation

$$\dot{x} = 1 + \frac{1}{2} \cos x. \tag{2.1}$$

Note that there are no fixed points for Equation (2.1) , so that these are not included in Figure 2.6.

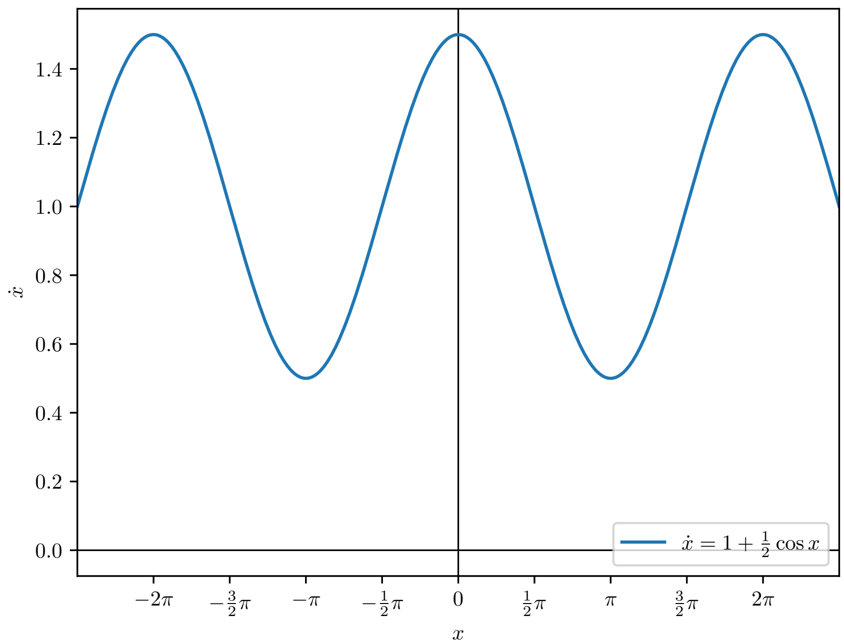

Figure 2.6: Vector field for $\dot{x} = 1 + \frac{1}{2}\cos x$

We separate equations to find

$$\begin{aligned} \frac{dx}{dt} &= 1 + \frac{1}{2}\cos x \\ \Leftrightarrow \int \frac{dx}{1 + \frac{1}{2}\cos x} &= \int dt. \end{aligned}$$

We introduce a new variable, $u = \tan\frac{x}{2}$, so that we have

$$\cos x = \frac{1-u^2}{1+u^2},$$

and

$$dx = \frac{2}{1+u^2}dt.$$

Then we have

$$\begin{aligned}
\int \frac{dx}{1+\frac{1}{2}\cos x} &= 4\int \frac{du}{3+u^2} \\
&= \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + c_1 \\
&= \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + c_1.
\end{aligned}$$

Thus our ordinary differential equation becomes

$$\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) = t + c_2,$$

so that we have

$$\begin{aligned}
\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) &= \frac{\sqrt{3}}{4}(t+c_2) \\
\Leftrightarrow \frac{\tan\frac{x}{2}}{\sqrt{3}} &= \tan\left(\frac{\sqrt{3}}{4}t + c_3\right) \\
\Leftrightarrow \tan\frac{x}{2} &= \sqrt{3}\tan\left(\frac{\sqrt{3}}{4}t + c_3\right) \\
\Leftrightarrow x &= 2\tan^{-1}\left[\sqrt{3}\tan\left(\frac{\sqrt{3}}{4}t + c_3\right)\right],
\end{aligned}$$

where $c_3 = \frac{\sqrt{3}}{4}c_2$.

Substituting the initial condition $x(0) = 0$ yields

$$\begin{aligned}
\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{\tan 0}{\sqrt{3}}\right) &= c_2 \\
\Leftrightarrow \frac{4}{\sqrt{3}}\tan^{-1} 0 &= c_2 \\
\Leftrightarrow 0 &= c_2,
\end{aligned}$$

so that $c_3 = 0$, and we have a solution corresponding to the initial condition $x(0) = 0$ given by

$$x(t) = 2\tan^{-1}\left[\sqrt{3}\tan\left(\frac{\sqrt{3}}{4}t\right)\right].$$

In Figure 2.7, we have a plot of this solution.

2.2.6

In Figure 2.8, we have a plot of the vector field and fixed points for the equation

$$\dot{x} = 1 - 2\cos x$$

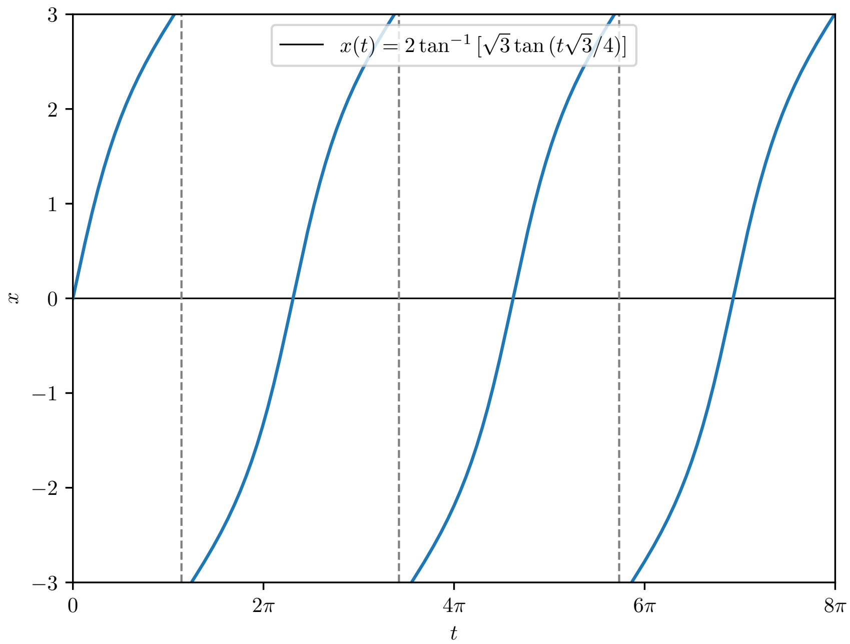

Figure 2.7: Graph of a solution of $\dot{x} = 1 + \frac{1}{2}\cos x$

2.2.7

In Figure 2.9, we have a plot of the functions e^x and $\cos x$. From this figure, we can estimate that the fixed points are given approximately by $x = -1.25$ and $x = 0.0$.

2.2.8

We are looking for an equation which is consistent with a phase portrait with a half-stable fixed point at $x = -1$, a stable fixed point at $x = 0$, and an unstable fixed point at $x = 2$. One equation which satisfies these conditions is given by

$$\begin{aligned} f(x) &= x(x+1)^2(x-2) \\ &= x^4 - 3x^2 - 2x. \end{aligned}$$

We plot the vector field and fixed points for this equation in Figure 2.10.

2.2.9

From Figure 2, we see that we are looking for an equation $\dot{x} = f(x)$ with a stable fixed point at $x = 0$ and an unstable fixed point at $x = 1$, whose solutions $x(t)$

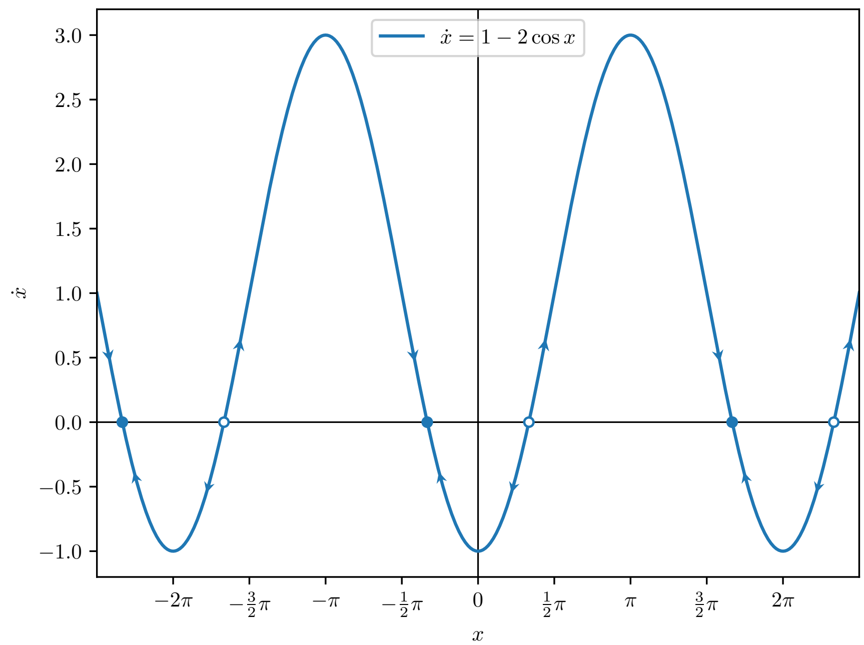

Figure 2.8: Vector field and fixed points for $\dot{x} = 1 - 2\cos x$

satisfy the following three conditions:

(i) $x_1(t)$ is a particular solution satisfying $x_1(0) > 1$, where

$$\lim_{t\to\infty} x_1(t) = \infty,$$

(ii) $x_2(t)$ is a particular solution satisfying $0 < x_2(0) < 1$, where

$$\lim_{t\to\infty} x_2(t) = 0,$$

(iii) $x_3(t)$ is a particular solution satisfying $x_3(0) = -1$, where

$$\lim_{t\to\infty} x_3(t) = 0.$$

One such equation is $\dot{x} = x^2 - x$, which we solve as follows. First, we

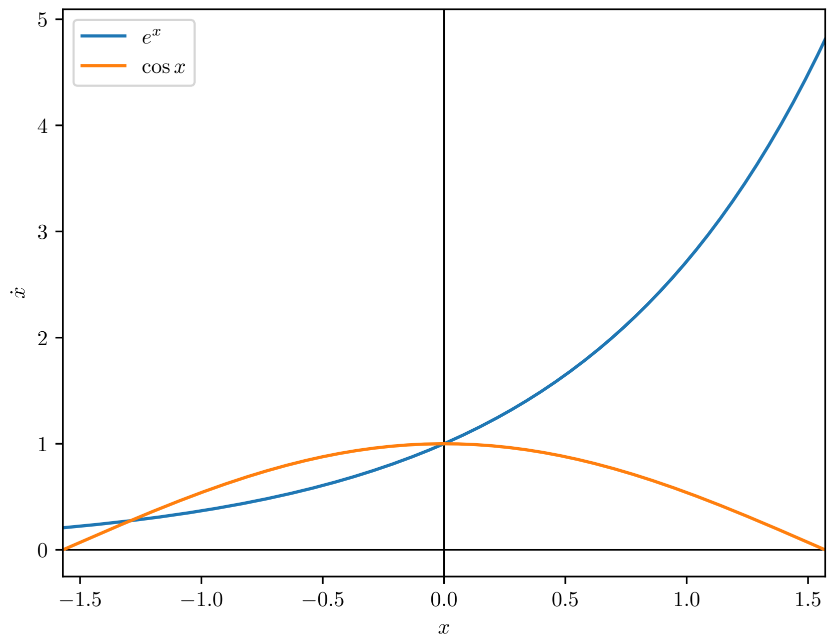

Figure 2.9: Graphs of e^x and $\cos x$

integrate both sides with respect to x and t to find

$$\begin{aligned}
\frac{dx}{dt} &= x^2 - x \\
\Leftrightarrow \int \frac{dx}{x^2 - x} &= \int dt \\
\Leftrightarrow \int \frac{dx}{x-1} - \int \frac{dx}{x} &= t + c_1 \\
\Leftrightarrow \ln|x-1| - \ln|x| &= t + c_1 \\
\Leftrightarrow \ln\left|\frac{x-1}{x}\right| &= t + c_1 \\
\Leftrightarrow \frac{x-1}{x} &= \pm e^{t+c_1}.
\end{aligned}$$

If we replace $\pm e^{c_1}$ by c and solve for y, we get the one-parameter family of solutions

$$x = \frac{1}{1 - ce^t}.$$

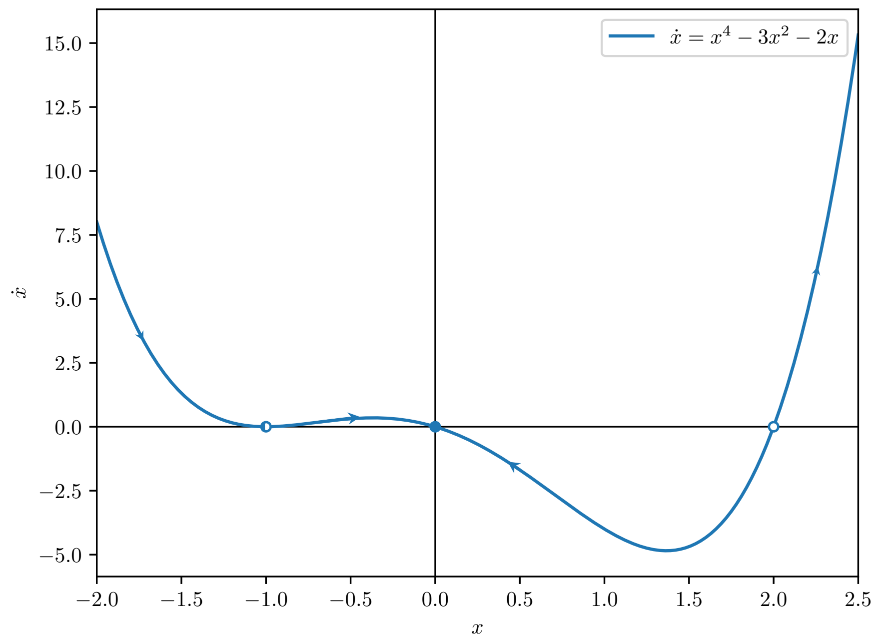

Figure 2.10: Vector field and fixed points for $\dot{x} = x^4 - 3x^2 - 2x$

Thus, for $x_1(0) > 1$, say $x_1(0) = \frac{3}{2}$, we have

$$\begin{aligned} \frac{1}{1-c} &= \frac{3}{2} \\ \Leftrightarrow 1 - c &= \frac{2}{3} \\ \Leftrightarrow c &= \frac{1}{3}, \end{aligned}$$

so that

$$x_1(t) = \frac{1}{1 - e^t/3}.$$

2.2.10

(a) For the function $\dot{x} = f(x) = 0$, every real number is a fixed point, i.e., for $x \in \mathbb{R}, f(x) = 0$.

(b) For the function $\dot{x} = f(x) = \sin \pi x$, every integer is a fixed point, since $\forall x \in \mathbb{R}$, $\sin \pi x = 0$.

(c) Such a function cannot exist, since $f(x^*) = 0$ implies that either $f(x^* - \varepsilon) < 0$ and $f(x^* + \varepsilon) > 0$ (or vice versa) for some $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, or $\dot{x} = f(x) = 0 \ \forall x \in \mathbb{R}$, by the intermediate value theorem. Thus there cannot be exactly three stable fixed points of a continuous function.

(d) $\dot{x} = f(x) = 1$ has no fixed points, since $\forall x \in \mathbb{R}$, $f(x) \neq 0$.

(e) The polynomial

$$\dot{x} = f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_1 00),$$

for $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq \alpha_3$, $\ldots$, $\alpha_2 \neq \alpha_3$, $\ldots$, $\alpha_{99} \neq \alpha_{100}$ has exactly 100 fixed points.