1 Descriptive Analysis

1.1 Data Description

For this assignment, the main source of data will be the Federal Reserve Economics Data (FRED) database. The outcome variable (Y_t) is the growth rate of Canada's quarterly Real Gross Domestic Product (GDP). Canada's real GDP is the total value of goods and services produced in Canada for a given period, as adjusted for inflation and has been growing steadily since the 1960s with a few breaks of the trend. The primary variable (X_t) is the growth rate of Canada's general government final consumption expenditure which has also been growing at a steady state up to date. This analysis will cover the period from the 1st quarter of 1961 to the 3rd quarter of 2023 with a total of 251 observations for each variable. The outcome variable (Y_t) is denoted as GDPGR and the primary variable (X_t) as FCEGR

Research Questions

- Does past GDP growth rate have a significant effect on forecasting future GDP growth rate?
- Is there a significant pattern between past and future periods in general government final consumption expenditure growth rate?
- Are the past GDP and general government final consumption growth rates jointly significant in predicting future values of GDP growth rate?

Hypotheses:

- H0,1 : Past GDPGR does not have statistical significance in forecasting GDPGR in the future
- H1,1 : Past GDPGR has statistical significance in forecasting future GDPGR.
- H0,2 :Past FCEGR does not have statistical significance in forecasting GDPGR in the future

- H1,2 : Past FCEGR has statistical significance in forecasting the GDPGR in the future

1.2 Time Series Plot

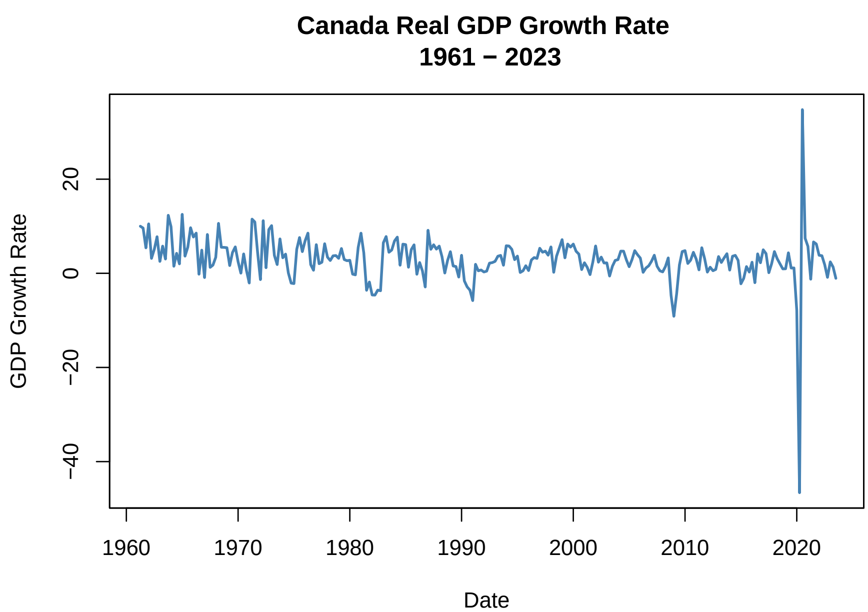

Figure 1: Canada Real GDP growth 1961 - 2023

The figure above shows Canada's real GDP growth rate time series which exhibits the same pattern for most of the periods, but there is a break of pattern between the second and third quarter of 2020 which is an outlier. This was reported by Statistics Canada that the Covid-19 pandemic caused the real GDP growth rate to contract compared to any other crisis faced by the economy. As the outlier is at one end of the time series, we can simply exclude it by removing the subset from our dataset.

1.2.1 Canada Real GDP growth rate between 1961 Q1 and 2019

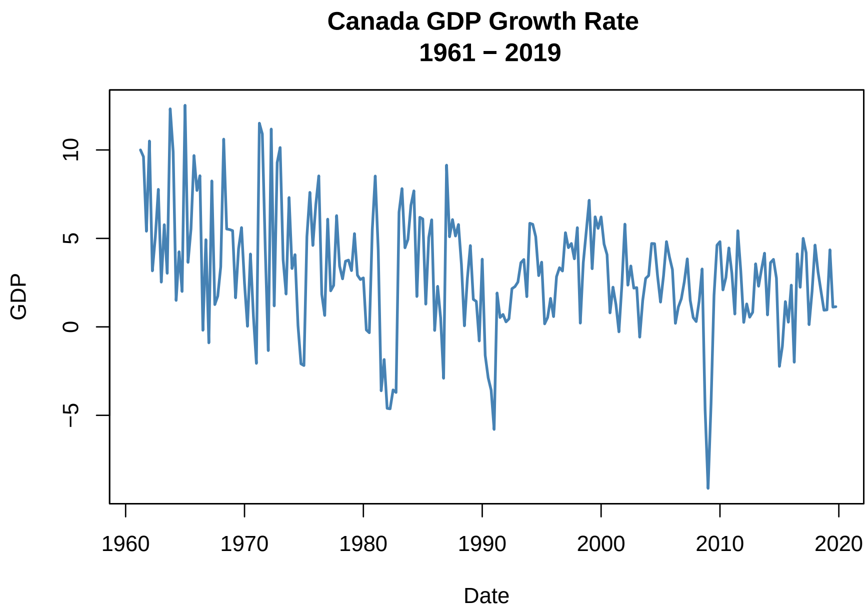

Figure 2: Canada Real GDP growth rate between 1961 and 2019 without outliers

From the figure above, there also seems to exist a break of trend right before 1980 and in the year 2000 which I cannot confirm now but will be tested later in this analysis using a QLR test.

1.3 Autoregression Analysis of a Time Series

1.3.1 Estimate an AR (1) Model

The AR (1) model estimates a regression model where the outcome variable is $GDPGR_t$ which is regressed on its own lag.

$$GDPGR_t = \beta_0 + \beta_1 GDPGR_{t-1} + u_t \tag{1}$$

Where:
$GDPGR_t$ is the GDP growth rate in time t.
β_0 is the intercept.
β_1 is the coefficient of the 1st lag of GDP growth rate.
$GDPGR_{t-1}$ is the 1st lag of the GDP growth rate.
u_t is the error term.

The estimated GDPGR AR(1) model is given by:

$$GD\widehat{PG}R_t = 1.8918 + 0.3802 \times GDPGR_{t-1} \tag{2}$$

1.3.2 BIC

The BIC results:

P	BIC(p)
1	2.3016
2	2.3152
3	2.3362
4	2.3473

Table 1: BIC Results

The smallest BIC (1) value is 2.3016 which tells us that the best forecast of GDPGR will be estimated using only 1 lag of $GDPGR_{t-1}$. Therefore, after testing all the four models, the best model is the AR (1) model.

AR(p) Model: From the results above, there will be no other model run other than the AR (1) Model out of the four models.

2 UNIT ROOT TEST

The estimated first difference of GDP growth rate AR(1) model is given by:

$$\Delta GDPGR_t = \beta_0 + \delta GDPGR_{t-1} + u_t \tag{3}$$

Where:
ΔGDPgr$_t$ is the first difference of the GDP growth rate at time t.
β_0 is the intercept.
δ is the coefficient of the 1st lag of GDP growth rate.
$GDPgr_{t-1}$ is the GDP growth rate at time $_{t-1}$.
u$_t$ is the error term.

$\widehat{\text{H}}$ypotheses:
H_0 : GDPgr has a unit root, meaning $\delta = 0$
H$_1$: GDPgr is stationary
If $\delta = 0$, then $0 = \beta_1 - 1$ which means that $\beta_1 = 1$.

Dickey-Fuller Test Analysis:

$$\widehat{GDPGR}_t = 1.8918 - 0.6198 GDPGR_{t-1} + u_t \tag{4}$$

Where:
$\beta_0 = 1.8918$ is the intercept.
$\delta = -0.6198$ is the coefficient of the 1st lag of GDP growth rate.

2.0.1 Dickey Fuller Critical Values:

1pct	5pct	10pct
-3.46	-2.88	-2.57

Table 2: DF values

T-statistic for = -10.31, this is shows that the is significant at all levels as shown above. Basing on the evidence above, I reject the null hypothesis and conclude that GDPGR does not have a unit root and that it is stationary.

2.0.2 TEST FOR A BREAK

Visualizing a break in the time series plot doesn't not enable clear conclusion that the break occurred, this leads to running of a QLR test which runs chow tests and returns the highest F-statistic.

The regression model is given by:

$$GDPGR_t = \beta_0 + \beta_1 GDPGR_{t-1} + \gamma_0 D_t(\tau) + \gamma_1[D_t(\tau) * GDPGR_{t-1}] \quad (5)$$

where:
D_t is the interaction term that is assigned the value
0 when $t < \tau$ and 1 when $t \geq \tau$.

The hypotheses are: H_0 : GDPGR has no break, i.e., $\gamma_0 = \gamma_1 = 0$.H$_1$: GDPGR has a break.

	Dependent variable:
	GDPGR_xts
lag(GDPGR_xts, 1)	−0.075
	(0.105)
D	−4.387***
	(0.748)
lag(GDPGR_xts, 1):D	0.556***
	(0.128)
Constant	5.659***
	(0.693)
Observations	234
R^2	0.258
Adjusted R^2	0.248
Residual Std. Error	2.905 (df = 230)
F Statistic	26.641*** (df = 3; 230)
Note:	*p<0.1; **p<0.05; ***p<0.01

QLR result finds the period with the highest F-statistics as 1973 Q4. On estimating the chow test model with and having conducted a joint hypothesis test on D and D($GDPGR_t1$), we find that an F-statistic of 17.1897. Therefore, we reject the null hypothesis and conclude that the relationship between GDPGR and its lag exhibit a shift in movement after the break.

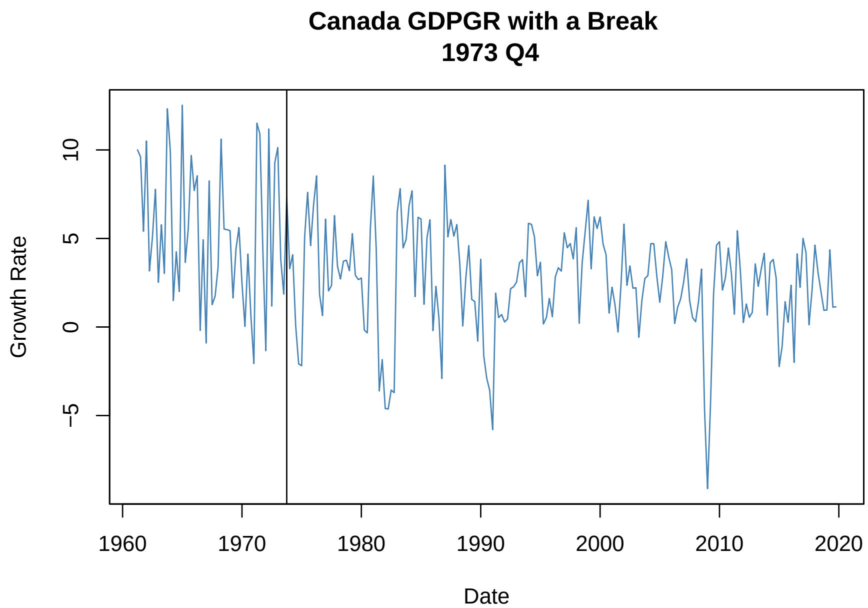

Figure 3: Canada Real GDP showing the break position

With this conclusion, we shall use only the period past the break event (1973 Q4) for the rest of the analysis of this assignment.

2.0.3 Report estimated coefficients from both the AR(1) and AR(p

Table 3: AR(1) Model Result

	Dependent variable:
	GDPGR_xts
lag(GDPGR_xts)	0.380^{***}
	(0.060)
Constant	1.892^{***}
	(0.276)
Observations	234
R^2	0.147
Adjusted R^2	0.143
Residual Std. Error	3.101 (df = 232)
F Statistic	39.966^{***} (df = 1; 232)
Note:	*p<0.1; **p<0.05; ***p<0.01

2.0.4 Interpretation of the coefficients of $GDPGR_{t-1}$

In the AR(1) model result that is presented in table 3 above, the coefficient statistically significant an also positive.This means that by estimation the predicted value of $GDPGR_t|GDPGR_t1$:

$$\widehat{GDPGR}_t = 1.892 + 0.380GDPGR_{t-1}$$

That is, the value of GDPGR in the past has an impact of 0.380 in predicting GDPGR in the future.The AR(p) model is same as AR(1) model, so this same argument still holds.

Table 4: AR(1) No Break Model

	Dependent variable:
	GDPgr_withoutBreak
lag(GDPgr_withoutBreak, 1)	0.485***
	(0.065)
Constant	1.269***
	(0.250)
Observations	183
R^2	0.235
Adjusted R^2	0.231
Residual Std. Error	2.571 (df = 181)
F Statistic	55.690*** (df = 1; 181)
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 4 shows that the lag of GDPGR growth rate has significant impact in predicting GDPGR in the future but with a stronger impact of 0.485 compared to 0.380 in the first instance. The analysis in table 3 may have been misleading because of the break that existed in the time series. This is because when there is a break in our time series,it affects the way our data behaves from one period toanother, and using such data may produce results that may be misleading to the users of such data.

3 The ADL (1,1) model is given by:

$$GDPGR_t = \beta_0 + \beta_1 GDPGR_{t-1} + \beta_2 FCEGR_{t-1} + u_t \tag{6}$$

where:

- $GDPGR_t$ is the GDP growth rate in period t.
- β_0 is the intercept.
- β_1 is the coefficient of the 1st lag of GDPgr.
- $GDPGR_{t-1}$ is the 1st lag of GDPGR.
- β_2 is the coefficient of the lag of FCEGR.
- $FCEGR_{t-1}$ is the lag of FCEGR.
- u_t is the error term.

3.0.1 Lag Selection using BIC for ADL(p,p)

BIC on ADL(1:4,1:4)

P	BIC
2	1.9128
4	1.9718
6	2.0174
8	2.0686

Table 5: Results

The model with the lowest BIC is when P is 2 which is the ADL(1,1) model, therefore being the model that best predicts GDPGR.

3.0.2 Granger causality test

To test whether the lags of the explanatory variable FCEGR are jointly significant predictors of GDPR.

As the ADL(p, p) model is ADL(1,1) model then there is no need to carry out the granger causality test on the lags of FCEGR since there is only one lag of FCEGR in the model.Testing whether the lag of FCEGR is significant in predicting $GDPGR_t$ with a t-statistic of -2.5991 which is sufficiently enough.

NOTE: Since the ADL(1,1) using ADL(2,2) to demonstrate how the Granger causality test is carried out.

3.0.3 Test whether the lags of GDPGR and FCEGR are jointly significant:

Null hypothesis is: $H_0 : \beta_1 = \beta_2 = 0$
$GDPGR_{t-1}$ and $F\,\mathrm{CEGR}_{t-1}$ are not jointly significant on predicting GDPGR_t

Analysis and Interpretation: We reject the null hypothesis and conclude that statistically, the lags of GDPGR and FCEGR are jointly significant in predicting GDPGR basing on the F-statistic of the ADL(1,1). Furthermore, our result shows that we have enough evidence to be convinced that at one, five and ten percent levels of significance, the first lags of the two variables jointly predict GDPGR.

Table 6: ADL(1,1) model

	Dependent variable:
	GDPgr_l
GDPgr_lag1	0.497^{***}
	(0.064)
FCEgr_lag1	-0.170^{***}
	(0.056)
Constant	1.549^{***}
	(0.261)
Observations	183
R^2	0.273
Adjusted R^2	0.264
Residual Std. Error	2.514 (df = 180)
F Statistic	33.722^{***} (df = 2; 180)
Note:	*p<0.1; **p<0.05; ***p<0.01

The predicted model from the ADL(1,1) estimation as shown in the table above is: The equation is given by:

$$GDPgr_l = 1.549 + 0.497 GDPgr_l ag_1 - 0.170 FCEgr_l ag_1 \tag{7}$$

Both regressors are individually significant in forecasting GDP at a 5 percent level of significance. Though the coefficient of the lag of GDPGR shows a positive relationship between past GDPGR in the past and future GDPGR, meaning that a positive GDPGR in past would impact GDPGR positively in the future. The coefficient of of the lag of government final consumption expenditure is a negative. This tells explains that the impact of past FCEGR on future GDPGR is weaker than the impact of lag of GDPGR and the nature of relationship is negative - meaning that all things being equal, positive FCEGR in the past will impact GDPGR negatively in the future. Also, From the F-statistics on the ADL(1,1) model, the conclusion iis that the lags of first lags GDPGR and FCEGR are jointly significant in predicting GDPGR.

4 Pseudo Out-Of-Sample Forecast Performance of Real GDPGR

The standard error of our regression from 2008: Q4 to 2019: Q4 is

- Within Sample SER is 2.486287
- Approximate RMSFE is 2.605025

The t-statistic testing if our mean out of sample forecast error is -1.2838.The estimated root mean square forecasted error is larger than the with-sample errors but by a small difference. So, do not reject the null hypothesis given the t-statistic for a t-test with 44 degrees of freedom which suggests that the model can predict out of sample. For better understanding, also look at the plot below of how the model predicts Canada's GDPGR.

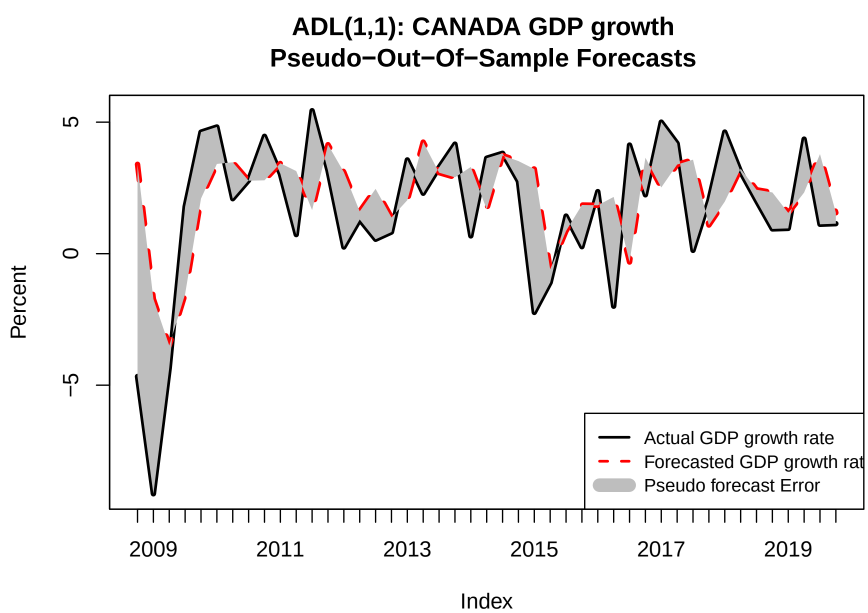

Figure 4: Canada Real GDPGR Forecasting

Figure 4 above shows that our forecasted GDPGR prediction followed the same pattern with the actual GDPGR.

5 Dynamic Causal Effects

Using the GLS to estimate the dynamic multipliers for a distributed lag model: First, we estimate the DL (3): The equation is given by:

$$GDPGR_t = \beta_0 + \beta_1 FCEGR_t + \beta_2 FCEGR_{t-1} + \beta_3 FCEGR_{t-2} + u_t \tag{8}$$

where:

- $GDPGR_t$ is the GDPGR in period t.
- β_0 is the intercept.
- β_1, β_2, and β_3 are dynamic multipliers.
- $FCEGR_t$ is FCEGR at time t.
- $FCEGR_{t-1}$ is the 1st lag of FCEGR.
- $FCEGR_{t-2}$ is the 2nd lag of FCEGR.
- u_t is the error term.

The next step is extracting residuals from the model we estimated and use it to estimate an AR (1) model of the residuals: The equation is given by:

$$\hat{u}_t = \hat{\phi}_1 \hat{u}_{t-1} + \tilde{u}_t \tag{9}$$

where:

- $\hat{u}_t$ is the estimated error term at time t.
- $\hat{\phi}_1$ is the estimated coefficient for the lagged error term.
- $\hat{u}_{t-1}$ is the estimated error term at time $t-1$.
- $\tilde{u}_t$ is the residual error term.

Use the ˆ1 value from the equation above to estimate the quasi difference of our variables and then estimate the GLS model. The equations are given by:

$$F\widehat{CEG}R_t = FCEGR_t - \hat{\phi}_1 FCEGR_{t-1}$$
$$F\widehat{CEG}R_{t-1} = FCEGR_{t-1} - \hat{\phi}_1 FCEGR_{t-2}$$
$$F\widehat{CEG}R_{t-2} = FCEGR_{t-2} - \hat{\phi}_1 FCEGR_{t-3}$$
$$G\widehat{DP}GR_t = GDPGR_t - \hat{\phi}_1 GDPGR_{t-1}$$

where:

- $F\widehat{CEG}R_t$, $F\widehat{CEG}R_{t-1}$, and $F\widehat{CEG}R_{t-2}$ are the predicted values of FCEGR at time t, $t-1$, and $t-2$ respectively.
- $G\widehat{DP}GR_t$ is the predicted value of GDPGR at time t.
- $FCEGR_t$, $FCEGR_{t-1}$, $FCEGR_{t-2}$, and $FCEGR_{t-3}$ are the actual values of FCEGR at time t, $t-1$, $t-2$, and $t-3$ respectively.

- $GDPGR_t$ and $GDPGR_{t-1}$ are the actual values of GDPGR at time t and $t-1$ respectively.
- $\hat{\phi}_1$ is the estimated coefficient for the lagged variable.

Then estimate the GLS model below: The equation is given by:

$$GD\hat{P}GR_t = \alpha_0 + \beta_1 FC\hat{E}GR_t + \beta_2 FC\hat{E}GR_{t-1} + \beta_3 FC\hat{E}GR_{t-2} + \hat{u}_t \quad (10)$$

where:

- $G\widehat{DP}GR_t$ is the predicted value of GDPGR at time t.
- $F\widehat{CE}GR_t$, $F\widehat{CE}GR_{t-1}$, and $F\widehat{CE}GR_{t-2}$ are the predicted values of FCEGR at time t, $t-1$, and $t-2$ respectively.
- α_0 is the intercept.
- β_1, β_2, and β_3 are the coefficients of the predicted FCEGR at times t, $t-1$, and $t-2$ respectively.
- $\hat{u}_t$ is the estimated residual error term.

Since future final government spending has no bearing on current final government spending, it is simple to assume rigorous exogeneity in this case. An illustration of this would be a situation in which people know that government spending will increase in the upcoming period, and this leads to GDP growth because people spend more money now on capital goods in preparation for the anticipated boost to the economy that government spending will provide in the upcoming period. Because expected government expenditure in the upcoming time has affected GDP growth in the current period, this breaks stringent exogeneity.

Implications of the dynamic multipliers: The model's greatest dynamic multiplier, FCEGR's initial lag, is the only one that is significant at the 5 percent significance level. At the five percent significant level, the remainder is deemed inconsequential. over a coefficient of -0.217, the first dynamic multiplier's influence is diminishing over time.

Table 7: DL3 Model

	Dependent variable:
	GDPgr_tilde_xts
FCEgr_tilde_xts	0.022
	(0.059)
FCEgr_tilde_lag1_xts	-0.166^{**}
	(0.068)
FCEgr_tilde_lag2_xts	-0.093
	(0.057)
Constant	1.440^{***}
	(0.226)
Observations	181
R^2	0.053
Adjusted R^2	0.037
Residual Std. Error	2.517 (df = 177)
F Statistic	3.333^{**} (df = 3; 177)
Note:	*p<0.1; **p<0.05; ***p<0.01

6 Multiperiod Forecasts

6.1 Iterated multiperiod Forecasting

To run a multiperiod forecast for our ADL (1,1) model, we will make use of VAR(p=1) to estimate GDPGR in time t and FCEGR in time t. From the regression of these models, we will extract our coefficients that will be used to forecast $GDP\ GR$ and $F\ CEGR$ in the future. Models:

The equations are given by:

$$\text{GDPGR}_t = \beta_{1,0} + \beta_{1,1} GDPGR_{t-1} + \beta_{1,2} FCEGR_{t-1} + u_t$$

$$\text{FCEGR}_t = \beta_{2,0} + \beta_{2,1} GDPGR_{t-1} + \beta_{2,2} FCEGR_{t-1} + u_t$$

where:

- $GDPGR_t$ $FCEGR_t$ are the GDPGR and FCEGR at time t respectively.
- $\beta_{1,0}$ and $\beta_{2,0}$ are intercepts.
- $\beta_{1,1}$ and $\beta_{2,1}$ are coefficients for the lagged GDPGR at time $t-1$.
- $\beta_{1,2}$ and $\beta_{2,2}$ are coefficients for the lagged FCEGR at time $t-1$.
- u_t is the error term.

Forecast output in the tenth period:

1. Iterated forecast (GDPGR): 2.453973
2. Direct Multiperiod forecast (GDPGR): 2.434877
3. Iterated forecast (FCEGR): 1.84012
4. Direct Multiperiod forecast (FCEGR): 1.875772

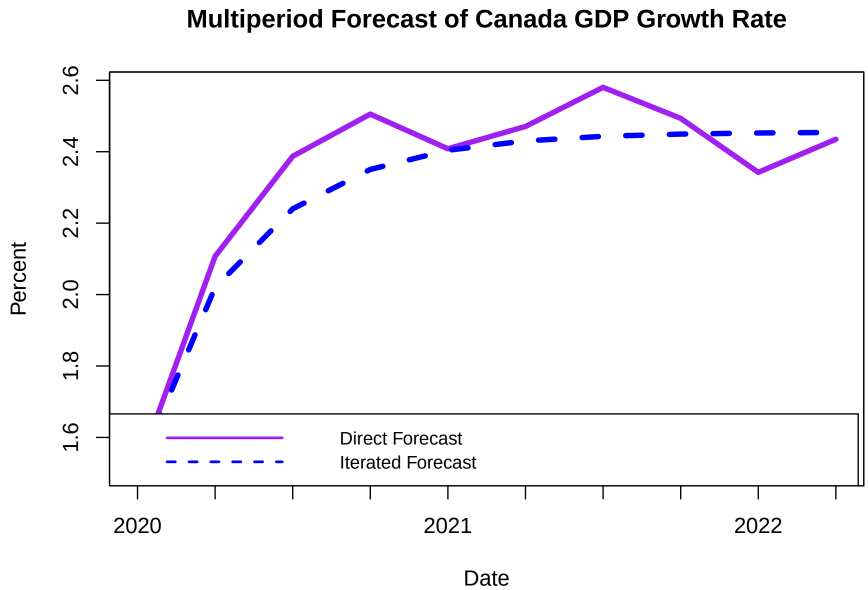

Figure 5: Canada real GDP Direct and Iterated Forecasts

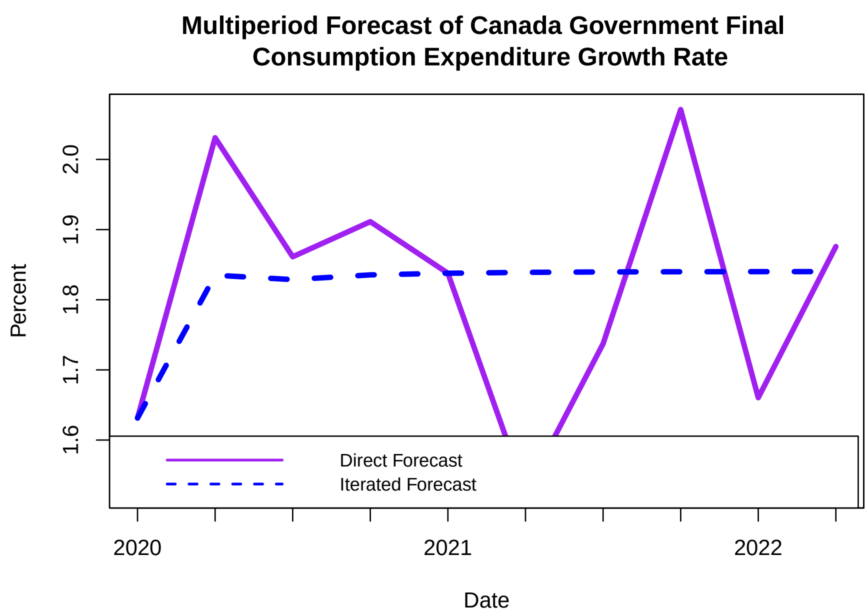

Figure 6: Canada FCE growth rate direct and iterated forecasts

7 Cointegration

The conditions for cointegration require that the GDPGR and FCEGR follow a unit root process and that using a differencing factor, both become stationary. Conducting a cointegration test on GDPGR and FCEGR is not viable since they are stationary. Look at the figure on the next page.

7.1 Cointegration test

As this time series does not follow a trend and is unknown, then the next step is to estimate it by selecting the coefficient of FCEGR when GDPGR is regressed on FCEGR The equation is given by:

$$\hat{z}_t = GDPGR_t - \hat{\theta} FCEGR_t \tag{11}$$

where:

- $\hat{z}_t$ is the estimated value of z at time t.
- $GDPGR_t$ is the GDPGR at time t.
- $\hat{\theta}$ is the estimated coefficient for F CEGR.
- $FCEGR_t$ is the FCEGR at time t.

Upon doing the GLS Dickey-Fuller test on $\hat{z}_t, thet-statisticvaluecomesouttobe-$ 4.5018$.Thisindicatesthatthenullhypothesisshouldberejected, indicatingthat \hat{z}_t isstationaryandthatGDPGRan$

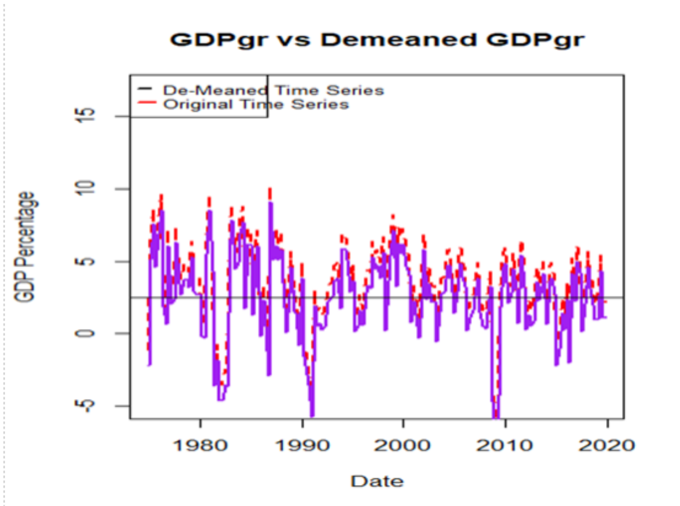

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Figure 7: GDPGR vs Deamed GDPGR

7.2 Volatility Clustering Analysis

Running the ARMA (1,1) – GARCH (1,1) model requires first estimating the models below, The equations are given by:
$\mathrm{Y}_t = \mu + \sigma_t \varepsilon_t + \phi \mathrm{Y}_{t-1} + \theta_1 \varepsilon_{t-1}$
$\sigma_t^2 = \omega + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2$ where:

- Y_t is the value of Y at time t.
- μ is the intercept.
- σ_t is the standard deviation at time t.
- ε_t is the error term at time t.
- ϕ is the coefficient for the lagged value of Y.
- θ_1 is the coefficient for the lagged error term.
- ω is a constant.
- α_1 and β_1 are coefficients.
- u_{t-1}^2 is the squared value of the error term at time $t-1$.

GARCH MODEL RESULTS

	Dependent variable:
	GDPgr_withoutBreak
mu	1.619***
	(0.607)
ar1	0.365
	(0.224)
ma1	0.031
	(0.265)
omega	1.902**
	(0.830)
alpha1	0.290***
	(0.108)
beta1	0.424**
	(0.173)
Observations	184
Log Likelihood	424.245
Akaike Inf. Crit.	4.677
Bayesian Inf. Crit.	4.781
Note:	*p<0.1; **p<0.05; ***p<0.01

From the Garch analysis above, the coefficients mu, omega, alpha1 and beta1 are significant at a 5 percent level. This indicates that while variation has an impact on variance today, prior errors do not significantly affect errors today. It follows that extended periods of volatility will have a greater overall impact. The graph below similarly displays this behaviour.

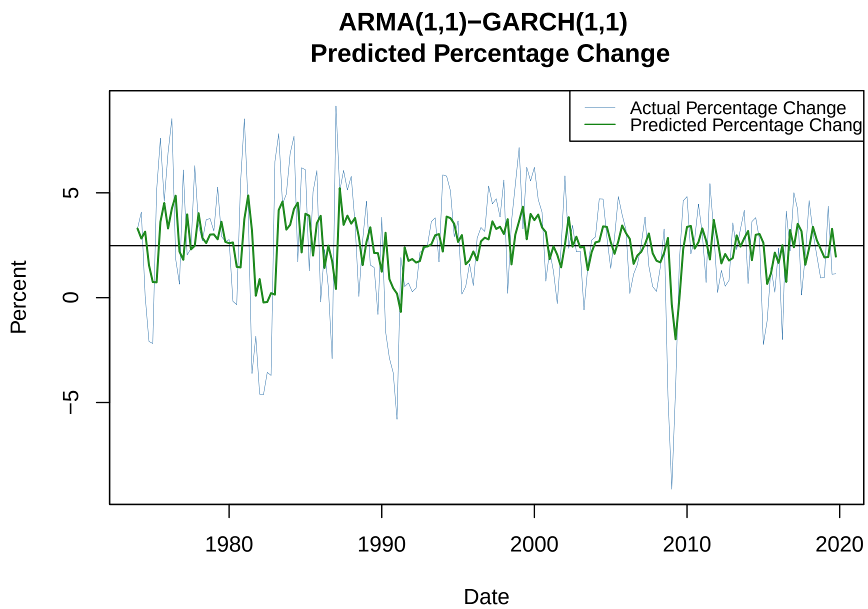

Figure 8: Predicted change