An Introduction to Measure Theory

GrammCyao

Update: Wednesday 1st December, 2021

Contents

CHAPTER I

Measure theory

§1.1 Prologue: The problem of measure

1.1.1 Elementary measure

Before we discuss Jordan measure, we discuss the even simpler notion of *elementary measure*, which allows one to measure a very simple class of sets, namely the *elementary sets* (finite unions of boxes).

Definition 1.1.1 (Intervals, boxes, elementary sets)**.** An *interval* is a subset of $\mathbf{R}$ of the form $[a, b] := \{x \in \mathbf{R} : a \leq x \leq b\}$, $[a, b) := \{x \in \mathbf{R} : a \leq x < b\}$, $(a, b] := \{x \in \mathbf{R} : a < x \leq b\}$, or $(a, b) := \{x \in \mathbf{R} : a < x < b\}$, where $a \leq b$ are real numbers. We define the *length* $|I|$ of an interval $I = [a, b], [a, b), (a, b], (a, b)$ to be $|I| := b - a$. A *box* in $\mathbf{R}^d$ is a Cartesian product $B := \prod_{i=1}^d I_i$ of d intervals $I_1, \cdots, I_d$, thus for instance, an interval is a one-dimensional box. The *volume* $|B|$ of such a box B is defined as $|B| := \prod_{i=1}^d |I_i|$. An *elementary set* is any subset of $\mathbf{R}^d$ which is the union of a finite number of boxes.

Proposition 1.1.2 (Boolean closure). *If $E, F \subset \mathbf{R}^d$ are elementary sets, then the union $E \cup F$, the intersection $E \cap F$, and the set theoretic difference $E \setminus F := \{x \in E : x \notin F\}$, and the symmetric difference $E \triangle F := (E \setminus F) \cup (F \setminus E)$ are also elementary. If $x \in \mathbf{R}^d$, then the translate $E + x := \{y + x : y \in E\}$ is also an elementary set.*

Proof. Suppose that E and F can be represented as the finite union of $A_1, \cdots, A_m$ and the finite union of $B_1, \cdots, B_n$, respectively, where $m, n \leq d$.

Then we have

$$E \cup F = \Big(\bigcup_{i=1}^{m} A_i\Big) \cup \Big(\bigcup_{j=1}^{n} B_j\Big).$$

Thus $E \cup F$ is an elementary set for that it equals to the union of $m + n$ boxes.

For the intersection $E \cap F$, we have

$$E \cap F = \Big(\bigcup_{i=1}^{m} A_i\Big) \cap \Big(\bigcup_{j=1}^{n} B_j\Big) = \bigcup_{(i,j) \in \{1,\cdots,m\} \times \{1,\cdots,n\}} (A_i \cap B_j).$$

Let $A_i := I_{i,1} \times \cdots \times I_{i,d}$ and $B_j := J_{j,1} \times \cdots \times J_{j,d}$. We have

$$\begin{aligned} A_i \cap B_j &= (I_{i,1} \times \cdots \times I_{i,d}) \cap (J_{j,1} \times \cdots \times J_{j,d}) \\ &= (I_{i,1} \cap J_{j,1}) \times \cdots \times (I_{i,d} \cap J_{j,d}). \end{aligned}$$

Thus $A_i \cap B_j$ are boxes for all $(i, j) \in \{1, \cdots, m\} \times \{1, \cdots, n\}$. Therefore, $E \cap F$ is an elementary set for that it equals to the union of $m \times n$ boxes.

For the difference $E \setminus F$, for any $1 \leq j \leq n$ we have

$$E \setminus B_j = \Big(\bigcup_{i=1}^{m} A_i\Big) \setminus B_j = \bigcup_{i=1}^{m} (A_i \setminus B_j).$$

We want to show that if A and B are boxes, then $A \setminus B$ is elementary. For this, let $A := I_1 \times \cdots \times I_d$ and $B := J_1 \times \cdots \times J_d$. Then

$$A \setminus B = A \cap B^c$$

$$
\begin{aligned}
&= A \cap (K_1 \times \cdots \times K_d) \\
&= (I_1 \cap K_1) \times \cdots \times (I_d \cap K_d),
\end{aligned}
$$

where K_n equals to the union of bounded or unbounded intervals, then $I_n \cap K_n$ equals to the union of bounded intervals. Hence $A \setminus B$ equals to the union of some Cartesian sets, so that elementary.

Thus $E \setminus B_j$ is an elementary. Since

$$
E \setminus F = E \setminus \Big(\bigcup_{j=1}^{n} B_j \Big) = \bigcap_{j=1}^{n} (E \setminus B_j),
$$

by conclusion above, $E \setminus F$ is an elementary.

The symmetric difference $E \triangle F$ is an elementary set is immediately comes from above conclusions.

For the translation $E + x$, this is easy to see that

$$
E + x = \bigcup_{i=1}^{m} (A_i + x),
$$

where $A_i + x$ are boxes for all $1 \leq i \leq m$. □

We now give each elementary set a measure.

Lemma 1.1.3 (Measure of an elementary set)**.** *Let $E \subset \mathbf{R}^d$ be an elementary set.*

(i) *E can be expressed as the finite union of disjoint boxes.*

(ii) *If E is partitioned as the finite union $B_1 \cup \cdots \cup B_k$ of disjoint boxes, then the quantity $m(E) := |B_1| + \cdots + |B_k|$ is independent of the partition. In other words, given any partition $B_1' \cup \cdots \cup B_{k'}'$ of E, one has $|B_1| + \cdots + |B_k| = |B_1'| + \cdots + |B_{k'}'|$.*

We refer to $m(E)$ as the elementary measure of E.

Proof. We first prove (i) in the one-dimensional case $d = 1$. Given any finite collection of intervals $I_1, \cdots, I_k$, one can place the $2k$ endpoints of these intervals in increasing order (discarding repetitions). Looking at the open intervals between these endpoints, together with the endpoints themselves (viewed as intervals of length zero), we see that there exists a finite collection of disjoint intervals $J_1, \cdots, J_{k'}$ such that each of the $I_1, \cdots, I_k$ are a union of some subcollection of the $J_1, \cdots, J_{k'}$. This already gives (i) when $d = 1$. To prove the higher dimensional case, we express E as the union $B_1, \cdots, B_k$ of boxes $B_i = I_{i,1} \times \cdots \times I_{i,d}$. For each $j = 1, \cdots, d$, we use the one-dimensional argument to express $I_{1,j}, \cdots, I_{k,j}$ as the union of subcollections of a collection $J_{1,j}, \cdots, J_{k'_j,j}$ of disjoint intervals. Taking Cartesian products, we can express the $B_1, \cdots, B_k$ as finite unions of boxes $J_{i_1,1} \times \cdots \times J_{i_d,d}$, where $1 \le i_j \le k'_j$ for all $1 \le j \le d$. Such boxes are all disjoint, and the claim follows.

To prove (ii) we let $\mathcal{B}$ be the collection of $B_1, \cdots, B_k$ and $\mathcal{B}'$ be the collection of $B'_1, \cdots, B'_{k'}$. Then we can define the collection

$$\mathcal{B}\#\mathcal{B}' := \{B_i \cap B'_j : B_i \in \mathcal{B} \text{ and } B'_j \in \mathcal{B}'\}.$$

Define $|\mathcal{B}| := \sum_{B_i \in \mathcal{B}} |B_i|$, and define $|\mathcal{B}'|$ and $|\mathcal{B}\#\mathcal{B}'|$ by similar process. Now we want to show that $|\mathcal{B}| = |\mathcal{B}\#\mathcal{B}'|$ and $|\mathcal{B}'| = |\mathcal{B}\#\mathcal{B}'|$, this implies that $|\mathcal{B}| = |\mathcal{B}'|$.

Since

$$B_i = B_i \cap E = B_i \cap \left(\bigcup_{j=1}^{n} B'_j \right) = \bigcup_{j=1}^{n} (B_i \cap B'_j),$$

so that

$$E = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (B_i \cap B'_j),$$

where $B_i \cap B'_j$ are disjoint boxes for all $1 \le i \le m, 1 \le j \le n$. Then B_i can be expressed as the finite union of disjoint boxes $B_i \cap B'_j$, and we have

$$|B_j| = \sum_{j=1}^{n} |B_i \cap B'_j|.$$

Thus

$$m(E) = |B_1| + \cdots + |B_k| = \sum_{i=1}^{m} \sum_{j=1}^{n} |B_i \cap B'_j|,$$

i.e., $|\mathcal{B}| = |\mathcal{B}\#\mathcal{B}'|$. A similar argument shows that $|\mathcal{B}'| = |\mathcal{B}\#\mathcal{B}'|$. Thus $|\mathcal{B}| = |\mathcal{B}'|$, as desired. □

From definitions, the elementary measure obeys following properties:

Proposition 1.1.4 (The properties of elementary measure)**.**

(i) $m(\emptyset) = 0$.

(ii) *For all boxes B, we have $m(B) = |B|$.*

(iii) *(Non-negativity) For every elementary set E, we have $m(E) \geq 0$.*

(iv) *For every E and F are disjoint elementary sets, we have $m(E \cup F) = m(E) + m(F)$.*

(v) *(Finite additivity) Let $E_1, \cdots, E_k$ be a finite sequence of disjoint elementary sets, then $m(\bigcup_{i=1}^k E_i) = \sum_{i=1}^k m(E_i)$.*

(vi) *(Monotonicity) For every elementary sets $E \subset F$, we have $m(E) \leq m(F)$.*

(vii) *For arbitrary elementary sets E and F, we have $m(E \cup F) \leq m(E) + m(F)$.*

(viii) *(Finite subadditivity) Let $E_1, \cdots, E_k$ be a finite sequence of arbitrary elementary sets, then $m(\bigcup_{i=1}^k E_i) \leq \sum_{i=1}^k m(E_i)$.*

(ix) *(Translation invariance) For all elementary set E and $x \in \mathbf{R}^d$, we have $m(E + x) = m(E)$.*

Proof. Proof omitted. □

These properties in fact define elementary measure up to normalisation:

Theorem 1.1.5 (Uniqueness of elementary measure)**.** *Let $d \geq 1$. Let $m' : \mathcal{E}(\mathbf{R}^d) \to \mathbf{R}^+$ be a map from the collection $\mathcal{E}(\mathbf{R}^d)$ of elementary subsets of $\mathbf{R}^d$ to the non-negative reals that obeys the non-negativity, finite additivity, and translation invariance properties. Then there exists a constant $c \in \mathbf{R}^+$ such that $m'(E) = cm(E)$ for all elementary sets E.*

In particular, if we impose the additional normalisation $m'([0,1)^d) = 1$, then $m' \equiv m$.

Proof. We first prove the statement in the one-dimensional case $d = 1$. This will give an intuition about the proof.

Let m' be a map from $\mathcal{E}(\mathbf{R})$ to $\mathbf{R}^+$ which obeys non-negativity, finite additivity, and translation invariance. Set $c := m'([0,1))$. Since $[0,1) = \bigcup_{k=1}^{n}[\frac{k-1}{n}, \frac{k}{n})$ where $[\frac{k-1}{n}, \frac{k}{n})$ are disjoint intervals for all $k \in [0,n]$, and we also have $[0, \frac{1}{n}) = [\frac{k-1}{n}, \frac{k}{n}) - \frac{k-1}{n}$, hence by finite additivity and translation invariance, we have

$$\begin{aligned} m'([0,1)) &= m'\Big(\bigcup_{k=1}^{n}\Big[\frac{k-1}{n}, \frac{k}{n}\Big)\Big) \\ &= \sum_{k=1}^{n} m'\Big(\Big[\frac{k-1}{n}, \frac{k}{n}\Big)\Big) \\ &= n \times m'\Big(\Big[0, \frac{1}{n}\Big)\Big). \end{aligned}$$

Thus for every $E := [0, \frac{1}{n}) \in \mathcal{E}(\mathbf{R})$, there is a $c = m'([0,1))$ such that $m'(E) = m'([0,1)) \times \frac{1}{n} = cm(E)$.

Now we extend $[0, \frac{1}{n})$ to rational case $[0, \frac{p}{q})$, where $p, q \in \mathbf{Z}^+$. Because $[0,p) = \bigcup_{k=1}^{q}[\frac{(k-1)p}{q}, \frac{kp}{q})$ and $[0, \frac{p}{q}) = [\frac{(k-1)p}{q}, \frac{kp}{q}) - \frac{(k-1)p}{q}$, by the finite additivity and translation invariance, we have

$$\begin{aligned} m'([0,p)) &= m'\Big(\bigcup_{k=1}^{q}\Big[\frac{(k-1)p}{q}, \frac{kp}{q}\Big)\Big) \\ &= \sum_{k=1}^{q} m'\Big(\Big[\frac{(k-1)p}{q}, \frac{kp}{q}\Big)\Big) \\ &= q \times m'\Big(\Big[0, \frac{p}{q}\Big)\Big). \end{aligned}$$

Thus $m'([0, \frac{p}{q})) = \frac{1}{q} m'([0,p))$. By finite additivity and translation invariance,

$$m'([0,p)) = m'\Big(\bigcup_{k=1}^{p}[k-1,k)\Big) = \sum_{k=1}^{p} m'([k-1,k)) = p \times m'([0,1)).$$

Thus for $E = [0, \frac{p}{q}) \in \mathcal{E}(\mathbf{R})$ we have $m'(E) = \frac{p}{q} m'([0,1)) = cm(E)$.

Furthermore, we extend to real interval $[0, x]$ where $x \in \mathbf{R}^+$. From non-negativity and finite additivity, we conclude the monotonicity property $m'(E) \leq m'(F)$ whenever $E \subset F$. Then for every $x \in \mathbf{R}^+$, there is a $r, s \in \mathbf{Q}^+$ where $0 < r < s$ such that $r \leq x \leq s$. ($\mathbf{Q}$ is dense in $\mathbf{R}$.) By monotonicity and conclusion above, we have

$$m'([0, r)) \leq m'([0, x)) \leq m'([0, s)),$$

this implies

$$cr \leq m'([0, x)) \leq cs.$$

Recall that we define reals as the limit of Cauchy sequence of rationals, then for $\varepsilon > 0$, we have

$$c(x - \varepsilon) \leq cr \leq m'([0, x)) \leq cs \leq c(x + \varepsilon).$$

Since ε is arbitrary, we have $m'([0, x)) = cx = cm([0, x))$. This is easy to see that every $E \in \mathcal{E}(\mathbf{R})$ can be expressed as the disjoint union of real intervals, using finite additivity and translation invariance, we have $m'(E) = cm(E)$ for all elementary set E, as desired.

Now we briefly show the higher dimensional case. Set $c := m'([0, 1)^d)$. Consider d-dimensional box $[0, 1)^d \in \mathcal{E}(\mathbf{R}^d)$, by Definition 1.1.1, it can be expressed as the Cartesian product of intervals $[0, 1)$. We dividing $[0, 1)$ into n disjoint intervals, then $[0, 1)^d$ is divided into n^d disjoint boxes (see Lemma 1.1.3). Hence by finite additivity and translation invariance,

$$m'([0, 1)^d) = n^d \times m'\Big(\Big[0, \frac{1}{n}\Big)^d\Big),$$

and we have $m'([0, \frac{1}{n})^d) = \frac{1}{n^d} m'([0, 1)^d) = cm([0, \frac{1}{n})^d)$.

Similarly, we have

$$m'([0, p)^d) = q^d \times m'\Big(\Big[0, \frac{p}{q}\Big)^d\Big)$$

and

$$m'([0, p)^d) = p^d \times m'([0, 1)^d).$$

This implies that $m'([0,\frac{p}{q})^d) = (\frac{p}{q})^d m'([0,1)^d) = cm([0,\frac{p}{q})^d)$.

And for real box case, we have

$$c(x-\varepsilon)^d \le cr^d \le m'([0,x)^d) \le cs^d \le c(x+\varepsilon)^d,$$

this implies $m'([0,x)^d) = cx^d = cm([0,x)^d)$.

Because every elementary set $E \in \mathcal{E}(\mathbf{R}^d)$ can be expressed as the disjoint union of $[0,x)^d$, we have $m'(E) = cm(E)$. In particular, if we normalize $m'([0,1)^d) = 1$, we have $m' \equiv m$. □

Lemma 1.1.6. *Let $d_1, d_2 \ge 1$, and let $E_1 \subset \mathbf{R}^{d_1}, E_2 \subset \mathbf{R}^{d_2}$ be elementary sets. Then $E_1 \times E_2 \subset \mathbf{R}^{d_1+d_2}$ is elementary, and $m^{d_1+d_2}(E_1 \times E_2) = m^{d_1}(E_1) \times m^{d_2}(E_2)$.*

Proof. Let E_1 equals to the finite disjoint union of $A_1, \cdots, A_k$ and E_2 equals to the finite disjoint union of $B_1, \cdots, B_{k'}$. Then for $1 \le i \le k$, we have

$$A_i \times E_2 = A_i \times \bigcup_{j=1}^{k'} B_j = \bigcup_{j=1}^{k'} (A_i \times B_j).$$

By Definition 1.1.1, we have $|A_i \times B_j| = |A_i| \times |B_j|$. Then by Proposition 1.1.4(ii), we have

$$\begin{aligned} m^{d_1+d_2}(A_i \times E_2) &= \sum_{j=1}^{k'} m^{d_1+d_2}(A_i \times B_j) \\ &= \sum_{j=1}^{k'} |A_i \times B_j| \\ &= \sum_{j=1}^{k'} (|A_i| \times |B_j|) \\ &= |A_i| \sum_{j=1}^{k'} |B_j| \\ &= m^{d_1}(A_i) \sum_{j=1}^{k'} m^{d_2}(B_j) \end{aligned}$$

$$= m^{d_1}(A_i)m^{d_2}(E_2).$$

Thus

$$\begin{aligned} m^{d_1+d_2}(E_1 \times E_2) &= m^{d_1+d_2}\Big(\bigcup_{i=1}^{k}(A_i \times E_2)\Big) \\ &= \sum_{i=1}^{k} m^{d_1}(A_i)m^{d_2}(E_2) \\ &= m^{d_2}(E_2)\sum_{i=1}^{k} m^{d_1}(A_i) \\ &= m^{d_1}(E_1)m^{d_2}(E_2), \end{aligned}$$

as desired. □

1.1.2 Jordan measure

The elementary sets are a very restrictive class of sets, far too small for most applications. For instance, a solid triangle or disk in the plane will not be elementary, or even a rotated box. On the other hand, as essentially observed long ago by Archimedes, such sets E can be approximated from within and without by elementary sets $A \subset E \subset B$, and the inscribing elementary set A and the circumscribing elementary set B can be used to give lower and upper bounds on the putative measure of E. As one makes the approximating sets A, B increasingly fine, one can hope that these two bounds eventually match. This gives rise to the following definitions.

Definition 1.1.7 (Jordan measure)**.** Let $E \subset \mathbf{R}^d$ be a bounded set.

- The *Jordan inner measure* $m_{*,(J)}(E)$ of E is defined as

$$m_{*,(J)}(E) := \sup_{A \subset E, A \text{ elementary}} m(A).$$

- The *Jordan outer measure* $m^{*,(J)}(E)$ of E is defined as

$$m^{*,(J)}(E) := \inf_{B \supset E, B \text{ elementary}} m(B).$$

- If $m_{*,(J)}(E) = m^{*,(J)}(E)$, then we say that E is *Jordan measurable*, and call $m(E) := m_{*,(J)}(E) = m^{*,(J)}(E)$ then *Jordan measure* of E. As before, we write $m(E)$ as $m^d(E)$ when we wish to emphasise the dimension d.

Jordan measurable sets are those sets which are "almost elementary" with respect to Jordan outer measure. More precisely, we have

Proposition 1.1.8 (Characterisation of Jordan measurability)**.** *Let $E \subset \mathbf{R}^d$ be bounded. Then following are equivalent:*

(i) *E is Jordan measurable.*

(ii) *For every $\varepsilon > 0$, there exist elementary sets $A \subset E \subset B$ such that $m(B \setminus A) \leq \varepsilon$.*

(iii) *For every $\varepsilon > 0$, there exists an elementary set A such that $m^{*,(J)}(A \triangle E) \leq \varepsilon$.*

Proof. (i) $\Rightarrow$ (ii). Since E is Jordan measurable, by Definition 1.1.7, there is elementary sets $A \subset E \subset B$ such that $m(E) \leq m(A) + \varepsilon/2$ and $m(E) \geq m(B) - \varepsilon/2$. Then $m(B \setminus A) = m(B) - m(A) \leq \varepsilon$, as desired.

(ii) $\Rightarrow$ (iii). There exist elementary sets $A \subset E \subset B$ such that $E \setminus A \subset B \setminus A$. Then by Definition 1.1.7, we have

$$m^{*,(J)}(A \triangle E) = m^{*,(J)}(E \setminus A) = \inf_{C \supset E \setminus A} m(B \setminus A) \leq m(B \setminus A) \leq \varepsilon.$$

(iii) $\Rightarrow$ (i). From definition, we obviously have $m_{*,(J)}(E) \leq m^{*,(J)}(E)$. This is sufficient to show that $m^{*,(J)}(E) \leq m_{*,(J)}(E)$ so that E is Jordan measurable with $m(E) = m_{*,(J)}(E) = m^{*,(J)}(E)$.

By hypothesis, for every $\varepsilon > 0$, there exists an elementary set A such that $m^{*,(J)}(A\triangle E) \leq \varepsilon$. Since

$$\begin{aligned} m^{*,(J)}(A\triangle E) &= m^{*,(J)}((A \setminus E) \cup (E \setminus A)) \\ &= \inf_{U\cup V \supset A\triangle E: U\supset A\setminus E, V\supset E\setminus A} m(U \cup V) \\ &\geq \inf_{V\supset E\setminus A} m(V) \\ &= m^{*,(J)}(E \setminus A), \end{aligned}$$

we have $m^{*,(J)}(E \setminus A) \leq \varepsilon$ where $E \setminus A \subset E$.

Since $A \cap E \subset E$, by definition, we have

$$m_{*,(J)}(E) + \varepsilon \geq m_{*,(J)}(A \cap E) + m^{*,(J)}(E \setminus A).$$

Now we want to show that $m_{*,(J)}(A \cap E) = m^{*,(J)}(A \cap E)$, i.e., $A \cap E$ is Jordan measurable. From $m(A\triangle E) \leq \varepsilon$, there exists an elementary set $S \supset A\triangle E$ such that $m(S) \leq \varepsilon$ (Why this is true? Use the Jordan outer measure of $A\triangle E$). Since we have $A \setminus S \subset A \cap E \subset A \cup S$, then

$$\begin{aligned} m^{*,(J)}(A \cap E) - m_{*,(J)}(A \cap E) &\leq m(A \cup S) - m(A \setminus S) \\ &\leq m(A) + m(S) - m(A) + m(S) \\ &\leq 2\varepsilon, \end{aligned}$$

for arbitrary ε. Thus we have $m_{*,(J)}(A \cap E) = m^{*,(J)}(A \cap E)$. Together our conclusions, we have

$$m_{*,(J)}(E) + \varepsilon \geq m^{*,(J)}(A \cap E) + m^{*,(J)}(E \setminus A).$$

Then we show that $m^{*,(J)}(A \cap E) + m^{*,(J)}(E \setminus A) \geq m^{*,(J)}(E)$. We can see that $A \cap E$ and $E \setminus A$ are disjoint, then for every elementary set D containing E can be divided into the union of elementary sets D_1 and D_2 where $D_1 \supset A \cap E$ and $D_2 \supset E \setminus A$. Notice that D_1 and D_2 are not necessary to be disjoint. Thus

$$\begin{aligned} m^{*,(J)}(E) &= \inf_{D\supset E} m(D) \\ &\leq \inf_{D_1\cup D_2\supset E} (m(D_1) + m(D_2)) \end{aligned}$$

$$
\begin{aligned}
&= \inf_{D_1 \supset A \cap E} m(D_1) + \inf_{D_2 \supset E \setminus A} m(D_2) \\
&= m^{*,(J)}(A \cap E) + m^{*,(J)}(E \setminus A).
\end{aligned}
$$

Therefore, we have $m_{*,(J)}(E) + \varepsilon \geq m^{*,(J)}(E)$. Because ε is arbitrary, we have $m_{*,(J)}(E) = m^{*,(J)}(E)$, as desired. This shows that E is Jordan measurable and we complete the proof. □

Remark. As one corollary of this proposition, we see that every elementary set E is Jordan measurable, and that Jordan measure and elementary measure coincide for such sets; this justifies the use of $m(E)$ to denote both. In particular, we still have $m(\emptyset) = 0$.

Jordan measurability inherits many of properties of elementary measure:

Proposition 1.1.9 (The properties of Jordan measure)**.** *Let $E, F \subset \mathbf{R}^d$ be Jordan measurable sets.*

(i) *(Boolean closure) $E \cup F$, $E \cap F$, $E \setminus F$, and $E \triangle F$ are Jordan measurable.*

(ii) *(Non-negativity) $m(E) \geq 0$.*

(iii) *(Finite additivity) If E, F are disjoint, then $m(E \cup F) = m(E) + m(F)$.*

(iv) *(Monotonicity) If $E \subset F$, then $m(E) \leq m(F)$.*

(v) *(Finite subadditivity) $m(E \cup F) \leq m(E) + m(F)$.*

(vi) *(Translation invariance) For any $x \in \mathbf{R}^d$, $E + x$ is Jordan measurable, and $m(E + x) = m(E)$.*

Proof. (i) To show the union $E \cup F$ is Jordan measurable. Since E and F are Jordan measurable, by Proposition 1.1.8(ii), for every $\varepsilon > 0$ there are elementary sets $A \subset E \subset B$ and $C \subset F \subset D$ such that $m(B \setminus A) \leq \varepsilon/4$ and $m(D \setminus C) \leq \varepsilon/4$. Since $A \cup C \subset E \cup F \subset B \cup D$, and

$$
\begin{aligned}
(B \cup D) \setminus (A \cup C) &= ((B \cup D) \setminus A) \cap ((B \cup D) \setminus C) \\
&= ((B \setminus A) \cup (D \setminus A)) \cap ((B \setminus C) \cup (D \setminus C))
\end{aligned}
$$

$$
\begin{aligned}
&= ((B \setminus A) \cap (B \setminus C)) \cup ((D \setminus A) \cap (D \setminus C)) \\
&\quad \cup ((B \setminus A) \cap (D \setminus C)) \cup ((D \setminus A) \cap (B \setminus C)) \\
&\subset (B \setminus A) \cup (D \setminus C) \cup (B \setminus A) \cup (D \setminus C),
\end{aligned}
$$

by monotonicity and finite subadditivity, there exist $A \cup C \subset E \cup F \subset B \cup D$ such that

$$m((B \cup D) \setminus (A \cup C)) \leq 2m(B \setminus A) + 2m(D \setminus C) \leq \varepsilon.$$

Thus by Proposition 1.1.8(ii), $E \cup F$ is Jordan measurable.

To show the intersection $E \cap F$ is Jordan measurable. From Proposition 1.1.8(ii), for every $\varepsilon > 0$ there is elementary sets $A \subset E \subset B$ and $C \subset F \subset D$ such that $m(B \setminus A) \leq \varepsilon/2$ and $m(D \setminus C) \leq \varepsilon/2$. Since $A \cap C \subset E \cap F \subset B \cap D$, and

$$
\begin{aligned}
(B \cap D) \setminus (A \cap C) &= ((B \cap D) \setminus A) \cup ((B \cap D) \setminus C) \\
&= ((B \setminus A) \cap (D \setminus A)) \cup ((B \setminus C) \cap (D \setminus C)) \\
&\subset (B \setminus A) \cup (D \setminus C),
\end{aligned}
$$

by monotonicity and finite subadditivity, there exist $A \cap C \subset E \cap F \subset B \cap D$ such that

$$m((B \cup D) \setminus (A \cup C)) \leq m(B \setminus A) + m(D \setminus C) \leq \varepsilon.$$

Thus by Proposition 1.1.8(ii), $E \cap F$ is Jordan measurable.

To show the difference $E \setminus F$. From Proposition 1.1.8(ii), for every $\varepsilon > 0$ there is elementary sets $A \subset E \subset B$ and $C \subset F \subset D$ such that $m(B \setminus A) \leq \varepsilon$ and $m(D \setminus C) \leq \varepsilon$. Then $A \setminus C \subset E \setminus F \subset B \setminus D$. Since

$$(B \setminus D) \setminus (A \setminus C) = (B \setminus A) \setminus (D \setminus C) \subset B \setminus A,$$

by monotonicity, there exist $A \setminus C \subset E \setminus F \subset B \setminus D$ such that

$$m((B \setminus D) \setminus (A \setminus C)) \leq m(B \setminus A) \leq \varepsilon.$$

Thus by Proposition 1.1.8(ii), $E \setminus F$ is Jordan measurable.

The symmetric difference $E \triangle F$ is Jordan measurable is immediately comes from above.

(ii) Since for every elementary set A we have $m(A) \geq 0$, by Definition 1.1.7, $m_{*,(J)}(E) = \sup_{A \subset E} m(A) \geq 0$. Thus for every Jordan measurable set E, we have $m(E) \geq 0$.

Since E and F are Jordan measurable, for every $\varepsilon > 0$, there exist elementary sets $A \subset E$ and $B \subset F$ such that $m(E) \leq m(A) + \varepsilon/2$ and $m(F) \leq m(B) + \varepsilon/2$. By finite additivity of elementary measure, we have

$$\begin{aligned} m(E \cup F) &= m_{*,(J)}(E \cup F) \\ &= \sup_{A \cup B \subset E \cup F} m(A \cup B) \\ &= \sup_{A \cup B \subset E \cup F} m(A) + m(B) \\ &\geq m(E) + m(F) - \varepsilon. \end{aligned}$$

For the other hand, for every $\varepsilon > 0$, there exist elementary sets $E \subset C$ and $F \subset D$ such that $m(E) \geq m(C) - \varepsilon/2$ and $m(F) \geq m(D) - \varepsilon/2$. Then

$$\begin{aligned} m(E \cup F) &= m^{*,(J)}(E \cup F) \\ &= \inf_{E \cup F \subset C \cup D} m(C \cup D) \\ &\leq \inf_{E \cup F \subset C \cup D} m(C) + m(D) \\ &\leq m(E) + m(F) + \varepsilon. \end{aligned}$$

Thus we conclude that for arbitrary $\varepsilon > 0$ we have

$$m(E) + m(F) - \varepsilon \leq m(E \cup F) \leq m(E) + m(F) + \varepsilon.$$

This means that $m(E \cup F) = m(E) + m(F)$, as desired.

(iv) This immediately comes from non-negativity and finite additivity of Jordan measure.

(v) Finite subadditivity immediately comes from monotonicity and finite additivity.

(vi) By Proposition 1.1.8(ii), for every $\varepsilon > 0$, there exist elementary sets $A \subset E \subset B$ such that $m(B \setminus A) \leq \varepsilon$. Then by translation invariance of elementary measure, for $x \in \mathbf{R}^d$, there exists elementary sets $A + x \subset E + x \subset B + x$ such that

$$m((B + x) \setminus (A + x)) = m(B \setminus A + x) = m(B \setminus A) \leq \varepsilon.$$

Thus $E + x$ is Jordan measurable. □

Now we give some examples of Jordan measurable sets:

Example 1.1.10 (Regions under graphs are Jordan measurable)**.** Let B be a closed box in $\mathbf{R}^d$, and let $f : B \to \mathbf{R}$ be a continuous function.

(1) Show that the graph $\{(x, f(x)) : x \in B\} \subset \mathbf{R}^{d+1}$ is Jordan measurable in $\mathbf{R}^{d+1}$ with Jordan measure zero.

(2) Show that the set $\{(x, t) : x \in B; 0 \leq t \leq f(x)\} \subset \mathbf{R}^{d+1}$ is Jordan measurable.

Proof. (1) Denote the graph of f as graph(f). Since B is closed and bounded (see Definition 1.1.1), by Heine-Borel theorem, B is compact.

For the first assertion, since f is defined on a compact metric space, continuity of f is equivalent to uniform continuity. Then by the definition of uniform continuity, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon$ whenever $x, x' \in B$ are such that $|x - x'| < \delta$. This means that every element $x \in B$ lies in some open ball $B_n := \{x \in B : |x_n - x| < \delta\}$, and we have $B = \bigcup_{n=1}^{\infty} B_n$. Since B is compact, there exists an N such that $B = \bigcup_{n=1}^{N} B_n$.

Then for every $\varepsilon > 0$, there exist a $\delta > 0$ and $N > 0$ such that $\emptyset \subset \text{graph}(f) \subset G_n$, where

$$G_n := \bigcup_{n=1}^{N} B_n \times (f(x_n) - \varepsilon, f(x_n) + \varepsilon),$$

such that

$$m(G_n \setminus \emptyset) = m(G_n) \leq 2\varepsilon m(B).$$

Thus by Proposition 1.1.8(ii), graph(f) is Jordan measurable. Since ε is arbitrary, by non-negativity and monotonicity, we have $m(\text{graph}(f)) \leq m(G_n) = 0$. Thus $m(\text{graph}(f)) = 0$, as desired.

(2) Denote the set as S. Since f is uniformly continuous, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\bigcup_{n=1}^{N} B_n \times [0, f(x_n) - \varepsilon] \subset S \subset \bigcup_{n=1}^{N} B_n \times [0, f(x_n) + \varepsilon]$$

such that

$$\begin{aligned}
& m\Big(\bigcup_{n=1}^{N} B_n \times [0, f(x_n) + \varepsilon] \setminus \bigcup_{n=1}^{N} B_n \times [0, f(x_n) - \varepsilon]\Big) \\
\leq\ & m\Big(\bigcup_{n=1}^{N} B_n \times \bigcup_{n=1}^{N} [0, f(x_n) + \varepsilon]\Big) - m\Big(\bigcup_{n=1}^{N} B_n \times \bigcup_{n=1}^{N} [0, f(x_n) - \varepsilon]\Big) \\
\leq\ & m(B) \sum_{n=1}^{N} (f(x_n) + \varepsilon) - m(B) \sum_{n=1}^{N} (f(x_n) - \varepsilon) \\
=\ & 2\varepsilon m(B).
\end{aligned}$$

Because ε is arbitrary, S is Jordan measurable. □

Example 1.1.11. Let A, B, C be three points in $\mathbf{R}^2$.

(1) Show that the solid triangle with vertices A, B, C is Jordan measurable.

(2) Show that the Jordan measure of the solid triangle is equal to $\frac{1}{2}|(B - A) \wedge (C - A)|$, where $|(a, b) \wedge (c, d)| := |ad - bc|$.

Proof. (1) Let $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$. Suppose that $x_1 < x_2 < x_3$ and $y_1 < y_3$. We first consider that the edge AB is horizontal to x-axis, i.e., $A = (x_1, y_1), B = (x_2, y_1)$. Define the edges AB, AC, and BC as the lines pass through the points and restricting the domain on the corresponding intervals. The solid triangle is the area under the graph as following:

$$\triangle ABC := \{(x, t) : x \in [x_1, x_3]; AB \leq t \leq AC, BC \leq t \leq AC\}.$$

Since

$$\begin{aligned}
U_1 &:= \{(x,t) : x \in [x_1, x_2]; 0 \le t \le AC\},\\
U_2 &:= \{(x,t) : x \in [x_1, x_2]; 0 \le t < AB\},\\
U_3 &:= \{(x,t) : x \in [x_2, x_3]; 0 \le t \le AC\},\\
U_4 &:= \{(x,t) : x \in [x_2, x_3]; 0 \le t < BC\},
\end{aligned}$$

are Jordan measurable from Exercise 1.1.10(2), we have

$$\triangle ABC = (U_1 \setminus U_2) \cup (U_3 \setminus U_4)$$

is Jordan measurable by Proposition 1.1.9.

Without loss of generality, let $AB : [x_1, y_1] \to \mathbf{R}$, $AC : [x_2, y_2] \to \mathbf{R}$, $BC : [x_3, y_3] \to \mathbf{R}$ be the lines segments pass through the corresponding vertices. Define

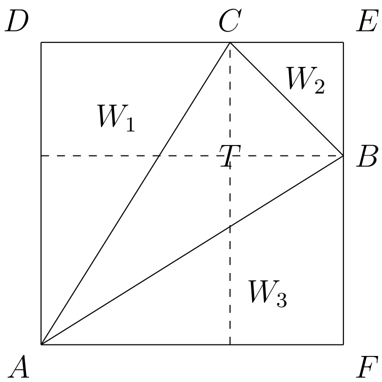
$$\begin{aligned}
V_1 &:= \{(x,t) : x \in [x_1, y_1] \cap [x_2, y_2]; t \in [AC, AB] \cup [AB, AC]\},\\
V_2 &:= \{(x,t) : x \in [x_2, y_2] \cap [x_3, y_3]; t \in [AC, BC] \cup [BC, AC]\},\\
V_3 &:= \{(x,t) : x \in [x_1, y_1] \cap [x_3, y_3]; t \in [AB, BC] \cup [BC, AC]\},
\end{aligned}$$

where V_1, V_2, V_3 are Jordan measurable (use Example 1.1.10). Thus

$$\triangle ABC = V_1 \cup V_2 \cup V_3$$

is Jordan measurable from Proposition 1.1.9.

(2) Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$. Let $a = x_2 - x_1$, $b = y_2 - y_1$, $c = x_3 - x_1$, $d = y_3 - y_1$. Then by translation invariance, we have $A = (0,0)$, $B = (a,b)$, $C = (c,d)$, it would not change the areas. We have shown that the triangle is Jordan measurable, denote the square $ADEF$ as S, then we can see that $S = W_1 \cup W_2 \cup W_3 \cup T$ which is Jordan measurable. Since the edges have zero measure, we can just ignore them and $T = S \setminus (W_1 \cup W_2 \cup W_3)$.



This is easy to see that $m(S) = ad$, $m(W_1) = \frac{1}{2}cd$, $m(W_2) = \frac{1}{2}(a - c)(d - b)$, $m(W_3) = \frac{1}{2}ab$. Since $m(W_1 \cap W_2 \cap W_3) = 0$, we have

$$\begin{aligned} m(W_1 \cup W_2 \cup W_3) &= m(W_1) + m(W_2) + m(W_3) - m(W_1 \cap W_2 \cap W_3) \\ &= \frac{1}{2}(cd + (a-c)(d-b) + ab) \\ &= \frac{1}{2}(ad + cb). \end{aligned}$$

Thus

$$\begin{aligned} m(T) &= m(S \setminus (W_1 \cup W_2 \cup W_3)) \\ &= ad - \frac{1}{2}(ad + cb) \\ &= \frac{1}{2}(ad - cb) \\ &= \frac{1}{2}|(B - A) \wedge (C - A)|. \end{aligned}$$

Here, notice that $m(T \cap (W_1 \cup W_2 \cup W_3)) = 0$. □

Example 1.1.12. Show that every compact convex polytope[a] in $\mathbf{R}^d$ is Jordan measurable.

[a] A *closed convex polytope* is a subset of $\mathbf{R}^d$ formed by intersecting together finitely many closed half-spaces of the form $\{x \in \mathbf{R}^d : x \cdot v \leq c\}$, where $v \in \mathbf{R}^d, c \in$

R, and $\cdot$ denotes the usual dot product on $\mathbf{R}^d$. A *compact convex polytope* is a closed convex polytope which is also bounded.

Example 1.1.13.

(1) Show that all open and closed Euclidean balls $B(x,r) := \{y \in \mathbf{R}^d : |y-x| < r\}$, $\overline{B(x,r)} := \{y \in \mathbf{R}^d : |y-x| \leq r\}$ in $\mathbf{R}^d$ are Jordan measurable, with Jordan measure $c_d r^d$ for some constant $c_d > 0$ depending only on d

(2) Establish the crude bounds

$$\left(\frac{2}{\sqrt{d}}\right)^d \leq c_d \leq 2^d.$$

(An exact formula for c_d is $c_d = \frac{1}{d}\omega_d$, where $\omega_d := \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is volume of the unit sphere $S^{d-1} \subset \mathbf{R}^d$ and Γ is the *Gamma function*, but we will not derive this formula here.)

Example 1.1.14. Define a *Jordan null set* to be a Jordan measurable set of Jordan measure zero. Show that any subset of a Jordan null set is a Jordan null set.

Example 1.1.15. Show that

$$m(E) := \lim_{N\to\infty} \frac{1}{N^d} \#(E \cap \frac{1}{N}\mathbf{Z}^d)$$

where $\frac{1}{N}\mathbf{Z} := \{\frac{n}{N} : n \in \mathbf{Z}\}$ and $\#A$ denotes the cardinality of a finite set A, holds for all Jordan measurable $E \subset \mathbf{R}^d$.

Theorem 1.1.16 (Uniqueness of Jordan measure)**.** *Let $d \geq 1$. Let $m' : \mathcal{J}(\mathbf{R}^d) \to \mathbf{R}^+$ be a map from the collection $\mathcal{J}(\mathbf{R}^d)$ of Jordan-measurable subsets of $\mathbf{R}^d$ to the non-negative reals that obeys the non-*

negativity, finite additivity, and translation invariance properties. Then there exists a constant $c \in \mathbf{R}^+$ such that $m'(E) = cm(E)$ for all Jordan measurable sets E. In particular, if we impose the additional normalisation $m'([0,1]^d) = 1$, then $m' \equiv m$.

Proof. Set $c := m'([0,1))$. Since every elementary sets are Jordan measurable, if E is an elementary set, then from Theorem 1.1.5 we have $m'(E) = cm(E)$ (recall that elementary measure and Jordan measure are coincide for each other for elementary sets). For arbitrary elementary sets A, B that $A \subset E \subset B$, we have $m(E) = \sup_{A \subset E} m(A) = \inf_{B \supset E} m(B)$. This implies that

$$cm(E) = \sup_{A \subset E} m'(A) = \inf_{B \supset E} m'(B).$$

This is trivial to show that m' obeys monotonicity, then for every $\varepsilon > 0$ there exist elementary sets A, B such that

$$\sup_{A \subset E} m'(A) - \varepsilon \leq m'(A) \leq m'(E) \leq m'(B) \leq \inf_{B \supset E} m'(B) + \varepsilon.$$

Because ε is arbitrary, thus we have $m'(E) = cm(E)$ for all Jordan measurable sets. □

Proposition 1.1.17. *Let $d_1, d_2 \geq 1$, and let $E_1 \subset \mathbf{R}^{d_1}, E_2 \subset \mathbf{R}^{d_2}$ be Jordan measurable sets. Then $E_1 \times E_2 \subset \mathbf{R}^{d_1+d_2}$ is Jordan measurable, and $m^{d_1+d_2}(E_1 \times E_2) = m^{d_1}(E_1) \times m^{d_2}(E_2)$.*

Proof. For every $\varepsilon > 0$, there exist elementary sets $A_1 \subset E_1 \subset B_1$, $A_2 \subset E_2 \subset B_2$ such that $m(B_1 \setminus A_1) \leq \varepsilon m(B_2)/2$ and $m(B_2 \setminus A_2) \leq \varepsilon m(B_1)/2$. Since we have $A_1 \times A_2 \subset E_1 \times E_2 \subset B_1 \times B_2$, we have

$$\begin{aligned} m((B_1 \times B_2) \setminus (A_1 \times A_2)) &= m([(B_1 \setminus A_1) \times B_2] \cup [B_1 \times (B_2 \setminus A_2)]) \\ &\leq m((B_1 \setminus A_1) \times B_2) + m(B_1 \times (B_2 \setminus A_2)) \\ &= m(B_1 \setminus A_1)m(B_2) + m(B_1)m(B_2 \setminus A_2) \\ &\leq \varepsilon \end{aligned}$$

Thus $E_1 \times E_2$ is Jordan measurable.

By Lemma 1.1.6 there exist $A_1 \subset E_1$ and $A_2 \subset E_2$ such that

$$\begin{aligned} m^{d_1+d_2}(E_1 \times E_2) &\geq m^{d_1+d_2}(A_1 \times A_2) \\ &= m^{d_1}(A_1)m^{d_2}(A_2) \\ &\geq (m^{d_1}(E_1) - \varepsilon)(m^{d_2}(E_2) - \varepsilon). \end{aligned}$$

For the other hand, there is $B_1 \supset E_1$ and $B_2 \supset E_2$ such that

$$\begin{aligned} m^{d_1+d_2}(E_1 \times E_2) &\leq m^{d_1+d_2}(B_1 \times B_2) \\ &= m^{d_1}(B_1)m^{d_2}(B_2) \\ &\leq (m^{d_1}(E_1) + \varepsilon)(m^{d_2}(E_2) + \varepsilon). \end{aligned}$$

Because ε is arbitrary, we have $m^{d_1+d_2}(E_1 \times E_2) = m^{d_1}(E_1)m^{d_2}(E_2)$. □

Example 1.1.18. Let $E \subset \mathbf{R}^d$ be a bounded set.

(1) Show that E and the closure[a] $\overline{E}$ of E have the same Jordan outer measure.

(2) Show that E and the interior[b] E° of E have the same Jordan inner measure.

(3) Show that E is Jordan measurable if and only if the topological boundary ∂E of E has Jordan outer measure zero.

(4) Show that the *bullet-riddled square* $[0,1]^2 \setminus \mathbf{Q}^2$, and set of bullets $[0,1]^2 \cap \mathbf{Q}^2$, both have Jordan inner measure zero and Jordan outer measure one. In particular, both sets are not Jordan measurable.

[a] The *closure* of E is defined as the intersection of all closed sets containing E.

[b] The *interior* of E is defined as the union of all open sets contained in E.

1.1.3 Connection with the Riemann integral

Definition 1.1.19 (Riemann integrability)**.** Let $[a,b]$ be an interval of positive length, and let $f : [a,b] \to \mathbf{R}$ be a function. A *tagged partition* $\mathcal{P} = ((x_0, x_1, \cdots, x_n), (x_1^*, \cdots, x_n^*))$ of $[a,b]$ is a finite sequence of real numbers $a = x_0 < x_1 < \cdots < x_n = b$, together with additional numbers $x_{i-1} \leq x_i^* \leq x_i$ for each $i = 1, \cdots, n$. We abbreviate $x_i - x_{i-1}$ as δx_i. The quantity $\Delta(\mathcal{P}) := \sup_{1 \leq i \leq n} \delta x_i$ will be called the *norm* of the tagged partition. The *Riemann sum* $\mathcal{R}(f, \mathcal{P})$ of f with respect to the tagged partition $\mathcal{P}$ is defined as

$$\mathcal{R}(f, \mathcal{P}) := \sum_{i=1}^{n} f(x_i^*) \delta x_i.$$

We say that f is *Riemann integrable* on $[a, b]$ if there exists a real number, denote $\int_a^b f(x)dx$ and referred to as the *Riemann integral* of f on $[a, b]$, for which we have

$$\int_a^b f(x)dx = \lim_{\Delta(\mathcal{P}) \to 0} \mathcal{R}(f, \mathcal{P})$$

by which we mean that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathcal{R}(f, \mathcal{P}) - \int_a^b f(x)dx| \leq \varepsilon$ for every tagged partition $\mathcal{P}$ with $\Delta(\mathcal{P}) \leq \delta$.

If $[a, b]$ is an interval of zero length, we adopt the convention that every function $f : [a, b] \to \mathbf{R}$ is Riemann integrable, with a Riemann integral of zero.

Proposition 1.1.20 (Piecewise constant functions)**.** *Let $[a, b]$ be an interval. A* piecewise constant function *$f : [a, b] \to \mathbf{R}$ is a function for which there exists a partition of $[a, b]$ into finitely many intervals $I_1, \cdots, I_n$ such that f is equal to a constant c_i on each of the intervals I_i. If f is piecewise constant, then the expression*

$$\sum_{i=1}^{n} c_i |I_i|$$

is independent of the choice of partition used to demonstrate the piecewise constant nature of f. We denote this quantity by p.c. $\int_a^b f(x)dx$, and refer to it as the piecewise constant integral *of f on $[a, b]$.*

Proof. Since $[a,b]$ is elementary, by Proposition 1.1.3(ii), it is independent of the partition. Let $J_1, \cdots, J_m$ be a different partition of $[a,b]$ from $I_1, \cdots, I_n$ such that f be a piecewise constant function. Then $\{I_i \cap J_j : 1 \le i \le n, 1 \le j \le m\}$ is also a partition of $[a,b]$, and we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m} |I_i \cap J_j| = \sum_{i=1}^{n} |I_i| = \sum_{j=1}^{m} |J_j|.$$

We have $c_i = c_j$ for all $I_i \cap I_j$. Thus

$$\begin{aligned}
\sum_{i=1}^{n} c_i|I_i| &= \sum_{i=1}^{n} c_i \left(\sum_{j=1}^{m} |I_i \cap J_j|\right) \\
&= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} c_i|I_i \cap J_j|\right) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} c_j|I_i \cap J_j|\right) \\
&= \sum_{j=1}^{m} c_j \left(\sum_{i=1}^{n} |I_i \cap J_j|\right) \\
&= \sum_{j=1}^{m} c_j|J_j|.
\end{aligned}$$

Thus $\sum_{i=1}^{n} c_i|I_i|$ is independent of the choice of partition and we denote this quantity as $\mathrm{p.c.}\int_a^b f(x)dx := \sum_{i=1}^{n} c_i|I_i|$. □

Theorem 1.1.21 (Basic properties of the piecewise constant integral)**.** *Let $[a,b]$ be an interval, and let $f, g : [a,b] \to \mathbf{R}$ be piecewise constant functions.*

(i) *(Linearity) For any real number c, cf and $f+g$ are piecewise constant, with $\mathrm{p.c.}\int_a^b cf(x)dx = c\,\mathrm{p.c.}\int_a^b f(x)dx$ and $\mathrm{p.c.}\int_a^b f(x) + g(x)dx = \mathrm{p.c.}\int_a^b f(x) + \mathrm{p.c.}\int_a^b g(x)dx$.*

(ii) *(Monotonicity) If $f \le g$ pointwise (i.e., $f(x) \le g(x)$ for all $x \in [a,b]$), then $\mathrm{p.c.}\int_a^b f(x)dx \le \mathrm{p.c.}\int_a^b g(x)dx$.*

(iii) *(Indicator) If E is an elementary subset of $[a,b]$, then the indicator function $1_E : [a,b] \to \mathbf{R}$ (defined by setting $1_E(x) := 1$ when $x \in E$ and $1_E(x) := 0$ otherwise) is piecewise constant, and* $\mathrm{p.c.}\int_a^b 1_E(x)dx = m(E)$.

Proof. (i) Let $I_1, \cdots, I_n$ and $J_1, \cdots, J_m$ be partitions of f and g, respectively. Then f and g are piecewise constant functions with respect to the partition $\{I_i \cap J_j : 1 \le i \le n, 1 \le j \le m\}$. In each $I_i \cap J_j$, f equals to a_i and g equals to b_j.

We can see that cf is a piecewise constant and in each $I_i \cap J_j$ equals to $c \cdot a_i$, and

$$\mathrm{p.c.}\int_a^b cf(x)dx = \sum_{i=1}^n c \cdot a_i |I_i| = c \sum_{i=1}^n a_i |I_i| = c \cdot \mathrm{p.c.}\int_a^b f(x)dx.$$

We also have $f + g$ is a piecewise constant and in each $I_i \cap J_j$ equals to $a_i + b_j$, and

$$\begin{aligned}\mathrm{p.c.}\int_a^b f(x) + g(x)dx &= \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j)|I_i \cap I_j| \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i |I_i \cap I_j| + \sum_{j=1}^m \sum_{i=1}^n b_j |I_i \cap I_j| \\ &= \sum_{i=1}^n a_i |I_i| + \sum_{j=1}^m b_j |I_j| \\ &= \mathrm{p.c.}\int_a^b f(x)dx + \mathrm{p.c.}\int_a^b g(x)dx.\end{aligned}$$

(ii) Since $f \le g$, we have $a_i \le b_j$ for all $I_i \cap J_j$. Then

$$\begin{aligned}\mathrm{p.c.}\int_a^b f(x)dx &= \sum_{i=1}^n \sum_{j=1}^m a_i |I_i \cap I_j| \\ &\le \sum_{j=1}^m \sum_{i=1}^n b_j |I_i \cap I_j| \\ &= \mathrm{p.c.}\int_a^b g(x)dx.\end{aligned}$$

(iii) Because E is elementary, then $[a,b]\setminus E$ is also elementary. Then there is disjoint sequences $I_1,\cdots,I_n$ and $J_1,\cdots,J_m$ such that $E=\bigcup_{i=1}^n I_i$ and $[a,b]\setminus E=\bigcup_{j=1}^m J_j$ (see Lemma 1.1.3). Then $I_1,\cdots,I_n,J_1,\cdots,J_m$ is a partition of $[a,b]$, and 1_E is a piecewise constant function. Thus

$$\text{p. c.}\int_a^b 1_E(x)dx=\sum_{i=1}^n 1\cdot|I_i|+\sum_{j=1}^m 0\cdot|J_j|=\sum_{i=1}^n |I_i|=m(E).$$

$\square$

Definition 1.1.22 (Darboux integral)**.** Let $[a,b]$ be an interval, and $f:[a,b]\to\mathbf{R}$ be a bounded function. The *lower Darboux integral* $\underline{\int_a^b} f(x)dx$ of f on $[a,b]$ is defined as

$$\underline{\int_a^b} f(x)dx := \sup_{g\le f,\text{piecewise constant}} \text{p. c.}\int_a^b g(x)dx,$$

where g ranges over all piecewise constant functions that are pointwise bounded above by f. (The hypothesis that f is bounded ensures that the supremum is over a non-empty set.) Similarly, we define the *upper Darboux integral* $\overline{\int_a^b} f(x)dx$ of f on $[a,b]$ by the formula

$$\overline{\int_a^b} f(x)dx := \inf_{h\ge f,\text{piecewise constant}} \text{p. c.}\int_a^b h(x)dx.$$

Clearly $\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx$. If these two quantities are equal, we say that f is *Darboux integrable*, and refer to this quantity as the *Darboux integral* of f on $[a,b]$.

Remark. Note that the upper and lower Darboux integrals are related by the reflection identity

$$\overline{\int_a^b} -f(x)dx = -\underline{\int_a^b} f(x)dx.$$

Proposition 1.1.23. *Let $[a,b]$ be an interval, and $f:[a,b]\to\mathbf{R}$ be a bounded function. We say that f is Riemann integrable if and only if it is Darboux integrable, in which case the Riemann integral and Darboux integrals are equal.*

Proof. First we suppose that f is Riemann integrable. Since f is bounded, define $\underline{f}:[a,b]\to\mathbf{R}$ as $\underline{f}(x)=\inf_{x\in[x_{i-1},x_i]} f(x)$ and $\overline{f}:[a,b]\to\mathbf{R}$ as $\overline{f}(x)=\sup_{x\in[x_{i-1},x_i]} f(x)$. We can see that $\underline{f}$ and $\overline{f}$ are piecewise constant functions with respect to $\mathcal{P}$ such that $\underline{f}\le f\le\overline{f}$. Then for arbitrary piecewise constant function g that minorize f, we have $\mathrm{p.c.}\int_a^b g(x)dx\le\mathcal{R}(\underline{f},\mathcal{P})$, and piecewise constant function h that majorize f, we have $\mathrm{p.c.}\int_a^b h(x)dx\ge\mathcal{R}(\overline{f},\mathcal{P})$.

Since f is Riemann integrable, by Definition 1.1.19, for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\left|\mathcal{R}(f,\mathcal{P})-\int_a^b f(x)dx\right|\le\varepsilon$$

for every $\mathcal{P}$ with $\Delta(\mathcal{P})\le\delta$. Then

$$\mathcal{R}(\underline{f},\mathcal{P})-\varepsilon\le\mathcal{R}(f,\mathcal{P})-\varepsilon\le\int_a^b f(x)dx\le\mathcal{R}(f,\mathcal{P})+\varepsilon\le\mathcal{R}(\overline{f},\mathcal{P})+\varepsilon,$$

so that

$$\mathrm{p.c.}\int_a^b g(x)dx-\varepsilon\le\int_a^b f(x)dx\le\mathrm{p.c.}\int_a^b h(x)dx+\varepsilon$$

Because ε is arbitrary, taking the supremum and infimum, we have

$$\underline{\int_a^b} f(x)dx\le\int_a^b f(x)dx\le\overline{\int_a^b} f(x)dx.$$

For the other hand, we have $\underline{\int_a^b} f(x)dx\le\mathcal{R}(\underline{f},\mathcal{P})$ and $\overline{\int_a^b} f(x)dx\ge\mathcal{R}(\overline{f},\mathcal{P})$ from $\mathrm{p.c.}\int_a^b g(x)dx\le\mathcal{R}(\underline{f},\mathcal{P})$ and $\mathrm{p.c.}\int_a^b h(x)dx\ge\mathcal{R}(\overline{f},\mathcal{P})$. Then

$$\underline{\int_a^b} f(x)dx-\overline{\int_a^b} f(x)dx\le\mathcal{R}(\underline{f},\mathcal{P})-\mathcal{R}(\overline{f},\mathcal{P})$$

$$
\begin{aligned}
&\leq \int_a^b f(x)dx + \varepsilon - \int_a^b f(x)dx + \varepsilon \\
&= 2\varepsilon
\end{aligned}
$$

for every $\mathcal{P}$ with $\Delta(\mathcal{P}) \leq \delta$. Since ε is arbitrary, we have $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$. Thus

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$$

and f is Darboux integrable.

Conversely, suppose that f is Darboux integrable. By Definition 1.1.22, we have

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

We want to show that

$$\underline{\int_a^b} f(x)dx = \sup(\mathcal{R}(\underline{f}, \mathcal{P}))$$

and

$$\overline{\int_a^b} f(x)dx = \inf(\mathcal{R}(\overline{f}, \mathcal{P})).$$

Then Darboux integrability implies that

$$\int_a^b f(x)dx = \sup(\mathcal{R}(\underline{f}, \mathcal{P})) = \inf(\mathcal{R}(\overline{f}, \mathcal{P})).$$

This is sufficient to show that

$$\int_a^b f(x)dx = \lim_{\Delta(\mathcal{P}) \to 0} \mathcal{R}(f, \mathcal{P})$$

and f is Riemann integrable.

We already show that $\underline{\int_a^b} f(x)dx \leq \mathcal{R}(\underline{f}, \mathcal{P})$ and $\overline{\int_a^b} f(x)dx \geq \mathcal{R}(\overline{f}, \mathcal{P})$, taking the supremum and infimum, we have

$$\underline{\int_a^b} f(x)dx \leq \sup(\mathcal{R}(\underline{f}, \mathcal{P}))$$

and

$$\overline{\int_a^b} f(x)dx \geq \inf(\mathcal{R}(\overline{f}, \mathcal{P})).$$

To prove in the other direction, suppose for sake of contradiction that $\underline{\int_a^b} f(x)dx < \sup(\mathcal{R}(\underline{f}, \mathcal{P}))$. Then there exists some $\mathcal{P}$ such that

$$\underline{\int_a^b} f(x)dx < \mathcal{R}(\underline{f}, \mathcal{P}).$$

Since $\underline{f}$ minorizes f (i.e., $\underline{f} \leq f$ for all x), by Definition 1.1.22, we have

$$\mathcal{R}(\underline{f}, \mathcal{P}) \leq \underline{\int_a^b} f(x)dx,$$

a contradiction. Thus $\underline{\int_a^b} f(x)dx \geq \sup(\mathcal{R}(\underline{f}, \mathcal{P}))$. This implies that

$$\underline{\int_a^b} f(x)dx = \sup(\mathcal{R}(\underline{f}, \mathcal{P})).$$

A similar argument shows that

$$\overline{\int_a^b} f(x)dx = \inf(\mathcal{R}(\overline{f}, \mathcal{P})),$$

as desired. □

Proposition 1.1.24. *Any continuous function $f : [a, b] \to \mathbf{R}$ is Riemann integrable. More generally, any bounded, piecewise continuous*[a] *function $f : [a, b] \to \mathbf{R}$ is Riemann integrable.*

[a] A function $f : [a, b] \to \mathbf{R}$ is *piecewise continuous* if one can partition $[a, b]$ into finitely many intervals, such that f is continuous on each interval.

Proof. Since f is continuous and defined on $[a, b]$, it is bounded. By Proposition 1.1.23, we only need to show that f is Darboux integrable. From continuity, for every ε there exists a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ for all $x, y \in [a, b]$ such that $|x - y| \leq \delta$. By Archimedes principle, there exists an N such that $(b-a)/N < \delta$. Let $I_k = [a + \frac{(b-a)(k-1)}{N}, a + \frac{(b-a)k}{N}]$. Then $I_1, \cdots, I_N$ is a partition of $[a,b]$ with $|I_k| = (b-a)/N$.

From $\underline{\int_a^b} f(x)dx = \sup(\mathcal{R}(\underline{f}, \mathcal{P}))$ and $\overline{\int_a^b} f(x)dx = \inf(\mathcal{R}(\overline{f}, \mathcal{P}))$ we have

$$\underline{\int_a^b} f(x)dx \geq \sum_{k=1}^{N} (\inf_{x \in I_k} f(x))|I_k|$$

and

$$\overline{\int_a^b} f(x)dx \leq \sum_{k=1}^{N} (\sup_{x \in I_k} f(x))|I_k|,$$

so

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq \sum_{k=1}^{N} (\sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x))|I_k|.$$

Since $|f(x) - f(y)| \leq \varepsilon$ is hold for all $x, y \in I_k$ for that $|x-y| \leq |I_k| < \delta$, we have

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq \sum_{k=1}^{N} \varepsilon |I_k| = \varepsilon(b-a).$$

Since ε is arbitrary, we have

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Thus by Definition 1.1.22 and Proposition 1.1.23, f is Riemann integrable. □

Now we connect the Riemann integral to Jordan measure in two ways. First, we connect the Riemann integral to one-dimensional Jordan measure:

Theorem 1.1.25 (Basic properties of the Riemann integral)**.** *Let $[a,b]$ be an interval, and let $f, g : [a,b] \to \mathbf{R}$ be Riemann integrable.*

(i) *(Linearity) For any real number c, cf and $f+g$ are Riemann integrable, with $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ and $\int_a^b f(x) + g(x)dx =$*

$\int_a^b f(x) + \int_a^b g(x)dx$.

(ii) *(Monotonicity) If $f \leq g$ pointwise (i.e., $f(x) \leq g(x)$ for all $x \in [a,b]$) then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.*

(iii) *(Indicator) If E is a Jordan measurable of $[a,b]$, then the indicator function $1_E : [a,b] \to \mathbf{R}$ (defined by setting $1_E(x) := 1$ when $x \in E$ and $1_E(x) := 0$ otherwise) is Riemann integrable, and $\int_a^b 1_E(x)dx = m(E)$.*

These properties uniquely define the Riemann integral, in the sense that the function $f \mapsto \int_a^b f(x)dx$ is the only map from the space of Riemann integrable functions on $[a,b]$ to $\mathbf{R}$ which obeys all three of the above properties.

Proof. (i) Since f is Riemann integrable, we have

$$\int_a^b f(x)dx = \lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^n f(x_i^*)\delta x_i.$$

Then for cf

$$\int_a^b cf(x)dx = \lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^n cf(x_i^*) = c \lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^n f(x_i^*)\delta x_i$$

which is convergent. Thus cf is Riemann integrable with $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

Similarly, we have

$$\begin{aligned}\int_a^b f(x)dx + \int_a^b g(x)dx &= \lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^n f(x_i^*)\delta x_i + \lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^n g(x_i^*)\delta x_i \\ &= \lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^n (f(x_i^*) + g(x_i^*))\delta x_i\end{aligned}$$

which is convergent. Thus $f+g$ is Riemann integrable with $\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b f(x) + g(x)dx$.

(ii) Since $f \leq g$, we have

$$\lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^n f(x_i^*)\delta x_i \leq \lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^n g(x_i^*)\delta x_i,$$

thus $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

(iii) Since 1_E is bounded, by Proposition 1.1.23, we show that 1_E is Darboux integrable. By Theorem 1.1.21(iii) and Definition 1.1.22, for arbitrary elementary set $A \subset E$,

$$m(A) = \text{p. c.} \int_a^b 1_A(x)dx \leq \underline{\int_a^b} 1_E(x)dx.$$

Taking the supremum, we have $m_{*,(J)}(E) \leq \underline{\int_a^b} 1_E(x)dx$ by Definition 1.1.7. Similarly, for every elementary set $B \supset E$ where $B \subset [a,b]$, we have

$$m(B) = \text{p. c.} \int_a^b 1_B(x)dx \geq \overline{\int_a^b} 1_E(x)dx.$$

Taking the infimum, we have $m^{*,(J)}(E) \geq \overline{\int_a^b} 1_E(x)dx$.

Because E is Jordan measurable, from

$$m_{*,(J)}(E) \leq \underline{\int_a^b} 1_E(x)dx \leq \overline{\int_a^b} 1_E(x)dx \leq m^{*,(J)}(E)$$

and $m_{*,(J)}(E) = m^{*,(J)}(E)$ we have $\underline{\int_a^b} 1_E(x)dx = \overline{\int_a^b} 1_E(x)dx$. Hence 1_E is Riemann integrable.

Finally, we prove that these three properties uniquely define the Riemann integral. Let $\mathcal{R}([a,b] \to \mathbf{R})$ be the space of Riemann integrable functions on $[a,b]$ to $\mathbf{R}$. The Riemann integral is the map $\mathcal{M} : \mathcal{R}([a,b] \to \mathbf{R}) \to \mathbf{R}$. Suppose that there is another map $\mathcal{M}' : \mathcal{R}([a,b] \to \mathbf{R}) \to \mathbf{R}$ satisfying above three properties where $\mathcal{M} \neq \mathcal{M}'$. Then the property (i) means that for every Riemann integrable functions $f, g \in \mathcal{R}([a,b] \to \mathbf{R})$ and constants $a, b \in \mathbf{R}$ we have $\mathcal{M}(af + bg) = a\mathcal{M}(f) + b\mathcal{M}(g)$. This is also holds for $\mathcal{M}'$.

We already know that every piecewise constant function f is Riemann integrable, i.e., $f \in \mathcal{R}([a,b] \to \mathbf{R})$. Let f be p.c. with respect to $I_1, \cdots, I_n$ with $c_1, \cdots, c_n$. Then we have

$$f(x) = \sum_{i=1}^n c_i 1_{I_i}(x),$$

so that

$$\begin{aligned}
\mathcal{M}'(f) &= \mathcal{M}'\Big(\sum_{i=1}^{n} c_i 1_{I_i}(x)\Big) \\
&= \sum_{i=1}^{n} c_i \mathcal{M}'(1_{I_i}(x)) \\
&= \sum_{i=1}^{n} c_i m(I_i) \\
&= \sum_{i=1}^{n} c_i \mathcal{M}(1_{I_i}(x)) \\
&= \sum_{i=1}^{n} \mathcal{M}(c_i 1_{I_i}(x)) \\
&= \mathcal{M}\Big(\sum_{i=1}^{n} c_i 1_{I_i}(x)\Big) \\
&= \mathcal{M}(f).
\end{aligned}$$

Since the Riemann integral depends on the piecewise constant integral, we have $\mathcal{M}(f) = \mathcal{M}'(f)$ for arbitrary Riemann integrable function, a contradiction. □

Next, we connect the integral to two-dimensional Jordan measure:

Proposition 1.1.26 (Area interpretation of the Riemann integral)**.** *Let $[a, b]$ be an interval, and let $f : [a, b] \to \mathbf{R}$ be a bounded function. Then f is Riemann integrable if and only if the sets $E_+ := \{(x, t) : x \in [a, b]; 0 \leq t \leq f(x)\}$ and $E_- := \{(x, t) : x \in [a, b]; f(x) \leq t \leq 0\}$ are both Jordan measurable in $\mathbf{R}^2$, in which case one has*

$$\int_a^b f(x)dx = m^2(E_+) - m^2(E_-),$$

where m^2 denotes two-dimensional Jordan measure.

Proof. We first suppose that f is non-negative, then $m^2(E_-) = 0$.

Suppose that f is Riemann integrable. Let $I_1, \cdots, I_n$ be a partition of $[a,b]$, let $\overline{f}(x) = \sup_{x\in I_k} f(x)$ and $\underline{f}(x) = \inf_{x\in I_k} f(x)$. Then we have

$$E_+ \subset \overline{E} := \bigcup_{k=1}^{n} (I_k \times [0, \overline{f}(x)])$$

and

$$E_+ \supset \underline{E} := \bigcup_{k=1}^{n} (I_k \times [0, \underline{f}(x)]).$$

By Proposition 1.1.17, we have

$$m^2(\overline{E}) = \sum_{k=1}^{n} \overline{f}(x)|I_k|$$

and

$$m^2(\underline{E}) = \sum_{k=1}^{n} \underline{f}(x)|I_k|$$

Since f is Riemann integrable, we have

$$m^2(\underline{E}) = m^2(\overline{E}) = \lim_{\Delta(\mathcal{P})\to 0} \sum_{k=1}^{n} f(x_i^*)|I_k| = \int_a^b f(x)dx.$$

Thus from monotonicity of Jordan measure, we have $m^2(E_+) = \int_a^b f(x)dx$.

Conversely, suppose that

$$\int_a^b f(x)dx = m^2(E_+).$$

Let $\underline{E}, \overline{E}$ be arbitrary elementary sets such that $\underline{E} \subset E_+ \subset \overline{E}$.

Since the elementary sets' expression is independent of the choice of partition (see Proposition 1.1.3), we express elementary sets $\underline{E}$ and $\overline{E}$ as disjoint unions $\underline{E} = \bigcup_{i=1}^{n} A_i$ and $\overline{E} = \bigcup_{j=1}^{m} B_j$, where $A_i = I_{i,1} \times I_{i,2}$ and $B_j = J_{j,1} \times J_{j,2}$, such that $I_{1,1}, \cdots, I_{n,1}$ and $J_{1,1}, \cdots, J_{m,1}$ be two partition of $[a,b]$. Then we have $|I_{i,2}| \leq f(x)$ and $|J_{j,2}| \geq f(y)$ for all $x \in I_{i,1}$ and $y \in J_{j,1}$.

Now we define piecewise constant functions $g, h : [a,b] \to \mathbf{R}$ as $g(x) := |I_{i,2}|$ for $x \in I_{i,1}$ and $h(x) := |J_{j,2}|$ for $x \in J_{j,1}$. Then

$$\underline{\int_a^b} f(x)dx = \sup_{g \leq f} \text{p.c.} \int_a^b g(x)dx = \sup_{g \leq f} \sum_{i=1}^n g(x)|I_{i,1}| = \sup_{\underline{E} \subset E_+} m(\underline{E}).$$

and

$$\overline{\int_a^b} f(x)dx = \inf_{h \geq f} \text{p.c.} \int_a^b h(x)dx = \inf_{h \geq f} \sum_{i=1}^n h(x)|J_{i,1}| = \inf_{\overline{E} \supset E_+} m(\overline{E}).$$

Therefore,

$$\inf_{\overline{E} \supset E_+} m(\overline{E}) = \overline{\int_a^b} f(x)dx \leq \underline{\int_a^b} f(x)dx = \sup_{\underline{E} \subset E_+} m(\underline{E}).$$

Because E_+ is Jordan measurable, by Definition 1.1.7, there exist elementary sets $\underline{E} \subset E_+ \subset \overline{E}$ such that

$$m(E_+) = \sup_{\underline{E} \subset E_+} m(\underline{E}) = \inf_{\overline{E} \supset E_+} m(\overline{E}).$$

Thus f is Riemann integrable.

A similar argument shows that when f is negative, we have

$$\int_a^b f(x)dx = -m^2(E_-)$$

if and only if f is Riemann integrable.

Define functions f^+ and f^- from $[a,b]$ to $\mathbf{R}$ by

$$f^+(x) := \begin{cases} f(x), & \text{if } f(x) \geq 0, \\ 0, & \text{if } f(x) < 0, \end{cases}$$

and

$$f^-(x) := \begin{cases} 0, & \text{if } f(x) \geq 0, \\ -f(x), & \text{if } f(x) < 0. \end{cases}$$

Then $f = f^+ - f^-$ and

$$\int_a^b f(x)dx = \int_a^b f^+(x)dx - \int_a^b f^-(x)dx = m^2(E_+) - m^2(E_-),$$

as desired. □

§1.2 Lebesgue measure

Following example shows that not all sets are Jordan measurable, even if one restricts attention to bounded sets.

Example 1.2.1. The countable union $\bigcup_{n=1}^{\infty} E_n$ or countable intersection $\bigcap_{n=1}^{\infty} E_n$ of Jordan measurable sets $E_1, E_2, \cdots \subset \mathbf{R}$ need not be Jordan measurable, even when bounded.

Proof. We know that $[0,1] \setminus \mathbf{Q}$ is not Jordan measurable. Since $\mathbf{Q}$ is countable, then $[0,1] \setminus \mathbf{Q}$ can be represented as the countable union of open interval where each interval is Jordan measurable.

For the second assertion, suppose that $[0,1] \setminus \mathbf{Q} = \bigcup_{n=1}^{\infty} E_n$ for Jordan measurable sets $E_1, E_2, \cdots$. Then $[0,1] \setminus \bigcup_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty}([0,1] \setminus E_n)$. Since $[0,1] \setminus E_n$ are Jordan measurable for all $n \geq 1$ and $[0,1] \setminus \bigcup_{n=1}^{\infty} E_n = [0,1] \cap \mathbf{Q}$ which is not Jordan measurable. $\square$

Definition 1.2.2 (Lebesgue measurability)**.** The *Lebesgue outer measure $m^*(E)$* of E is defined as

$$m^*(E) := \inf_{\bigcup_{n=1}^{\infty} B_n \supset E; B_n \text{ boxes}} \sum_{n=1}^{\infty} |B_n|.$$

A set $E \subset \mathbf{R}^d$ is said to be *Lebesgue measurable* if, for every $\varepsilon > 0$, there exists an open set $U \subset \mathbf{R}^d$ containing E such that $m^*(U \setminus E) \leq \varepsilon$. If E is Lebesgue measurable, we refer to $m(E) := m^*(E)$ as the Lebesgue measure of E. We also write $m(E)$ as $m^d(E)$ when we wish to emphasise the dimension d.

1.2.1 Properties of Lebesgue outer measure

We begin by studying the Lebesgue outer measure m^*, which was defined earlier, and takes values in the extended non-negative real axis $[0, +\infty]$. We first record three easy properties of Lebesgue outer measure, which we will use repeatedly in the sequel without further comment:

Proposition 1.2.3 (The outer measure axioms)**.**

(i) *(Empty set)* $m^*(\emptyset) = 0$.

(ii) *(Monotonicity) If* $E \subset F \subset \mathbf{R}^d$, *then* $m^*(E) \leq m^*(F)$.

(iii) *(Countable subadditivity) If* $E_1, E_2, \cdots \subset \mathbf{R}^d$ *is a countable sequence of sets, then* $m^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

Proof. (i) Since empty set is subset of any set, in particular, we have $B_n = \emptyset$ for all $n \geq 1$. By Definition 1.2.2, $m^*(\emptyset) = \sum_{n=1}^{\infty} |B_n| = 0$.

(ii) If $E \subset F$, every collection of boxes cover F must cover E, thus $m^*(E) \leq m^*(F)$.

(iii) If $m^*(E_n) = +\infty$ for some $n \geq 1$, the conclusion is trivial. We suppose that $m^*(E_n)$ is finite for all $n \geq 1$. Let $\varepsilon > 0$. For every $n \geq 1$ there exists a cover such that $E_n \subset \bigcup_{k=1}^{\infty} A_{n,k}$ where $A_{n,k}$ are boxes. By Definition 1.2.2, for each E_n, there is an k such that

$$\sum_{k=1}^{\infty} A_{n,k} \leq m^*(E_n) + \frac{\varepsilon}{2^k}.$$

Then we have $\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{n,k}$ and

$$\begin{aligned} m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} A_{n,k} \\ &\leq \sum_{n=1}^{\infty} \Big(m^*(E_n) + \frac{\varepsilon}{2^k}\Big) \\ &= \sum_{n=1}^{\infty} m^*(E_n) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we have

$$m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} m^*(E_n),$$

as desired. ☐

Remark. Note that countable subadditivity, when combined with the empty set axiom, gives as a corollary the finite subadditivity property

$$m^*(E_1 \cup \cdots \cup E_k) \leq m^*(E_1) + \cdots + m^*(E_k)$$

for any $k \geq 0$.

It is natural to ask whether Lebesgue outer measure has the *finite additivity property*, that is to say that $m^*(E \cup F) = m^*(E) + m^*(F)$ whenever $E, F \subset \mathbf{R}^d$ are disjoint. The answer to this question is somewhat subtle: as we shall see later, we have finite additivity (and even countable additivity) when all sets involved are Lebesgue measurable, but that finite additivity (and hence also countable additivity) can break down in the non-measurable case.

Following lemma says that if disjoint sets E and F have a positive separation from each other, then the Lebesgue outer measure is finitely additive:

Lemma 1.2.4 (Finite additivity for separated sets)**.** *Let* $E, F \subset \mathbf{R}^d$ *be such that* $\operatorname{dist}(E, F) > 0$*, where*

$$\operatorname{dist}(E, F) := \inf\{|x - y| : x \in E, y \in F\}$$

is the distance[a] *between* E *and* F*. Then* $m^*(E \cup F) = m^*(E) + m^*(F)$*.*

[a] Recall from the preface that we use the usual Euclidean metric $|(x_1, \cdots, x_d)| := \sqrt{x_1^2 + \cdots + x_d^2}$ on $\mathbf{R}^d$.

Proof. Proof omitted. □

In general, disjoint sets E, F need not have a positive separation from each other (e.g. $E = [0, 1)$ and $F = [1, 2]$). But the situation improves when E, F are closed, and at least one of E, F is compact:

Lemma 1.2.5. *Let $E, F \subset \mathbf{R}^d$ be disjoint closed sets, with at least one of E, F being compact. Then* $\operatorname{dist}(E, F) > 0$.

Proof. Let E be compact and F closed. Suppose for sake of contradiction that $\operatorname{dist}(E, F) = 0$. Then there exist sequences $(x_n)_{n=1}^\infty$ in E and $(y_n)_{n=1}^\infty$ in F such that $\lim_{n\to\infty} |x_n - y_n| = 0$. Since E is compact, there is a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ which converges to x. Then

$$\lim_{k\to\infty} |x - y_{n_k}| \leq \lim_{k\to\infty} (|x - x_{n_k}| + |x_{n_k} - y_{n_k}|) = 0.$$

This means that x is an adherent point of F. Since F is closed, it contains all of its adherent points, thus we have $x \in F$ so that $E \cap F \neq \emptyset$, a contradiction.

For the counterexample, since every singleton is closed in $\mathbf{R}^d$. We can see that $\mathbf{Z}^+$ and $E := \{n + \frac{1}{n} : n \in \mathbf{Z}^+\}$ are closed and disjoint. But $\operatorname{dist}(\mathbf{Z}^+, E) = \inf_{n\in\mathbf{N}} \frac{1}{n} = 0$. □

From definition we know that countable sets have Lebesgue outer measure zero. Now we start computing the outer measure of some other sets. We begin with *elementary sets*:

Lemma 1.2.6 (Outer measure of elementary sets)**.** *Let E be an elementary set. Then the Lebesgue outer measure $m^*(E)$ of E is equal to the elementary measure $m(E)$ of E, i.e., $m^*(E) = m(E)$.*

Proof. Proof omitted. □

Remark. The above lemma allows us to compute the Lebesgue outer measure of a finite union of boxes. From this and monotonicity we conclude that the Lebesgue outer measure of any set is bounded below by its Jordan inner measure. As it is also bounded above by the Jordan outer measure, we have

$$m_{*,(J)}(E) \leq m^*(E) \leq m^{*,(J)}(E) \tag{1.1}$$

for every $E \subset \mathbf{R}^d$.

Now we turn to *countable unions of boxes*. First we define almost disjoint:

Definition 1.2.7 (Almost disjoint)**.** We say that two boxes are *almost disjoint* if their interiors are disjoint.

With Lemma 1.2.6 has the following consequence:

Lemma 1.2.8 (Outer measure of countable unions of almost disjoint boxes)**.** *Let $E = \bigcup_{n=1}^{\infty} B_n$ be a countable union of almost disjoint boxes $B_1, B_2, \cdots$. Then*

$$m^*(E) = \sum_{n=1}^{\infty} |B_n|.$$

Thus, for instance, $\mathbf{R}^d$ itself has an infinite outer measure.

Proof. Proof omitted. □

Lemma 1.2.9. *If a set $E \subset \mathbf{R}^d$ is expressible as the countable union of almost disjoint boxes, then the Lebesgue outer measure of E is equal to the Jordan inner measure: $m^*(E) = m_{*,(J)}(E)$, where*

$$m_{*,(J)}(E) := \sup_{\bigcup_{n=1}^{k} B_n \subset E; B_n \, boxes} \sum_{n=1}^{k} |B_n|,$$

and $m_{*,(J)}(E) = +\infty$ when E is unbounded.

Proof. Let $E = \bigcup_{n=1}^{\infty} B_n$ be a countable union of almost disjoint boxes $B_1, B_2, \cdots$. Then for all $k \in \mathbf{N}$ we have $E \supset \bigcup_{n=1}^{k} B_n$. From Lemma 1.2.8 we have

$$m^*(E) = \sum_{n=1}^{\infty} |B_n| \geq \sum_{n=1}^{k} |B_n|. \tag{1.2}$$

Since the inequality is hold for all $k \in \mathbf{N}$, taking the supremum, we have $m^*(E) \geq m_{*,(J)}(E)$.

For the other hand, we want to show that $m^*(E) \leq m_{*,(J)}(E)$. Since

$$\sum_{n=1}^{k} |B_n| \leq m_{*,(J)}(E)$$

for all k. Letting $k \to \infty$, we have $m^*(E) = m_{*,(J)}(E)$, as desired. □

Lemma 1.2.10. *Let $E \subset \mathbf{R}^d$ be an open set. Then E can be expressed as the countable union of almost disjoint boxes (and, in fact, as the countable union of almost disjoint closed cubes).*

Proof. Proof omitted. □

Lemma 1.2.11 (Outer regularity)**.** *Let $E \subset \mathbf{R}^d$ be an arbitrary set. Then one has*

$$m^*(E) = \inf_{E \subset U; U \text{ open}} m^*(U). \tag{1.3}$$

Proof. Proof omitted. □

1.2.2 Lebesgue measurability

We now define the notion of a Lebesgue measurable set as one which can be efficiently contained in open sets in the sense of Definition 1.2.2, and set out their basic properties.

First, we show that there are plenty of Lebesgue measurable sets.

Lemma 1.2.12 (Existence of Lebesgue measurable sets)**.**

(i) *Every open set is Lebesgue measurable.*

(ii) *Every closed set is Lebesgue measurable.*

(iii) *Every set of Lebesgue outer measure zero is measurable. (Such sets are called* null sets*.)*

(iv) *The empty set* $\emptyset$ *is Lebesgue measurable.*

(v) *If* $E \subset \mathbf{R}^d$ *is Lebesgue measurable, then so is its complement* $\mathbf{R}^d \setminus E$*.*

(vi) *If* $E_1, E_2, \cdots \subset \mathbf{R}^d$ *are a sequence of Lebesgue measurable sets, then the union* $\bigcup_{n=1}^{\infty} E_n$ *is Lebesgue measurable.*

(vii) *If* $E_1, E_2, \cdots \subset \mathbf{R}^d$ *are a sequence of Lebesgue measurable sets, then the intersection* $\bigcap_{n=1}^{\infty} E_n$ *is Lebesgue measurable.*

Theorem 1.2.13 (Criteria for measruability)**.** *Let* $E \subset \mathbf{R}^d$*. The following are equivalent:*

(i) E *is Lebesgue measurable.*

(ii) *(Outer approximation by open) For every* $\varepsilon > 0$*, there exists an open set* $U \supset E$ *with* $m^*(U \setminus E) \leq \varepsilon$

(iii) *(Almost open) For every* $\varepsilon > 0$*, there exists an open* U *such that* $m^*(U \triangle E) \leq \varepsilon$*. (In other words,* E *differs from an open set by a set of outer measure at most* ε*.)*

(iv) *(Inner approximation by closed) For every* $\varepsilon > 0$*, there exists a closed set* $F \subset E$ *with* $m^*(E \setminus F) \leq \varepsilon$*.*

(v) *(Almost closed) For every* $\varepsilon > 0$*, there exists a closed set* F *such that* $m^*(F \triangle E) \leq \varepsilon$*. (In other words,* E *differs from a closed set by a set of outer measure at most* ε*.)*

(vi) *(Almost measurable) For every* $\varepsilon > 0$*, there exists a Lebesgue measurable set* E_ε *such that* $m^*(E_\varepsilon \triangle E) \leq \varepsilon$*. (In other words,* E *differs from a measurable set by a set of outer measure at most* ε*.)*

Proof. By Definition 1.2.2, (i) and (ii) are equivalent.

(ii) $\Rightarrow$ (iii). For every $\varepsilon > 0$, there exists an open set $U \supset E$ with $m^*(U \setminus E) \leq \varepsilon$. Then

$$m^*(U \triangle E) = m^*((U \setminus E) \cup (E \setminus U)) = m^*(U \setminus E) \leq \varepsilon.$$

(iii) $\Rightarrow$ (vi). By Lemma 1.2.12(i), U is Lebesgue measurable set, then let $E_\varepsilon := U$ we have $m^*(E_\varepsilon \triangle E) \le \varepsilon$.

(vi) $\Rightarrow$ (i). For every $n \in \mathbf{Z}^+$ and every $\varepsilon > 0$, there exists a Lebesgue measurable set $E_{\varepsilon/2^n}$ such that $m^*(E_{\varepsilon/2^n} \triangle E) \le \varepsilon/2^n$. Let $F := \bigcup_{n=1}^\infty E_{\varepsilon/2^n}$. We have

$$m^*(E \setminus F) = m^*\Big(E \setminus \bigcup_{n=1}^\infty E_{\varepsilon/2^n}\Big) \le m^*(E \setminus E_{\varepsilon/2^n}) \le m^*(E_{\varepsilon/2^n} \triangle E) \le \varepsilon/2^n,$$

since ε is arbitrary and the inequality is hold for all $n \ge 1$, we have $m^*(E \setminus F) = 0$. For the other hand, we have

$$\begin{aligned} m^*(F \setminus E) &= m^*\Big(\bigcup_{n=1}^\infty E_{\varepsilon/2^n} \setminus E\Big) \\ &= m^*\Big(\bigcup_{n=1}^\infty (E_{\varepsilon/2^n} \setminus E)\Big) \\ &\le m^*\Big(\bigcup_{n=1}^\infty (E_{\varepsilon/2^n} \triangle E)\Big) \\ &\le \sum_{n=1}^\infty m^*(E_{\varepsilon/2^n} \triangle E) \\ &= \varepsilon \end{aligned}$$

for arbitrary ε, thus $m^*(F \setminus E) = 0$. Hence, by Lemma 1.2.12(iii), $E \setminus F$ and $F \setminus E$ are Lebesgue measurable.

Now we want to show that there exists an open set $U \supset E$ with $m^*(U \setminus E) \le \varepsilon$. Because $F \cup E = F \cup (E \setminus F)$, by Lemma 1.2.12(vi), $F \cup E$ is Lebesgue measurable. Thus for every $\varepsilon > 0$ there exists an open set $U \supset F \cup E$ with $m^*(U \setminus (F \cup E)) \le \varepsilon$. Obviously, we have $U \supset E$. Then

$$\begin{aligned} m^*(U \setminus E) &= m^*((U \setminus (F \cup E)) \cup (F \setminus E)) \\ &\le m^*(U \setminus (F \cup E)) + m^*(F \setminus E) \\ &= 2\varepsilon. \end{aligned}$$

Thus E is Lebesgue measurable.

(iv) $\Rightarrow$ (v). For every $\varepsilon > 0$, there exists a closed set $F \subset E$ with $m^*(E \setminus F) \leq \varepsilon$. Then

$$m^*(F \triangle E) \leq m^*(E \setminus F) \leq \varepsilon.$$

(v) $\Rightarrow$ (vi). By Lemma 1.2.12(ii), F is Lebesgue measurable. Then we have $m^*(E_\varepsilon \triangle E) \leq \varepsilon$ for $E_\varepsilon := F$.

(i) $\Rightarrow$ (iv). By Lemma 1.2.12(v), E is Lebesgue measurable implies that $\mathbf{R}^d \setminus E$ is also Lebesgue measurable. Then there exists an open set $U \supset \mathbf{R}^d \setminus E$ with

$$m^*(U \setminus (\mathbf{R}^d \setminus E)) = m^*(E \setminus (\mathbf{R}^d \setminus U)),$$

where $\mathbf{R}^d \setminus U$ is closed and $\mathbf{R}^d \setminus U \subset E$. This complete the proof. □

Proposition 1.2.14. *Every Jordan measurable set is Lebesgue measurable.*

Proof. By Definition 1.2.2, elementary set is Lebesgue measurable. By Proposition 1.1.8(iii), for every Jordan measurable set E, there is an elementary set A such that $m^{*,(J)}(A \triangle E) \leq \varepsilon$. Then by Definition 1.2.2, there exists an elementary set (which is Lebesgue measurable) such that

$$m^*(A \triangle E) \leq m^{*,(J)}(A \triangle E) \leq \varepsilon.$$

Here, we use the inequality that $m^*(F) \leq m^{*,(J)}(F)$ for every $F \subset \mathbf{R}^d$. Thus by Theorem 1.2.13(vi), E is Lebesgue measurable. □

Every countable set has outer measure 0. A reasonable question arises about whether the converse holds. In other words, is every set with outer measure 0 countable? unfortunately, this is not always true, the *Cantor set* gives a counterexample:

Example 1.2.15 (Middle thirds Cantor set)**.** Let $I_0 := [0, 1]$ be the unit interval, let $I_1 := [0, 1/3] \cup [2/3, 1]$ be I_0 with the interior of the middle third interval removed, let $I_2 := [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup$

$[8/9, 1]$ be I_1 with the interior of the middle third of each of the two intervals of I_1 removed, and so forth. More formally, write

$$I_n := \bigcup_{a_1,\cdots,a_n \in \{0,2\}} \left[\sum_{i=1}^{n} \frac{a_i}{3^i}, \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n} \right].$$

Let $C := \bigcap_{n=1}^{\infty} I_n$ be the intersection of all the elementary sets I_n. Then C is compact, uncountable, and a null set.

Proof. We first show that C is compact. Since every I_n is the union of finitely many closed intervals, I_n is closed for all $n \in \mathbf{N}$. Thus C is closed. For every $a_1, \cdots, a_n \in \{0, 2\}$, we have

$$0 \leq \sum_{i=1}^{n} \frac{a_i}{3^i} \leq \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n} \leq 1.$$

So that $I_n \subset [0, 1]$, and C is also bounded by $[0, 1]$. Because C is closed and bounded, it is compact.

Then we show that C is uncountable. Consider the function $\Lambda : \{0, 2\}^{\mathbf{N}} \to [0, 1]$ where $(a_i)_{i=1}^{\infty} \in \{0, 2\}^{\mathbf{N}}$ (recall that $a_n : \mathbf{N} \to \{0, 2\}$) defined as

$$\Lambda((a_i)_{i=1}^{\infty}) := \sum_{i=1}^{\infty} \frac{a_i}{3^i}.$$

Because the series is convergent, function Λ is well-defined. We can see that for every $n \in \mathbf{N}$ we have $\sum_{i=1}^{n} \frac{a_i}{3^i} \in I_n$. Then

$$\Lambda((a_i)_{i=1}^{\infty}) = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{a_i}{3^i} \in \bigcap_{n=1}^{\infty} I_n = C.$$

We want to show that Λ is injective. Suppose for sake of contradiction that there exists two different sequences $(a_i)_{i=1}^{\infty}, (b_i)_{i=1}^{\infty} \in \{0, 2\}^{\mathbf{N}}$ such that $\Lambda((a_i)_{i=1}^{\infty}) = \Lambda((b_i)_{i=1}^{\infty})$. Suppose that these two sequences are different from the kth term, i.e., $a_k \neq b_k$. Let $a_k > b_k$. Then from

$$\sum_{i=1}^{\infty} \frac{a_i}{3^i} - \sum_{i=1}^{\infty} \frac{b_i}{3^i} = \frac{2}{3^k} + \sum_{i=k+1}^{\infty} \frac{a_i - b_i}{3^i} = 0,$$

we have

$$\frac{2}{3^k} = \sum_{i=k+1}^{\infty} \frac{b_i - a_i}{3^i} \leq \sum_{i=k+1}^{\infty} \frac{2}{3^i}$$

This implies that

$$1 \leq \sum_{i=1}^{\infty} \frac{1}{3^i} = \frac{1}{2},$$

a contradiction, Λ is injective. Since $\{0,2\}^{\mathbf{N}}$ is uncountable, $\Lambda(\{0,2\}^{\mathbf{N}}) \subset C$ is also uncountable. Thus C is uncountable.

Now we show that C is a null set. Since $C \subset I_n$ for all $n \in \mathbf{N}$, we have $m(C) \leq m(I_n)$. We can see that I_n is the union of disjoint closed intervals, we have $m(I_n) = \frac{n}{3^n}$. As $n \to \infty$, we have $m(C) \leq m(I_n) = 0$. Thus C is a null set. □

Example 1.2.16. The half-open interval $[0,1)$ cannot be expressed as the countable union of disjoint closed *intervals*. In general, $[0,1)$ cannot be expressed as the countable union of disjoint closed *sets*.

Now we look at the Lebesgue measure $m(E)$ of a Lebesgue measurable set E, which is defined to equal its Lebesgue outer measure $m^*(E)$. If E is Jordan measurable, we see from (1.2) that the Lebesgue measure and the Jordan measure of E coincide, thus Lebesgue measure extends Jordan measure. This justifies the use of the notation $m(E)$ to denote both Lebesgue measure of a Lebesgue measurable set, and Jordan measure of a Jordan measurable set (as well as elementary measure of an elementary set).

Lebesgue measure obeys significantly better properties than Lebesgue outer measure, when restricted to lebesgue measurable sets:

Lemma 1.2.17 (The measure axioms)**.**

(i) *(Empty set)* $m(\emptyset) = 0$.

(ii) *(Countable additivity) If* $E_1, E_2, \dots \subset \mathbf{R}^d$ *is a countable sequence* *of disjoint Lebesgue measurable sets, then*

$$m\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} m(E_n).$$

Proof. Proof omitted. $\square$

Theorem 1.2.18 (Monotone convergence thm for measurable sets)**.**

(i) *(Upward monotone convergence) Let $E_1 \subset E_2 \subset \cdots \subset \mathbf{R}^n$ be a countable non-decreasing sequence of Lebesgue measurable sets. Then we have $m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$.*

(ii) *(Downward monotone convergence) Let $\mathbf{R}^d \supset E_1 \supset E_2 \supset \cdots$ be a countable non-increasing sequence of Lebesgue measurable sets. If at least one of the $m(E_n)$ is finite, then we have $m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$.*

Proof. (i) Let $A_1 := E_1$ and let $A_n := E_n \setminus E_{n-1}$. Then for every $n \geq 1$, A_n are disjoint. Since $E_n = \bigcup_{i=1}^{n} A_i$ and $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n$, by Lemma 1.2.17(ii),

$$\begin{aligned} m\Big(\bigcup_{n=1}^{\infty} E_n\Big) &= m\Big(\bigcup_{n=1}^{\infty} A_n\Big) \\ &= \sum_{n=1}^{\infty} m(A_n) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} m(A_i) \\ &= \lim_{n\to\infty} m\Big(\bigcup_{i=1}^{n} A_i\Big) \\ &= \lim_{n\to\infty} m(E_n). \end{aligned}$$

(ii) Let $A_1 := \emptyset$ and $A_n = E_1 \setminus A_n$. Then we have $A_n \subset A_{n+1}$ for all

$n \geq 1$. From above conclusion, we have

$$m\Big(\bigcup_{n=1}^{\infty} A_n\Big) = m\Big(\bigcup_{n=1}^{\infty}(E_1 \setminus E_n)\Big) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} m(E_1 \setminus E_n).$$

Since $\bigcup_{n=1}^{\infty}(E_1 \setminus E_n) = E_1 \setminus \bigcap_{n=1}^{\infty} E_n$, we have

$$m\Big(\bigcup_{n=1}^{\infty}(E_1 \setminus E_n)\Big) = m\Big(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} m(E_1 \setminus E_n).$$

Use additivity of Lebesgue measure, we see that RHS of the second equality above can be written as

$$\lim_{n\to\infty} m(E_1 \setminus E_n) = \lim_{n\to\infty} (m(E_1) - m(E_n)) = m(E_1) - \lim_{n\to\infty} m(E_n),$$

and LHS of the second equality can be written as

$$m\Big(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\Big) = m(E_1) - m\Big(\bigcap_{n=1}^{\infty} E_n\Big).$$

Thus we have

$$m(E_1) - \lim_{n\to\infty} m(E_n) = m(E_1) - m\Big(\bigcap_{n=1}^{\infty} E_n\Big).$$

This means that $m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$, as desired. $\square$

Corollary 1.2.19. *Any map $E \mapsto m(E)$ from Lebesgue measurable sets to elements of $[0, +\infty]$ that obeys the above empty set and countable additivity axioms will also obey the monotonicity and countable subadditivity axioms from Lemma 1.2.3, when restricted to Lebesgue measurable sets of course.*

Proof. Let μ be arbitrary map from Lebesgue measurable sets to $[0, +\infty]$ obeys the Lemma 1.2.17. Monotonicity and countable subadditivity are trivial when there is unbounded set. We assume that all of sets are bounded, so that with finite measure.

For Lebesgue measurable sets E, F where $E \subset F$, we have E and $F \setminus E$ are two disjoint Lebesgue measurable sets. From finite additivity (which is the implication of countable additivity), we have

$$\mu(F) = \mu(E \cup (F \setminus E)) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

For the countable subadditivity, let $E_1, E_2, \cdots$ be the sequence of Lebesgue measurable sets. Let $E := \bigcup_{n=1}^{\infty} E_n$. For every E_n we can find a subset $F_n \subset E_n$ which is defined as $F_n := E_n \setminus \bigcup_{i=1}^{n-1} E_i$. Then $(F_n)_{n=1}^{\infty}$ consisting a disjoint sequence and we have

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \left(E_n \setminus \bigcup_{i=1}^{n-1} E_i \right) = \bigcup_{n=1}^{\infty} E_n,$$

this holds for that $\bigcup_{i=1}^{0} E_i = \emptyset$.

Now consider the sequence $(E \cap E_n)_{n=1}^{\infty}$, we have

$$\begin{aligned} \mu(E) &= \mu\left(\bigcup_{n=1}^{\infty} E \cap E_n \right) \\ &\le \mu\left(\bigcup_{n=1}^{\infty} E_n \right) \\ &= \mu\left(\bigcup_{n=1}^{\infty} F_n \right) \\ &= \sum_{n=1}^{\infty} \mu(F_n) \\ &= \sum_{n=1}^{\infty} \mu\left(E_n \setminus \bigcup_{i=1}^{n-1} E_i \right) \\ &\le \sum_{n=1}^{\infty} \mu(E_n), \end{aligned}$$

where the second line holds from the monotonicity; the fourth and fifth lines hold from the countable additivity. □