Half range Fourier Series :- If only cosine term or only sine term present in the Fourier series, then it is called Half range Forier series :

Half range cosine series:- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$; $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

Half range sine series:

$f(x) = \cancel{\frac{a_0}{2}} \cancel{+} \sum_{n=1}^{\infty} b_n \sin nx\,dx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$.

Since the function is generally defined on $(0,\pi)$ which is half of $(-\pi,\pi)$. For this reason, it is called half range Fourier series.

Fourier series in Different Intervals:

(i) $f(x)$ is defined in $(0,\pi)$ ~~a then~~ with period π:

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,dx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$;

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$.

(ii) $f(x)$ is defined on $(0, 2\pi)$ with period 2π.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$; $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx.$

(iii) $f(x)$ in $(-l, l)$ with period $2l$.

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$, where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\,dx$

Parseval's Identity for Fourier Series

If the Fourier series for $f(x)$ converges in $(-l, l)$

then $\frac{1}{l}\int_{-l}^{l} \{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$; where

a_0, a_n, b_n are Fourier constants.

Proof :- we know

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \text{——— (1)}$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx.$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\,dx.$

Now multiplying both sides of (i) by $f(x)$, then integrating from $-l$ to l.

$$\int_{-l}^{l}\{f(x)\}^2\,dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx\right.$$

$$\left. + b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right).$$

$$= \frac{a_0}{2}.a_0 l + \sum_{n=1}^{\infty}(a_n \cdot a_n l + b_n \cdot b_n l)$$

or, $\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$. Proved.

Prob: 1 : Expand in Fourier sine and cosine series of the function $f(x) = x + x^2$ in the interval $-\pi < x < \pi$. Also show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

A:- We have $f(x) = x + x^2$; $-\pi < x < \pi$

In the interval $-\pi < x < \pi$, Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{—— (1)}$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2)\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{3} \right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3} \right) \right]$$

$$= \frac{1}{\pi} \cdot \frac{2\pi^3}{3} = \frac{2}{3}\pi^2.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx\, dx.$$

$$= \frac{1}{\pi} \left[(x + x^2) \cdot \frac{\sin nx}{n} - \int_{-\pi}^{\pi} (1 + 2x) \cdot \frac{\sin nx}{n}\, dx \right]$$

$$= \frac{1}{\pi} \left[(x + x^2) \frac{\sin nx}{n} + \frac{\cos nx}{n^2} - \int_{-\pi}^{\pi} 2x \frac{\sin nx}{n}\, dx \right]$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} + \frac{\cos nx}{n^2} - \left\{2x\cdot\frac{\cos nx}{n^2} - \int_{-\pi}^{\pi} 2\cdot\left(-\frac{\cos nx}{n^2}\right)dx\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} + \frac{\cos nx}{n^2} + 2x\cdot\frac{\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(0 + \frac{\cos n\pi}{n^2} + \frac{2\pi\cos n\pi}{n^2} + 0\right) - \left(0 + \frac{\cos n\pi}{n^2} - \frac{2\pi\cos n\pi}{n^2} + 0\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + 2\pi\cdot\frac{(-1)^n}{n^2} - \frac{(-1)^n}{n^2} + 2\pi\frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\cdot 4\pi\frac{(-1)^n}{n^2} = \frac{4(-1)^n}{n^2}$$

Now

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[-(x+x^2)\frac{\cos nx}{n} + \int_{-\pi}^{\pi}(1+2x)\cdot\frac{\cos nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[-(x+x^2)\frac{\cos nx}{n} + \frac{\sin nx}{n^2} + \left\{2x\cdot\frac{\sin nx}{n^2} - \int_{-\pi}^{\pi}2\cdot\frac{\sin nx}{n^2}dx\right\}\right]$$

$$= \frac{1}{\pi}\left[-(x+x^2)\frac{\cos nx}{n} + \frac{\sin nx}{n^2} + 2x\frac{\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{-(\pi+\pi^2)\frac{\cos n\pi}{n} + 0 + 0 + \frac{2\cos n\pi}{n^3}\right\} - \left\{-(-\pi+\pi^2)\frac{\cos n\pi}{n} + 0 + 0 + \frac{2\cos n\pi}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{(-1)^n}{n} + \frac{2}{n^3}(-1)^n + (-\pi+\pi^2)\frac{(-1)^n}{n} - \frac{2}{n^3}(-1)^n\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n}(-\pi-\pi^2-\pi+\pi^2)\right]$$

$$= \frac{1}{\pi}(-2\pi)\cdot\frac{(-1)^n}{n} = -2\frac{(-1)^n}{n}.$$

Now putting a_0, a_n, b_n in (i);

$$f(x) = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\pi^2 + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx - \frac{2(-1)^n}{n}\sin nx\right]$$

$$= \frac{\pi^2}{3} + 4\left\{-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right\}$$
$$-2\left\{-\frac{\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \cdots\right\}$$

$$f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$
$$+ 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right] \quad \ldots (2)$$

__2nd part__: At the end points $x = \pm\pi$.

$$f(x) = \frac{1}{2}[f(x)+f(x)]$$

~~$f(\pi) = \frac{1}{2}[f(-\pi+0) + f(\pi-0)]$~~

$$\therefore f(\pm\pi) = \frac{1}{2}[f(-\pi+0) + f(\pi-0)]$$

$$= \frac{1}{2}[-\pi+\pi^2+\pi+\pi^2] \qquad \text{Since } f(x) = x+x^2$$

$$= \frac{2\pi^2}{2} = \pi^2.$$

Now putting $x = \pi$ in (2) ;

$$f(\pi) = \frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \cdots\cdots\right)$$
$$+ 2(0)$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots\right)$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots\right)$$

$$2\frac{\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots\right)$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\cdots$$ proved