Fourier

Q: Expand $f(x) = x$, $0 < x < 2$ in half range Fourier sine series and after integrating it express $f(x) = x^2$, $0 < x < 2$ as Fourier series. Also show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

A:- Let half-range sine series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} dx \quad \cdots\cdots (1)$$

here $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$

$$= \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} dx \text{ ; since } l = 2.$$

$$= \int_0^2 x \sin \frac{n\pi x}{2} dx = \left[x\left\{ -\frac{\cos n\pi x/2}{n\pi/2} \right\} - \int_0^2 1 \cdot -\frac{\cos n\pi x/2}{n\pi/2} dx \right]$$

$$= \left[x\left\{ -\frac{\cos(n\pi x/2)}{n\pi/2} \right\} - 1 \cdot \left\{ -\frac{\sin(n\pi x/2)}{(n\pi/2)^2} \right\} \right]_0^2$$

$$= \left[-\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right]_0^2$$

$$= \frac{-2 \cdot 2}{n\pi} \cos n\pi + 0 = -\frac{4(-1)^n}{n\pi}$$

From (1); ~~f(x)~~

$$x = -\sum_{n=1}^{\infty} \frac{4(-1)^n}{n\pi} \sin \frac{n\pi x}{2}$$

$$= -\frac{4}{\pi} \left(-\frac{1}{1} \sin \frac{\pi x}{2} + \frac{1}{2} \sin \frac{2\pi x}{2} - \frac{1}{3} \sin \frac{3\pi x}{2} + \cdots\cdots \right)$$

$$x = \frac{4}{\pi} \left(\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} \right.$$

(2)

Now integrating (2) between the limits 0 and x

$$\int_0^x x\,dx = \frac{4}{\pi}\int_0^x\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\cdots\right)dx.$$

$$\left[\frac{x^2}{2}\right]_0^x = -\frac{4}{\pi}\left[\frac{2}{\pi}\cos\frac{\pi x}{2} - \frac{2}{4\pi}\cos\frac{2\pi x}{2} + \frac{2}{9\pi}\cos\frac{3\pi x}{2} - \cdots\cdots\right]_0^x$$

$$\frac{x^2}{2} = -\frac{4}{\pi}\cdot\frac{2}{\pi}\left[\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\cdots\right) - \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots\right)\right]$$

$$x^2 = -\frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\cdots\right) + \frac{16}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots\right) \quad (3)$$ Ans

This series (3) is a Fourier cosine series of $f(x) = x^2$; $0 < x < 2$. So

$$\frac{16}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots\right) = \frac{a_0}{2}.$$

$$\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{1}{2}\cdot\frac{2}{2}\int_0^2 f(x)\,dx.$$

$$\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{1}{2}\int_0^2 x^2\,dx$$

$$= \frac{1}{2}\cdot\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\cdot\frac{8}{3}$$

or, $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{16}\cdot\frac{1}{2}\cdot\frac{8}{3} = \frac{\pi^2}{12}.$$ Proved.

Q: Solve the one dimensional heat equation

$$\frac{\partial T}{\partial t} = h^2 \frac{\partial^2 T}{\partial x^2}$$ subject to the boundary conditions

$T(0,t)=0$, $T(s,t)=0$ and initial conditions

$T = F(x)$ for $t=0$, $T \neq \infty$ for $t=\infty$.

A:-

$$\frac{\partial T}{\partial t} = h^2 \frac{\partial^2 T}{\partial x^2} \quad \text{——— (1)}$$

Let we have a bar of length s of uniform section. The surface is impervious to heat so that there is no radiation from the sides. Let the initial temp. of the bar be given & let its ends be kept constant temp. zero. If we take one end of the bar at the origin and distances along the bar be x, then

$$\frac{\partial T}{\partial t} = h^2 \frac{\partial^2 T}{\partial x^2} \quad \text{——— (1)}$$ with b.c.

$T(0,t) = T(s,t) = 0$ and the initial conditions

$T = F(x)$ for $t=0$, $T \neq \infty$ for $t=\infty$.

To solve the equation (1), let us try a solution is of the form

$$T(x,t) = e^{mt} u(x) \quad \text{——— (2)}$$

where m is constant. putting this $T(x)$ in (1),

$$m e^{mt} u(x) = h^2 \frac{\partial^2}{\partial x^2}\left[e^{mt} u(x)\right]$$

$$m e^{mt} u(x) = h^2 \cdot e^{mt} \frac{d^2 u(x)}{dx^2}$$ [$\because$ one dim. so $\frac{d}{dx}$ in place $\frac{\partial}{\partial x}$]

$$m u = h^2 \frac{d^2 u}{dx^2} \quad \text{——— (3)}$$

$$\frac{d^2 u}{dx^2} = \frac{m}{h^2} u$$

$$\frac{d^2 u}{dx^2} + a^2 u = 0 \quad \text{——— (4)}$$ putting $a^2 = -\frac{m}{h^2}$

Let $u = e^{mx}$

$\frac{du}{dx} = me^{mx}$; $\frac{d^2u}{dx^2} = m^2 e^{mx}$.

$(4) \Rightarrow (m^2+a^2)e^{mx} = 0$

$\therefore m^2+a^2 = 0$ $[\because e^{mx} \neq 0$

$m = \pm ia$

$\therefore$ The solⁿ. of (4) is

$u = c_1 e^{iax} + c_2 e^{-iax}$

$= c_1(\cos ax + i \sin ax) + c_2(\cos ax - i \sin ax)$

$= (c_1+c_2)\cos ax + i(c_1-c_2)\sin ax$

The general solⁿ. of the eqⁿ (4) is

$u = A\cos ax + B\sin ax$ — (5) where $A = c_1+c_2$, $B = i(c_1-c_2)$

Now u must satisfy boundary conditions. The first condition is at $x=0$, $0 = A\cdot 1 + B\cdot 0 \Rightarrow A = 0$

at $x = s$, $B\sin as = 0$ $[\because$ at $x=s$, $T=0]$

or, $as = r\pi$ — (6)

$\therefore a = \frac{r\pi}{s}$; $r = 0, 1, 2, 3 \ldots$ for a non-trivial solution.

To each value of r there corresponds a solution of (4) of the form

$u_r = B_r \sin \frac{r\pi x}{s}$, where B_r is an arbitrary constant.

The possible values of m are can be written by (6) & $a^2 = -m/h^2$ as

$m_r = -\left(\frac{r\pi}{s}\right)^2 h^2$

$= -\left(\frac{hr\pi}{s}\right)^2$.

To each value of r there corresponds a solution of d.e (i) of the form [by (2)]

$$T_r = e^{-\left(\frac{hr\pi}{s}\right)^2 t} B_r \sin \frac{r\pi x}{s}$$

$$T_r = B_r e^{-\left(\frac{hr\pi}{s}\right)^2 t} \sin \frac{r\pi x}{s}$$; that satisfies the boundary conditions.

By summing over all values of r, we construct the general solution

$$T = \sum_{r=1}^{\infty} B_r e^{-\left(\frac{hr\pi}{s}\right)^2 t} \sin \frac{r\pi x}{s}. \qquad (7)$$

To evaluate the arbitrary constant B_r, we place $t=0$ in (7) and using the initial conditions

$$F(x) = \sum_{r=1}^{\infty} B_r \sin \frac{r\pi x}{s}$$

we must expand $F(x)$ in a half range sine series as $B_r = \frac{2}{s}\int_0^s F(x) \sin \frac{r\pi x}{s}\, dx$.

Hence equation (7) gives ~~the~~ with these values of the constants B_r gives ~~of~~ the solution of the problem.