Q: Solve the one dimensional heat equation

$\frac{\partial T}{\partial t} = h^2 \frac{\partial^2 T}{\partial x^2}$ subject to the boundary conditions

$T(0,t)=0$, $T(s,t)=0$ and initial conditions

$T = F(x)$ for $t=0$, $T \neq \infty$ for $t=\infty$.

A:-

$$\frac{\partial T}{\partial t} = h^2 \frac{\partial^2 T}{\partial x^2} \quad \text{——— (1)}$$

Let we have a bar of length s of uniform section. The surface is impervious to heat so that there is no radiation from the sides. Let the initial temp. of the bar be given & let its ends be kept constant temp. zero. If we take one end of the bar at the origin and distances along the bar be x, then

$$\frac{\partial T}{\partial t} = h^2 \frac{\partial^2 T}{\partial x^2} \quad \text{——— (1)} \quad \text{with b.c.}$$

and the initial conditions

$T(0,t) = T(s,t) = 0$

$T = F(x)$ for $t=0$, $T \neq \infty$ for $t=\infty$.

To solve the equation (1), let us try a solution is of the form

$$T(x,t) = e^{mt} u(x) \quad \text{——— (2)}$$

where m is constant. putting this $T(x)$ in (1),

$$m e^{mt} u(x) = h^2 \frac{\partial^2}{\partial x^2}\left[e^{mt} u(x)\right]$$

$$m e^{mt} u(x) = h^2 . e^{mt} \frac{d^2 u(x)}{dx^2} \quad \left[\because \text{one dim. so } \frac{d}{dx} \text{ in place } \frac{\partial}{\partial x}\right]$$

$$m u = h^2 \frac{d^2 u}{dx^2} \quad \text{——— (3)}$$

$$\frac{d^2 u}{dx^2} = \frac{m}{h^2} u$$

$$\frac{d^2 u}{dx^2} + a^2 u = 0 \quad \text{——— (4)} \quad \text{putting } a^2 = -\frac{m}{h^2}$$

Let $u = e^{mx}$

$\frac{du}{dx} = me^{mx}$; $\frac{d^2u}{dx^2} = m^2 e^{mx}$.

(4) $\Rightarrow$ $(m^2 + a^2) e^{mx} = 0$

$\therefore m^2 + a^2 = 0$ $\quad [\because e^{mx} \neq 0$

$m = \pm ia$

$\therefore$ The solⁿ. of (4) is

$u = c_1 e^{iax} + c_2 e^{-iax}$

$= c_1(\cos ax + i\sin ax) + c_2(\cos ax - i\sin ax)$

$= (c_1 + c_2)\cos ax + i(c_1 - c_2)\sin ax$

The general solⁿ. of the eqn (4) is

$u = A\cos ax + B\sin ax$ — (5) where $A = c_1 + c_2$, $B = i(c_1 - c_2)$

Now u must satisfy boundary conditions. The first condition is at $x = 0$, $0 = A \cdot 1 + B \cdot 0 \Rightarrow A = 0$

at $x = s$, $B\sin as = 0$ $\quad [\because$ at $x = s$, $T = 0]$

or, $as = r\pi$

$a = \frac{r\pi}{s}$; $r = 0, 1, 2, 3 \ldots$ — (6) for a non-trivial solution.

To each value of r there corresponds a solution of (4) of the form

$u_r = B_r \sin \frac{r\pi x}{s}$, where B_r is an arbitrary constant.

The possible values of m are can be written by (6) & $a^2 = -m/h^2$ as

$m_r = -\left(\frac{r\pi}{s}\right)^2 h^2$

$= -\left(\frac{hr\pi}{s}\right)^2$.

To each value of r there corresponds a solution of d.e (i) of the form [by (2)]

$$T_r = e^{-\left(\frac{hr\pi}{s}\right)^2 t} B_r \sin \frac{r\pi x}{s}$$

$$T_r = B_r e^{-\left(\frac{hr\pi}{s}\right)^2 t} \sin \frac{r\pi x}{s}$$; that satisfies the boundary conditions.

By summing over all values of r, we construct the general solution

$$T = \sum_{r=1}^{\infty} B_r e^{-\left(\frac{hr\pi}{s}\right)^2 t} \sin \frac{r\pi x}{s} \quad \text{——— (7)}$$

To evaluate the arbitrary constant B_r, we place $t=0$ in (7) and using the initial conditions

$$F(x) = \sum_{r=1}^{\infty} B_r \sin \frac{r\pi x}{s}$$

we must expand $F(x)$ in a half range sine series as $B_r = \frac{2}{s}\int_0^s F(x) \sin \frac{r\pi x}{s}\, dx$.

Hence equation (7) gives ~~the~~ with these values of the constants B_r gives ~~of~~ the solution of the problem.

Q. Solve the one dimensional wave equation subject to the boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(0,t) = u(1,t) = 0$, $u(x,0) = \lambda \sin \pi x$, $\left(\frac{du}{dt}\right)_{t=0} = 0$

Ans:- $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ——— (1)

$u(0,t) = u(1,t) = 0$ ——— (2)

$u(x,0) = \lambda \sin \pi x$ ——— (3)

$\left(\frac{du}{dt}\right)_{t=0} = 0$ ——— (4)

Let $u = YT$, when $Y = f(x)$ only & $T = f(t)$ only.

$\therefore \frac{\partial u}{\partial t} = YT'$ & $\frac{\partial^2 u}{\partial t^2} = YT''$.

$\frac{\partial u}{\partial x} = Y'T$ & $\frac{\partial^2 u}{\partial x^2} = Y''T$.

(1) $\Longrightarrow$ $YT'' = c^2 Y'' T$.

or, $\frac{T''}{c^2 T} = \frac{Y''}{Y}$

Let $\frac{T''}{c^2 T} = \frac{Y''}{Y} = -k^2$ (suppose)

or, $T'' = -k^2 c^2 T$ & $Y'' = -k^2 Y$

or, $T'' + c^2 k^2 T = 0$ ——— (5) & $Y'' + k^2 Y = 0$ ——— (6)

~~From (5),~~ $\cancel{(D^2 + c^2 k^2) T = 0}$ ~~where $D = \frac{d}{dt}$~~.

~~A.E is~~ let $T = e^{mt}$ in (5),

then $\frac{dT}{dt} = m e^{mt}$

$\frac{d^2 T}{dt^2} = m^2 e^{mt}$.

$(5) \Rightarrow \quad m^2 e^{mt} + c^2k^2 e^{mt} = 0$

$(m^2+c^2k^2) e^{mt} = 0$

$a_1 \quad m^2+c^2k^2=0 \Rightarrow m^2 = -c^2k^2$

$m = \pm ick.$

$\therefore T = a_1 e^{ickt} + a_2 e^{-ickt}$

$= a_1(\cos ckt + i\sin ckt) + a_2(\cos ckt - i\sin ckt)$

$= (a_1+a_2)\cos ckt + i(a_1-a_2)\sin ckt$

$T = A_1 \cos ckt + A_2 \sin ckt$ where $A_1 = a_1+a_2$, $A_2 = i(a_1-a_2)$

Similarly from (6); $Y = B_1 \cos kx + B_2 \sin kx$

Therefore, the solution of (1) is

$u(x,t) = YT$

$u(x,t) = (B_1 \cos kx + B_2 \sin kx)(A_1 \cos ckt + A_2 \sin ckt)$ — (7)

$(7) \Rightarrow \quad u(0,t) = B_1(A_1 \cos ckt + A_2 \sin ckt)$

$0 = B_1(A_1 \cos ckt + A_2 \sin ckt)$; by (2)

$\Rightarrow B_1 = 0 \quad (\because A_1 \cos ckt + A_2 \sin ckt \neq 0)$

$\therefore (7) \Rightarrow u(x,t) = B_2 \sin kx (A_1 \cos ckt + A_2 \sin ckt)$ — (8)

Differentiating both sides of (8) w.r.to t.

$\frac{du}{dt} = B_2 \sin kx(-A_1 ck \sin ckt + A_2 ck \cos ckt)$

$\left.\frac{du}{dt}\right|_{t=0} = B_2 \sin kx \cdot (A_2 ck)$

$0 = A_2 B_2 ck \sin kx$. by (4).

For nonzero solution, we must have $A_2 = 0$ [Because if $B_2=0$, then (8); $u(x,t)=0$]

Hence from (8); $u(x,t) = B_2 \sin kx \cdot A_1 \cos ckt.$

$u(x,t) = A_1 B_2 \sin kx \cos ckt$.

$u(x,t) = A \sin kx \cos ckt$ ——— (9) where $A = A_1 B_2$

$u(1,t) = A \sin k \cos ckt$

$0 = A \sin k \cos ckt$ by (2) .

For non-zero solution, $\sin k = 0 \Rightarrow k = \pi$.

From (9); $u(x,t) = A \sin \pi x \cos c\pi t$

$u(x,0) = A \sin \pi x$

$\lambda \sin \pi x = A \sin \pi x$ [By (3)

$\therefore A = \lambda$.

Hence from (9); the required solution is

$u(x,t) = \lambda \sin \pi x \cos c\pi t$. Ans.