Fourier series

Trigonometric series: The infinite series of the form
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$ is called Trigonometric series where a_0, a_n & b_n are constants.

periodic function: when $f(x+T) = f(x)$; $f(x)$ is periodic.

eg: $\sin x = f(x)$; $f(x+2\pi) = \sin(2\pi + x) = \sin x = f(x)$
period 2π,

$f(x) = \tan x$; $f(x+\pi) = \tan(x+\pi) = \tan x = f(x)$
$\therefore$ period π.

Fourier Series :- Let $f(x)$ be defined in the interval $(-\pi, \pi)$ with period 2π which can be expanded in a trigonometric series:
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$ is called Fourier series where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx \quad \& \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx.$$

here a_0, a_n, b_n are Fourier coefficients.

Important Integrals (To remember):

(i) $\int_{-\pi}^{\pi} \sin nx\,dx = 0$, (ii) $\int_{-\pi}^{\pi} \cos nx\,dx = 0$; (iii) $\int_{-\pi}^{\pi} \sin nx \cos nx\,dx = 0$

(iv) $\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0$ (v) $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0$
when $m \neq n$.

(vi) $\int_{-\pi}^{\pi} \cos^2 nx\,dx = \pi$, (vii) $\int_{-\pi}^{\pi} \sin^2 nx\,dx = \pi$.

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__Even function__: If $f(-x) = f(x)$, then $f(x)$ is Even.

exp: $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x = f(x)$.

$\therefore$ $\cos x$ is even function.

__odd function__: If $f(-x) = -f(x)$; then $f(x)$ is odd.

exp: $f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x = -f(x)$

$\therefore$ $\sin x$ is odd.

__Theorem__: If $f(x)$ is an even function, then show that show that (i) $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ (ii) $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

(iii) $b_n = 0$

Proof:- (i) We know that $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right]. \quad \text{—①}$$

put $x = -y$, then $dx = -dy$ in the first integral

x	$-\pi$	0
y	π	0

$$\therefore \int_{-\pi}^{0} f(x)\,dx = -\int_{\pi}^{0} f(-y)\,dy$$

$$= \int_0^{\pi} f(-y)\,dy$$

$$= \int_0^{\pi} f(-x)\,dx \qquad \left[\because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right]$$

$$= \int_0^{\pi} f(x)\,dx \qquad \left[\because f(x) \text{ is even}\right]$$

$\therefore$ From (i); $\quad a_0 = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right]$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \,. \quad \text{proved.}$$

(ii) $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_0^{\pi} f(x)\cos nx\,dx\right] \quad \text{--- (i)}$$

In first integral, put $x = -y$, $dx = -dy$

$$\therefore \int_{-\pi}^{0} f(x)\cos nx\,dx = -\int_{\pi}^{0} f(-y)\cos(-ny)\,dy$$

x	$-\pi$	0
y	π	0

$$= \int_0^{\pi} f(-y)\cos ny\,dy$$

$$= \int_0^{\pi} f(-x)\cos nx\,dx$$

$$= \int_0^{\pi} f(x)\cos nx\,dx \,. \quad [\because f(x) \text{ is even.}$$

From (i); $\quad a_n = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_0^{\pi} f(x)\cos nx\,dx\right]$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad \text{proved.}$$

(iii) $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right] \quad \text{--- (ii)}$$

Now $\int_{-\pi}^{0} f(x) \sin nx \, dx = -\int_{\pi}^{0} f(-y) \sin(-ny) \, dy$

$$= \int_{0}^{\pi} f(-y) \sin(-ny) \, dy$$

$$= -\int_{0}^{\pi} f(-y) \sin ny \, dy$$

$$= -\int_{0}^{\pi} f(-x) \sin nx \, dx$$

$$= -\int_{0}^{\pi} f(x) \sin nx \, dx \quad \because f(x) \text{ is even}$$

$\therefore$ From (i); $b_n = \frac{1}{\pi}\left[-\int_{0}^{\pi} f(x) \sin nx \, dx + \int_{0}^{\pi} f(x) \sin nx \, dx\right] = 0$

proved.

Fourier coefficients for odd function :- If $f(x)$ is an odd function, then (i) $a_0 = 0$, (ii) $a_n = 0$ ~~(iii)~~

(iii) $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$.

Ans:- Same process. [only put $f(-x) = -f(x)$]

Half range Fourier Series :- If only cosine term or only sine term present in the Fourier series, then it is called Half range Forier series :

Half range cosine series:- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$; $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$

Half range sine series:

$$f(x) = \cancel{\frac{a_0}{2}} \cancel{+} \sum_{n=1}^{\infty} b_n \sin nx\,dx$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx$.

Since the fuction is generally defined on $(0, \pi)$ which is half of $(-\pi, \pi)$. For this reason, it is called half range Fourier series.

Fourier series in Different Intervals :

(i) $f(x)$ is defined in $(0, \pi)$ ~~a term~~ with period π :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,dx$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$;

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx.$$

(ii) $f(x)$ is defined on $(0, 2\pi)$ with period 2π.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$; $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx$.

(iii) $f(x)$ in $(-l, l)$ with period $2l$.

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$, where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\,dx$

Parseval's Identity for Fourier Series

If the Fourier series for $f(x)$ converges in $(-l, l)$

then $\frac{1}{l}\int_{-l}^{l} \{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$; where

a_0, a_n, b_n are Fourier constants.

Proof:- we know

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \text{—— (1)}$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\,dx$.

Now multiplying both sides of (i) by $f(x)$, then integrating from $-l$ to l.

$$\int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx\right.$$

$$\left. + b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right).$$

$$= \frac{a_0}{2}\cdot a_0 l + \sum_{n=1}^{\infty}(a_n\cdot a_n l + b_n\cdot b_n l)$$

or, $\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$. proved.