Cauchy's theorem: If $f(z)$ is an analytic function whose derivative $f'(z)$ exists is continuous at each point within and on the closed ~~curve~~ contour C, then

$$\oint_C f(z)\,dz = 0$$

Elematary proof:- Let R be the closed domain which consists of all points within and on C.

> This proof is based on two-dim green's thm & it requires the assumption that $f'(z)$ is continuous

We know $z = x+iy$; $f(z) = u+iv$ is analytic and has continuous derivatives,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad [\text{By C-R equation}]$$

It follows that partial derivatives $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ are continuous within and on C.

Now
$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy)$$

$$= \oint_C \{(u\,dx - v\,dy) + i(v\,dx + u\,dy)\} \quad \text{———} (1)$$

But by green's theorem, we can write (1) as

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right)dx\,dy$$

> green's thm
> $$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

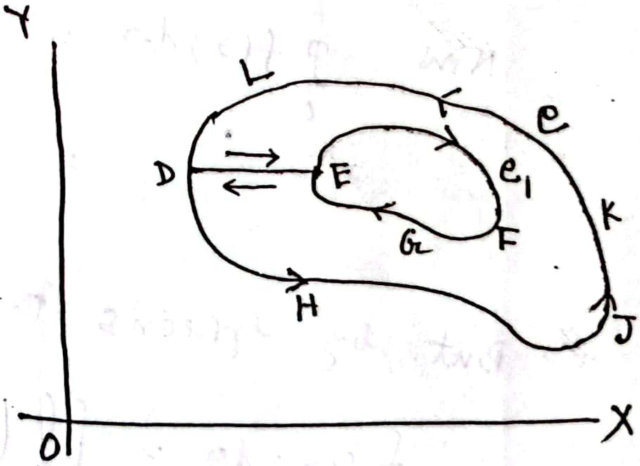
$$= 0 + i0 = 0$$

$$\therefore \oint_C f(z)\,dz = 0 \quad \text{proved.}$$

Remarks: Goursat (French Mathematician 1850 – 1936) first showed that it is necessary to assume the continuity of $f'(z)$ and that Cauchy's thm is true if it is only assumed that $f'(z)$ exists at each point within ~~and~~ or on c. Goursat first proved that without assuming the continuity of $f'(z)$ and so in his honor the more general form of this theorem is usually known as Cauchy-Goursat theorem.

Theorem: If $f(z)$ is analytic in a region by two simple closed curve c & c_1 (where c_1 lies inside c) and on this curve, then $\oint_c f(z)\,dz = \oint_{c_1} f(z)\,dz$, where c & c_1 are both traversed in the positive sense relative to their interiors.

Proof:- Let us consider a crosscut DE, since $f(z)$ is analytic in the region R, we have ~~the~~ by Cauchy's theorem,



$$\oint f(z)\,dz = \oint_{DEFGEDHJKLD} f(z)\,dz = 0.$$

or, $$\int_{DE} f(z)\,dz + \int_{EFGE} f(z)\,dz + \int_{ED} f(z)\,dz + \int_{DHJKLD} f(z)\,dz = 0$$

or, $$\int_{DHJKLD} f(z)\,dz = -\int_{EFGE} f(z)\,dz$$

$$\oint_c f(z)\,dz = \int_{EGFE} f(z)\,dz.$$

[$\int_{DE}$ & $\int_{ED}$ are opposite in direction, so they can cancelled

$$\therefore \oint_c f(z)\,dz = \oint_{c_1} f(z)\,dz. \text{ Proved.}$$

Morera's theorem (Converse of Cauchy's thm) :- Let $f(z)$ be continuous in a simply connected region R and suppose that $\oint_C f(z)\,dz = 0$, around every simple closed curve C in R, then $f(z)$ is analytic in R.

Proof:- If $f(z)$ has continuous derivative in R, then by Green's Theorem,

$$\oint_C f(z)\,dz = \oint_C \{(u\,dx - v\,dy) + i(v\,dx + u\,dy)\}$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy \; i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy.$$

Given $\oint_C f(z)\,dz = 0$ around every closed curve is zero.

Then $\oint_C (u\,dx - v\,dy) = 0$, $\oint_C (v\,dx + u\,dy) = 0$

$$\iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = 0$$

or, $-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ or, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

& $i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$ i.e $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$

or, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

$\therefore$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ which are C-R equations indicating $f(z)$ is analytic.

Hence $f(z) = u + iv$ is analytic in R.