Automatic Control Project: Rotary Inverted Pendulum

Stefano Tonini
ID: 248413

July 1, 2024

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Abstract

This report addresses the stabilization of a rotary inverted pendulum using linear control techniques. The system is linearized, and controllers are designed to ensure the desired convergence rate. Simulations are performed to evaluate the performance of the designed controllers.

1 Introduction

The purpose of this project is to stabilize the zero position and zero velocity configuration of a rotary inverted pendulum system using linear control methods. The system dynamics is analyzed, linearized, and controllers are designed and evaluated based on their performance in simulations.

2 System Description

The rotary inverted pendulum consists of a pendulum attached to a rotary arm driven by a motor. The system is described by the following nonlinear equations:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + f_v(\dot{q}) + G(q) = \tau,$$

where $q = [\theta \;\; \alpha]^T$, and the terms $M(q)$, $C(q,\dot{q})$, $f_v(\dot{q})$, $G(q)$, and τ are defined as follows:

$$M(q) = \begin{bmatrix} J_r + m_p(L_r^2 + l_p^2(1-\cos^2(\alpha))) & m_p l_p L_r \cos(\alpha) \\ m_p l_p L_r \cos(\alpha) & J_p + m_p l_p^2 \end{bmatrix},$$

$$C(q,\dot{q}) = \begin{bmatrix} 2m_p l_p^2 \dot{\alpha} \sin(\alpha)\cos(\alpha) & -m_p l_p L_r \dot{\alpha} \sin(\alpha) \\ -m_p l_p^2 \dot{\theta} \sin(\alpha)\cos(\alpha) & 0 \end{bmatrix},$$

$$f_v(\dot{q}) = \begin{bmatrix} B_r \dot{\theta} \\ B_p \dot{\alpha} \end{bmatrix},$$

$$G(q) = \begin{bmatrix} 0 \\ -m_p l_p g \sin(\alpha) \end{bmatrix},$$

$$\tau = \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

3 Linearization and Stability Analysis

To linearize the system around the origin $x = [0\;0\;0\;0]^T$, we neglect higher-order terms and obtain the state-space representation $\dot{x} = Ax + Bu$. The matrices A and B are derived as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5.5880 & -17.7215 & 15.1899 \\ 0 & 30.4234 & 7.5949 & -82.7004 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 88.6076 \\ -37.9747 \end{bmatrix}.$$

The eigenvalues of matrix A are:

$$\{0, -84.7869, 0.3661, -16.0011\}$$

Since there are eigenvalues with positive real parts, the system is unstable. However, the controllability matrix is full rank, indicating that the system is controllable.

4 Controller Design and Simulation

The first gain matrix, K_1 is computed to achieve a convergence rate of $\alpha = 2$ in the closed loop system. The second gain matrix K_2 is computed ensuring the same convergence rate of the closed-loop system such that K2 has minimum norm. The first gain matrix K_1 is:

$$K_1 = \begin{bmatrix} 8.6542 & 5345.353 & 29.6770 & 66.9931 \end{bmatrix}$$

The second gain matrix K_2 with minimum norm is:

$$K_2 = \begin{bmatrix} 5.2863 & 3081.995 & 17.1956 & 40.2511 \end{bmatrix}$$

Simulations of the nonlinear closed-loop system for both controllers are performed starting from the initial condition $x(0) = [0.05\ 0\ 0.06\ 0]^T$. The results are compared to determine the better controller.

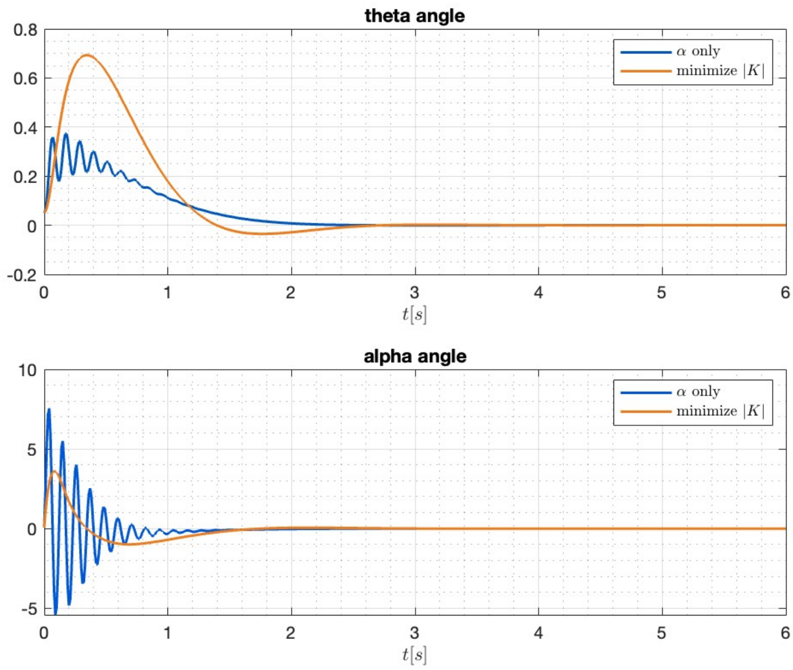

Figure 1: Theta and Alpha

5 Overshoot and Exponential Bound Analysis

The overshoot M for both controllers is computed, but the problem is found to be infeasible.

However, an estimated value of M can be found using the following formulas:

$$M_{\text{b}} = \sqrt{\frac{\max(\text{eig}(P_b))}{\min(\text{eig}(P_b))}} = 36173$$

for the first controller and

$$M_{\text{c}} = \sqrt{k_c} = 4079.3$$

for the second controller.
The exponential bound are plotted on the trajectories of the previous point.

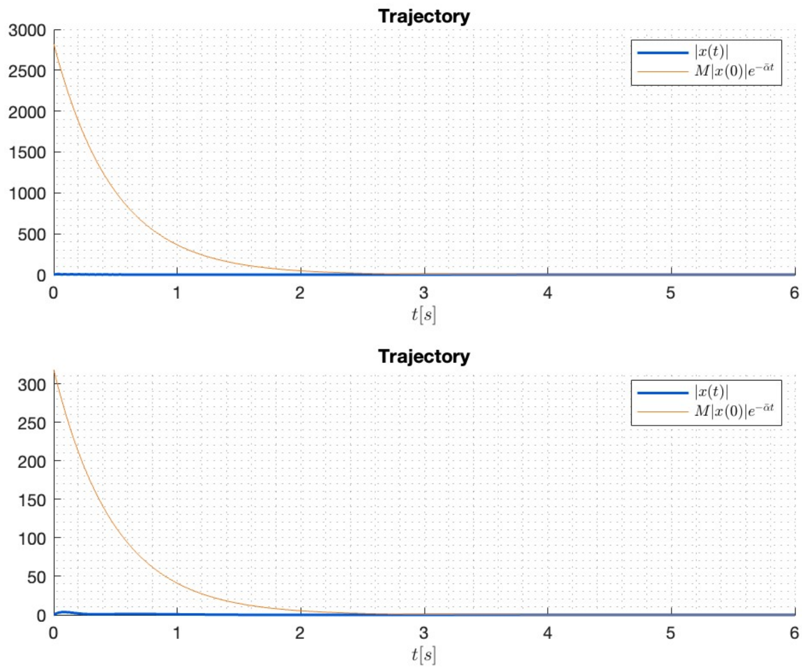

Figure 2: Exponential Bounds on trajectory

Assuming small angle deviations that allowed to use the linearized model confidently to predict the system behavior and to design control strategies based on it. The exponential bounds derived from the linearized system are likely to hold for the actual nonlinear system under these operating conditions. However, it is always advisable to validate these theoretical predictions with practical experiments or detailed simulations that can account for larger deviations and more complex nonlinear dynamics.

6 Conclusion

This project successfully demonstrated the application of linear control techniques to stabilize a rotary inverted pendulum. Through the process of system linearization, we derived appropriate control laws and implemented two different controllers, designated as K_1 and K_2.

Both controllers achieved the set objective of stabilizing the pendulum at the zero position and zero velocity. Controller K_1, with gains

$$K_1 = \begin{bmatrix} 8.6542 & 5345.353 & 29.6770 & 66.9931 \end{bmatrix}$$

and Controller K_2, with gains

$$K_2 = \begin{bmatrix} 5.2863 & 3081.995 & 17.1956 & 40.2511 \end{bmatrix}$$

were evaluated based on their convergence rates and the norms of their respective control inputs during simulations.

Controller K_2 was found to be more efficient due to its minimal norm, suggesting that it requires less energy input to maintain system stability, which is a significant advantage in practical applications. Additionally, the control input for K_2 shows smoother transitions and lower magnitude fluctuations, which are desirable characteristics for preventing wear and tear on mechanical components in real-world applications.

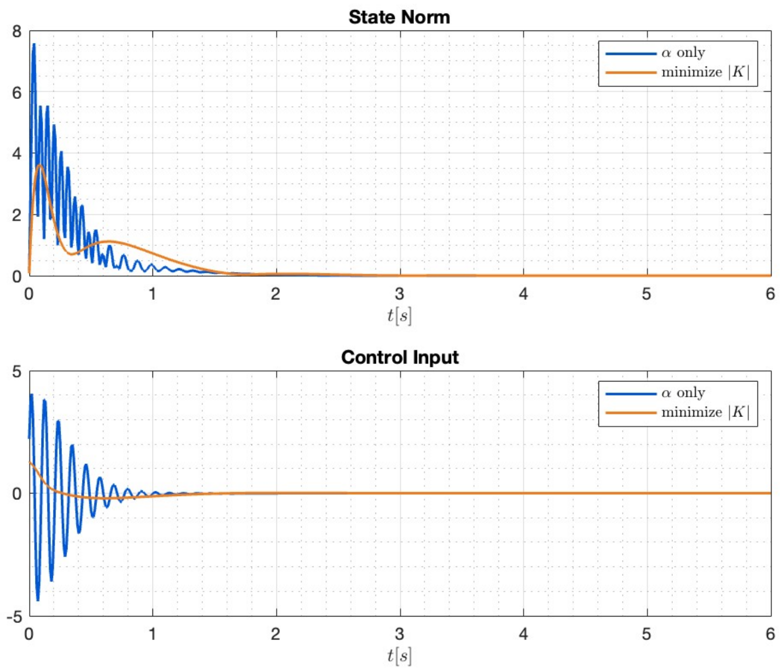

Figure 3: State Norm and Control Input

This project highlighted theoretical aspects of control system design and also provided valuable insights into practical considerations necessary for real-world applications.