Automatic Control Project: Rotary Inverted Pendulum

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1 Introduction

The purpose of this project is to stabilize the zero position and zero velocity configuration of a rotary inverted pendulum system using linear control methods. The system dynamics is analyzed, linearized, and controllers are designed and evaluated based on their performance in simulations.

2 System Description

The rotary inverted pendulum consists of a pendulum attached to a rotary arm driven by a motor. The system is described by the following nonlinear equations:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + f_v(\dot{q}) + G(q) = \tau,$$

where $q = [\theta \ \ \alpha]^T$, and the terms $M(q)$, $C(q,\dot{q})$, $f_v(\dot{q})$, $G(q)$, and τ are defined as follows:

$$M(q) = \begin{bmatrix} J_r + m_p(L_r^2 + l_p^2(1-\cos^2(\alpha))) & m_p l_p L_r \cos(\alpha) \\ m_p l_p L_r \cos(\alpha) & J_p + m_p l_p^2 \end{bmatrix},$$

$$C(q,\dot{q}) = \begin{bmatrix} 2m_p l_p^2 \dot{\alpha} \sin(\alpha)\cos(\alpha) & -m_p l_p L_r \dot{\alpha} \sin(\alpha) \\ -m_p l_p^2 \dot{\theta} \sin(\alpha)\cos(\alpha) & 0 \end{bmatrix},$$

$$f_v(\dot{q}) = \begin{bmatrix} B_r \dot{\theta} \\ B_p \dot{\alpha} \end{bmatrix},$$

$$G(q) = \begin{bmatrix} 0 \\ -m_p l_p g \sin(\alpha) \end{bmatrix},$$

$$\tau = \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

3 Linearization and Stability Analysis

To linearize the system around the origin $x = [0\ 0\ 0\ 0]^T$, we neglect higher-order terms and obtain the state-space representation $\dot{x} = Ax + Bu$. The matrices A and B are derived using the MATLAB "jacobian" function with the state and the equilibria value of the state. The matrices A and B are as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5.5880 & -17.7215 & 15.1899 \\ 0 & 30.4234 & 7.5949 & -82.7004 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 88.6076 \\ -37.9747 \end{bmatrix}.$$

And the linearized system in the state space representation is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5.5880 & -17.7215 & 15.1899 \\ 0 & 30.4234 & 7.5949 & -82.7004 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 88.6076 \\ -37.9747 \end{bmatrix} u$$

3.1 Stability

In order to analyse stability the eigenvalues of matrix A must be checked. The eigenvalues are as follows:

$$\{\lambda_1 = 0, \quad \lambda_2 = -84.7869, \quad \lambda_3 = 0.3661, \quad \lambda_4 = -16.0011\}$$

Since there are eigenvalues with positive real parts, the system is unstable.

3.2 Controllability

In order to study controllability, the controllability matrix C defined as follows was used:

$$C = \begin{bmatrix} B & BA & BA^2 & .. & BA^{(n-1)} \end{bmatrix}$$

The previous matrix must be full rank to have a fully controllable system. In this case, the controllability matrix is full rank, indicating that the system is controllable.

4 Controller Design and Simulation

Two feedback loops are designed to stabilize the system with two gain matrices. The first gain matrix, K_1 is computed to achieve a convergence rate of $\alpha = 2$ in the closed loop system $\dot{x} = A + BK_1$. The second gain matrix K_2 is computed ensuring the same convergence rate of the closed-loop system such that K2 has minimum norm.

4.1 K1 with convergence rate $\alpha = 2$

To find the first value, the following constraints must be imposed, which are necessary to find the gain matrix.

$$\begin{cases} W \geq 0 \\ \text{He}(AW + Bx) \leq -2\alpha W \end{cases}$$

In which "He" is the Hermitian operator defined as: $\text{He}(A) = A + A^T$. Furthermore, it is possible to solve the constraints and find the gain matrix using `mosek` solver with the `solvesdp` command.

The first gain matrix K_1 is:

$$K_1 = \begin{bmatrix} 8.6542 & 5345.353 & 29.6770 & 66.9931 \end{bmatrix}$$

This gain matrix ensures that the system becomes stable, this can be seen from the eigenvalues of the matrix $A + BK_1$:

$$\{\lambda_1 = -5.0815, \quad \lambda_2 = -5.0815, \quad \lambda_3 = -2.3464, \quad \lambda_4 = -2.3464\}$$

which are all negative.

4.2 K2 with convergence rate $\alpha = 2$ and minimum norm

The second gain matrix can be computed by adding a further constraint on the bound of K that ensure the minimum norm. The constraints used are the following:

$$\begin{cases} W \geq I_n \\ \text{He}(AW + Bx) \leq -2\alpha W \\ \begin{bmatrix} kI_n & X^T \\ X & kI_p \end{bmatrix} > 0 \end{cases}$$

The gain matrix K_2 that satisfies all the constraints and ensures system stability is:

$$K_2 = \begin{bmatrix} 5.2483 & 3054.8824 & 17.0476 & 39.8945 \end{bmatrix}$$

The system stability is ensured because the eigenvalues of the matrix $A + BK_2$:

$$\{\lambda_1 = -84.8143, \quad \lambda_2 = -16.0386, \quad \lambda_3 = -2.00104, \quad \lambda_4 = -2.00104\}$$

are all negative.

4.3 Simulation

Simulations of the nonlinear closed-loop system for both controllers are performed defing a function `f_NLDyna` for non-linear dynamics, setting the simulation parameters, the initial condition $x(0) = [0.05\ 0\ 0.06\ 0]^T$ and using the solver `ode45`. The results are compared to determine the better controller.

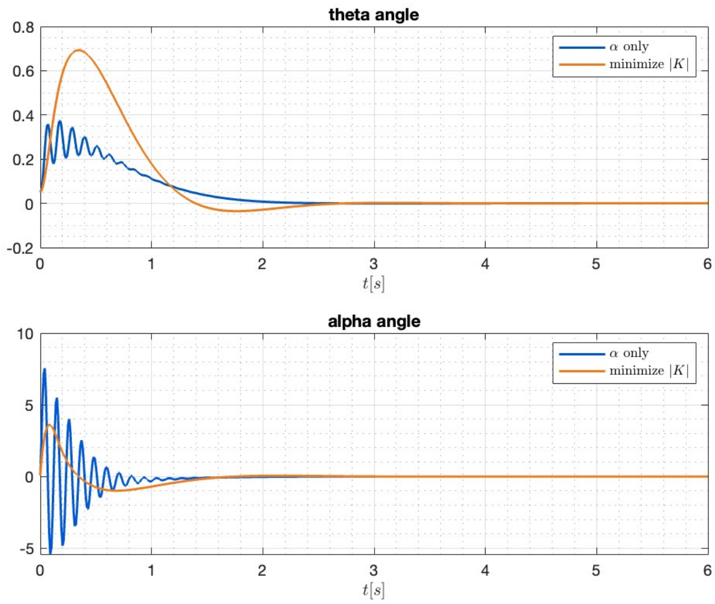

Figure 1: Theta and Alpha

The evaluation of the better controller is provided in the final section as well as state norm and input plot.

5 Overshoot and Exponential Bound Analysis

In order to compute the overshoot M corresponding to the closed-loop matrices for K_1 and K_2, an optimization problem is setted using the following constraints to find a minimized value of $\bar{M} = \sqrt{\bar{K}}$.

$$\bar{K} = \min_{K} \quad \text{s.t.} \quad \begin{cases} I \leq P \leq KI \\ A^T P + PA \leq -2\alpha P \end{cases}$$

with $K > 0$, $P = P^T$.

The overshoot M for both controllers is computed, but the problem is found to be infeasible.

However, an estimated initial value of M can be found using the following formulas in order to show a graphical solution:

$$M_b = \sqrt{\frac{\lambda_M}{\lambda_m}} = 36173$$

with λ_M the largest eigenvalue and λ_m the smallest of the matrix P_b, the inverse of the matrix W, that appears in the constraints of the first control formulation. The value of M for the second control K2

$$M_c = \sqrt{k_c} = 4079.3$$

The value of k is obtained from the set of constraints of the second control formulation.
Furthermore, using the following formula the exponential bound is computed and plotted on the trajectories of the previous point.

$$\text{Exponential Bound} = M|x(0)|e^{-\alpha t}$$

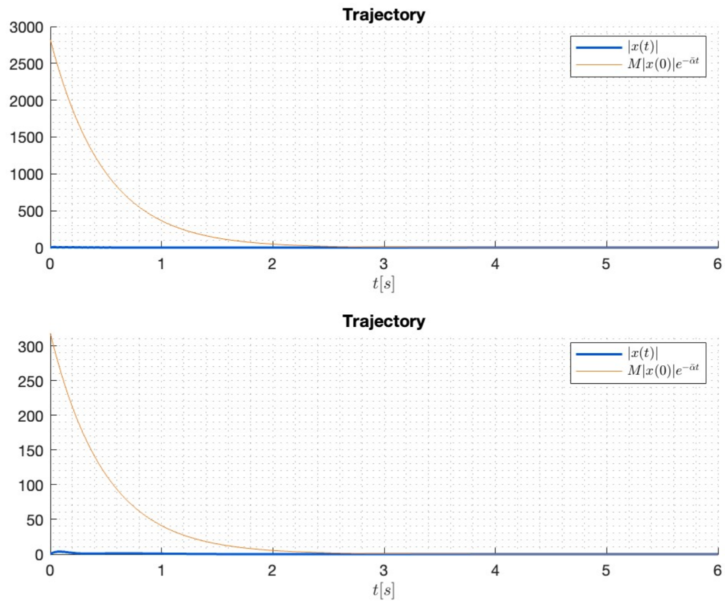

Figure 2: Exponential Bounds on trajectory

Assuming small angle deviations that allowed to use the linearized model confidently to predict the system behavior and to design control strategies based on it. The exponential bounds derived from the linearized system are likely to hold for the actual nonlinear system under these operating conditions.

6 Conclusion

Through the process of system linearization, we derived appropriate control laws and implemented two different controllers, designated as K_1 and K_2.

Both controllers achieved the set objective of stabilizing the pendulum at the zero position and zero velocity. Controller K_1, with gains

$$K_1 = \begin{bmatrix} 8.6542 & 5345.353 & 29.6770 & 66.9931 \end{bmatrix}$$

and Controller K_2, with gains

$$K_2 = \begin{bmatrix} 5.2863 & 3081.995 & 17.1956 & 40.2511 \end{bmatrix}$$

were evaluated based on their convergence rates and the norms of their respective control inputs during simulations.

The controller K_2 was found to be more efficient due to its minimal norm, suggesting that it requires less energy input to maintain system stability, which is a significant advantage in practical applications. Furthermore, the control input for K_2 shows smoother transitions and lower magnitude fluctuations, which are desirable characteristics to prevent wear and tear on mechanical components in real-world applications.

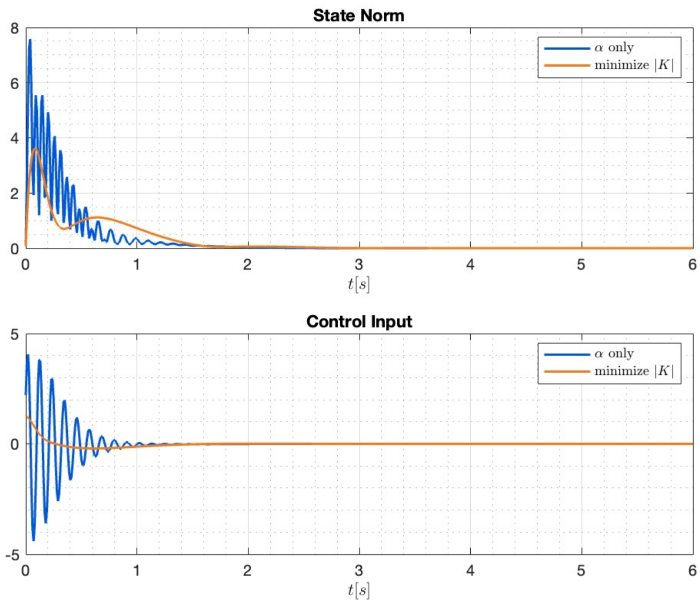

Figure 3: State Norm and Control Input

This project highlighted theoretical aspects of control system design and also provided valuable insights into practical considerations necessary for real-world applications.