Discovering the Foundations of Expansion and Spectral Graph Theory

(Fernando Granha Jeronimo)

$\boxed{\text{Under heavy construction}}$

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Inspired by Babai's approach and the hungarian school

<u>Van Gogh prize</u>

a symbolic prize for any student in the course that solves an important <u>open problem</u>

(like van Gogh you are not going to receive anything other than have done something amazing)

Warning: these open problems can be challenging

Spectral Lens

Let $G=(V,E)$ be a d-regular graph on n vertices.

Let A be its adjacency matrix, i.e.,

$$A \in \mathbb{R}^{n\times n}, \quad A_{u,v} = \mathbb{1}_{[\{u,v\}\in E]}.$$

Study the spectral theorem.

Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of A with corresponding orthonormal eigenvectors $\varphi_1, \dots, \varphi_n \in \mathbb{R}^n$.

$$(A\varphi_i = \lambda_i \varphi_i)$$

1] Prove that d is an eigenvalue of A.

Def $\langle x, x \rangle = \sum_{i=1}^{n} \bar{x}_i x_i$

Ex $x \in \mathbb{R}^n$ and $e_1, \dots, e_n \in \mathbb{R}^n$ ONB (orthonormal basis)

We can write

$$x = \sum_{i=1}^{n} \alpha_i e_i \quad \text{with } \alpha_i = \langle e_i, x \rangle.$$

Ex $\langle x, x \rangle = \sum_{i=1}^{n} \alpha_i^2$ [Parseval]

Ex: $\langle x, Ax \rangle = \sum_{i=1}^{n} \alpha_i^2 \lambda_i$

Ex A sym $\Rightarrow$ A has real eigenvalues

Def Rayleigh quotient $\dfrac{\langle x, Ax \rangle}{\langle x, x \rangle}$ (for $x \neq 0$)

Ex $\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$

Ex $\mathrm{Tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k$

Ex $\mathrm{Tr}(A^2) = 2|E|$

Ex If G is simple with deg ≥ 1

$\Downarrow$

$\lambda_n < 0$

Ex Prove that $\frac{1}{n} \sum_{i=1}^{n} \lambda_i^2 = d$

Def $J_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$ all ones $n \times n$ matrix

Ex Compute eigenvalues of J_n/n

Def K_n complete graph on n vertices

Def $K_{a,b}$ complete bipartite graph
$G = (V = L \sqcup R, E)$ with $|L| = a$, $|R| = b$

Ex Compute the spectrum of K_n

Ex Compute the spectrum of $K_{n,n}$

Ex Compute 〃 〃 of $K_{1,d}$

Ex 〃 〃 〃 of $K_{a,b}$

Ex $\lambda_1 \geq \max_{i \geq 2} |\lambda_i|$

Ex G bipartite $\iff$ $\mathrm{Spec}(G) = -\mathrm{Spec}(G)$.

Some Notions of Expansion

Def $\partial(S) = E(S, \bar{S})$ "Edge boundary"

Def $\Phi(S) = \dfrac{|\partial(S)|}{d|S|}$ "Conductance"

Def $\Phi(G) = \min\limits_{\substack{\emptyset \neq S \subseteq V \\ |S| \leq n/2}} \Phi(S)$ "Cheeger's constant"

Def: $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$

two-sided spectral expansion

Def $\lambda^{\text{one-sided}} = \lambda_2$ one-sided spectral expansion

Def $e(S,T) = |\{(s,t) \mid \{s,t\} \in E\}|$

Ex Prove that

$$\left| e(S,T) - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}.$$

[Expander Mixing Lemma]

Ex Improve the error bound $\lambda\sqrt{|S||T|}$.

Def $\alpha(G)$ = independence number

Ex Prove that $\frac{\alpha(G)}{n} \leq \frac{-\lambda_n}{d-\lambda_n}$

[Hoffman's bound]

Ex Prove that $d_{avg} \leq \lambda_1 \leq \Delta(G)$

Def $\chi(G)$ is the chromatic number

Ex Prove that $\chi(G) \leq \lambda_1 + 1$

[Wilf's bound]

Mixing Bounds

Def $\vec{1}$ is the all ones vector

Def $R = \frac{1}{d} A$ is the random walk matrix

Ex $R \vec{1}/n = \vec{1}/n$

Ex Prove $\| R^r p - \vec{1}/n \|_1 \leq \left(\frac{\lambda}{d}\right)^r \sqrt{n}$ for any distribution p.

[Mixing bound in l_1]

Study the Perron-Frobenius theorem (useful for understanding more general Markov chains)

Eigenvalues as an optimization problem

Let $V_k = \mathrm{span}\{\varphi_1, \dots, \varphi_k\}$

$W_k = \mathrm{span}\{\varphi_k, \dots, \varphi_n\}$

Ex $$\lambda_k = \min_{0 \neq x \in V_k} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} = \max_{0 \neq x \in W_k} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$$

Ex Prove the min-max variational theorem

$$\lambda_k = \max_{\substack{V \subseteq \mathbb{R}^n \\ \dim(V) = k}} \min_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} = \min_{\substack{V \subseteq \mathbb{R}^n \\ \dim(V) = n-k+1}} \max_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$$

[Courant-Fischer-Weyl]

The Magic of Interlacing

Ex Eigenvalue Interlacing

Let $A \in \mathbb{R}^{n \times n}$ be real symmetric matrix and B be a $(n-1) \times (n-1)$ principal submatrix

$$\mathrm{eig}(A) = \{ \lambda_1 \geq \dots \geq \lambda_n \}$$

$$\mathrm{eig}(B) = \{ \tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_{n-1} \}$$

Prove $\lambda_1 \geq \tilde{\lambda}_1 \geq \lambda_2 \geq \dots \geq \tilde{\lambda}_{n-1} \geq \lambda_n$

Hint: use min-max theorem for eigenvalues

Extend to $r \times r$ principal submatrix B with $1 \leq r < n$

Ex: $\lambda_j \geq \tilde{\lambda}_j \geq \lambda_{j+n-r}$ for $j \in \{1, \dots, r\}$

[Cauchy Interlacing Thm]

Refresher on PSDness

Def A real sym matrix M is <u>positive semi-definite</u> (PSD) if

$$\forall x \in \mathbb{R}^n, \quad x^t M x \geq 0.$$

Ex Prove: The following are equivalent

1) M is PSD
2) M has non-negative eigenvalues
3) $\exists$ a matrix W s.t. $M = W^t W$

<u>Notation</u> We write $M \succeq 0$ if M is PSD

We write $M_1 \succeq M_2$ if $M_1 - M_2 \succeq 0$.

This gives a partial order (Loewner order)

Laplacian Matrix

Def $L = dI - A$ [Laplacian Matrix]

Let $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ be the eigenvalues of L

Ex For d-regular G, we have

$$\underbrace{\mu_1 = d - \lambda_1}_{=0}, \quad \dots \quad , \mu_n = d - \lambda_n .$$

Ex Prove that $\langle x, Lx \rangle = \sum_{i \sim j} (x_i - x_j)^2$

Ex Conclude that $L \succcurlyeq 0$ (PSD)

Ex Prove that $\langle 1_S, L 1_S \rangle = |E(S, \bar{S})|$

Ex G connected $\Longleftrightarrow$ $\mu_2 > 0$

Ex If G is connected, then $\mu_2 \geq \frac{1}{n \operatorname{diam}(G)}$

Ex $|\{k \mid \mu_k = 0\}|$ = # of connected components

Ex G bipartite iff $\mu_n = 2d$

Ex Prove that $\frac{\mu_2}{2d} \leq \Phi(G)$

$\boxed{CH^*}$ Prove that $\Phi(G) \leq O\left(\sqrt{\frac{\mu_2}{d}}\right)$

Hint: use eigenvector to μ_2 to find a cut ["rounding"]

[Cheeger's Inequality]

$$\frac{\mu_2}{2d} \leq \Phi(G) \leq \sqrt{\frac{2\mu_2}{d}}$$

Characteristic Polynomial

Def Characteristic polynomial $\det(\lambda \underline{I} - A) =: ch(\lambda)$

The roots of $ch(\lambda)$ are the eigenvalues of A

Cayley-Hamilton Theorem: $ch(A) = 0$

Ex G has diam $= K \Rightarrow A$ has at least
connected $\quad K+1$ distinct eigenvalues

Hint: [Cayley-Hamilton] minimal polynomial

Ex Let A, B be two real sym matrices with eig $(\lambda_1 \geq \dots \geq \lambda_n)$
$(\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_n)$

Compute the eigenvalues of $A \otimes B$.

Def $s : E \to \{\pm 1\}$ is an edge signing

Def $(A_s)_{u,v} = \begin{cases} s(\{u,v\}) & \text{if } \{u,v\} \in E \\ 0 & \text{otherwise} \end{cases}$

Ex Prove $\lambda_1(A_s) \leq \Delta(G)$ for any signing s.

Limitations on Spectral Expansion

Ex G d-regular $\Rightarrow \lambda \geq \sqrt{d}\,(1 - o_n(1))$

Ex If G has diam $\geq 4 \Rightarrow \lambda_2 \geq \sqrt{d}$
[Hint: look for the stars]

Ex $\lambda_2 < 0 \iff G = K_n$
[Hint: interlacing]

Ex Suppose G is connected. G has a unique positive eigenvalue iff G is a complete k-partite graph
[Hint: interlacing]

$\boxed{Ch^*}$ $\lambda_2 \geq 2\sqrt{d-1}\left(1 - O\left(\frac{1}{diam}\right)\right)$

[Alon-Boppana bound]

(Cor) If $d = O(1)$

$\Downarrow$

$$\lambda_2 \geq 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\log n}\right)\right)$$

Def G is <u>Ramanujan</u> if $\lambda \leq 2\sqrt{d-1}$

a.k.a. "Optimal" Spectral Expanders

OP [van Gogh Prize]

Construct infinite families of Ramanujan graphs for <u>every</u> $d \geq 3$

Vertex Expansion

Def $N(S) = \{u \mid \exists s \in S, \{u,s\} \in E\}$

Def Vertex (or Losslen) Expansion

$$\Phi^V(S) = \frac{|N(S)|}{d|S|}$$

$$\Phi^V_\varepsilon(G) = \min_{\substack{\phi \neq S \subseteq V \\ |S| \leq \varepsilon n}} \Phi^V(G).$$

OP [van Gogh prize]

Construct explicit family with

$$\Phi^V_\varepsilon(G) > \frac{1}{2} \quad \text{for } \varepsilon = \Omega(1)$$

(or two-sided bipartite losslen)

On the Complexity of Expansion

(Hypothesis) $\forall \eta \in (0,1)$ $\exists \delta \in (0,1)$ s.t. it is NP-hard to distinguish given input graph $G=(V,E)$

(Yes) $\exists S \subseteq V$ with $|S| \leq \delta n$ and $\bar{\Phi}(S) \leq \eta$.

(No) $\forall S \subseteq V$ with $|S| \leq \delta n$, we have $\bar{\Phi}(S) \geq 1-\eta$

OP van Gogh prize

Prove or refute the above hypothesis