# An Invitation to Expander Graphs

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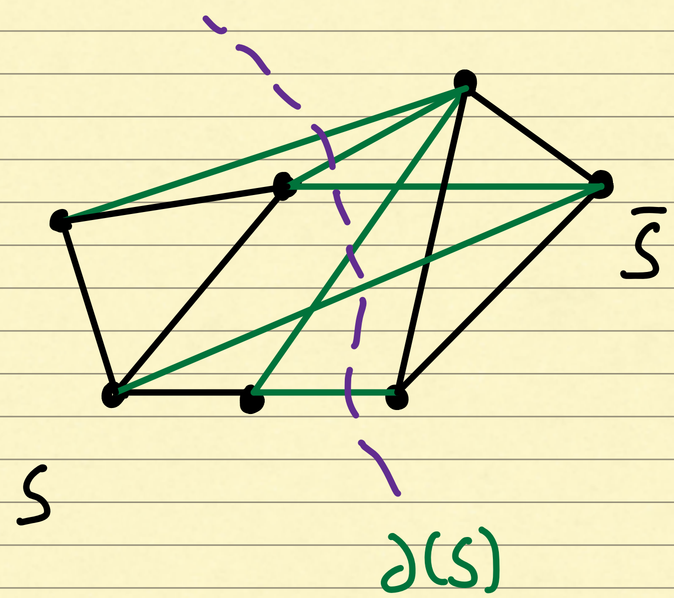

# Plan

1) What is an expander?

2) Warm up: Expander Mixing Lemma

3) Some Applications

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1) What is an expander?

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3) Some Applications

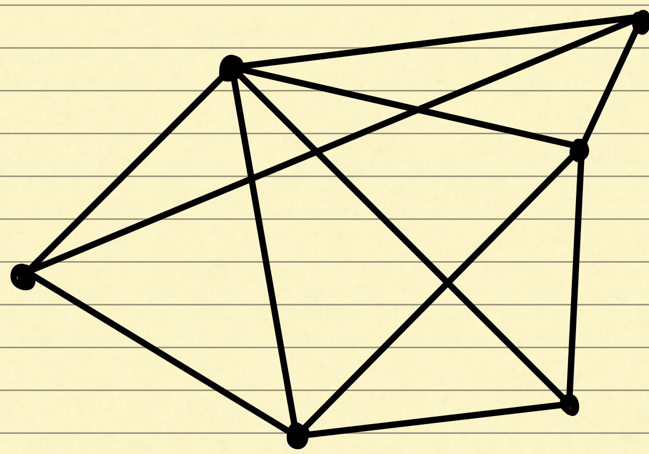
4) Zig-Zag Product

5) Near-optimal Expanders

# What is an expander?

**Informal Def:** A well connected yet sparse graph

Graph $\qquad G = (V, E)$



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"perspectives"

- Combinatorial
- Algebraic
- Random Walk

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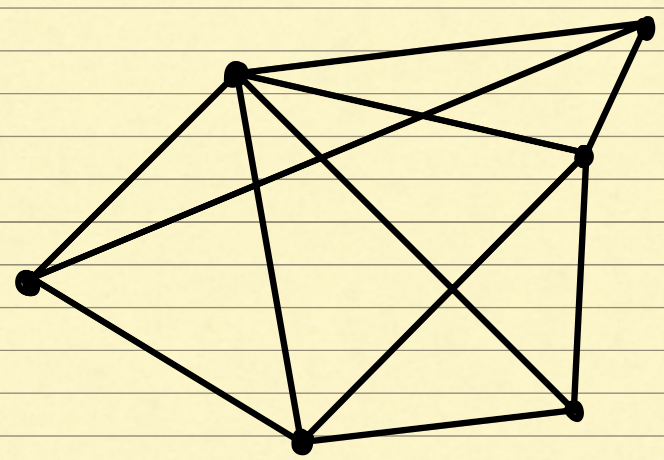

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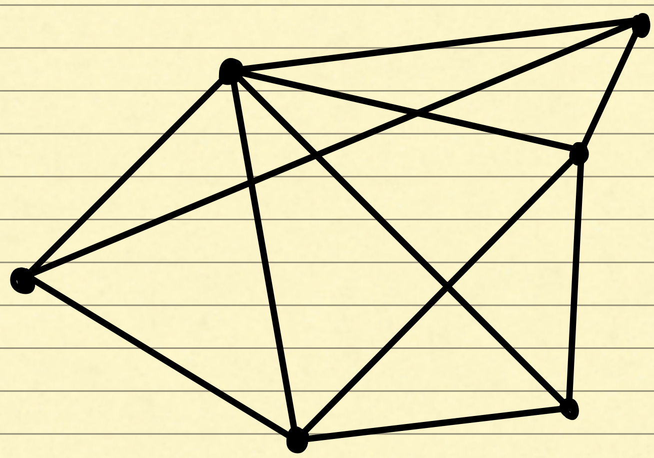
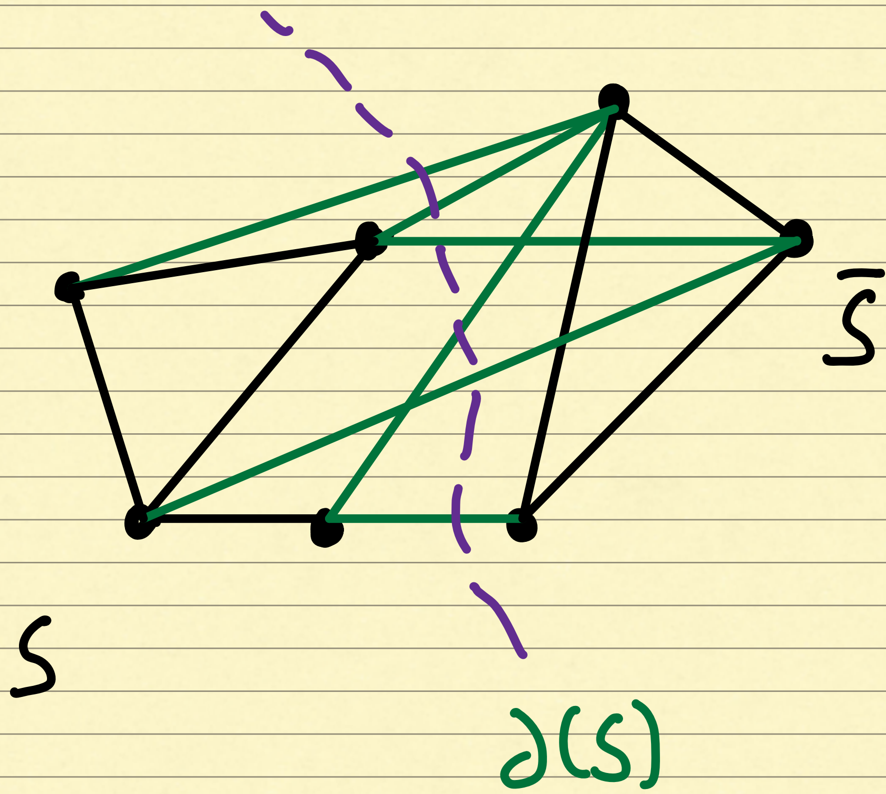
Graph $G = (V, E)$ $|E|$ small



# Expanders: Combinatorial Def.

Graph $G=(V,E)$ d-regular n-vtx

"No Small Cuts"



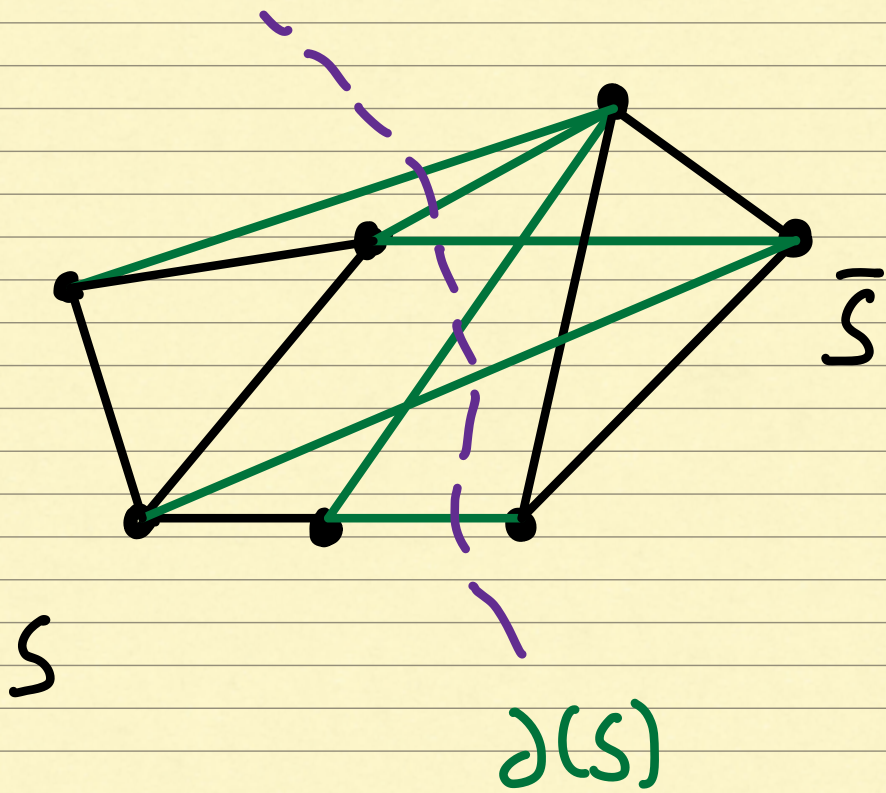
# Expanders: Combinatorial Def.

Graph $G=(V,E)$ $d$-regular $n$-vtx

"No Small Cuts"

(Edge) Boundary of $S \subseteq V$

$\partial(S) := \{(s,v) \in E \mid s \in S, v \in \bar{S}\}$



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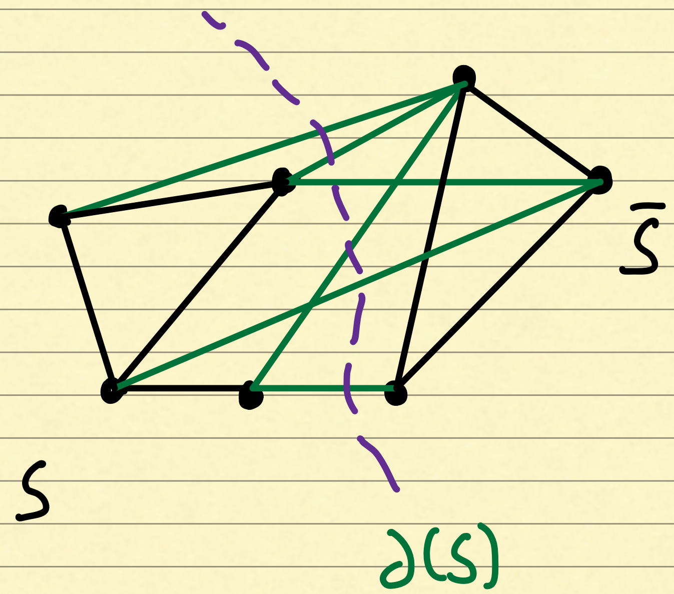


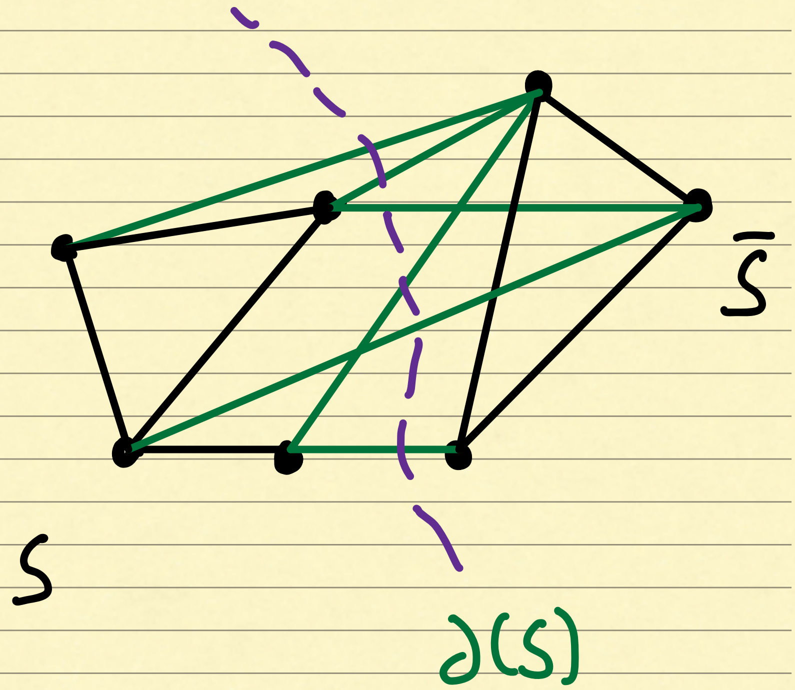

$$\boxed{\begin{array}{l}\text{Conductance of } S \\ \Phi_G(S) := \dfrac{|\partial(S)|}{d|S|}\end{array}}$$

# Expanders: Combinatorial Def.

Graph $G=(V,E)$ d-regular n-vtx

**Comb. Expansion**

$$\Phi(G) := \min_{\substack{S \subseteq V \\ 1 \le |S| \le \frac{n}{2}}} \Phi_G(S)$$

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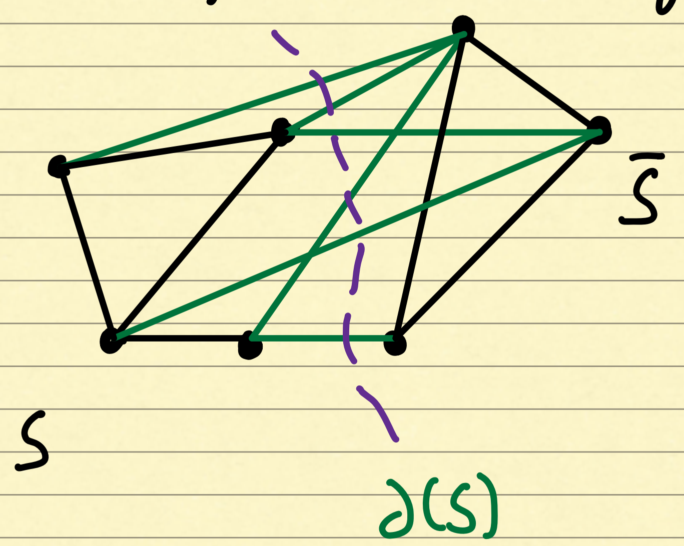


**Conductance of $S$**

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Want $\Phi(G) \geq \varepsilon > 0$ for a constant $\varepsilon$

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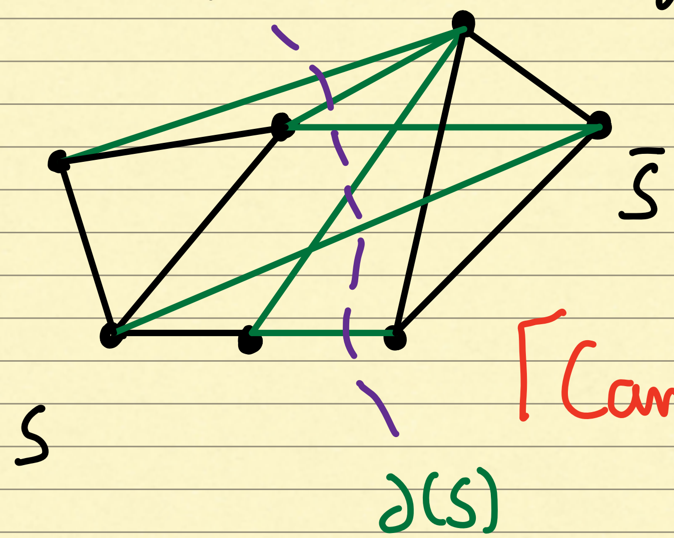


[Can we check this ??]

**Conductance of $S$**

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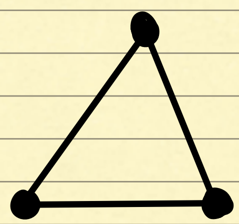
# Expanders: Algebraic Def.

Graph $G = (V, E)$ $d$-regular $n$-vtx

Adjacency Matrix $A \in \mathbb{R}^{n \times n}$ of $G$

$$A_{u,v} = \mathbb{1}[u \sim_G v]$$

Ex:

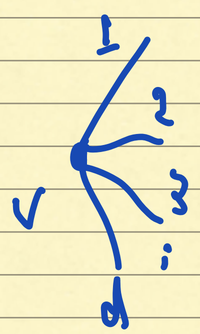


$G$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$A$

$$\vec{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad A\vec{1} = d\,\vec{1}$$



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Graph $G=(V,E)$ $d$-regular $n$-vtx

Adjacency Matrix $A \in \mathbb{R}^{n\times n}$ of $G$

$$A_{u,v} = \mathbb{1}_{[u \sim_G v]}$$

$A$ is symmetric (over $\mathbb{R}$)

$$\Downarrow \text{ Spectral Theorem}$$

$$A = \sum_{j=1}^{n} \lambda_j v_j v_j^T \quad \text{with } \{v_j\}_{j\in[n]} \text{ ONB}$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$\lambda_1 = d$

$$v_1 = \tilde{\mathbb{1}}$$

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$$A = \sum_{j=1}^{n} \lambda_j v_j v_j^T$$

$$\begin{array}{ccccccc} \lambda_1 & \geq & \lambda_2 & \geq & \cdots & \geq & \lambda_n \\ \parallel & & & & & & \vee\!\!\! / \\ d & & & & & & -d \end{array} \qquad v_1 = \vec{1}/\sqrt{n}$$

Algebraic Expansion

$$\lambda(G) := \max\{\underline{|\lambda_2|}, \underline{|\lambda_n|}\}$$

Want $\underline{\lambda(G)}$ small $\ll d$

# Expanders: Algebraic Def.

Graph $G = (V, E)$ $d$-regular $n$-vtx

Adjacency Matrix $A \in \mathbb{R}^{n \times n}$ of $G$

$$A = \sum_{j=1}^{n} \lambda_j v_j v_j^T \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Algebraic Expansion

$$\lambda(G) := \max\{|\lambda_2|, |\lambda_n|\}$$

## Key Algebraic Perspective

$$A = \underbrace{\frac{d}{n} \vec{1} \vec{1}^T}_{\text{adj. matrix of complete graph with self loops}} + \overbrace{\lambda E}^{\text{error term}} \qquad \|E\|_{op} \leq 1$$

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$$\|E\|_{op} \leq 1$$

$$A \approx \frac{d}{n} J$$

complete graph with density $d/n$

# Expander Definitions

## Equivalent Views

Combinatorial: $\Phi(G)$ large

$\Updownarrow$ Cheeger's Ineq.

Algebraic: $\lambda(G)$ small

$\Updownarrow$

Random Walk: Fast Mixing

# Warm up: Expander Mixing Lemma

Graph $G=(V,E)$ $\quad (n, d, \lambda)$-graph

Suppose we want to compute (approx) $|E(S,\bar{S})|$?

$$|E(S,\bar{S})| = \mathbb{1}_S^T A \mathbb{1}_{\bar{S}}$$

# Warm up: Expander Mixing Lemma

Graph $G=(V,E)$ $\quad (n,d,\lambda)$

Suppose we want to compute $|E(S,\bar{S})|$?
(approx)

$$|E(S,\bar{S})| = \mathbb{1}_S^{T} A \mathbb{1}_{\bar{S}}$$

Recall that $A = \frac{d}{n} J + \lambda E$

$$\left( A \simeq \frac{d}{n} J \right)$$

$$|E(S,\bar{S})| \approx \frac{d}{n} \mathbb{1}_S^{T} J \mathbb{1}_{\bar{S}} = \frac{d}{n} |S| |\bar{S}|$$

# Warm up: Expander Mixing Lemma

Graph $G=(V,E)$ $(n, d, \lambda)$

$$|E(S,\bar{S})| = 1_S^T A 1_{\bar{S}}$$

Recall that $A = \frac{d}{n} J + \lambda E$

$$\left( A \simeq \frac{d}{n} J \right)$$

$$|E(S,\bar{S})| \simeq \frac{d}{n} |S||\bar{S}|$$

$\|E\|_{op} \leq 1$

More rigorously,

$$|E(S,\bar{S})| = \frac{d}{n} |S||\bar{S}| + \underbrace{\lambda 1_S^T E 1_{\bar{S}}}_{\pm \lambda \sqrt{|S||\bar{S}|}}$$

error term

# Some Applications

- Coding Theory
- Complexity Theory
- Hardness of Approximation

  ⋮

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- Coding Theory
- Complexity Theory
- Hardness of Approximation
- Pseudorandomness
- Sampling and Counting
- Algorithm Design
- Property Testing
- Metric Spaces
- Group Theory
- Number Theory

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# Some Applications

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⋮

- Your new application

# Some Applications

(an appetizer)

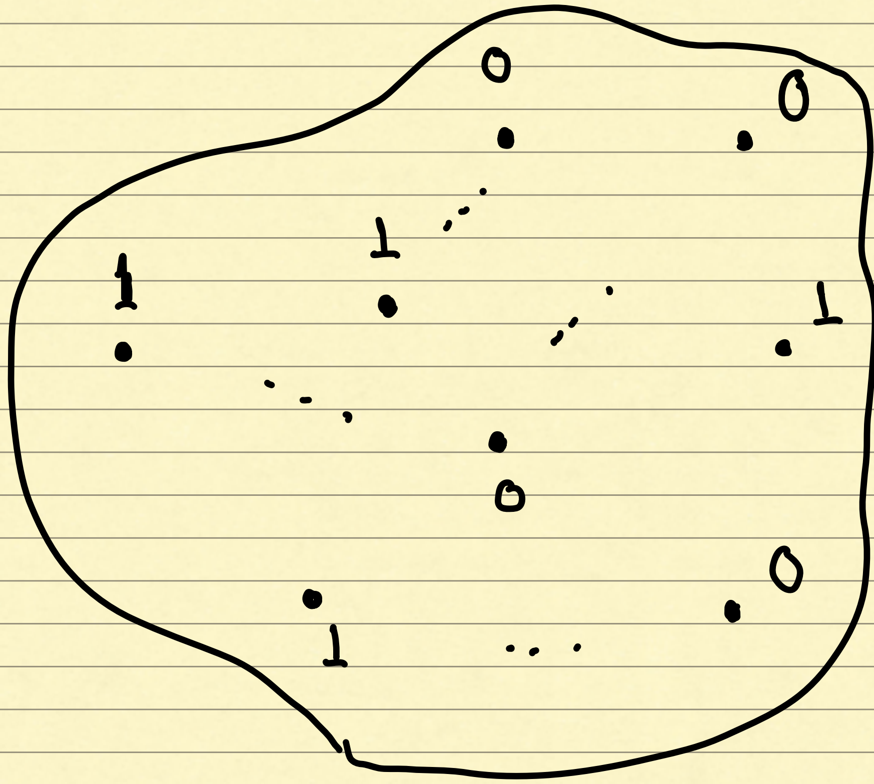
- Derandomization
- Complexity Theory

How does expansion appear?

# Chernoff Bound

$n$ 0/1 values

$$\mu := \frac{\#\text{ of 1's}}{n}$$



Sample $k$ indepent values uniformly

$$X_1, \dots, X_k$$

# Chernoff Bound

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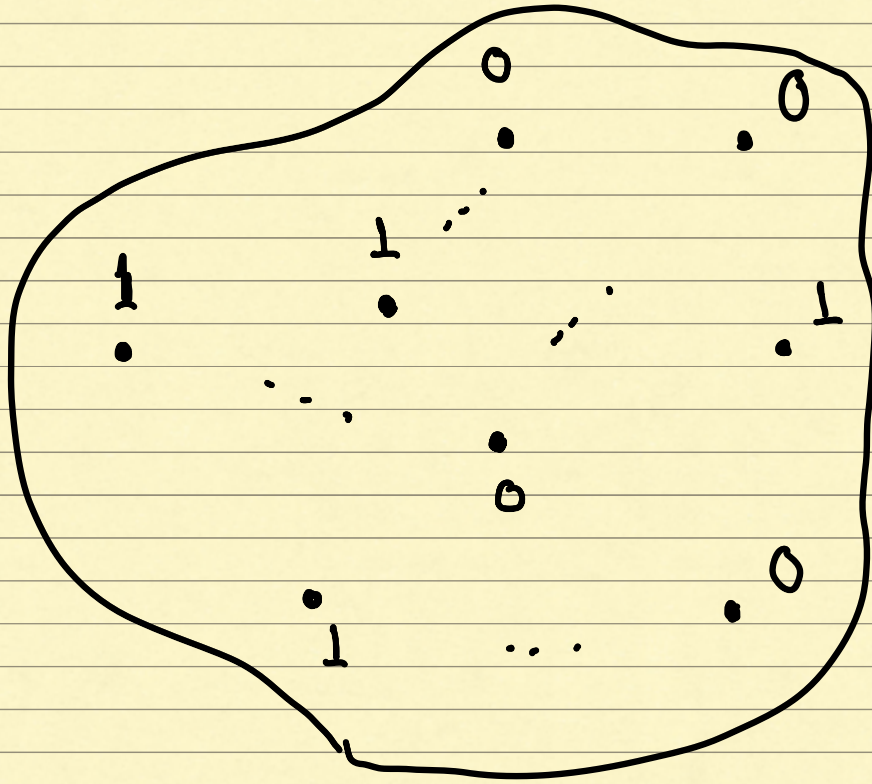


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$$\Pr\left[\left|\frac{1}{k}\sum X_i - \mu\right| > \varepsilon\right] \leq \exp\left(-\Omega(\varepsilon^2 k)\right)$$

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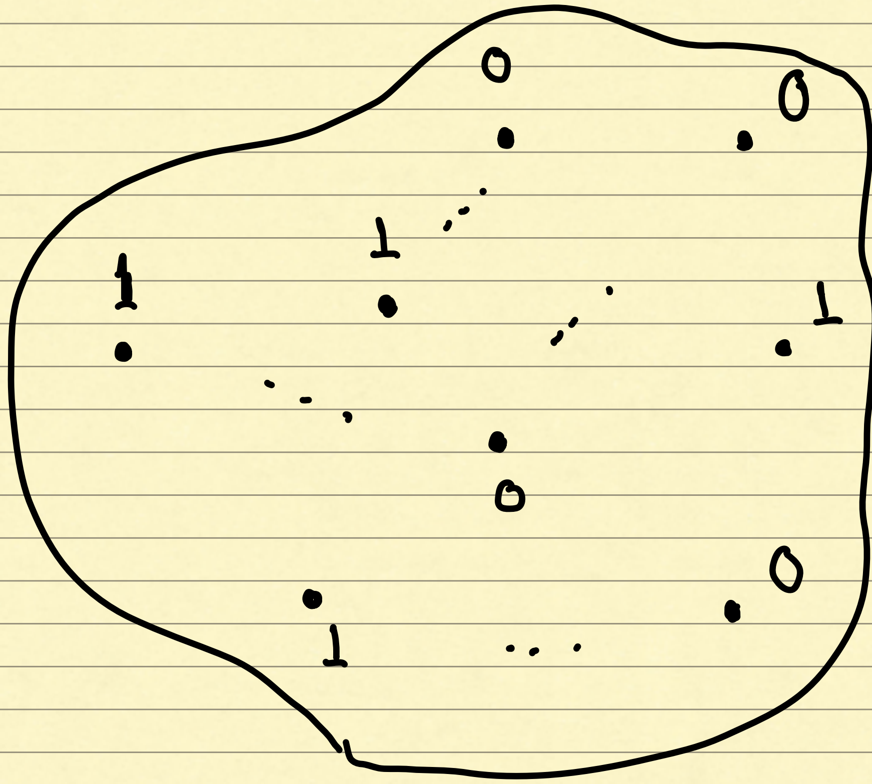

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\# of random bits $\approx k \cdot \log_2(n)$

# Expander Chernoff Bound

$G = (V = [n], E)$

$d$-regular $\quad d = O(1)$

$n \quad 0/1$ values

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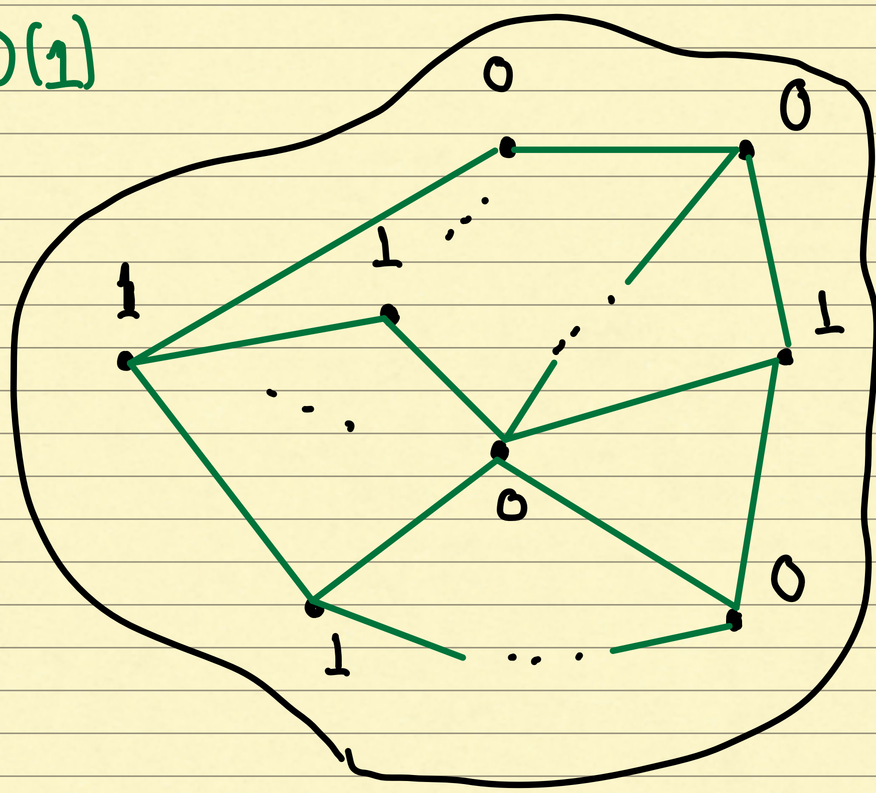

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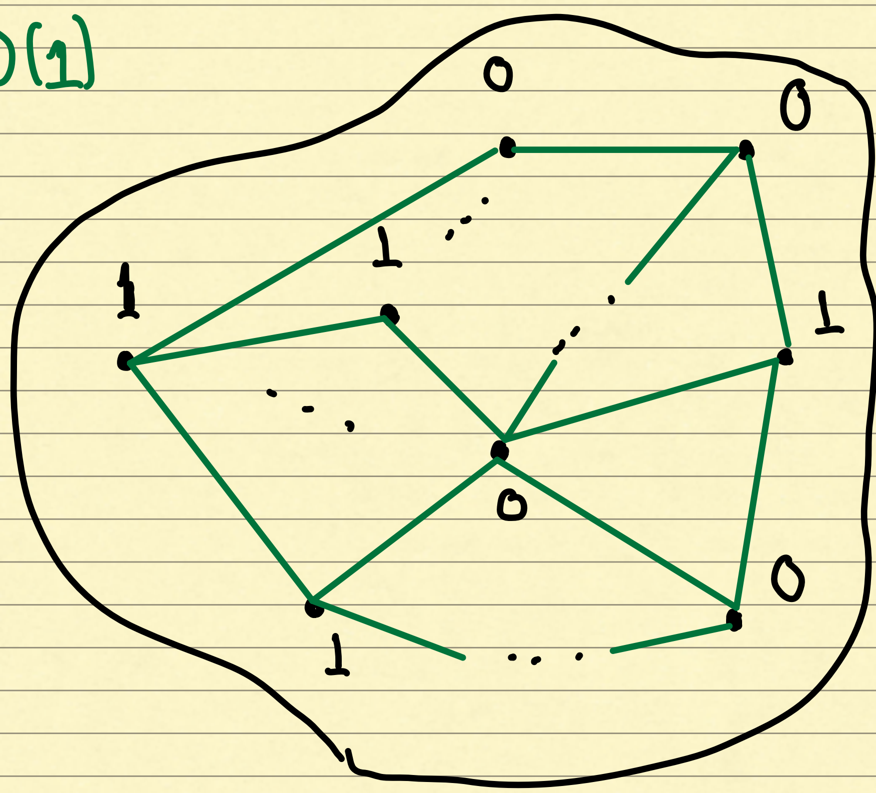


Sample $k$ values along a random walk on $G$

$X_1, \dots, X_k$

# Expander Chernoff Bound

$G = (V = [n], E)$

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$\boxed{\lambda(G) \leq \lambda}$

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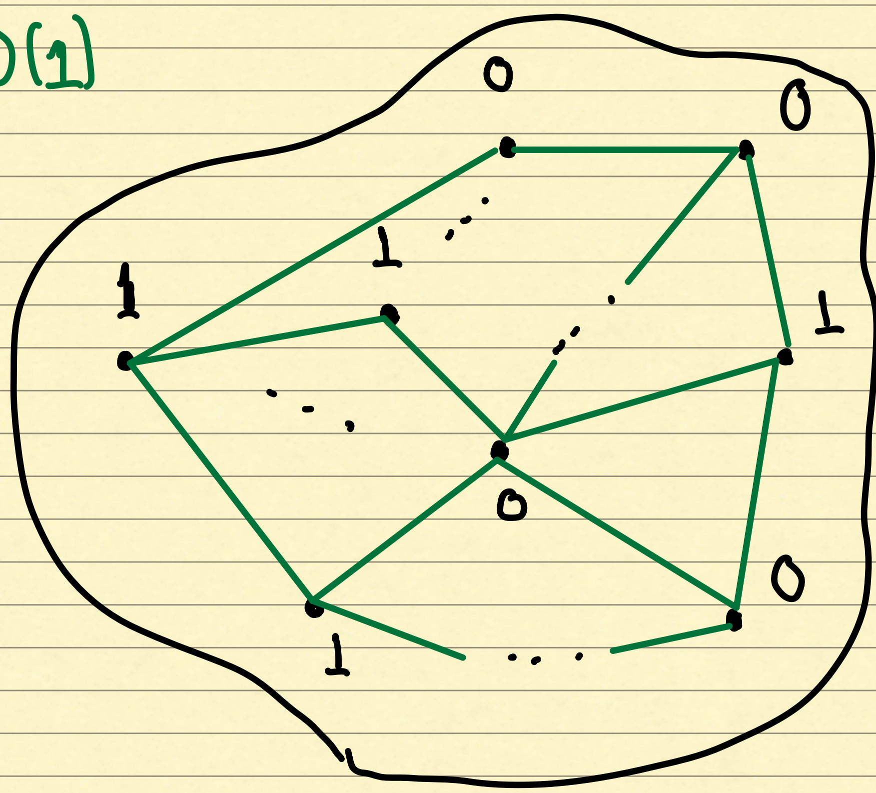


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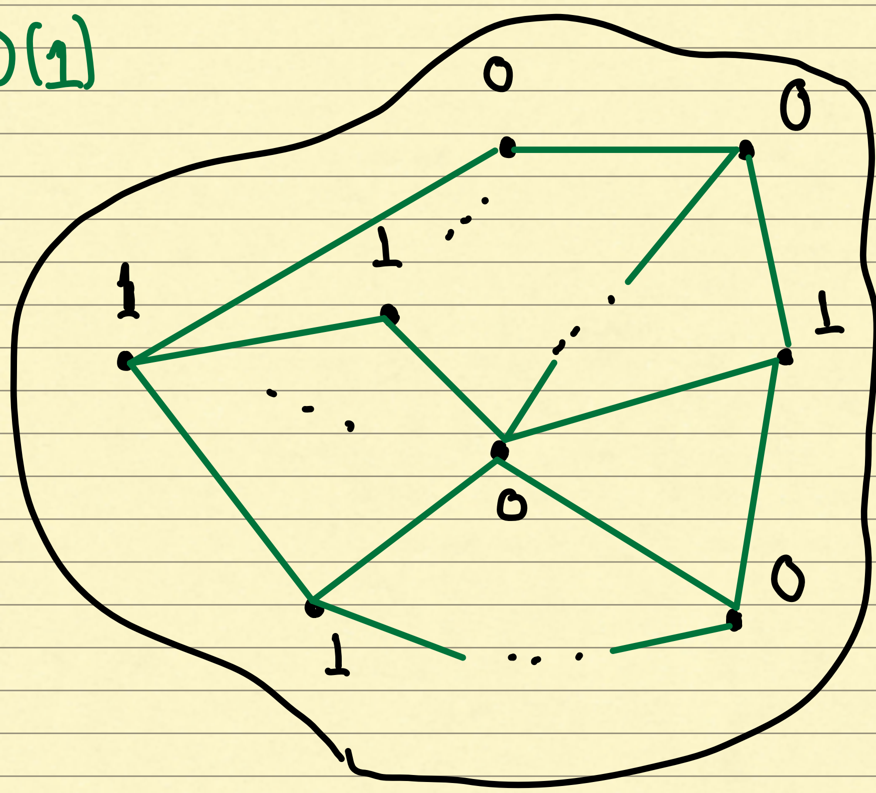


Sample $k$ values along a random walk on $G$

$X_1, \ldots, X_k$

Theorem [Gillman '93]

$$\Pr\left[\left|\frac{1}{k}\sum X_i - \mu\right| > \varepsilon\right] \leq \exp\left(-\Omega_\lambda(\varepsilon^2 k)\right)$$

$\#$ of random bits $\approx \cancel{k} \cdot \log_2(n) + O(k)$

# Complexity: $SL = L$

Input: $G=(V,E)$ an $n$-vtx graph,

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Question: $\exists$ $s$-$t$ path in $G$?

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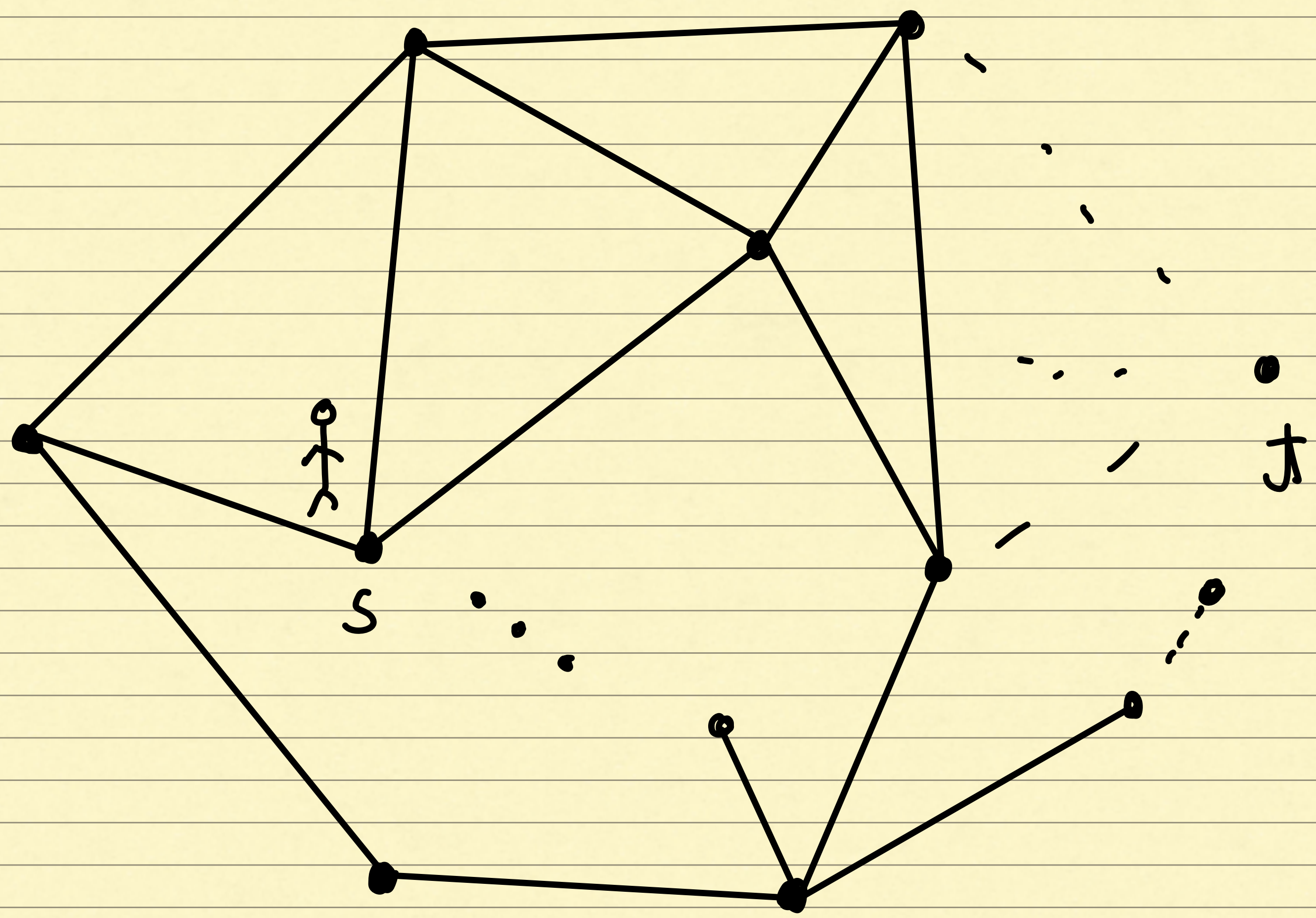

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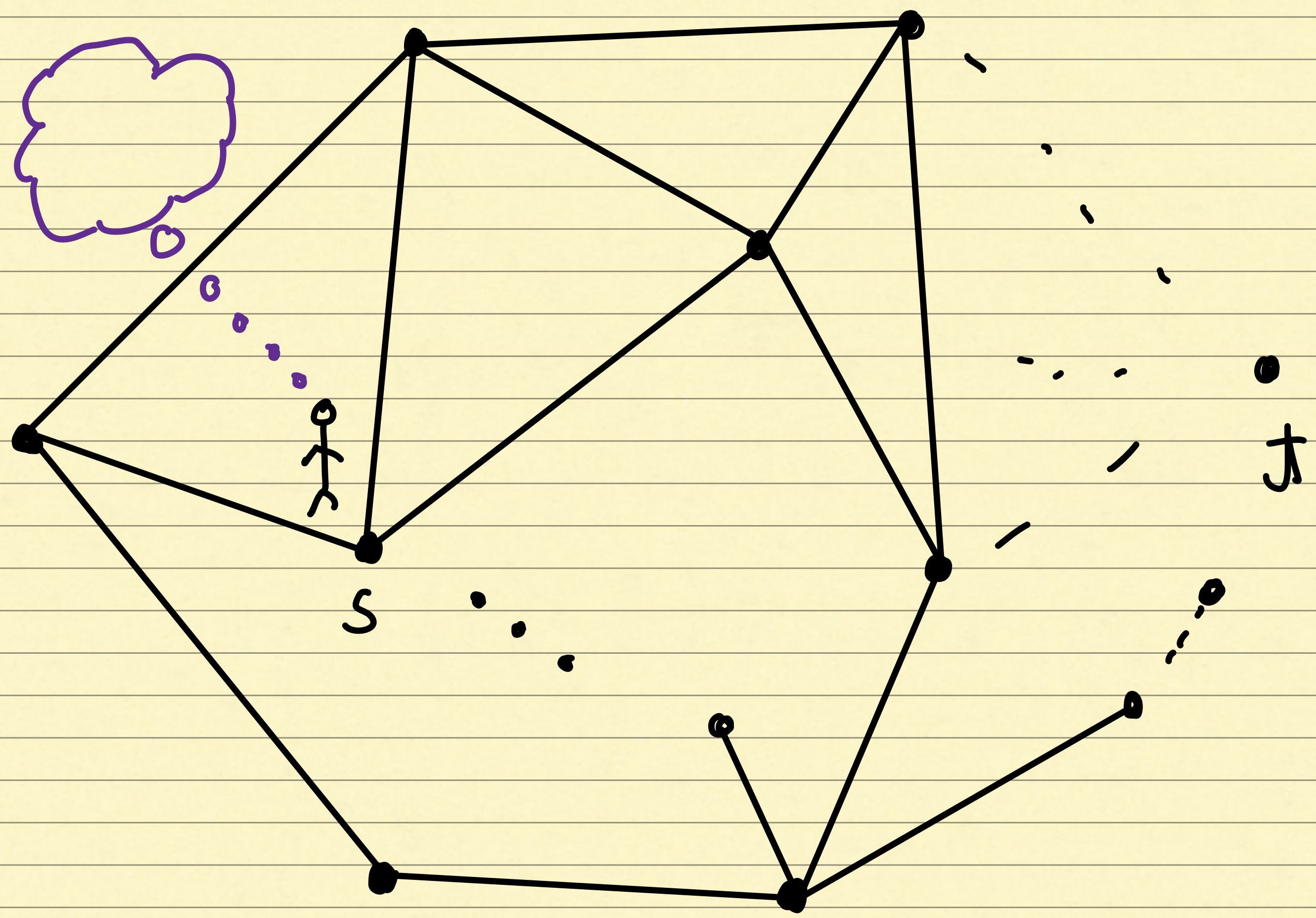
We can only remember $O(1)$ vertices



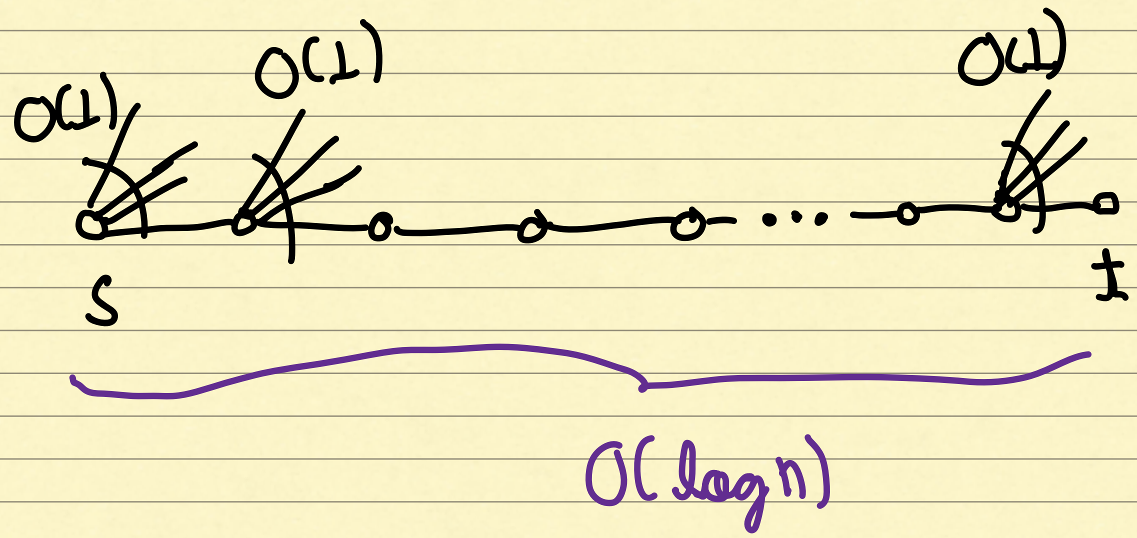
# Complexity: SL = L

Input: $G=(V,E)$ an $n$-vtx graph,
$s,t \in V$.

Question: $\exists$ $s$-$t$ path in $G$?

Req: deterministic algorithm with
$O(\log n)$ memory

Observation: This is easy if $G$ is a
constant degree expander
(diameter is $O(\log n)$)



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This can be done!

Transform (connected comp.) of $G$ into expander!

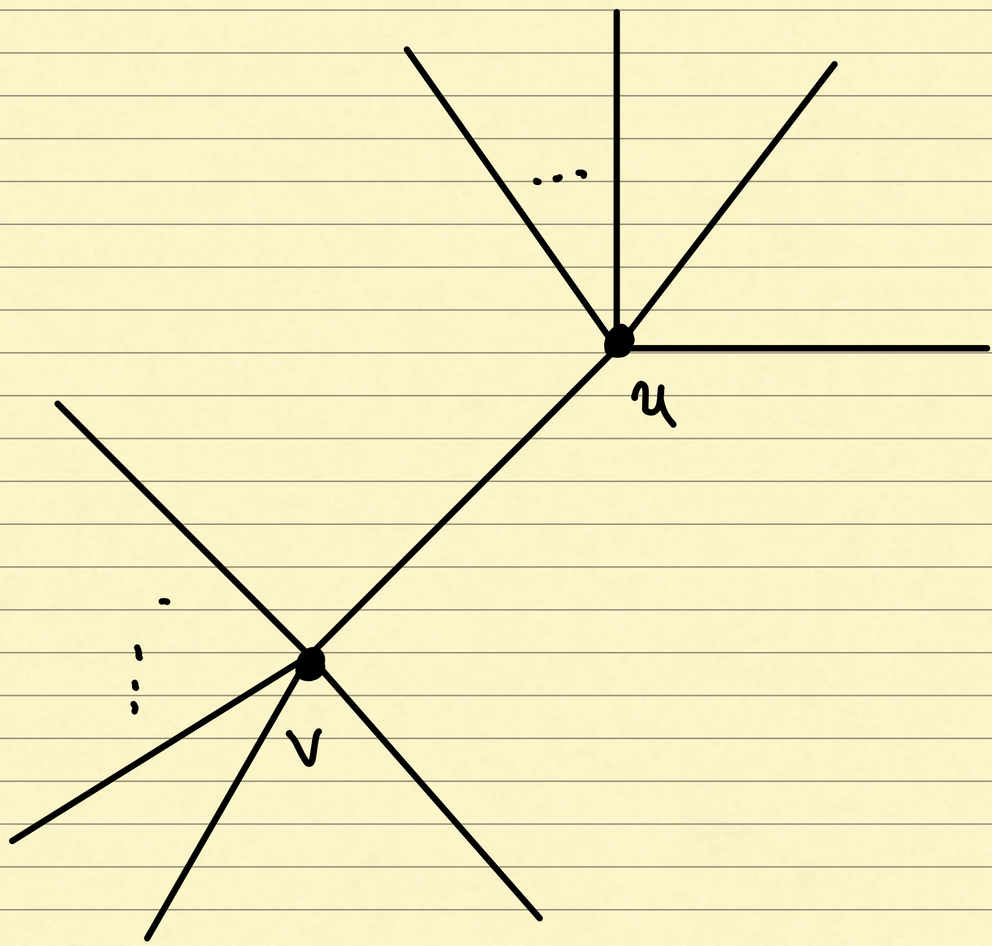
Thm [Reingold'05] $SL = L$

Key Technique: Zig-Zag product

# Towards Zig-Zag

First a simpler goal: Degree Reduction

Let G be n-vtx degree D ("large")



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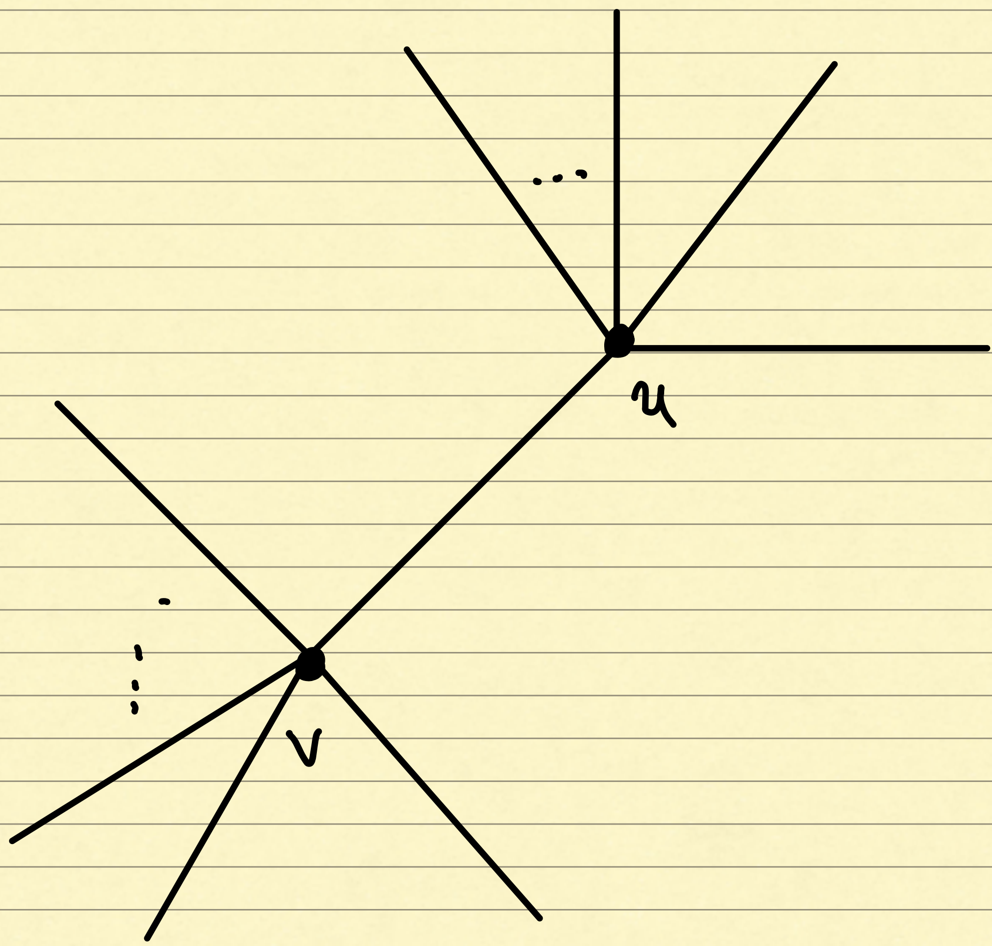


Hint: use a $D$-vtx graph $H$ of degree $d \ll D$

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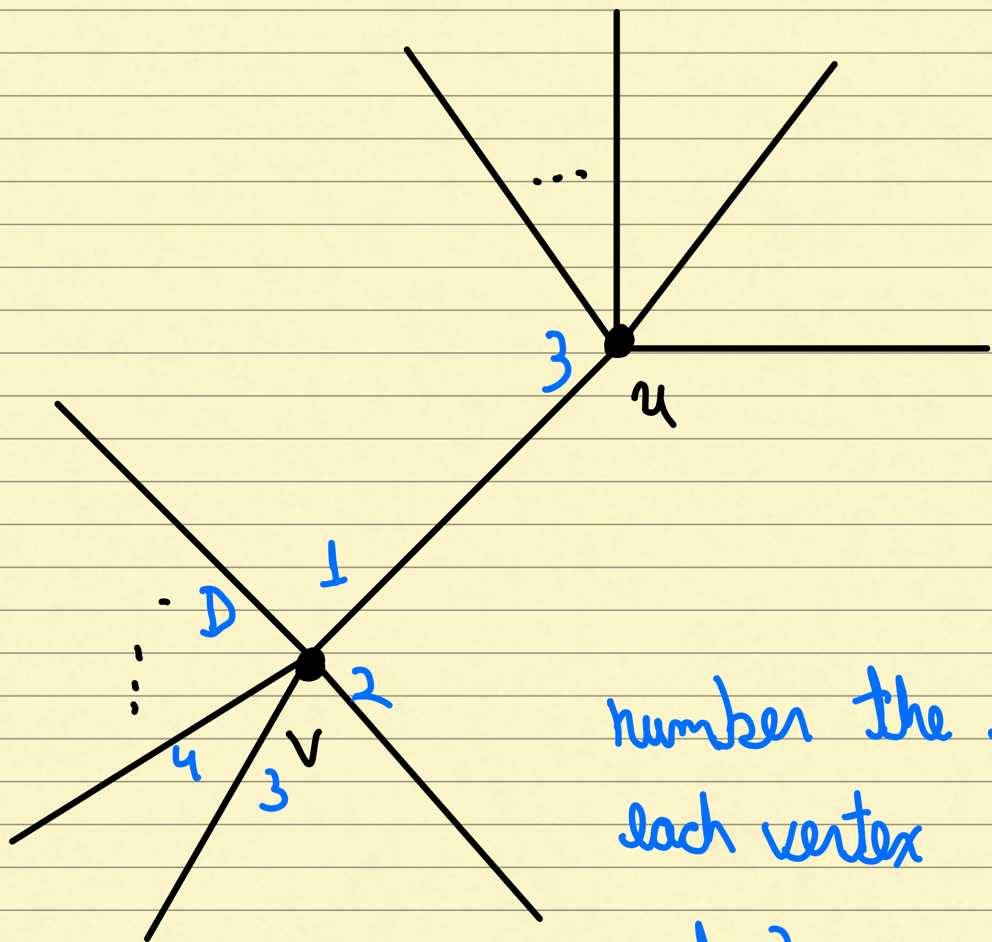



number the edges of each vertex using $1, 2, \ldots, D$

Hint: use a $D$-vtx graph $H$ of degree $d \ll D$

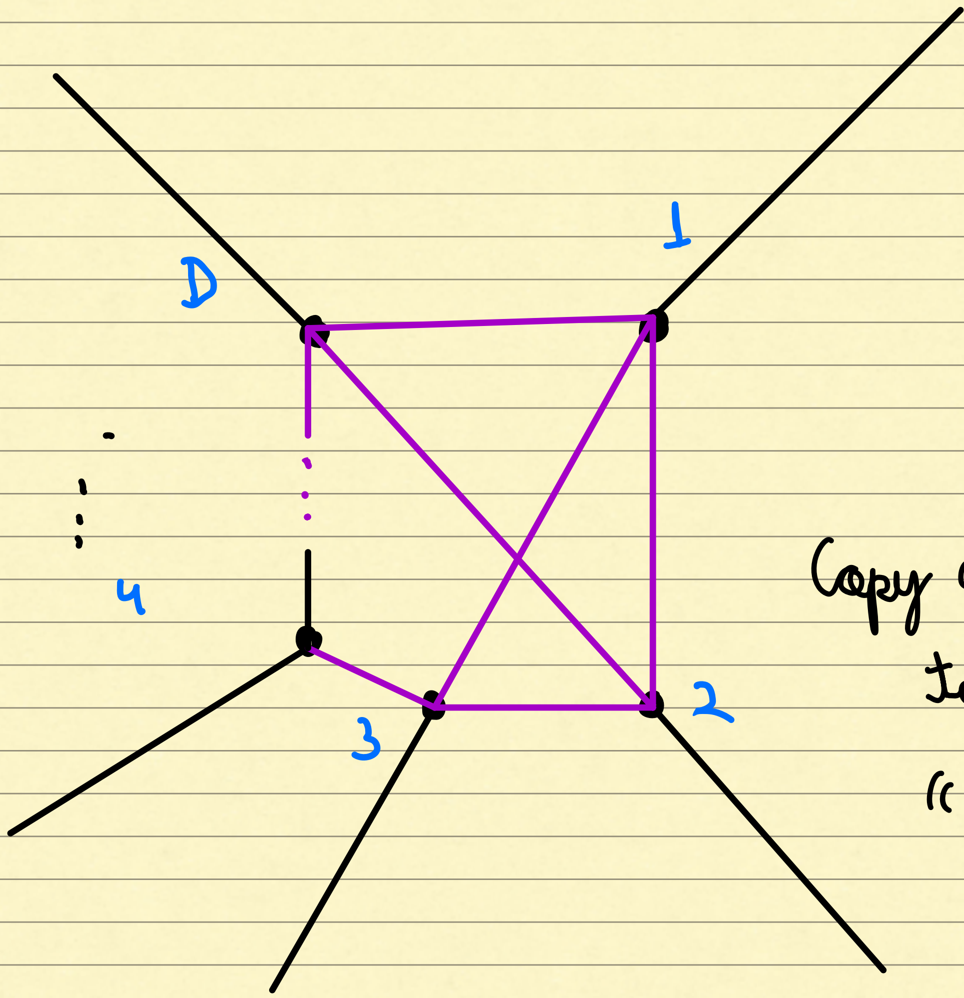
$$H = (\underline{V = [D]}, E)$$

# Towards Zig-Zag

First a simpler goal: Degree Reduction

Let $G$ be $n$-vtx degree $D$

Replace each vertex of $G$ by a copy of $H$




Copy of $H$ associated to $v$

"cloud"

$H = (V = [D], E)$ degree $d \ll D$

# Towards Zig-Zag

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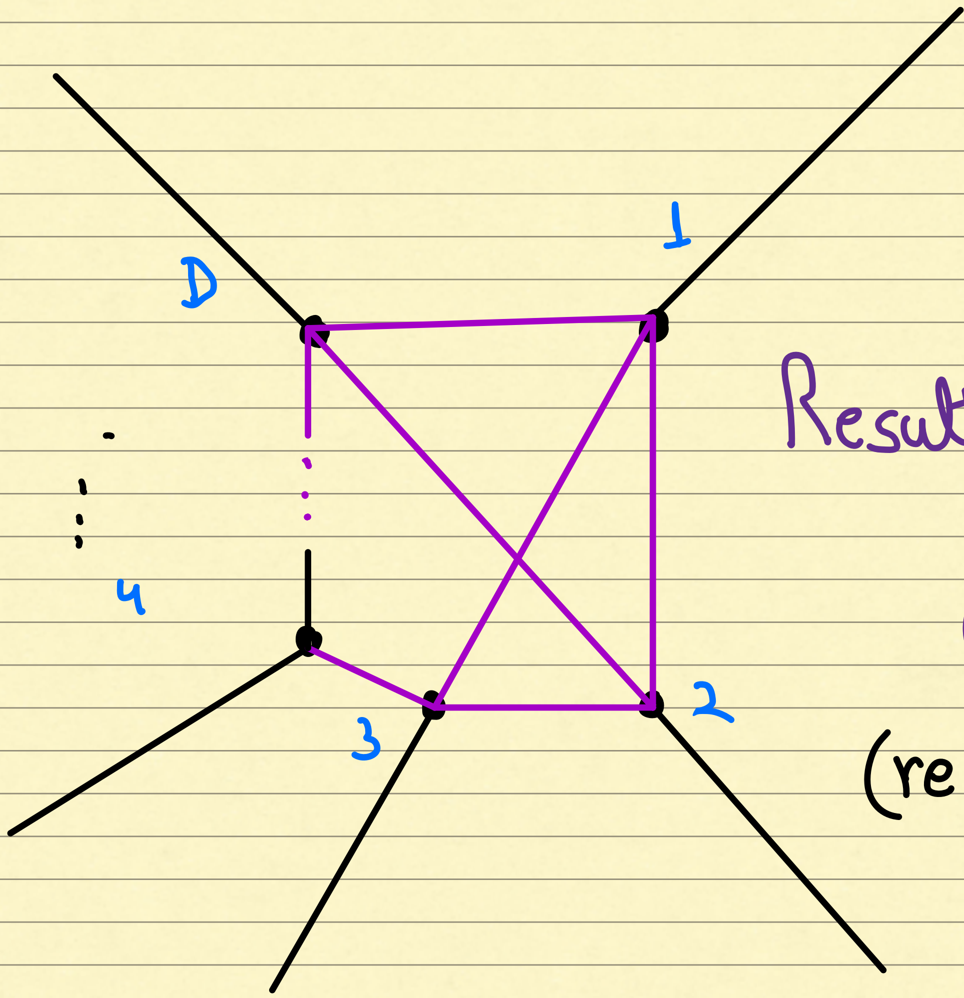


Resulting Graph:

$G \circledR H$

(replacement product)

deg: $d+1$

$H = (V = [D], E)$ degree $d \ll D$

# Zig-Zag Product

$G \textcircled{z} H$ is obtained from $G \textcircled{r} H$ by a "zig-zag walk"

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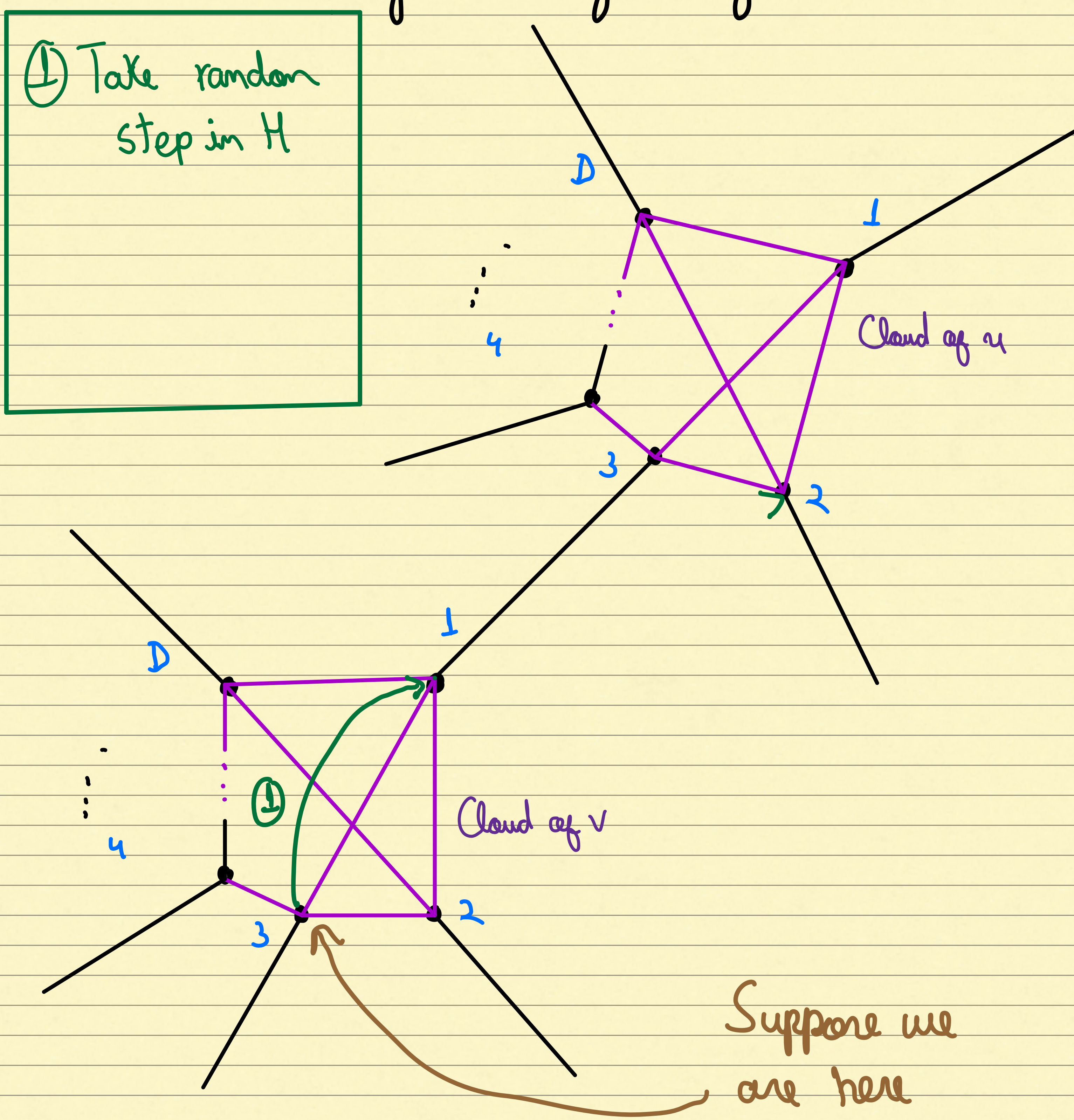
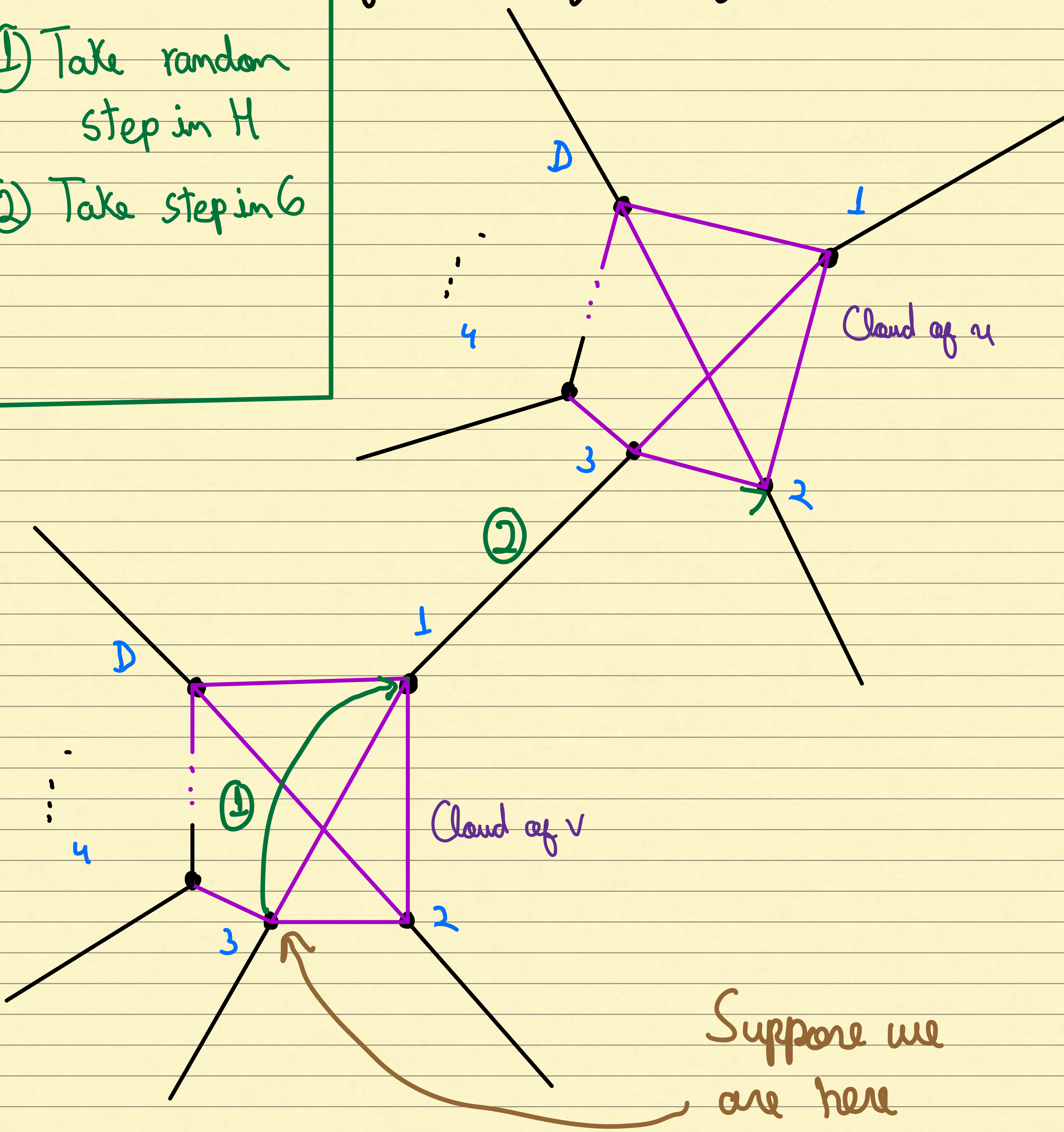


# Zig-Zag Product

$G \,\textcircled{z}\, H$ is obtained from $G \,\textcircled{r}\, H$
by a "zig-zag walk"

(1) Take random step in $H$

(2) Take step in $G$



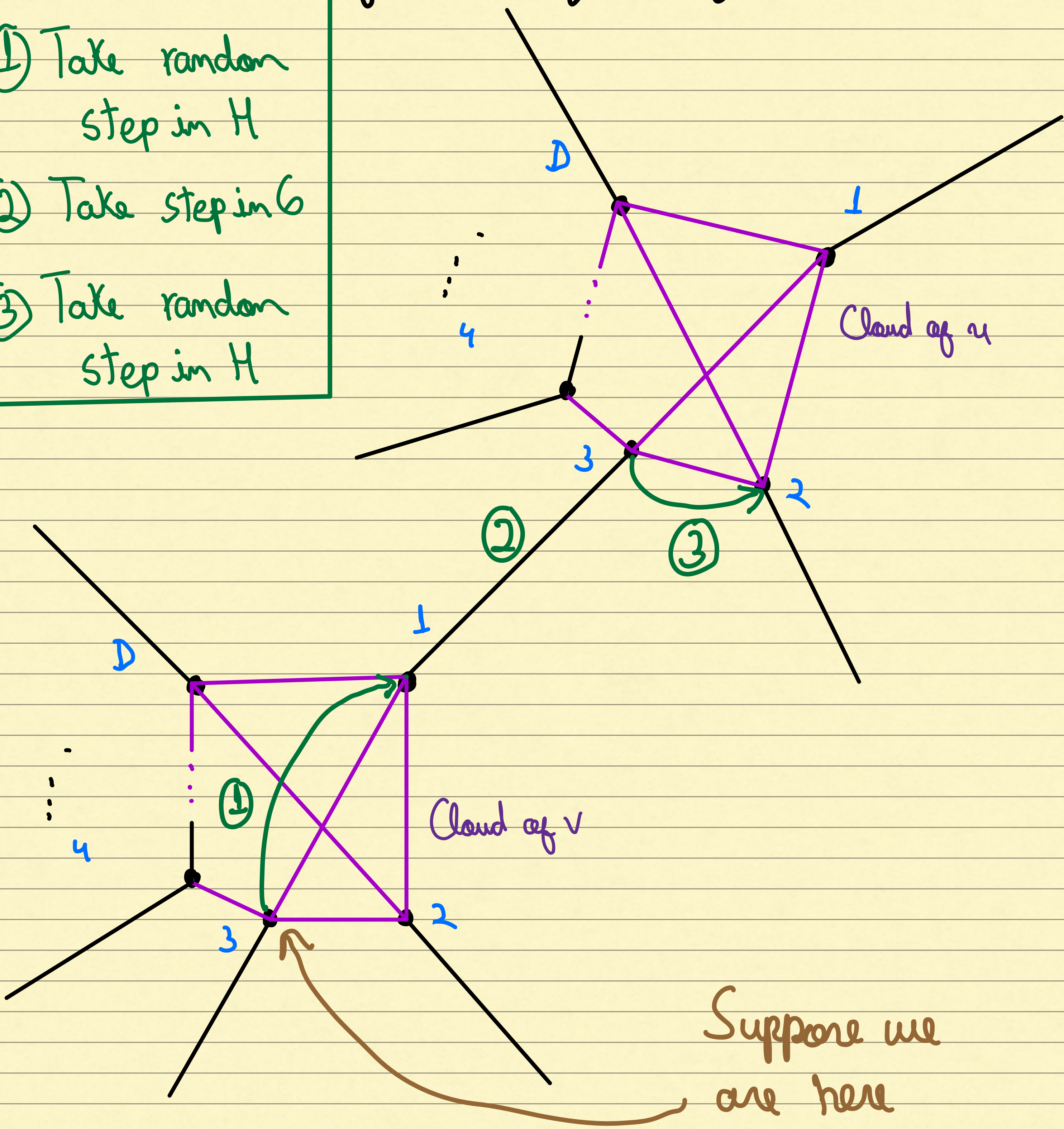
# Zig-Zag Product

$G \circledz H$ is obtained from $G \circledr H$ by a "zig-zag walk"

① Take random step in $H$

② Take step in $G$

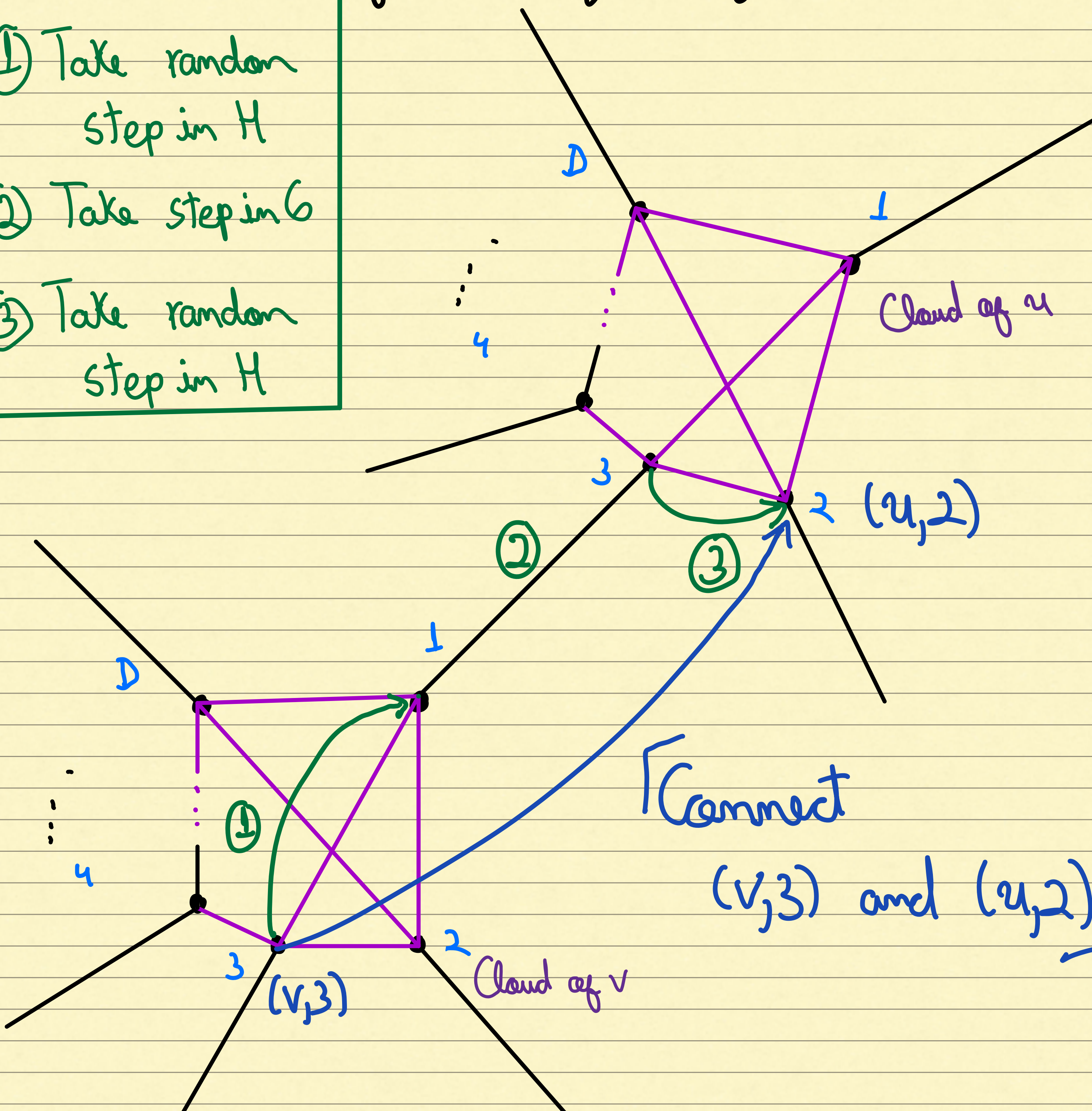
③ Take random step in $H$



# Zig-Zag Product

$G \,ⓩ\, H$ is obtained from $G \,ⓡ\, H$ by a "zig-zag walk"

1. Take random step in $H$
2. Take step in $G$
3. Take random step in $H$




[Connect (v,3) and (u,2)]

# Zig-Zag Product

Theorem [Reingold-Vadhan-Wigderson'00]

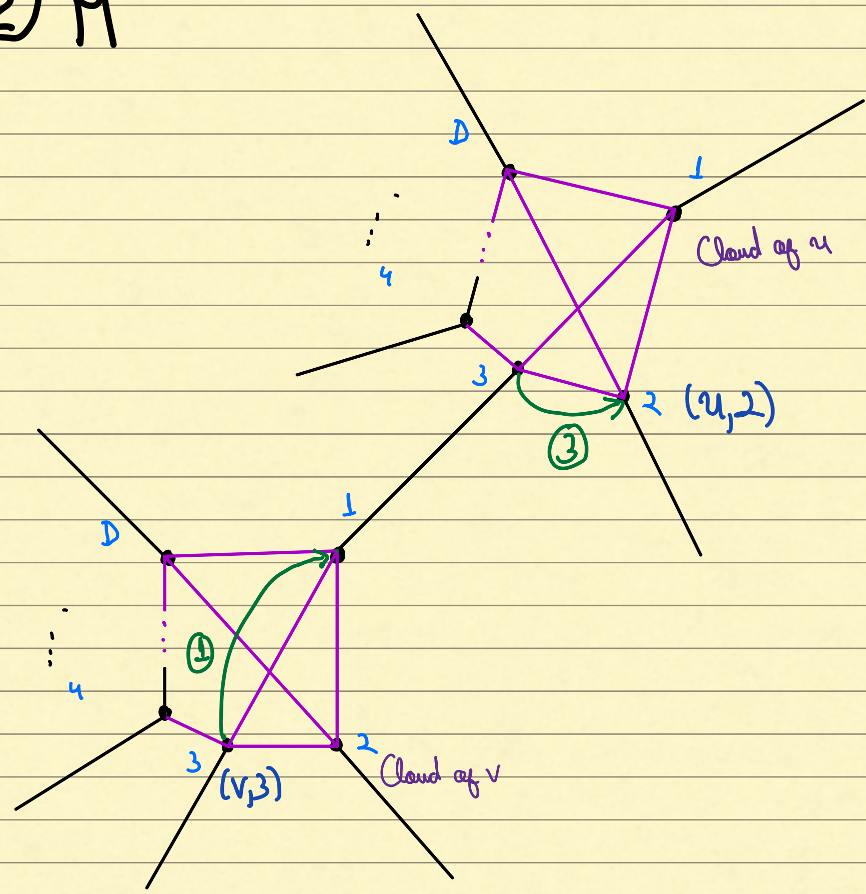
If $G$ is $(n, D, \lambda_1)$-graph and
$H$ is $(D, d, \lambda_2)$-graph, then

$G \textcircled{z} H$ is $(nD, d^2, f(\lambda_1, \lambda_2))$-graph

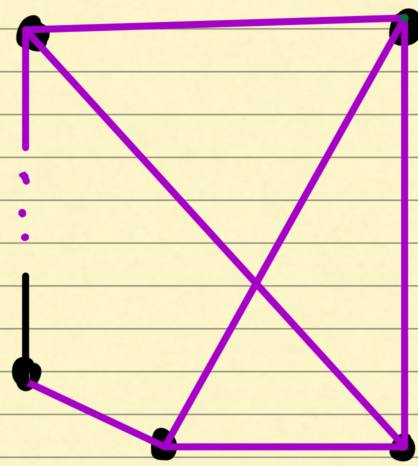
(We will bound $f(\lambda_1, \lambda_2)$ soon)

# Zig-Zag Product

$G \textcircled{z} H$

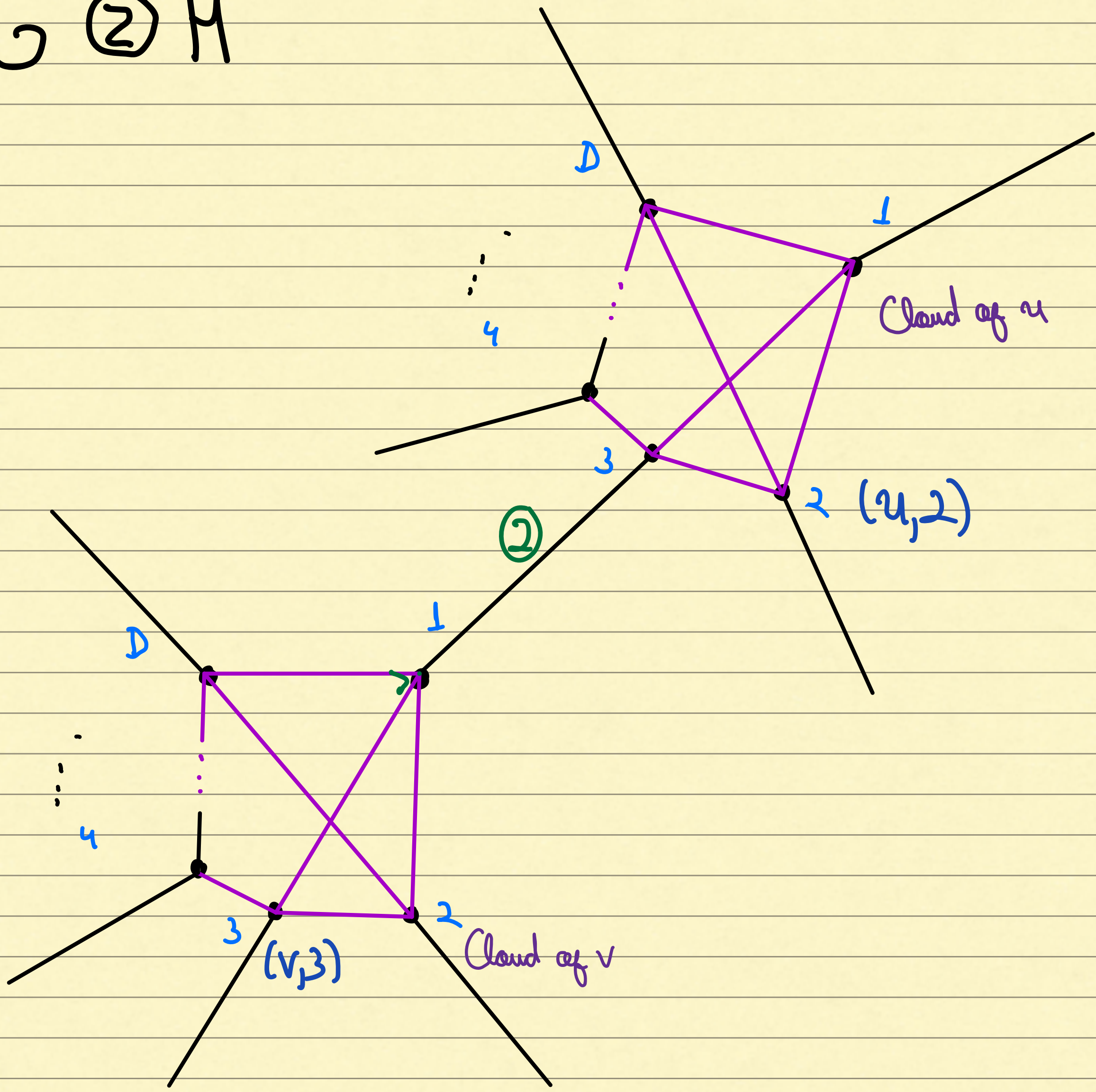



$\boxed{\text{Let } A_H \text{ be the normalized adj. matrix of } H}$



# Zig-Zag Product

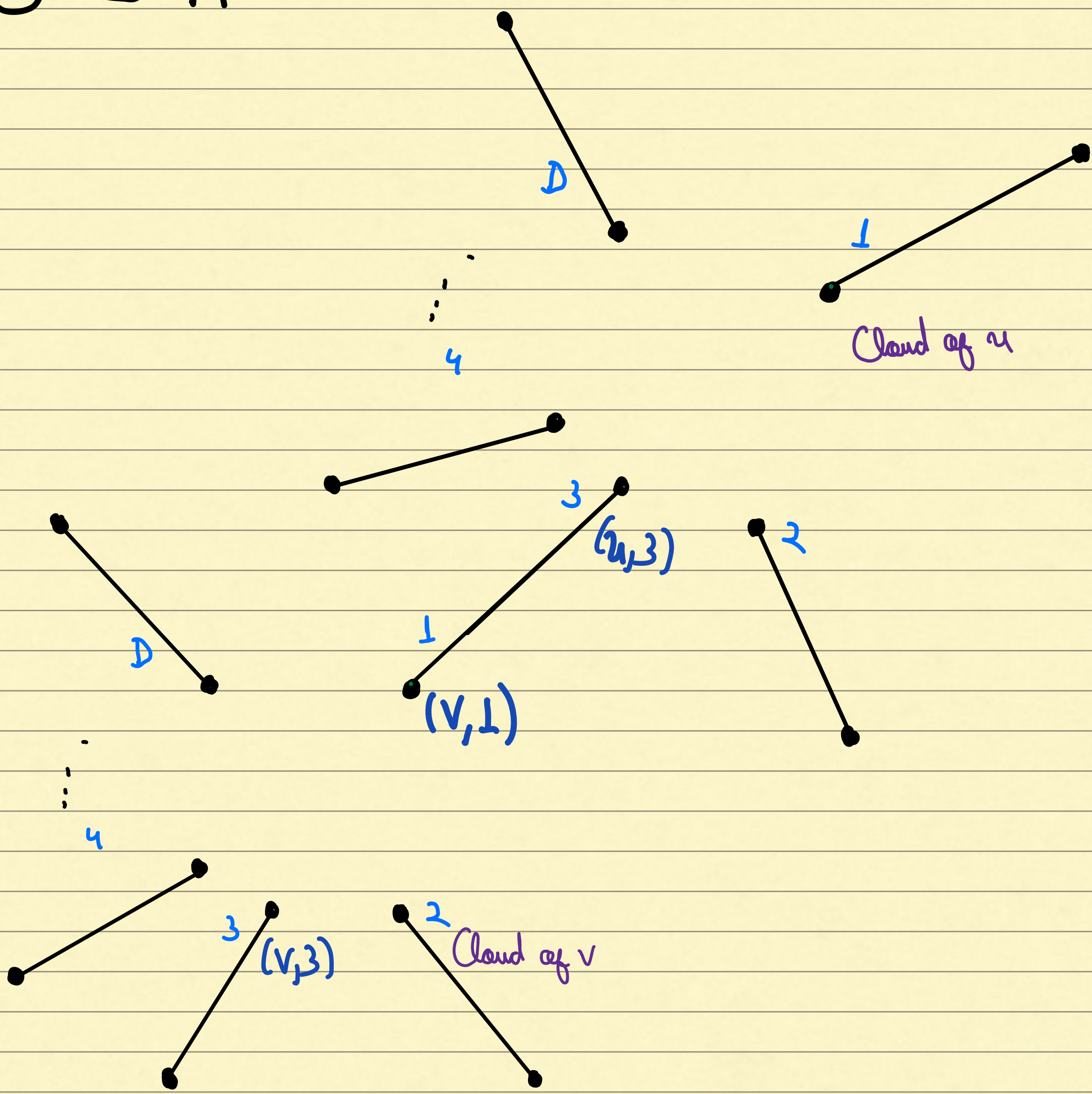
$G \textcircled{z} H$




$\boxed{\text{Let } \tilde{G} \text{ be the matrix of the action of } G \text{ on } \mathbb{R}^{V(G)\times V(H)}}$

# Zig-Zag Product

$G ⓩ H$

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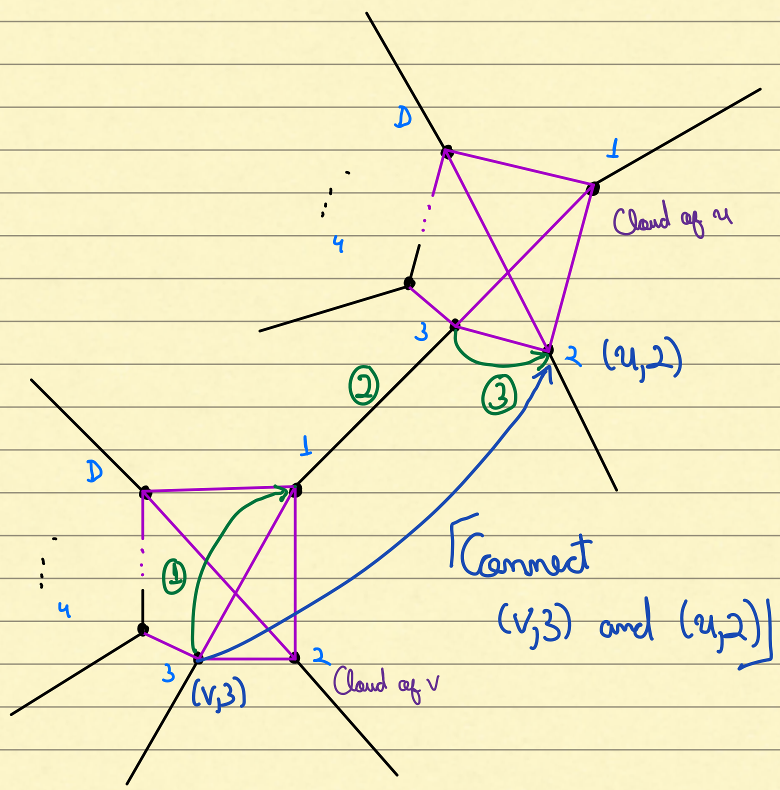
# Zig-Zag Product

$G \textcircled{z} H$

| | Operators |
|---|---|
| ① Take random step in H | $I \otimes A_H$ |
| ② Take step in G | $\tilde{G}$ |
| ③ Take random step in H | $I \otimes A_H$ |




Let $A_H$ be the normalized adj. matrix of H

Let $\tilde{G}$ be the matrix of the action of G on $\mathbb{R}^{V(G)\times V(H)}$

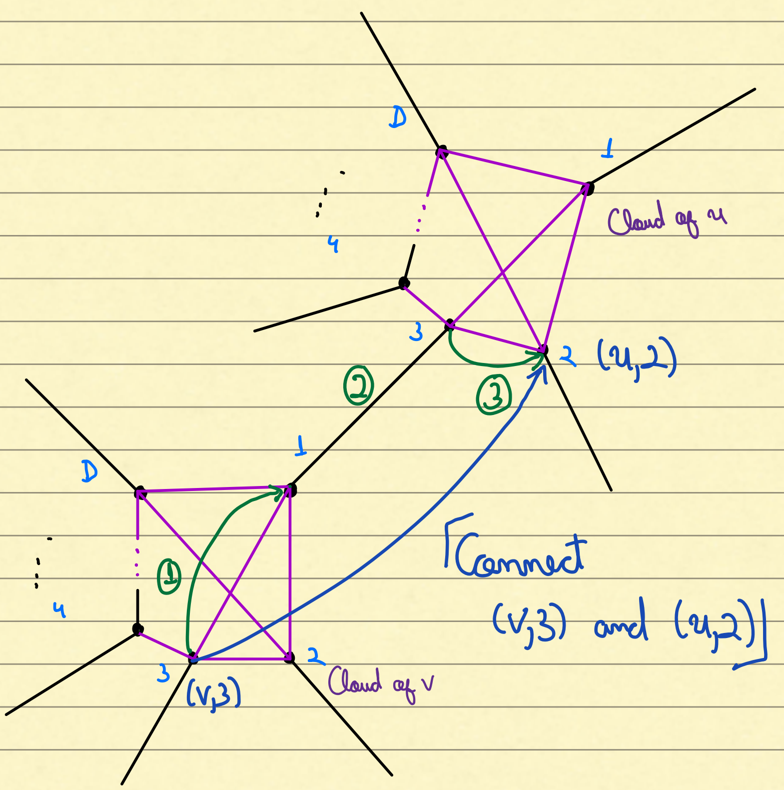
# Zig-Zag Product

$G \textcircled{z} H$

| | |
|---|---|
| ① Take random step in H | $I \otimes A_H$ |
| ② Take step in G | $\tilde{G}$ |
| ③ Take random step in H | $I \otimes A_H$ |

Adj. mat. $G \textcircled{z} H$

$$(I \otimes A_H)\tilde{G}(I \otimes A_H)$$




Let $A_H$ be the normalized adj. matrix of H

Let $\tilde{G}$ be the matrix of the action of G on $\mathbb{R}^{V(G) \times V(H)}$

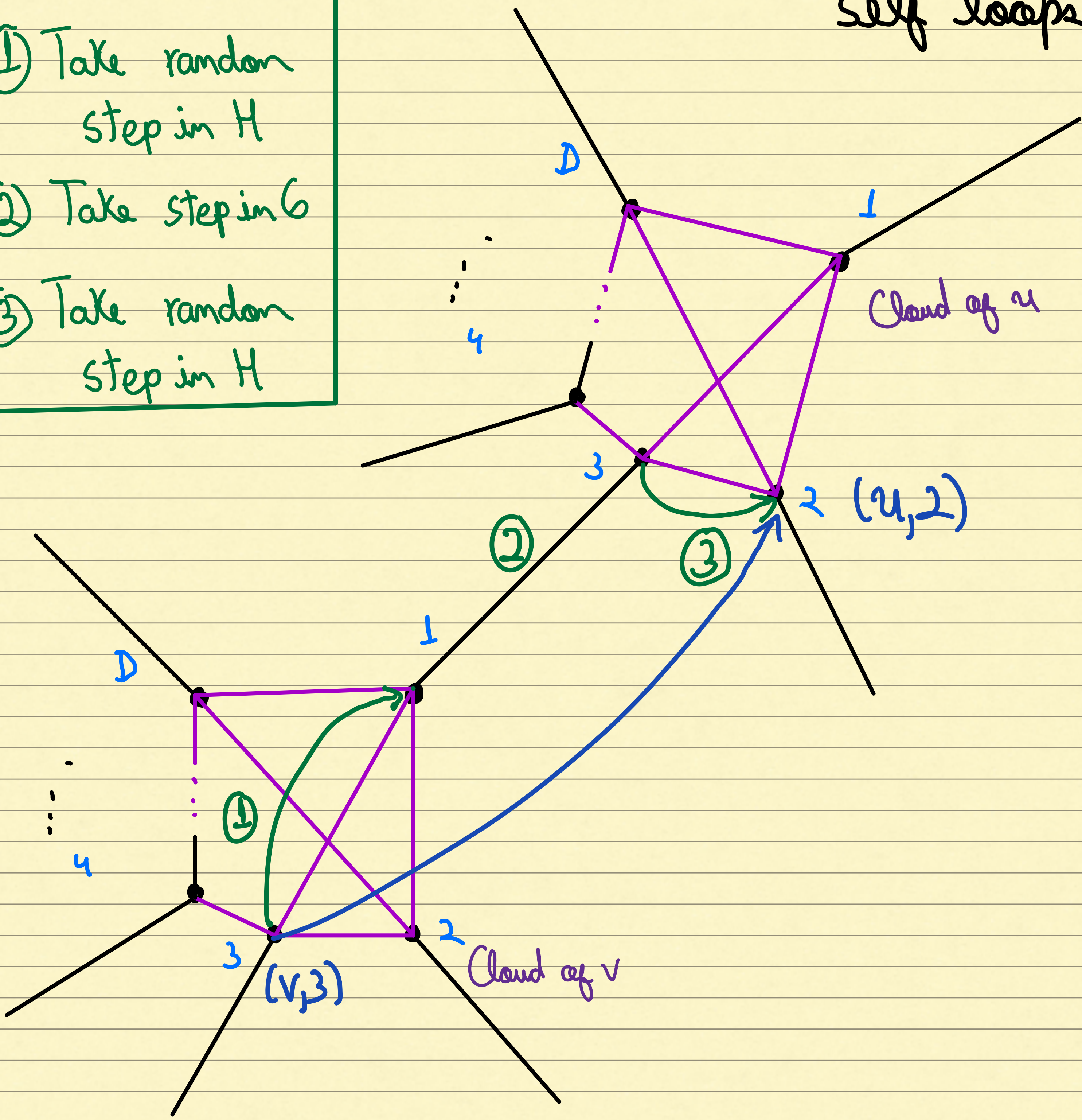
# Zig-Zag Product

Special Case: $G \text{ ⓩ } H$ with $H$ complete graph with self loops

# Zig-Zag Product

Special Case : $G \textcircled{z} H$ with $H$ complete graph with self loops

1. Take random step in H
2. Take step in G
3. Take random step in H



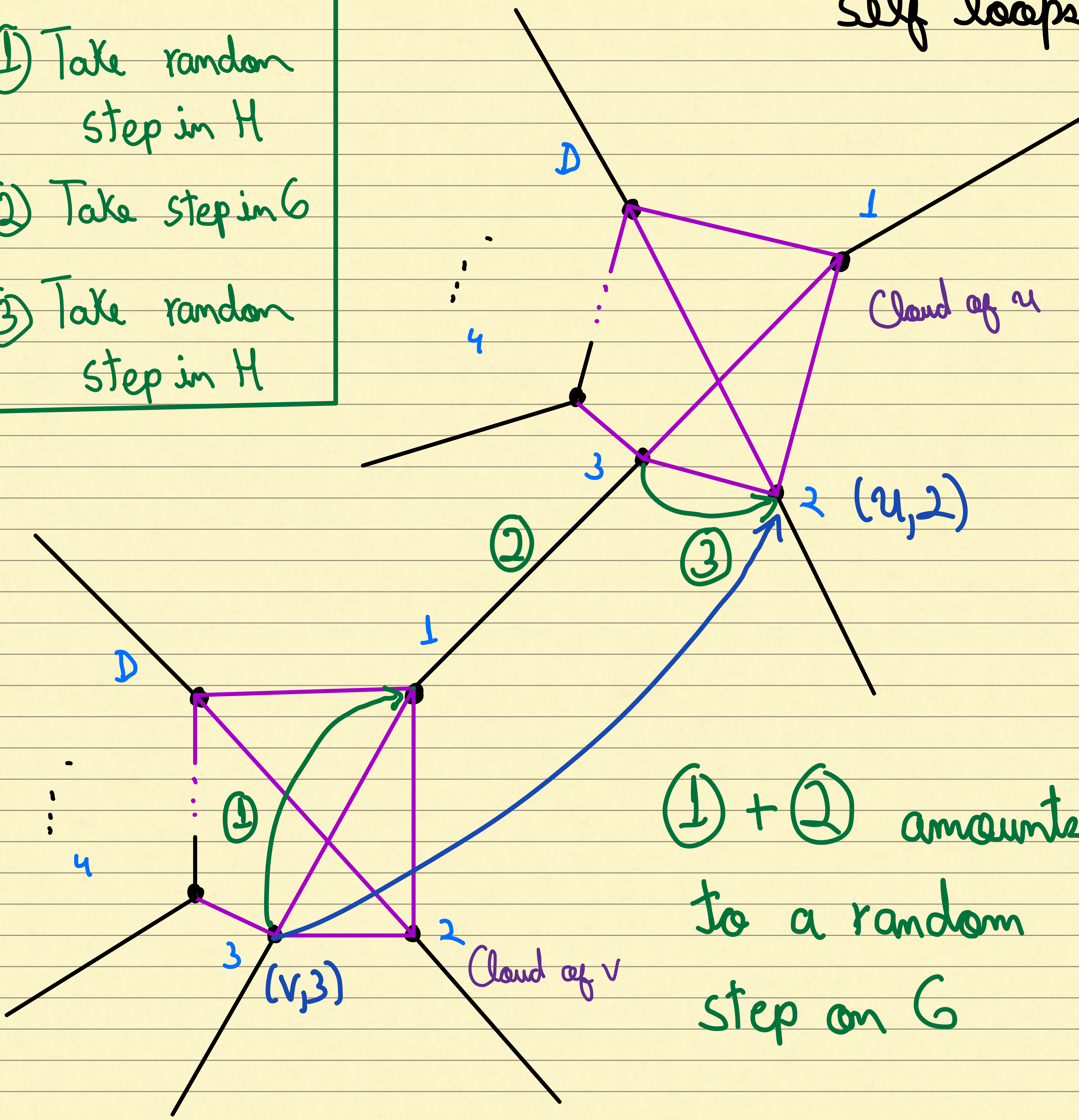
# Zig-Zag Product

Special Case : $G \textcircled{z} H$ with $H$ complete graph with self loops

① Take random step in $H$

② Take step in $G$

③ Take random step in $H$




① + ② amounts to a random step on $G$

# Zig-Zag Product

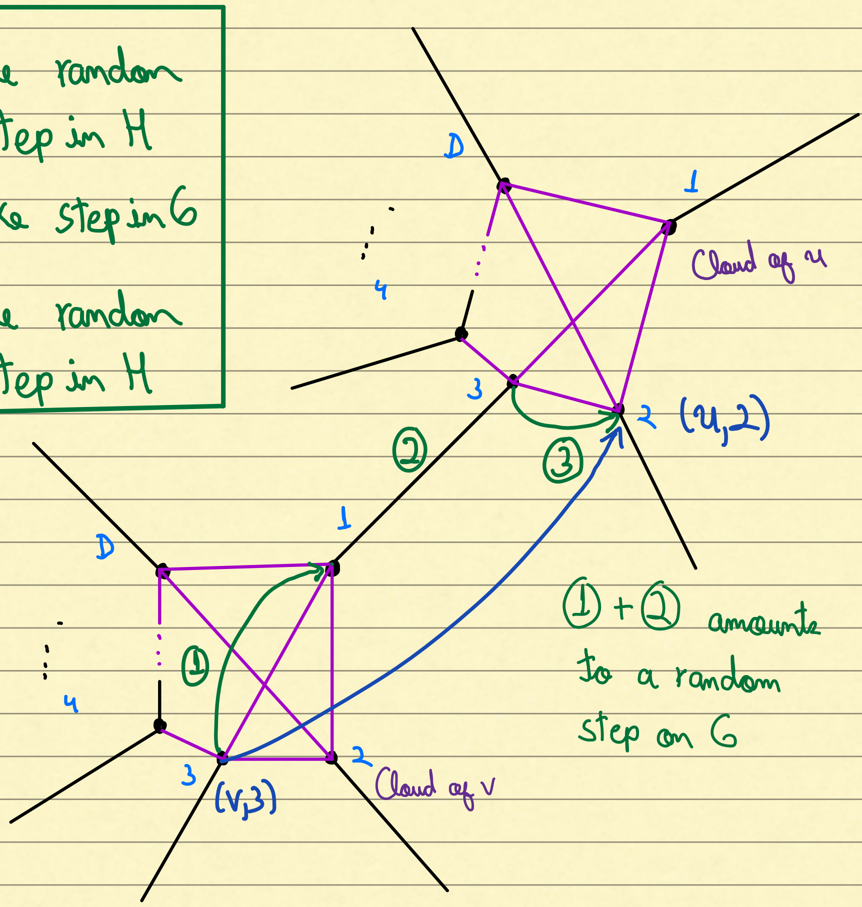
Special Case: $G \, ⓩ \, H$ with $H$ complete graph with self loops

$$\left(I \otimes \frac{J}{D}\right) \tilde{G} \left(I \otimes \frac{J}{D}\right) = A_G \otimes \frac{J}{D}$$

① Take random step in H

② Take step in G

③ Take random step in H




① + ② amounts to a random step on G

# Zig-Zag Product

Special Case : $G \text{ⓩ} H$ with $H$ complete graph with self loops

$$\boxed{\begin{aligned} &\text{Fact: } \left(I \otimes \tfrac{J}{D}\right) \tilde{G} \left(I \otimes \tfrac{J}{D}\right) = A_G \otimes \tfrac{J}{D} \\ &\lambda\left(A_G \otimes \tfrac{J}{D}\right) = \lambda(G) \end{aligned}}$$

# Zig-Zag Product

Special Case: $G \textcircled{z} H$ with $H$ complete graph with self loops

$$\boxed{\begin{aligned}&\text{Fact: } \left(I \otimes J/_D\right) \tilde{G} \left(I \otimes J/_D\right) = A_G \otimes J/_D \\ &\qquad \lambda\left(A_G \otimes J/_D\right) = \lambda(G)\end{aligned}}$$

$|V(G \textcircled{z} H)| = |V(G)| \cdot |V(H)|$ ✓

$\lambda(G \textcircled{z} H) = \lambda(G)$ ✓

Degree $G \textcircled{z} H$ is $D^2$ ✗

# Zig-Zag Product

$G \circledz H$ with $H$ an expander

Recall that $A_H \approx J/_D$

$$\boxed{\begin{array}{l} \text{Fact: } (I \otimes J/_D)\,\tilde{G}\,(I \otimes J/_D) = A_G \otimes J/_D \\ \qquad\qquad \lambda(A_G \otimes J/_D) = \lambda(G) \end{array}}$$

# Zig-Zag Product

$G \textcircled{z} H$ with $H$ an expander

Recall that $A_H \approx J/D$

$$A_H = \frac{J}{D} + \lambda E \qquad \text{with } \|E\|_{op} \leq 1$$

$$(I \otimes A_H)\tilde{G}(I \otimes A_H) =$$

$$(I \otimes \tfrac{J}{D})\tilde{G}(I \otimes J/D)$$

$$+ \lambda(I \otimes J/D)\tilde{G}(I \otimes E)$$

$$+ \lambda(I \otimes E)\tilde{G}(I \otimes J/D)$$

$$+ \lambda^2(I \otimes E)\tilde{G}(I \otimes E)$$

# Zig-Zag Product

$G \textcircled{z} H$ with $H$ an expander

Recall that $A_H \approx J/D$

$$A_H = \frac{J}{D} + \lambda E \quad \text{with } \|E\|_{op} \leq 1$$

$$(I \otimes A_H)\tilde{G}(I \otimes A_H) =$$

$$\left(I \otimes \frac{J}{D}\right)\tilde{G}\left(I \otimes J/D\right) \rightarrow \lambda_1$$

$$\left.\begin{array}{l} + \lambda\left(I \otimes J/D\right)\tilde{G}(I \otimes E) \\ + \lambda(I \otimes E)\tilde{G}(I \otimes J/D) \end{array}\right\} \rightarrow 2\lambda_2$$

$$+ \lambda^2 (I \otimes E)\tilde{G}(I \otimes E) \rightarrow \lambda_2^2$$

$$\boxed{\begin{array}{l} \text{Fact: } \left(I \otimes \frac{J}{D}\right)\tilde{G}\left(I \otimes J/D\right) = A_G \otimes J/D \\ \qquad \lambda(A_G \otimes J/D) = \lambda(G) \end{array}}$$

# Zig-Zag Product

$G \textcircled{z} H$ is obtained from $G \textcircled{r} H$ by a "zig-zag walk"

**Theorem** [Reingold-Vadhan-Wigderson '00]

If $G$ is $(n, D, \lambda_1)$-graph and $H$ is $(D, d, \lambda_2)$-graph, then

$G \textcircled{z} H$ is $(nD, d^2, \lambda_1 + 2\lambda_2 + \lambda_2^2)$-graph

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[Construct larger and larger expanders
starting from a constant size one]

# Near-optimal Spectral Expanders

Theorem [Alon-Boppana]

$$\lambda(G) \geq 2\sqrt{d-1} - o_n(1)$$

Ramanujan Graphs

$$\lambda(G) \leq 2\sqrt{d-1}$$

# Near-optimal Spectral Expanders

Ramanujan Graphs

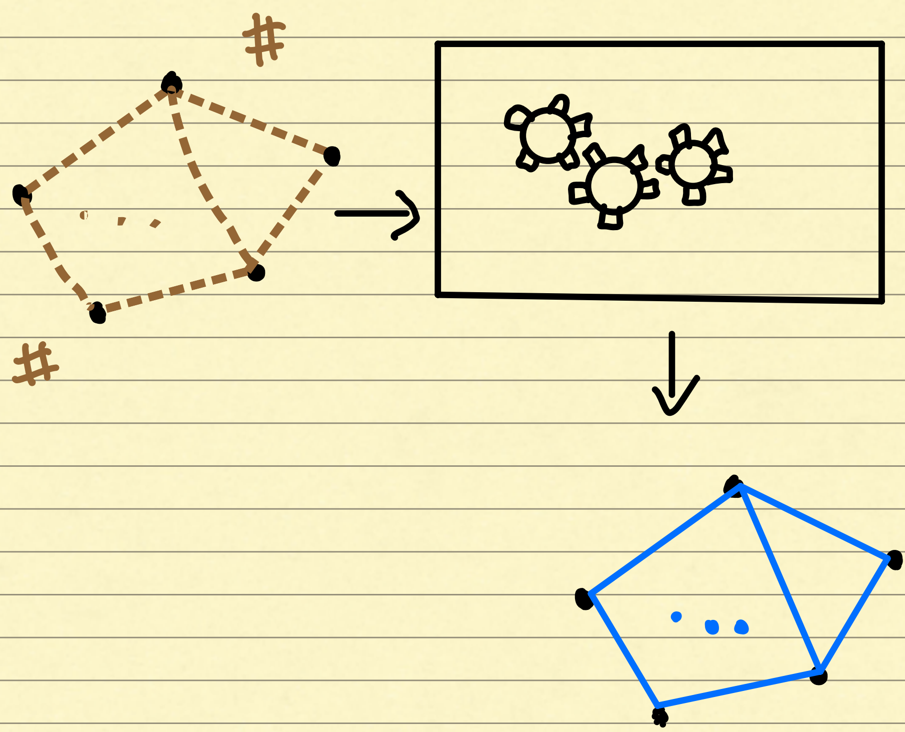
$$\lambda(G) \leq 2\sqrt{d-1}$$

What is required to be a near optimal expander?

# Near-optimal Spectral Expanders

Theorem [J-Mittal-Roy-Wigderson'22]

Any expander can be transformed into an almost Ramanujan one



# Near-optimal Spectral Expanders

Theorem [J-Mittal-Roy-Wigderson'22]
Any expander can be transformed into an almost Ramanujan one

Corollary [JMRW'22]
All "expanding groups" admit almost optimal expanders

Applications to

- "Generic" Expanders
- Cayley Graphs
- Quantum Expanders
- Monotone Expanders
- Dimension Expanders
- Codes

# Near-optimal Spectral Expanders

Theorem [J-Mittal-Roy-Wigderson'22]
Any expander can be transformed into an almost Ramanujan one

Key Technique: higher-order zig-zag for operators

(building on the work of Ben-Aroya and Ta-Shma)

Thank You!