Discovering the Foundations of Expansion and Spectral Graph Theory

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Under heavy construction

Last update: 09/13/24

Inspired by Babai's approach and the hungarian school

Van Gogh prize

a symbolic prize for any student in the course that solves an important open problem

(like van Gogh you are not going to receive anything other than have done something amazing)

Warning: These open problems can be challenging

Spectral Lens

Let $G=(V,E)$ be a d-regular graph on n vertices.

Let A be its adjacency matrix, i.e.,

$$A \in \mathbb{R}^{n\times n}, \qquad A_{u,v} = 1_{[\{u,v\}\in E]}.$$

Study the spectral theorem.

Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of A with corresponding orthonormal eigenvectors $\varphi_1, \dots, \varphi_n \in \mathbb{R}^n$.

$$(A\varphi_i = \lambda_i \varphi_i)$$

1] Prove that d is an eigenvalue of A.

Def $\langle x, x \rangle = \sum_{i=1}^{n} \bar{x}_i x_i$

Ex $x \in \mathbb{R}^n$ and $\varphi_1, \dots, \varphi_n \in \mathbb{R}^n$ ONB (orthonormal basis)

We can write
$x = \sum_{i=1}^{n} \alpha_i \varphi_i$ with $\alpha_i = \langle \varphi_i, x \rangle$.

Ex $\langle x, x \rangle = \sum_{i=1}^{n} \alpha_i^2$ [Parseval]

Ex: $\langle x, Ax \rangle = \sum_{i=1}^{n} \alpha_i^2 \lambda_i$

Ex A sym $\Rightarrow$ A has real eigenvalues

Def Rayleigh quotient $\dfrac{\langle x, Ax \rangle}{\langle x, x \rangle}$ (for $x \neq 0$)

Ex $Tr(A) = \sum_{i=1}^{n} \lambda_i$

Ex $Tr(A^K) = \sum_{i=1}^{n} \lambda_i^K$

Ex $Tr(A^2) = 2|E|$

Ex If G is simple with $\deg \geq 1$

$$\Downarrow$$

$$\lambda_n < 0$$

Ex Prove that $\frac{1}{n} \sum_{i=1}^{n} \lambda_i^2 = d$

Def $J_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$ all ones $n \times n$ matrix

Ex Compute eigenvalues of J_n/n

Def K_n complete graph on n vertices

Def $K_{a,b}$ complete bipartite graph
$G = (V = L \sqcup R, E)$ with $|L| = a, |R| = b$

Ex Compute the spectrum of K_n

Ex Compute the spectrum of $K_{n,n}$

Ex Compute 〃 〃 of $K_{1,d}$

Ex 〃 〃 〃 of $K_{a,b}$

Ex $\lambda_1 \geq \max_{i \geq 2} |\lambda_i|$

Ex G bipartite $\iff$ $\mathrm{Spec}(G) = -\mathrm{Spec}(G)$.

Some Notions of Expansion

Def $\partial(S) = E(S, \bar{S})$ "Edge boundary"

Def $\Phi(S) = \dfrac{|\partial(S)|}{d|S|}$ "Conductance"

Def $\Phi(G) = \min\limits_{\substack{\emptyset \neq S \subseteq V \\ |S| \leq n/2}} \Phi(S)$ "Cheeger's constant"

Def: $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$

two-sided spectral expansion

Def $\lambda^{\text{one-sided}} = \lambda_2$ one-sided spectral expansion

Def $e(S,T) = |\{(s,t) \mid \{s,t\} \in E\}|$

Ex Prove that

$$\left| e(S,T) - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}.$$

[Expander Mixing Lemma]

Ex Improve the error bound $\lambda\sqrt{|S||T|}$.

Def $\alpha(G)$ = independence number

Ex Prove that $\frac{\alpha(G)}{n} \leq \frac{-\lambda_n}{d - \lambda_n}$

[Hoffman's bound]

Ex Prove that $d_{avg} \leq \lambda_1 \leq \Delta(G)$

Def $\chi(G)$ is the chromatic number

Ex Prove that $\chi(G) \leq \lambda_1 + 1$

[Wilf's bound]

Mixing Bounds

Def $\vec{1}$ is the all ones vector

Def $R = \frac{1}{d} A$ is the random walk matrix

Ex $R \frac{\vec{1}}{n} = \frac{\vec{1}}{n}$

Ex Prove $\left\| R^r p - \frac{\vec{1}}{n} \right\|_1 \leq \left(\frac{\lambda}{d}\right)^r \sqrt{n}$ for any distribution p.

[Mixing bound in ℓ_1]

Study the Perron-Frobenius theorem (useful for understanding more general Markov chains)

Eigenvalues as an optimization problem

Let $V_k = \operatorname{span}\{e_1, \dots, e_k\}$

$W_k = \operatorname{span}\{e_k, \dots, e_n\}$

Ex $$\lambda_k = \min_{0 \neq x \in V_k} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} = \max_{0 \neq x \in W_k} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$$

Ex Prove the min-max variational theorem

$$\lambda_k = \max_{\substack{V \subseteq \mathbb{R}^n \\ \dim(V) = k}} \min_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} = \min_{\substack{V \subseteq \mathbb{R}^n \\ \dim(V) = n-k+1}} \max_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$$

[Courant-Fischer-Weyl]

The Magic of Interlacing

Ex Eigenvalue Interlacing

Let $A \in \mathbb{R}^{n \times n}$ be real symmetric matrix and B be a $(n-1) \times (n-1)$ <u>principal</u> submatrix

$$\mathrm{eig}(A) = \{\lambda_1 \geq \dots \geq \lambda_n\}$$

$$\mathrm{eig}(B) = \{\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_{n-1}\}$$

Prove $\lambda_1 \geq \tilde{\lambda}_1 \geq \lambda_2 \geq \dots \geq \tilde{\lambda}_{n-1} \geq \lambda_n$

Hint: use min-max theorem for eigenvalues

Extend to $r \times r$ principal submatrix B with $1 \leq r < n$

Ex: $\lambda_j \geq \tilde{\lambda}_j \geq \lambda_{j+n-r}$ for $j \in \{1, \dots, r\}$

[Cauchy Interlacing Thm]

Refresher on PSDness

Def A real sym matrix M is <u>positive semi-definite</u> (PSD) if

$\forall x \in \mathbb{R}^n$, $x^t M x \geq 0$.

Ex Prove: The following are equivalent

1) M is PSD
2) M has non-negative eigenvalues
3) $\exists$ a matrix W s.t. $M = W^t W$

<u>Notation</u> We write $M \succcurlyeq 0$ if M is PSD

We write $M_1 \succcurlyeq M_2$ if $M_1 - M_2 \succcurlyeq 0$.

This gives a partial order (Loewner order)

Laplacian Matrix

Def $L = dI - A$ [Laplacian Matrix]

Let $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ be the eigenvalues of L

Ex For d-regular G, we have

$$\underbrace{\mu_1}_{0} = d - \lambda_1, \quad \dots, \quad \mu_n = d - \lambda_n.$$

Ex Prove that $\langle x, Lx \rangle = \sum_{i \sim j} (x_i - x_j)^2$

Ex Conclude that $L \succeq 0$ (PSD)

Ex Prove that $\langle 1_S, L 1_S \rangle = |E(S, \bar{S})|$

Ex G connected $\iff \mu_2 > 0$

Ex If G is connected, then $\mu_2 \geq \frac{1}{n\,\mathrm{diam}(G)}$

Ex $|\{k \mid \mu_k = 0\}| = \#$ of connected components

Ex G bipartite iff $\mu_n = 2d$

Ex Prove that $\frac{\mu_2}{2d} \leq \Phi(G)$

$\boxed{CH^*}$ Prove that $\Phi(G) \leq O\left(\sqrt{\frac{\mu_2}{d}}\right)$

Hint: use eigenvector to μ_2 to find a cut ["rounding"]

[Cheeger's Inequality]

$$\frac{\mu_2}{2d} \leq \Phi(G) \leq \sqrt{\frac{2\mu_2}{d}}$$

Characteristic Polynomial

Def Characteristic polynomial $\det(\lambda \underline{I} - A) =: ch(\lambda)$

The roots of $ch(\lambda)$ are the eigenvalues of A

Cayley-Hamilton Theorem: $ch(A) = 0$

Ex G connected has diam $= K \Rightarrow$ A has at least $K+1$ distinct eigenvalues

Hint: [Cayley-Hamilton] minimal polynomial

Ex Let A, B be two real sym matrices with eig $(\lambda_1 \geq \dots \geq \lambda_n)$
$(\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_n)$

Compute the eigenvalues of $A \otimes B$.

Def $s: E \to \{\pm 1\}$ is an edge signing

Def $$(A_s)_{u,v} = \begin{cases} s(\{u,v\}) & \text{if } \{u,v\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Ex Prove $\lambda_1(A_s) \leq \Delta(G)$ for any signing s.

Limitations on Spectral Expansion

Ex G d-regular $\Rightarrow \lambda \geq \sqrt{d}\,(1 - o_n(1))$

Ex If G has diam $\geq 4 \Rightarrow \lambda_2 \geq \sqrt{d}$

[Hint: look for the star]

Ex $\lambda_2 < 0 \iff G = K_n$

[Hint: interlacing]

Ex Suppose G is connected. G has a unique positive eigenvalue iff G is a complete k-partite graph

[Hint: interlacing]

$\boxed{\text{Ch}^*}$ $\lambda_2 \geq 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\text{diam}}\right)\right)$

[Alon-Boppana bound]

Cor) If $d = O(1)$

$$\Downarrow$$

$$\lambda_2 \geq 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\log n}\right)\right)$$

Def G is <u>Ramanujan</u> if $\lambda \leq 2\sqrt{d-1}$

a.k.a. "Optimal" Spectral Expanders

OP [van Gogh Prize]

Construct infinite families of Ramanujan graphs for <u>every</u> $d \geq 3$

Vertex Expansion

Def $N(S) = \{u \mid \exists s \in S, \{u,s\} \in E\}$

Def Vertex (or Lossless) Expansion

$$\Phi^V(S) = \frac{|N(S)|}{d|S|}$$

$$\Phi^V_\varepsilon(G) = \min_{\substack{\phi \neq S \subseteq V \\ |S| \leq \varepsilon n}} \Phi^V(G).$$

OP [van Gogh prize]

Construct explicit family with

$$\Phi^V_\varepsilon(G) > \frac{1}{2} \quad \text{for } \varepsilon = \Omega(1)$$

(or two-sided bipartite lossless)

On the Complexity of Expansion

(Hypothesis) $\forall\, \eta \in (0,1)\ \exists\, \delta \in (0,1)$ s.t. it is NP-hard to distinguish given input graph $G=(V,E)$

(Yes) $\exists\, S \subseteq V$ with $|S| \leq \delta n$ and $\Phi(S) \leq \eta$.

(No) $\forall\, S \subseteq V$ with $|S| \leq \delta n$, we have $\Phi(S) \gtrsim 1-\eta$

OP van Gogh prize

Prove or refute the above hypothesis

Def: Boolean n-hypercube is the graph $H_n = (V, E)$ where

$$V = \mathbb{Z}_2^n$$

$$E = \{\{u, v\} \mid |u - v| = 1\}$$

or equivalently

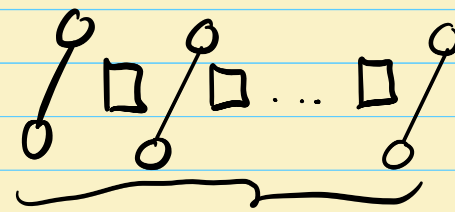
$$E = \{\{u, u + e_i\} \mid u \in \mathbb{Z}_2^n, i \in [n]\}$$

Ex Show that the adjacency matrix of H_n can be defined recursively as

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_n = \left(\begin{array}{c|c} A_{n-1} & I_{2^{n-1}} \\ \hline I_{2^{n-1}} & A_{n-1} \end{array}\right).$$

Def The cartesian product of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is defined as

$$G \square H = \Big(V = V(G) \times V(H),$$
$$E = \{\{(g_1, h_1), (g_2, h_2)\} \mid$$
$$(\{g_1, g_2\} \in E(G) \text{ and } h_1 = h_2) \text{ or}$$
$$(g_1 = g_2 \text{ and } \{h_1, h_2\} \in E(H))\}\Big)$$

Ex Prove that $H_n =$ 

n times

Ex Prove that "the" adjacency matrix of $G \square H$ can be written as

$$A_{G \square H} = A_G \otimes I_{|V(H)|} + I_{|V(G)|} \otimes A_H$$

Ex Prove that

$$\mathrm{Spec}(A_{G\square H}) = \{ \lambda + \tilde{\lambda} \mid \lambda \in \mathrm{Spec}(G),\ \tilde{\lambda} \in \mathrm{Spec}(H) \}$$

Ex Compute $\mathrm{Spec}(H_n)$

Ex Consider the recursive edge signing of H_n

$$B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_n = \left(\begin{array}{c|c} B_{n-1} & I_{2^{n-1}} \\ \hline I_{2^{n-1}} & -B_{n-1} \end{array}\right)$$

Prove that B_n has eigenvalues $\pm\sqrt{n}$ each with multiplicity 2^{n-1}.

[Hint: show $B_n B_n = n I_{2^n}$]

Def The induced subgraph of $G=(V,E)$ on $S \subseteq V$ is defined as

$$G[S] = \left(S, E' = \{\{u,v\} \in E \mid u,v \in S\}\right)$$

Ex Prove that

$$\left(\forall S \subseteq V(H_n) \text{ with } |S| \geq 2^{n-1}+1\right)$$

$$\left(\Delta(H_n[S]) \geq \sqrt{n}\right)$$

[Hint: Cauchy interlacing on B_n and $\lambda_1(G) \leq \Delta(G)$]

[Huang's theorem] implies the Sensitivity Conjecture via a known connection of Gotsman and Linial

OP van Gogh prize

Show that (families) of Ramanujan graphs exist for every degree $d \geq 3$.

OP van Gogh prize

Signing Conjecture [Bilu-Linial]

Every d-regular graph $G=(V,E)$ has an edge signing $s: E \to \{\pm 1\}$ such that the signed adjacency matrix satisfies $\mathrm{eig}(A_s) \subseteq [-2\sqrt{d-1}, 2\sqrt{d-1}]$.

(Positive answer would resolve the first OP on this page.)

Fourier Analysis

Def Let $S \subseteq [n]$. The character $\chi_S : \mathbb{Z}_2^n \to \{\pm 1\}$ is defined as

$$\chi_S(x) = \prod_{i \in S} (-1)^{x_i}.$$

Let $f, g : \mathbb{Z}_2^n \to \mathbb{R}$.

Def $\langle f, g \rangle = \mathbb{E}_{x \in \mathbb{Z}_2^n} f(x) g(x)$

(This just a convenient normalization for Fourier analysis (different from before).)

Ex $\langle \chi_S, \chi_T \rangle = \begin{cases} 1 & S = T \\ 0 & o/w \end{cases}$

Ex $\{\chi_S\}_{S\subseteq[n]}$ forms an ONB for the space of functions $\{f:\mathbb{Z}_2^n\to\mathbb{R}\}$.

Ex $\exists!$ Fourier decomposition

$$f = \sum_{S\subseteq[n]} \hat{f}(S)\,\chi_S$$

where $\hat{f}(S) := \langle f, \chi_S\rangle$.

Ex $\chi_S(x+y) = \chi_S(x)\,\chi_S(y)$

[homomorphism $\mathbb{Z}_2^n \to \{\pm 1\}$]

Ex $\langle f, f\rangle = \sum_{S\subseteq[n]} \hat{f}(S)^2$

[Parseval]

Ex $\chi_\emptyset = 1$

Ex $\hat{f}(\emptyset) = \mathbb{E}_{x \in \mathbb{Z}_2^n} f(x)$

Ex $\mathrm{Var}[f] = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} \hat{f}(S)^2$

Def The convolution of f and g is defined as

$$f * g(x) = \mathbb{E}_{y \in \mathbb{Z}_2^n} f(y)\, g(x-y).$$

Ex $\widehat{(f*g)}(S) = \hat{f}(S) \cdot \hat{g}(S)$

Def The degree of f is

$$\deg(f) = \max_{\hat{f}(S)\neq 0} |S| .$$

Def $\mathrm{dist}(f,g) = \Pr_{x\in\mathbb{Z}_2^n}[f(x)\neq g(x)]$

Ex If $f,g:\mathbb{Z}_2^n \to \{\pm 1\}$, then

$$\langle f,g\rangle = 1-2\,\mathrm{dist}(f,g) .$$

Ex Let A be the adjacency matrix of H_n. Prove that

$$A\,\chi_S = (n-2|S|)\,\chi_S$$

[Characters as eigenvectors]

Def 1 $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is linear if

$$f(x) = \sum_{i=1}^{n} c_i x_i \quad \text{for some } c \in \mathbb{Z}_2^n.$$

Def 2 $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is linear if

$$f(x) + f(y) = f(x+y) \quad \forall x, y \in \mathbb{Z}_2^n$$

Ex Def 1 $\iff$ Def 2.

Property Testing Model

Query

$$\underset{\mathbb{Z}_2^n}{\overset{x}{\cap}} \longrightarrow \boxed{f : \mathbb{Z}_2^n \to \mathbb{Z}_2} \longrightarrow f(x)$$

Def A property is a subset P of functions from $\{f : \mathbb{Z}_2^n \to \mathbb{Z}_2\}$

E.g. $P_{Lin} = \{$linear functions$\}$

Def $\text{dist}(f, P) = \min_{g \in P} \text{dist}(f, g)$

Meta Question: Decide

(1) f has property P or

(2) f is ε-far from P.

Consider the following 3-query tester for linearity

(1) Sample $x, y \in \mathbb{Z}_2^n$ uniformly

(2) Accept iff $f(x) + f(y) = f(x+y)$.

Ex If f is linear, then tester accepts with probability 1.

For convenience let's think f maps to $\{\pm 1\}$ instead of $\mathbb{Z}_2$.

Def $P_{rej}(f) = \Pr[\text{Tester rejects } f]$

Ex $\underset{x,y \in \mathbb{Z}_2^n}{\mathbb{E}} f(x) f(y) f(x+y) = 1 - 2 p_{rej}(f)$

Ex $p_{rej}(f) \geq dist(f, P_{Lin})$

[Hint: Fourier analysis and convolution]

[BLR linearity testing]

To be continued...