

Discovering the Foundations of Expansion and Spectral Graph Theory

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Under heavy construction

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Inspired by Babai's approach
and the Hungarian school

Van Gogh prize

a symbolic prize for any student
in the course that solves an
important open problem

(like van Gogh you are not going to receive anything)
other than have done something amazing

Warning: These open problems can be challenging

Spectral Lens

Let $G = (V, E)$ be a d -regular graph on n vertices.

Let A be its adjacency matrix, i.e.,

$$A \in \mathbb{R}^{n \times n}, \quad A_{u,v} = \begin{cases} 1 & [u,v] \in E \\ 0 & \text{otherwise} \end{cases}.$$

Study the spectral theorem.

Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of A with corresponding orthonormal eigenvectors $\varphi_1, \dots, \varphi_n \in \mathbb{R}^n$.

$$(A\varphi_i = \lambda_i \varphi_i)$$

1] Prove that d is an eigenvalue of A .

$$\text{Def } \langle x, x \rangle = \sum_{i=1}^n \bar{x}_i x_i$$

Ex $x \in \mathbb{R}^n$ and $\ell_1, \dots, \ell_n \in \mathbb{R}^n$

We can write

$$x = \sum_{i=1}^n \alpha_i \ell_i \quad \text{with } \alpha_i = \langle \ell_i, x \rangle.$$

$$\text{Ex } \langle x, x \rangle = \sum_{i=1}^n \alpha_i^2 \quad [\text{Parserval}]$$

$$\text{Ex: } \langle x, Ax \rangle = \sum_{i=1}^n \alpha_i^2 \lambda_i$$

Ex A sym $\Rightarrow A$ has real eigenvalues

Def Rayleigh quotient $\frac{\langle x, Ax \rangle}{\langle x, x \rangle}$ (for $x \neq 0$)

$$Ex \quad \text{Tr}(A) = \sum_{i=1}^n \lambda_i$$

$$Ex \quad \text{Tr}(A^k) = \sum_{i=1}^n \lambda_i^k$$

$$Ex \quad \text{Tr}(A^2) = 2|E|$$

Ex If G is simple with $\deg \geq 1$

$$\downarrow \\ \lambda_n < 0$$

$$Ex \quad \text{Prove that } \frac{1}{n} \sum_{i=1}^n \lambda_i^2 = d$$

Def $J_n = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$ all ones $n \times n$ matrix

Ex Compute eigenvalues of J_n/n

Def K_n complete graph on n vertices

Def $K_{a,b}$ complete bipartite graph

$G = (V = L \cup R, E)$ with $|L| = a, |R| = b$

Ex Compute the spectrum of K_n

Ex Compute the spectrum of $K_{n,n}$

Ex Compute " " of $K_{1,d}$

Ex " " " " of $K_{a,b}$

Ex $\lambda_1 \geq \max_{i \geq 2} |\lambda_i|$

Ex G bipartite $\Leftrightarrow \text{Spec}_{\parallel}(G) = \text{Spec}(G)$.

Some Notions of Expansion

$$\text{Def } \partial(S) = E(S, \bar{S})$$

"Edge boundary"

$$\text{Def } \Phi(S) = \frac{|\partial(S)|}{|S|}$$

"Conductance"

$$\text{Def } \Phi(G) = \min_{\substack{\emptyset \neq S \subseteq V \\ |S| \leq n/2}} \Phi(S)$$

"Cheeger's constant"

$$\text{Def: } \lambda = \max \{ |\lambda_2|, |\lambda_n| \}$$

two-sided spectral expansion

$$\text{Def } \lambda = \lambda_2 \quad \text{one-sided spectral expansion}$$

Def $e(S, T) = |\{(s, t) \mid \{s, t\} \in E\}|$

Ex Prove that

$$|e(S, T) - \frac{d|S||T|}{n}| \leq \lambda \sqrt{|S||T|}.$$

[Expander Mixing Lemma]

Ex Improve the error bound $\lambda \sqrt{|S||T|}$.

Def $\alpha(G)$ = independence number

Ex Prove that $\frac{\alpha(G)}{n} \leq -\frac{\lambda_n}{d-\lambda_n}$

Ex Prove that $\text{clawy} \leq \lambda_1 \leq \Delta(G)$

Def $\chi(G)$ is the chromatic number

Ex Prove that $\chi(G) \leq \lambda_1 + 1$

[Wig's bound]

Mixing Bounds

Def $\vec{1}$ is the all one vector

Def $R = \frac{1}{d} A$ is the random walk matrix

$$Ex \quad R \vec{\frac{1}{n}} = \vec{\frac{1}{n}}$$

Ex Prove $\|R^p - \vec{\frac{1}{n}}\|_1 \leq \left(\frac{\lambda}{d}\right)^p \sqrt{n}$ for any distribution p .

[Mixing bound]
in l_1

Study the Perron-Frobenius theorem (useful for understanding more general Markov chains)

Eigenvalues as an optimization problem

Let $V_K = \text{Span}\{\varphi_1, \dots, \varphi_K\}$

$W_K = \text{Span}\{\varphi_{K+1}, \dots, \varphi_n\}$

Ex $\lambda_K = \min_{0 \neq x \in V_K} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} = \max_{0 \neq x \in W_K} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$

Ex Prove the min-max variational theorem

$$\lambda_K = \max_{V \subseteq \mathbb{R}^n} \min_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} = \min_{V \subseteq \mathbb{R}^n} \max_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$$

$\dim(V) = K$ $\dim(V) = n - K + 1$

[Courant-Fischer-Weyl]

The Magic of Interlacing

Ex Eigenvalue Interlacing

Let $A \in \mathbb{R}^{n \times n}$ be real symmetric matrix

and B be a $(n-1) \times (n-1)$ principal submatrix

$$\text{eig}(A) = \{\lambda_1 \geq \dots \geq \lambda_n\}$$

$$\text{eig}(B) = \{\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_{n-1}\}$$

Prove $\lambda_1 \geq \tilde{\lambda}_1 \geq \lambda_2 \geq \dots \geq \tilde{\lambda}_{n-1} \geq \lambda_n$

Hint: use min-max theorem for eigenvalues

Extend to $r \times r$ principal submatrix B with $1 \leq r < n$

Ex: $\lambda_j \geq \tilde{\lambda}_j \geq \lambda_{j+n-r}$ for $j \in \{1, \dots, r\}$

[Cauchy Interlacing Thm]

Refresher on PSDness

Def A real sym matrix M is positive semi-definite (PSD) if

$$\forall x \in \mathbb{R}^n, \quad x^T M x \geq 0.$$

Ex Prove: The following are equivalent

1) M is PSD

2) M has non-negative eigenvalues

3) \exists a matrix W s.t. $M = W^T W$

Notation We write $M \succ 0$ if M is PSD

We write $M_1 \succ M_2$ if $M_1 - M_2 \succ 0$.

This gives a partial order (Lacumon order)

Laplacian Matrix

Def $L = dI - A$ [Laplacian Matrix]

Let $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ be the eigenvalues of L

Ex For d -regular G , we have

$$\mu_1 = d - \lambda_1, \dots, \mu_n = d - \lambda_n.$$

Ex Prove that $\langle x, Lx \rangle = \sum_{i \neq j} (x_j - x_i)^2$

Ex Conclude that $L \geq 0$ (PSD)

Ex Prove that $\langle \mathbf{1}_S, L \mathbf{1}_S \rangle = |E(S, \bar{S})|$

Ex G connected $\Leftrightarrow \mu_2 > 0$

Ex If G is connected, then $\mu_2 \geq \frac{1}{n} \text{diam}(G)$

$E \times K \times \{ \mu_k = 0 \} | = \# \text{ of connected components}$

Ex G bipartite iff $\mu_n = 2d$

Ex Prove that $\frac{\mu_2}{2d} \leq \Phi(\epsilon)$

CH* Prove that $\Phi(\epsilon) \leq O\left(\sqrt{\frac{\mu_2}{d}}\right)$

Mit: use eigenvector to μ_2 to find a cut ["rounding"]

[Cheeger's Inequality]

$$\frac{\mu_2}{2d} \leq \Phi(6) \leq \sqrt{\frac{2\mu_2}{d}}$$

Characteristic Polynomial

Def Characteristic polynomial $\det(\lambda I - A) =: ch(\lambda)$

The roots of $ch(\lambda)$ are the eigenvalues of A

Cayley-Hamilton Theorem: $ch(A) = 0$

Ex 6 has diam = $K \Rightarrow A$ has at least
connected $K+1$ distinct eigenvalues

Hint: [Cayley-Hamilton] minimal polynomial

Ex Let A, B be two real sym
matrices with $\text{eig}(A) \geq \dots \geq \lambda_n$
 $\text{eig}(B) \geq \dots \geq \tilde{\lambda}_n$

Compute the eigenvalues of $A \otimes B$.

Def $s: E \rightarrow \{-1\}$ is an edge
signing

Def $(A_s)_{u,v} = \begin{cases} s(u,v) & \text{if } \{u,v\} \in E \\ 0 & \text{otherwise} \end{cases}$

Ex Prove $\lambda_1(A_s) \leq \Delta(B)$ for any
signing s .

Limitations on Spectral Expansion

Ex G d-regular $\Rightarrow \lambda \geq \sqrt{d} (1 - o_n(1))$

Ex If G has diam $\geq 4 \Rightarrow \lambda_2 \geq \sqrt{d}$
[Hint: look for the stars]

Ex $\lambda_2 < 0 \iff G = K_n$
[Hint: interlacing]

Ex Suppose G is connected. G has a unique positive eigenvalue iff G is a complete k -partite graph
[Hint: interlacing]



$$\lambda_2 \geq 2\sqrt{d-1} \left(1 - O\left(\frac{1}{\text{diam}}\right)\right)$$

[Alon-Boppana bound]

(or) If $d = O(1)$

$$\lambda_2 \geq 2\sqrt{d-1} \left(1 - O\left(\frac{1}{\log n}\right)\right)$$

Def 6 is Ramanujan if $\lambda \leq 2\sqrt{d-1}$

a.k.a. "Optimal" Spectral Expanders

OP [van Gogh Prize]

Construct infinite families
of Ramanujan graphs for every
 $d \geq 3$

Vertex Expansion

Def $N(S) = \{u \mid \exists s \in S, \{u, s\} \in E\}$

Def Vertex (or Losen) Expansion

$$\Phi^V(S) = \frac{|N(S)|}{d|S|}$$

$$\underline{\Phi}_\epsilon^V(\epsilon) = \min_{\substack{\emptyset \neq S \subseteq V \\ |S| \leq \epsilon n}} \Phi^V(S).$$

OP [van Gogh prize]

Construct explicit family with

$$\underline{\Phi}_\epsilon^V(\epsilon) > \frac{1}{2} \quad \text{for } \epsilon = \Omega(1)$$

(on two-sided bipartite forest)

On the Complexity of Expansion

(Hypothesis) $\forall \eta \in (0, 1) \exists \delta \in (0, 1)$

s.t. it is NP-hard to distinguish
given input graph $G = (V, E)$

(Yes) $\exists S \subseteq V$ with $|S| \leq \delta n$ and
 $\bar{\Phi}(S) \leq \eta$.

(No) $\forall S \subseteq V$ with $|S| \leq \delta n$, we
have $\bar{\Phi}(S) \geq 1 - \eta$

OP Van Gogh ping

Prove or refute the above hypothesis

Def: Boolean n -hypercube is

the graph $H_n = (V, E)$ where

$$V = \mathbb{Z}_2^n$$

$$E = \{(u, v) \mid |u - v| = 1\}$$

or equivalently

$$E = \{(u, u + e_j) \mid u \in \mathbb{Z}_2^n, j \in [n]\}$$

Ex Show that the adjacency

matrix of H_n can be defined

recursively as

$$A_L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_n = \begin{pmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & A_{n-1} \end{pmatrix}.$$

Def The cartesian product of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is defined as

$$G \square H = (V = V(G) \times V(H),$$

$$E = \{(g_1, h_1), (g_2, h_2) \mid$$

$(g_1, g_2) \in E(G)$ and $h_1 = h_2$ or

$(g_1 = g_2 \text{ and } h_1, h_2) \in E(H)\}$

Ex Prove that $H_n = \underbrace{\bullet \square \bullet \square \dots \square \bullet}_{n \text{ times}}$

Ex Prove that "the adjacency matrix of $G \square H$ can be written as

$$A_{G \square H} = A_G \otimes I_{|V(H)|} + I_{|V(G)|} \otimes A_H$$

Ex Prove that

$$\text{Spec}(A_{G \square H}) = \left\{ \lambda + \tilde{\lambda} \mid \begin{array}{l} \lambda \in \text{Spec}(G), \\ \tilde{\lambda} \in \text{Spec}(H) \end{array} \right\}$$

Ex Compute $\text{Spec}(H_n)$

Ex Consider the recursive edge

Signing of H_n

$$B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} B_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -B_{n-1} \end{pmatrix}$$

Prove that B_n has eigenvalues $\pm \sqrt{n}$

each with multiplicity 2^{n-1} .

[Hint: show $B_n B_n = n I_{2^n}$]

Def The induced subgraph

of $G = (V, E)$ on $S \subseteq V$ is
defined as

$$G[S] = (S, E' = \{(u, v) \in E \mid u, v \in S\})$$

Ex Prove that

$$(\forall S \subseteq V(H_n) \text{ with } |S| \geq 2^{n-1} + 1)$$

$$(\Delta(H_n[S]) \geq \sqrt{n})$$

[Hint: Cauchy interlacing on B_n
and $\lambda_1(G) \leq \Delta(G)$]

[Huang's theorem] implies the
Sensitivity Conjecture via a known
connection of Gotsman and Linial

OP van Gogh prize
Show that (families) of Ramanujan graphs exist for every degree $d \geq 3$.

OP van Gogh prize
Signed Conjecture [Bilu-Linial]

Every d -regular graph $G = (V, E)$ has an edge signing $S : E \rightarrow \{-1, +1\}$ such that the signed adjacency matrix satisfies $\text{eig}(A_S) \subseteq [-2\sqrt{d-1}, 2\sqrt{d-1}]$.

(Positive answer would resolve)
(the first OP on this page.)

Fourier Analysis

Def Let $S \subseteq [n]$. The character $\chi_S: \mathbb{Z}_2^n \rightarrow \{\pm 1\}$ is defined as

$$\chi_S(x) = \prod_{j \in S} (-1)^{x_j}.$$

Let $f, g: \mathbb{Z}_2^n \rightarrow \mathbb{R}$.

$$\text{Def } \langle f, g \rangle = \mathbb{E}_{x \in \mathbb{Z}_2^n} f(x)g(x)$$

(This just a convenient normalization for Fourier analysis (different from before))

$$\text{Ex } \langle \chi_S, \chi_T \rangle = \begin{cases} 1 & S=T \\ 0 & \text{o/w} \end{cases}$$

$E_x \{X_S\}_{S \subseteq [n]}$ form an ONB

for the space of functions $\{f: \mathbb{Z}_2^n \rightarrow \mathbb{R}\}$.

$E_x \exists!$ Fourier decomposition

$$f = \sum_{S \subseteq [n]} \hat{f}(S) X_S$$

where $\hat{f}(S) := \langle f, X_S \rangle$.

$$E_x X_S(x+y) = X_S(x) X_S(y)$$

[homomorphism $\mathbb{Z}_2^n \rightarrow \{\pm 1\}$]

$$E_x \langle f, f \rangle = \sum_{S \subseteq [n]} \hat{f}(S)^2$$

[Parseval]

$$\mathbb{E}_x \chi_{\emptyset} = 1$$

$$\mathbb{E}_x \hat{f}(\emptyset) = \mathbb{E}_{x \in \mathbb{Z}_2^n} f(x)$$

$$\mathbb{E}_x \text{Var}[f] = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} \hat{f}(S)^2$$

Def The convolution of f and g is defined as

$$f * g(x) = \mathbb{E}_{y \in \mathbb{Z}_2^n} f(y) g(x-y).$$

$$\mathbb{E}_x (\widehat{f * g})(S) = \hat{f}(S) \cdot \hat{g}(S)$$

Def The degree of f is

$$\deg(f) = \max_{\substack{S \\ f(S) \neq 0}} |S|.$$

$$\text{Def } \text{dist}(f, g) = \Pr_{x \in \mathbb{Z}_2^n} [f(x) \neq g(x)]$$

Ex If $f, g: \mathbb{Z}_2^n \rightarrow \{\pm 1\}$, then

$$\langle f, g \rangle = 1 - 2 \text{dist}(f, g).$$

Ex Let A be the adjacency matrix
of H_n . Prove that

$$A \chi_S = (n - 2|S|) \chi_S$$

[Character as eigenvectors]

Def 1 $f: \mathbb{Z}_2^n \rightarrow \underline{\mathbb{Z}_2}$ is linear if

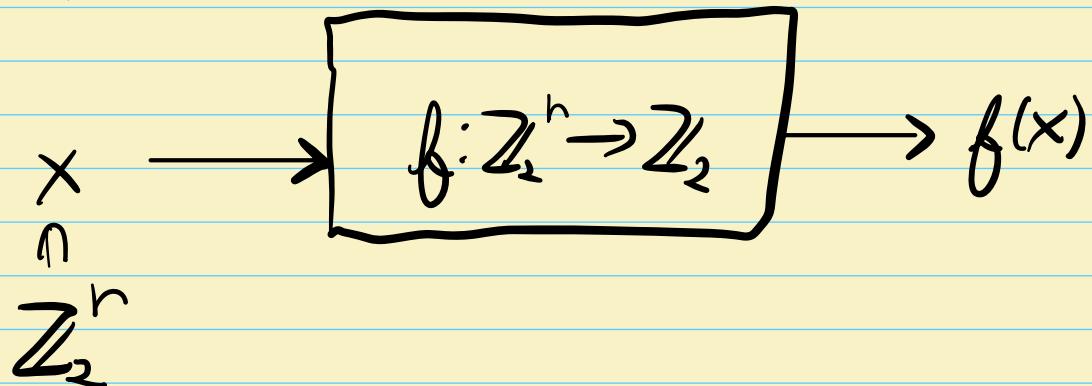
$$f(x) = \sum_{i=1}^n c_i x_i \quad \text{for some } c \in \mathbb{Z}_2^n.$$

Def 2 $f: \mathbb{Z}_2^n \rightarrow \underline{\mathbb{Z}_2}$ is linear if

$$f(x) + f(y) = f(x+y) \quad \forall x, y \in \mathbb{Z}_2^n$$

Ex Def 1 \Leftrightarrow Def 2.

Property Testing Model
Query



Def A property is a subset P of functions from $\{f: \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2\}$

E.g. $P_{\text{Lin}} = \{\text{linear functions}\}$

Def $\text{dist}(f, P) = \min_{g \in P} \text{dist}(f, g)$

Meta Question: Decide

(1) f has property P or

(2) f is ϵ -far from P .

Consider the following 3-query
tester for linearity

- (1) Sample $x, y \in \mathbb{Z}_2^n$ uniformly
- (2) Accept iff $f(x) + f(y) = f(x+y)$.

Ex If f is linear, then tester accepts
with probability 1.

For convenience let's think f maps
to $\{+1\}$ instead of \mathbb{Z}_2 .

If $\text{Proj}(f) = \Pr[\text{Tester rejects } f]$

$$\mathbb{E}_x \mathbb{E}_{\substack{x,y \in \mathbb{Z}_2^n \\ f(x), f(y)}} f(x) f(y) f(x+y) = 1 - 2 \text{Proj}(f)$$

$$\mathbb{E}_x \text{Proj}(f) \geq \text{dint}(f, P_{\text{Lim}})$$

[Hint: Fourier analysis and convolution]

[BLR linearity testing]

$$\mathbb{E}_x \chi_S \cdot \chi_T = \chi_{S \Delta T}$$

Complexity Measures

Let $f: \mathbb{Z}_2^n \rightarrow \{-1, 1\}$, $x \in \mathbb{Z}_2^n$

Def sensitivity of f at x

$$s(f, x) = |\{i \in [n] \mid f(x) \neq f(x + e_i)\}|$$

Def sensitivity of f

$$s(f) = \max_{x \in \mathbb{Z}_2^n} s(f, x)$$

Notation Let $B \subseteq [n]$. $x^B = x + \sum_{i \in B} e_i$

Def block sensitivity of f at x

$$\begin{aligned} bs(f, x) &= \max \{ k \mid \exists \text{ disjoint } B_1, \dots, B_k \subseteq [n] \\ &\quad \text{s.t. } f(x) \neq f(x^{B_i}) \forall i \in [k] \} \end{aligned}$$

Def block sensitivity of f

$$bs(f) = \max_{x \in \mathbb{Z}_2^n} bs(f)$$

$$\text{Ex } bs(f) \geq S(f)$$

Ex^* find f satisfying $bs(f) \geq \Omega(S(f)^2)$

Recall $\deg(f) = \max_{\substack{S: f(S) \neq 0}} |S|$

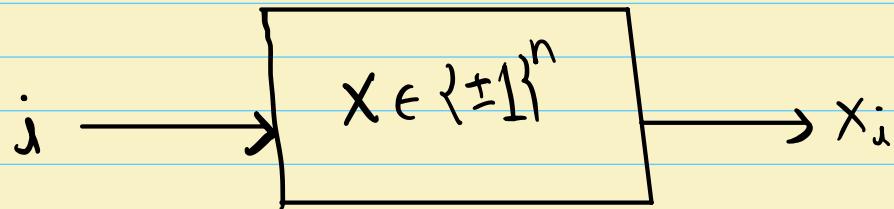
We say a polynomial $p(x_1, \dots, x_n)$ $\frac{1}{3}$ -approximates $f: \{-1, 1\}^n \rightarrow \{0, 1\}$ if

$$|p(x) - f(x)| \leq \frac{1}{3} \quad \forall x \in \{-1, 1\}^n$$

Def Approximate degree

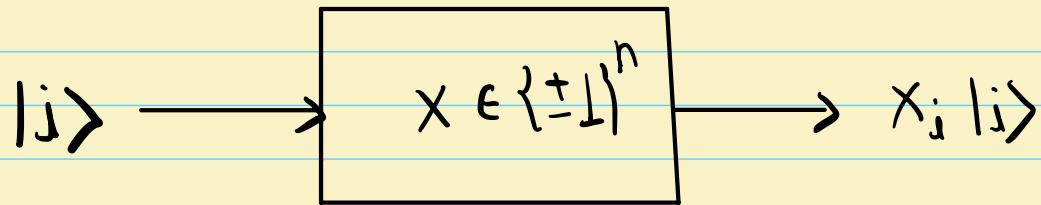
$$\tilde{\deg}(f) = \min \{ \deg(p) \mid p \text{ } \frac{1}{3}\text{-approximates } f \}$$

Classical query model (viewed as a string)



[Think of n as very large]

Quantum query model ("")



O_x unitary

$$O_x |i\rangle = x_i |i\rangle$$

Decision Tree: Simplified computational model

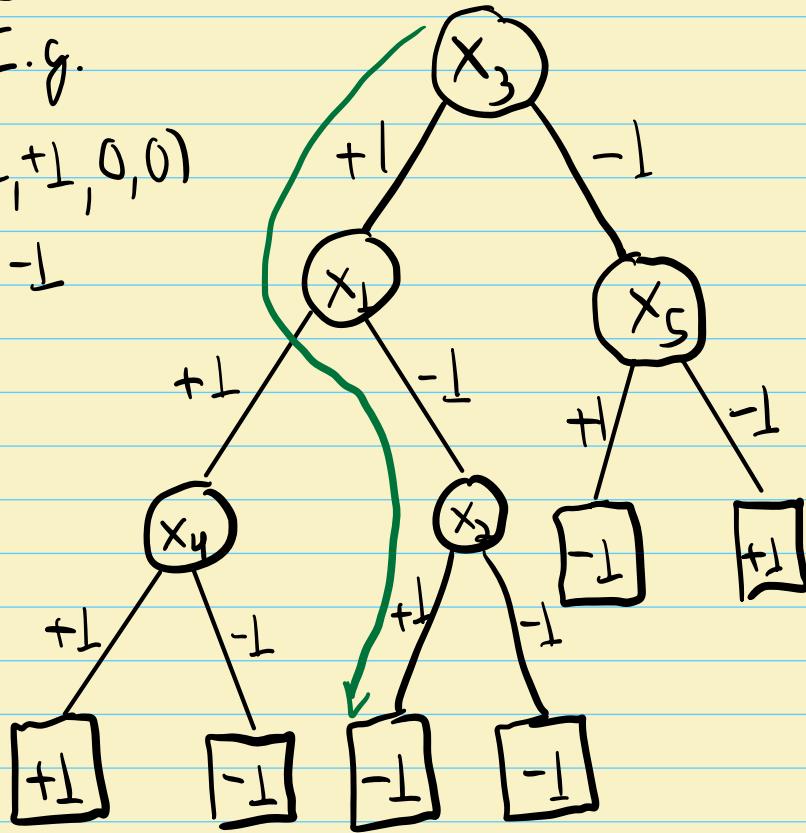
Input: $x = (x_1, \dots, x_n)$

Internal nodes label query variables

E.g.

$$x = (-1, +1, +1, 0, 0)$$

output: -1



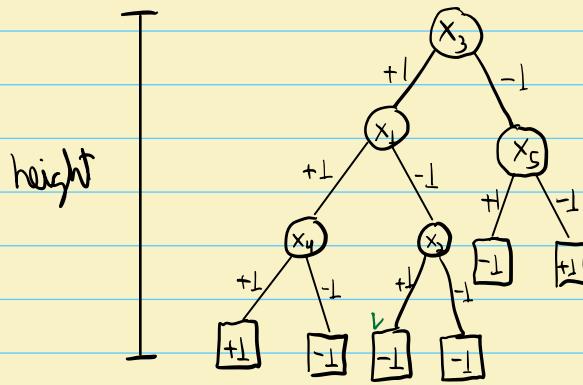
Start at the root
and query input
variables until
leaf is reached

Leaves are labelled with output value
of a computational path

Output: value of leaf reached by evaluating
the decision tree

Def $\text{height}(\text{Tree})$ = length of a longest path
from root to a leaf

E.g. for the previous example the height is 3



Def Decision tree complexity of f
(or classical query complexity)

$$D(f) = \min \{ \text{height}(T) \mid T \text{ is a tree computing } f \}$$

Ex $D(f) \geq \deg(f)$

Quantum Query Model

Given $f: \mathbb{Z}_2^n \rightarrow \{0,1\}$, a t -query protocol
specifies $t+1$ unitaries U_0, \dots, U_t ,

$$\left(U_t O_x \dots O_x U_1 O_x U_0 |0\rangle \begin{array}{c} \\ \swarrow \quad | \quad \searrow \\ t \text{ queries to the oracle} \end{array} \right)$$

\uparrow starting state

Π_{acc} : projector onto accepting subspace,

$$p_{\text{acc}}(x) := \|\Pi_{\text{acc}} U_t O_x \dots O_x U_1 O_x U_0 |0\rangle\|^2$$

such that

$$|p_{\text{acc}}(x) - f(x)| \leq \frac{1}{3} \quad \forall x \in \mathbb{Z}_2^n$$

$E_x p_{\text{acc}}(x)$ is a polynomial of degree $\leq 2t$
in x .

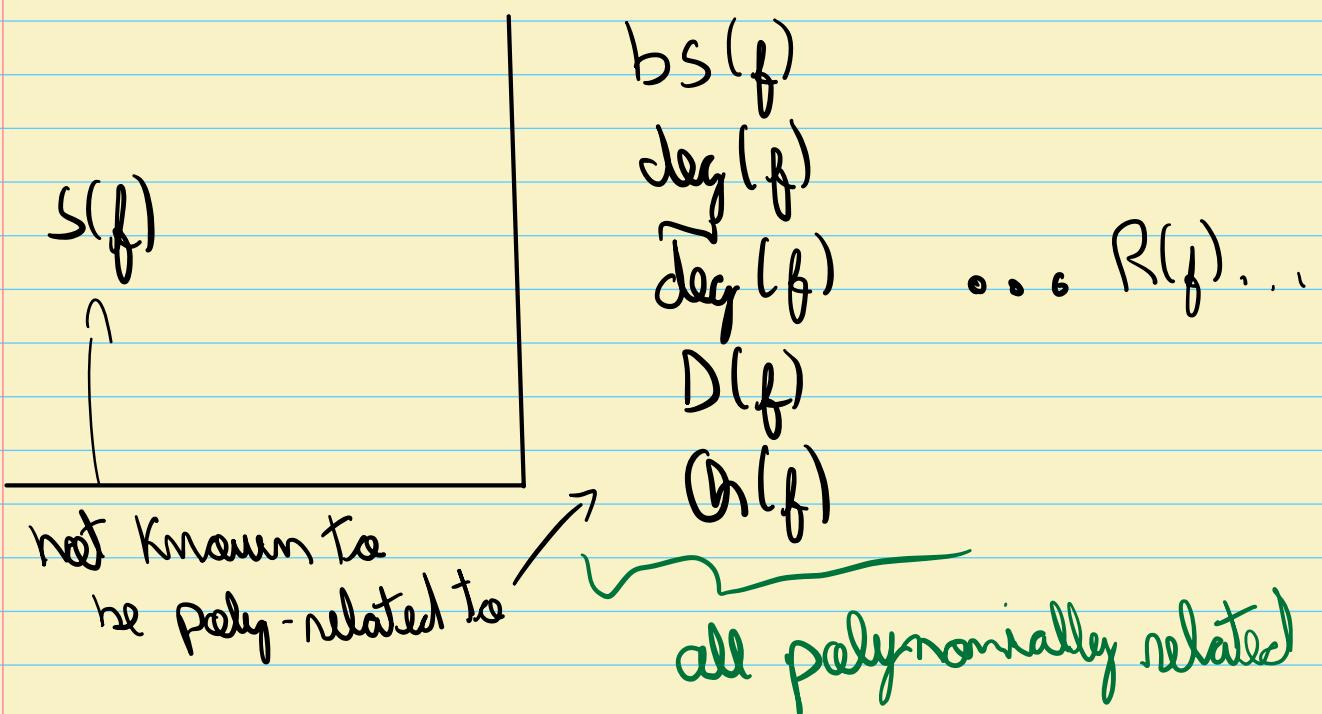
Def Quantum query complexity of f

$$\text{Q}_h(f) = \min \{t \mid \exists \text{ quantum } t\text{-query protocol}$$

for f

$$\text{Ex } \text{Q}_h(f) \geq \frac{\text{deg}(f)}{2}$$

Before 2018



NS'90 [Sensitivity Conjecture]

$$bs(f) \leq \text{poly}(S(f))$$

Proved by Muraug in 2018

$$E_x D(f) \geq O_h(f)$$

No Superpolynomial quantum speed-up
for total functions in the query model
of computation

CH If f is a total function, then

$$\mathcal{O}(f) \geq \text{poly}(\mathcal{D}(f)).$$

To be continued...