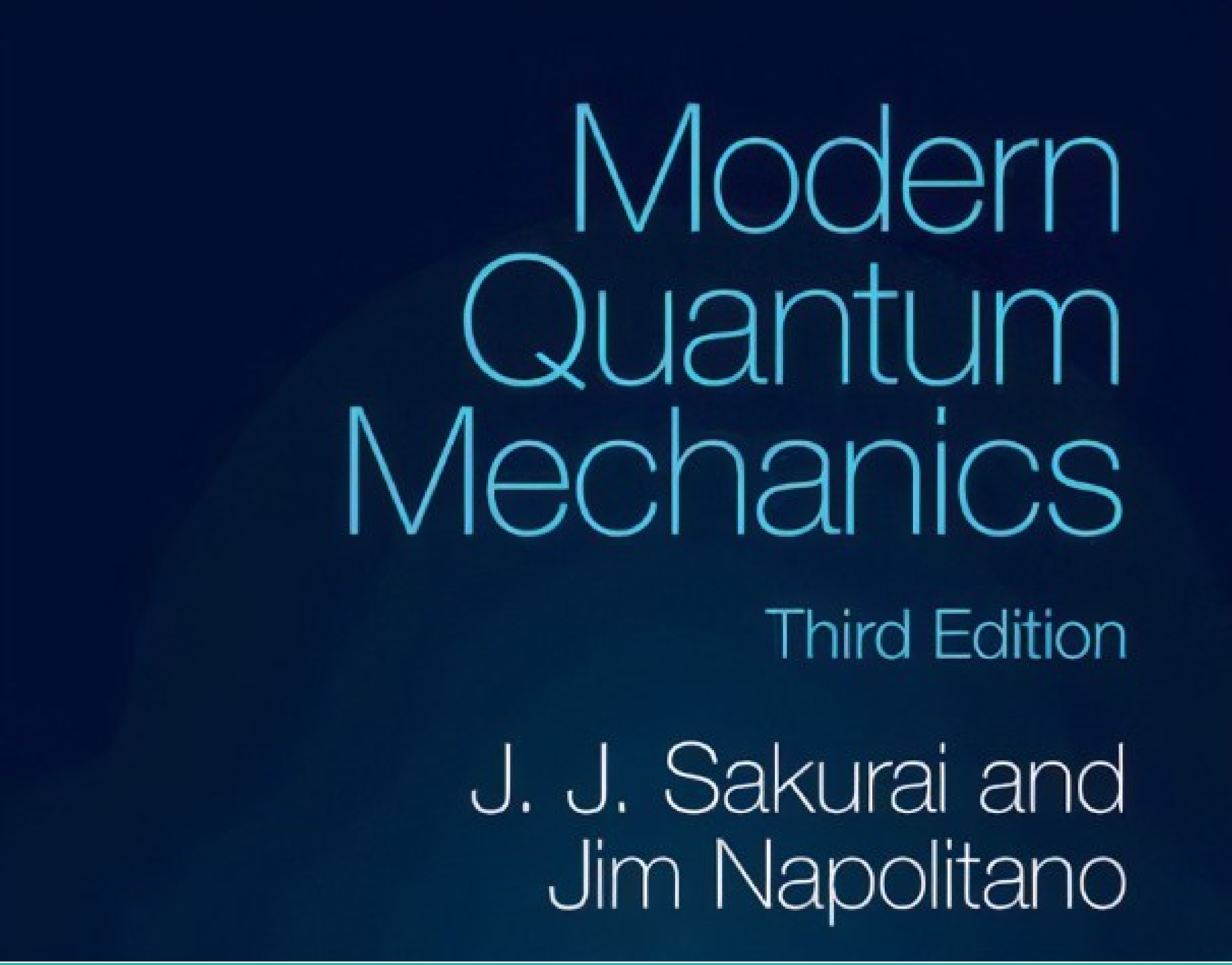

Modern Quantum Mechanics

–An abstract

Author: Grant

Institute: DEP, THU

Date: July 2, 2022

Contents

Chapter 1 Fundamental Concepts

Our knowledge of the physical world comes from making assumptions about nature, formulating these assumptions into postulates, deriving predictions from those postulates, and testing those predictions against experiment.

1.1 The Stern-Gerlach Experiment

It is conceived by O. Stern in 1921 and carried out in Frankfurt by him in collaboration with W. Gerlach in 1922.

In a certain sense, a two-state system of the Stern-Gerlach type is the least classical, most quantum-mechanical system.

1.1.1 Description of the Experiment

We now present a brief discussion of **the Stern-Gerlach experiment**.

1. Silver (Ag) atoms are heated in an oven, which has a small hole through which some of the silver atoms escape.
2. The beam goes through a collimator and is then subjected to an inhomogeneous magnetic field produced by a pair of pole pieces, one of which has a very sharp edge.

If we ignore the nuclear spin, the atom have an angular momentum of the single 47th (5s) electron. As a result, the atom possesses a magnetic moment equal to the spin magnetic moment of the 47th electron,

$$\boldsymbol{\mu} \propto \boldsymbol{S}, \tag{1.1}$$

where the precise propotionality factor turns out to be $e/m_e c$[1] to an accuracy of about 0.2%.

The z-component of the force is given by

$$F_z = \frac{\partial}{\partial z}(\boldsymbol{\mu} \cdot \boldsymbol{B}) \simeq \mu_z \frac{\partial B_z}{\partial z}, \tag{1.2}$$

where we have ignored the components in directions other than the z-direction. Obviously, the SG (Stren-Gerlach) apparatus "measures" the z-component of $\boldsymbol{\mu}$ or, equivalently, the z-component of $\boldsymbol{S}$ up to a proportionality factor.

The atoms in the oven are randomly oriented. If the electron were like a classical spinning object, we would expect all values of μ_z to be realized between $|\boldsymbol{\mu}|$ and $-|\boldsymbol{\mu}|$. This would lead us to expect a continuous bundle of beams coming out of the apparatus.

However, the apparatus splits the original silver beam from the oven into *two distinct* components. To the extent that $\boldsymbol{\mu}$ can be identified within a proportionality factor with the electron spin $\boldsymbol{S}$, only two possible values of the z-component of $\boldsymbol{S}$ are observed to be possible, which we call S_z+ and S_z-. Numerically it turns out that $|S_z| = \hbar/2$, where

$$\hbar = 1.0546 \times 10^{-27}\mathrm{erg/s} = 6.5822 \times 10^{-16}\mathrm{eV/s}. \tag{1.3}$$

This "quantization" of the electron spin angular momentum is the first important feature we deduce from the Stern-Gerlach experiment.

---

[1] $e < 0$ in this book

1.1.2 Sequential Stern-Gerlach Experiments

Now consider a **sequential Stern-Gerlach experiment**.

1. Through two SG$\hat{\boldsymbol{z}}$ apparatus
   Only one beam component coming out of the second apparatus, which is the same as the previous one.
2. Through SG$\hat{\boldsymbol{z}}$ apparatus, and then SG$\hat{\boldsymbol{x}}$ apparatus
   An S_x+ component and an S_x- component coming out with the same intensities.
3. Through an SG$\hat{\boldsymbol{z}}$ and an SG$\hat{\boldsymbol{x}}$ apparatus, and then an SG$\hat{\boldsymbol{z}}$ apparatus
   Both an S_z+ component and an S_z- component emerge from the third apparatus.

What a Surprise! This example is often used to illustrate that in quantum mechanics we cannot determine both S_z and S_x simultaneously, or the selection of the S_x+ beam by the second apparatus (SG$\hat{\boldsymbol{x}}$) completely destroys any *previous* information about S_z.

The peculiarities of quantum mechanics are imposed upon us by the experiment itself. The limitation is, in fact, inherent in microscopic phenomena.

1.1.3 Analogy with Polarization of Light

We now digress to consider **the polarization of light waves**.

Consider a monochromatic light wave propagating in the z-direction. We can definite an *x-polarized light* and *y-polarized light*,

$$\boldsymbol{E} = E_0 \hat{\boldsymbol{x}} \cos(kz - \omega t), \tag{1.4}$$

$$\boldsymbol{E} = E_0 \hat{\boldsymbol{y}} \cos(kz - \omega t). \tag{1.5}$$

As for filter, we now consider 2 examples.

1. Through an x-filter and an y-filter
   No light beam comes out.
2. Insert between the x-filter and the y-filter another Polaroid - x'-direction - that makes an angle of $45°$ with the x-direction in the xy plane
   There is a light beam coming out despite the fact that right after the beam went through the x-filter it did not have any polarization component in the y-direction.

We'll find these situations are quite analogous to the situations that we encountered earlier with the SG arrangement, provided that the following correspondence is made:

$$\begin{aligned} S_z \pm \text{atoms} &\leftrightarrow x-, y- \text{polarized light}, \\ S_x \pm \text{atoms} &\leftrightarrow x'-, y'- \text{polarized light}. \end{aligned} \tag{1.6}$$

We know the relation

$$\begin{aligned} E_0 \hat{\boldsymbol{x}}' \cos(kz - \omega t) &= E_0[\frac{1}{\sqrt{2}} \hat{\boldsymbol{x}} \cos(kz - \omega t) + \frac{1}{\sqrt{2}} \hat{\boldsymbol{y}} \cos(kz - \omega t)], \\ E_0 \hat{\boldsymbol{y}}' \cos(kz - \omega t) &= E_0[-\frac{1}{\sqrt{2}} \hat{\boldsymbol{x}} \cos(kz - \omega t) + \frac{1}{\sqrt{2}} \hat{\boldsymbol{y}} \cos(kz - \omega t)]. \end{aligned} \tag{1.7}$$

In the triple-filter arrangement the beam coming out of the first Polaroid is an $\hat{\boldsymbol{x}}$-polarized beam, which can be regarded as a linear combination of an x'-polarized beam and a y'-polarized beam. The second Polaroid selects the x'-polarized beam, which can in turn be regraded as a linear combination of an x-polarized and a y-polarized beam. And finally, the third Polaroid selects the y-polarized component.

We might be able to represent the spin state of a silver atom by some kind of vector in a new kind of two-dimensional vector space. Just represent the S_x+ state by a vector, which we call a ket in the Dirac notation.

So we conjecture

$$
\begin{aligned}
|S_x;+\rangle &= \frac{1}{\sqrt{2}}|S_z;+\rangle + \frac{1}{\sqrt{2}}|S_z;-\rangle \\
|S_z;+\rangle &= -\frac{1}{\sqrt{2}}|S_x;+\rangle + \frac{1}{\sqrt{2}}|S_x;-\rangle
\end{aligned}
\tag{1.8}
$$

in analogy with (1.7).

Thus the component coming out of the second apparatus is to be regarded as a superposition of S_z+ and S_z-. It is for this reason that two components emerge from the third apparatus.

Another question: **How to represent the $S_y\pm$ states?**

This time we consider a circularly polarized beam of light, which can be obtained by letting a linearly polarized light pass through a quarter-wave plate:

$$
\boldsymbol{E} = E_0[\frac{1}{\sqrt{2}}\hat{\boldsymbol{x}}\cos(kz-\omega t) + \frac{1}{\sqrt{2}}\cos(kz-\omega t+\frac{\pi}{2})]. \tag{1.9}
$$

It is more elegant to use complex notation:

$$
\boldsymbol{\epsilon} = [\frac{1}{\sqrt{2}}\hat{\boldsymbol{x}}e^{i(kz-\omega t)} + \frac{i}{\sqrt{2}}\hat{\boldsymbol{y}}e^{i(kz-\omega t)}], \tag{1.10}
$$

where we have used $i = e^{i\pi/2}$.

We can make the following analogy with the spin states of silver atoms:

$$
S_y + \text{atom} \leftrightarrow \text{right circularly polarized beam}, \tag{1.11}
$$

$$
S_y - \text{atom} \leftrightarrow \text{left circularly polarized beam}. \tag{1.12}
$$

Hence, if we are allowed to make the coefficients preceding base kets complex, there is no difficulty in accommodating the $S_y\pm$ atoms in our vector space formalism:

$$
|S_y;\pm\rangle = \frac{1}{\sqrt{2}}|S_z;+\rangle \pm \frac{i}{\sqrt{2}}|S_z;-\rangle. \tag{1.13}
$$

We thus see that the two-dimensional vector space needed to describe the spin states of silver atoms must be a *complex* vector space; an arbitrary vector in the vector space is written as a linear combination of the base vectors $|S_z;\pm\rangle$ with, in general, complex coefficients.

1.2 Kets, Bras, and Operators

In this and the following section we formulate the basic mathematics of vector spaces as used in quantum mechanics, which was developed by P. A. M. Dirac.

1.2.1 Ket Space

Definition 1.1 (Ket)

A ***state vector*** *in a complex vector space to represent a physical state in quantum mechanics.* ♣

This state ket is postulated to contain complete information about the physical state; everything we are allowed to ask about the state is contained in the ket.

If we multiply ket $|\alpha\rangle$ by a complex number c, the resulting product $c|\alpha\rangle$ is another ket. If c is zero, the resulting ket is said to be a **null ket**.

Postulate 1.1 (State vector)

$|\alpha\rangle$ *and* $c|\alpha\rangle$, *with* $c \neq 0$, *represent the same physical state.* ♡

In other words, only the "direction" in vector space is of significance.

Definition 1.2 (Operator)

A ***matrix*** *in the vector space to represent an* ***observable***. ♣

Quite generally, an operator acts on a ket *from the left*.

Definition 1.3 (Eigenkets, eigenvalues and eigenstate)

Kets are ***eigenkets*** *of operator* A *if* $A|\alpha\rangle$ *is a constant times* $|\alpha\rangle$, *and the constants are called* ***eigenvalues*** *of operator* A.
The physical state corresponding to an eigenket is called an ***eigenstate***. ♣

The dimensionality of the vector space is determined by the number of alternatives in Stern-Gerlach type experiments. More formally, we are concerned with an N-dimensional vector space spanned by the N eigenkets of observable A. Any arbitrary ket can be wriiten as

$$|\alpha\rangle = \sum_{N}^{i} c_i|\alpha_i\rangle. \tag{1.14}$$

1.2.2 Bra Space and Inner Products

Definition 1.4 (Bra and bra space)

Bra *is an element belong to* ***bra space*** *which is dual to corresponding ket space.* ♣

We call DC for **dual correspondence**.

Definition 1.5 (Inner product)

The ***inner product*** *of a bra and a ket is*

$$\langle\beta|\alpha\rangle = (\langle\beta|) \cdot (|\alpha\rangle). \tag{1.15}$$ ♣

This product is a complex number in general, and we postulate two fundamental properties of inner products below:

- $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.
- $\langle\alpha|\alpha\rangle \geq 0$.

Definition 1.6 (Orthogonal)

Two kets are said to be ***orthogonal*** *if*

$$\langle\alpha|\beta\rangle = 0. \tag{1.16}$$

♣

Definition 1.7 (Normalized ket and norm)

Given a ket which is not a null ket, we can form a ***normalized ket*** $|\tilde{\alpha}\rangle$*, where*

$$|\tilde{\alpha}\rangle = (\frac{1}{\sqrt{\langle\alpha|\alpha\rangle}})|\alpha\rangle, \tag{1.17}$$

with the property

$$\langle\tilde{\alpha}|\tilde{\alpha}\rangle = 1. \tag{1.18}$$

Generally, we call $\sqrt{\langle\alpha|\alpha\rangle}$ *the* ***norm*** *of* $|\alpha\rangle$.

♣

1.2.3 Operators

Definition 1.8 (Equality and null operator)

Operators are said to be ***equal*** *if*

$$X|\alpha\rangle = Y|\alpha\rangle \tag{1.19}$$

for an arbitrary ket in the ket space.

Operator is said to be the ***null operator*** *if, for any arbitrary ket* $|\alpha\rangle$*, we have*

$$X|\alpha\rangle = 0. \tag{1.20}$$

♣

Operators can be added and addition operations are **commutative** and **associative**.

Definition 1.9 (Hermitian adjoint and Hermitian)

We define the symbol $X^\dagger$ *which satisfies that* $\langle\alpha\rangle X^\dagger$ *is dual to* $X|\alpha\rangle$*, and the operator* $X^\dagger$ *is called the* ***Hermitian adjoint****, or simply the adjoint, of X.*

An operator X is said to be ***Hermitian*** *if*

$$X = X^\dagger. \tag{1.21}$$

♣

1.2.4 Multiplication

Operators can be multiplied. Multiplication is *noncommutative* but associative.

Notice that

$$(XY)^\dagger = Y^\dagger X^\dagger. \tag{1.22}$$

So far, we have considered the following products: $\langle\beta|\alpha\rangle, X|\alpha\rangle, \langle\alpha|X$, and XY. We now can define a new product.

Definition 1.10 (Outer product)

Outer product *of* $|\beta\rangle$ *and* $\langle\alpha|$ *is*

$$(|\beta\rangle) \cdot (\langle\alpha|) = |\beta\rangle\langle\alpha|. \tag{1.23}$$

♣

We should emphasize that $|\beta\rangle\langle\alpha|$ is to be regarded as an operator; hence it is fundamentally different from the inner product $\langle\beta|\alpha\rangle$, which is just a number.

1.2.5 The Associative Axiom

Postulate 1.2 (Associative axiom of multiplication)

When we are dealing with legal multiplications among kets, bras and operators, the ***associative property*** *is postulated.*

♡

Corollary 1.1

If $X = |\beta\rangle\langle\alpha|$, *then*

$$X^\dagger = |\alpha\rangle\langle\beta|. \tag{1.24}$$

♡

Corollary 1.2

Use the fundamental property of the inner product and the axiom, we find

$$\langle\beta|X|\alpha\rangle = \langle\alpha|X^\dagger|\beta\rangle^*. \tag{1.25}$$

For a Hermitian X we have

$$\langle\beta|X|\alpha\rangle = \langle\alpha|X|\beta\rangle^*. \tag{1.26}$$

♡

1.3 Base Kets and Matrix Representations

1.3.1 Eigenkets of an Observable

1.3.2 Eigenkets as Base Kets

1.3.3 Matrix Representations

1.3.4 Spin 1/2 Systems

1.4 Measurements, Observables, and the Uncertainty Relations

1.4.1 Measurements

1.4.2 Spin 1/2 Systems, Once Again

1.4.3 Compatible Observables

1.4.4 Incompatible Observables

1.4.5 The Uncertainty Relation

1.5 Change of Basis

1.5.1 Transformation Operator

1.5.2 Transformation Matrix

1.5.3 Diagonalization

1.5.4 Unitary Equivalent Observables

1.6 Position, Momentum, and Translation

1.6.1 Continuous Spectra

1.6.2 Position Eigenkets and Position Measurements

1.6.3 Translation

1.6.4 Momentum as a Generator of Translation

1.6.5 The Canonical Commutation Relations

1.7 Wave Functions in Position and Momentum Space

1.7.1 Position-Space Wave Function

1.7.2 Momentum Operator in the Position Basis

1.7.3 Momentum-Space Wave Function

1.7.4 Gaussian Wave Packets

1.7.5 Generalization to Three Dimensions

Bibliography

[1] Paul Adrien Maurice Dirac. *The principles of quantum mechanics*. Oxford university press, 1967.

[2] J Sakurai and J Napolitano. *Modern Quantum Mechanics. 3-nd edition*. Cambridge University Press, 2021.

Appendix A Mathematical Tools