Math 207 Section 1 Fall 2022

EXAM 3 INFO:

Time: Thursday, 12/1, 8:15–9:45pm

Location: Carver 205

Content: Sections 4.5–4.7, 5.1–5.5, and 6.1–6.4

Details:

- No calculators
- Closed book
- Closed notes

TOPICS AND SAMPLE PROBLEMS:

- **Topic 1:** Rank and nullity.
  1. Dimensions of vector spaces.
  2. Rank of a matrix and basis for row space.
  3. Apply the Rank Theorem (Rank-Nullity) and the Invertible Matrix Theorem.

  Sample problems:

  1) Find a basis for Row A, dim Row A, and rank A, where

  $$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}.$$

  2) If a 3×8 matrix A has rank 3, what are the dimensions of $\operatorname{Nul} A$ and $\operatorname{Row} A$, and what is the rank of $A^{\top}$?
  3) Is there a 4×9 matrix for which $\dim \operatorname{Nul} A = 4$?
  4) If A is a 6×4 matrix, what is the smallest possible dimension of $\operatorname{Nul} A$?

- **Topic 2:** Eigenvalues, eigenvectors, diagonalization.
  1. Compute eigenvalues and eigenspaces.
  2. Diagonalize a matrix, if possible.

  Sample problem: Diagonalize, if possible:

  $$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}.$$

- **Topic 3:** Change of coordinates, matrices of linear transformations in different coordinates.
  1. Find the matrix of change of coordinates from different bases.
  2. Calculate the matrix of a linear transformation in a given basis.
  3. Use the change of coordinates matrix to calculate the matrix of a linear transformation in another basis.

  Sample problem: Consider the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$, where as usual,

  $$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

  Also consider the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, where

  $$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

  1) Compute the coordinates $[\mathbf{x}]_{\mathcal{E}}$ and $[\mathbf{x}]_{\mathcal{B}}$ of the vector $\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ with respect to the bases $\mathcal{E}$ and $\mathcal{B}$, respectively.
  2) Find the change of coordinates matrix $P_{\mathcal{B}\leftarrow\mathcal{E}}$ which transforms coordinate vectors in the basis $\mathcal{E}$ to coordinate vectors in the basis $\mathcal{B}$. For the vector $\mathbf{x}$ above, verify that $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{E}}[\mathbf{x}]_{\mathcal{E}}$.
  3) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ for which $T(\mathbf{e}_1) = \mathbf{e}_2$, $T(\mathbf{e}_2) = \mathbf{e}_3$ and $T(\mathbf{e}_3) = 0$. Find the matrix $[T]_{\mathcal{E}}$ representing T in the standard basis $\mathcal{E}$.
  4) Find the matrix $[T]_{\mathcal{B}}$ that represents T in the basis $\mathcal{B}$.

- **Topic 4:** Orthogonal projections.
  1. Compute orthogonal projections onto subspaces.
  2. Find the closest vector in a subspace to a given vector.
  3. Find the distance between a vector and a subspace.

  Sample problem: Consider the subspace W of $\mathbb{R}^3$ spanned by

  $$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

  Find the vector $\mathbf{x}$ in W which minimizes the distance from the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to W, and compute the distance from $\mathbf{b}$ to W.

- **Topic 5:** Gram–Schmidt process
  1. Given a basis of a subspace, find an orthogonal or orthonormal basis of the same subspace.

  Sample problem: Find an orthonormal basis of the column space of the following matrix:

  $$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

  (Notice that the columns of A are linearly independent)

- **Topic 6:** Complex eigenvalues and eigenvectors.
  1. Compute complex eigenvalues and eigenvectors.
  2. Real and imaginary parts of complex eigenvalues and eigenvectors.

  Sample problem: Consider the matrix

  $$A = \begin{bmatrix} 0 & 20 \\ -1 & 4 \end{bmatrix}.$$

  (a) Find the (real and/or complex) eigenvalues.

  (b) Find an invertible (real) matrix P and a (real) matrix of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$. Alternatively, show that no such matrices exist.

SOLUTIONS:

- **Topic 1:**

1) A basis for Row A is given by the nonzero columns in any echelon form. Row reduce:

$$\begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 2 & -1 & 1 & -6 & 8 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$$

$$\underset{\sim}{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix} \underset{\sim}{R_3 = R_3 + 7R_1} \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & -6 & -18 & 24 & -24 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$$

$$\underset{\sim}{R_4 = R_4 - 4R_1} \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & -6 & -18 & 24 & -24 \\ 0 & 3 & 9 & -12 & 12 \end{bmatrix} \underset{\sim}{R_3 = R_3 + 2R_2} \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 9 & -12 & 12 \end{bmatrix}$$

$$\underset{\sim}{R_4 = R_4 - R_2} \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

One basis for Row A is

$$\{(1, -2, -4, 3, -2),\ (0, 3, 9, -12, 12)\}.$$

Since there are two basis vectors, $\dim \operatorname{Row} A = 2$. Finally, we count pivots in A to see that $\operatorname{rank} A = 2$. (If you kept row reducing all the way to reduced row echelon form, then you would have found a different basis. That is okay because a vector space has lots of different bases. You should still have gotten the same *dimension*, however.)

2) (a) Since there are $n = 8$ columns, the Rank Theorem (Rank-Nullity) says that $\operatorname{rank} A + \dim \operatorname{Nul} A = n = 8$. Subtract to find $\dim \operatorname{Nul} A = n - \operatorname{rank} A = 5$.

   (b) The dimension of Row A always equals $\operatorname{rank} A$, which in this case is 3.

   (c) The rank of $A^\top$ always equals $\operatorname{rank} A$, which in this case is 3.

3) Since A has $n = 9$ columns, the Rank Theorem (Rank-Nullity) says that $\operatorname{rank} A + \dim \operatorname{Nul} A = n = 9$. If $\dim \operatorname{Nul} A = 4$, then $\operatorname{rank} A = 5$. This is impossible since A only has 4 rows: $\operatorname{rank} A = \dim \operatorname{Col} A \leq 4$ since $\operatorname{Col} A$ is a subspace of $\mathbb{R}^4$.

4) In most problems like this, you should think about the Rank Theorem (Rank-Nullity). For this specific problem, it does not impose any constraints, and we can easily produce an example

where $\dim \operatorname{Nul} A = 0$. For instance,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix has $n = 4$ linearly independent columns, so $\dim \operatorname{Nul} A = n - \operatorname{rank} A = 4 - 4 = 0$.

- **Topic 2:**

Our goal is to find a basis for $\mathbb{R}^3$ consisting of eigenvectors for A.

The first step is to find the eigenvalues. Compute the characteristic polynomial and set it equal to zero:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 2-\lambda & 2 \\ -1 & -1 & -1-\lambda \end{vmatrix} \\
&= (1-\lambda) \begin{vmatrix} 2-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ -1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 2 & 2-\lambda \\ -1 & -1 \end{vmatrix} \\
&= (1-\lambda)\Big[(2-\lambda)(-1-\lambda) - 2(-1)\Big] - \Big[2(-1-\lambda) - 2(-1)\Big] + \Big[2(-1) - (2-\lambda)(-1)\Big] \\
&= (1-\lambda)(\lambda^2 - \lambda) - (-2\lambda) + (-\lambda) \\
&= -\lambda^3 + 2\lambda^2 \\
&= -\lambda^2(\lambda - 2).
\end{aligned}$$

The eigenvalues are $\lambda = 0$ (with algebraic multiplicity 2) and $\lambda = 2$ (with algebraic multiplicity 1).

The next step is to find bases for each of the eigenspaces. We start with the 0-eigenspace, $\operatorname{Nul}(A - 0I) = \operatorname{Nul}(A)$. It consists of solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, so we row reduce an augmented matrix:

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \overset{R_2 = R_2 - 2R_1}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \overset{R_3 = R_3 + R_1}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Reinterpret as equations, and solve for basic variables in terms of free variables:

$$x_1 + x_2 + x_3 = 0, \qquad x_1 = -x_2 - x_3.$$

Write $\mathbf{x}$ as a vector in parametric vector form, using the free variables as parameters:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the 0-eigenspace is given by:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Repeat to find a basis for the 2-eigenspace, $\mathrm{Nul}(A-2I)$. It consists of solutions to the homogeneous equation $(A-2I)\mathbf{x} = \mathbf{0}$, so we row reduce an augmented matrix:

$$\begin{bmatrix} A-2I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ -1 & -1 & -3 & 0 \end{bmatrix} \underset{R_2=R_2+2R_1}{\sim} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ -1 & -1 & -3 & 0 \end{bmatrix}$$

$$\underset{R_3=R_3-R_1}{\sim} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -2 & -4 & 0 \end{bmatrix} \underset{R_3=R_3+R_2}{\sim} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_2=\frac{1}{2}R_2}{\sim} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{R_1=R_1-R_2}{\sim} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_1=-R_1}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Reinterpret as equations, and solve for basic variables in terms of free variables:

$$\begin{cases} x_1 \quad + \ x_3 = 0 \\ \quad x_2 + 2x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = -x_3 \\ x_2 = -2x_3 \end{cases}$$

Write $\mathbf{x}$ as a vector in parametric vector form, using the free variables as parameters:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

A basis for the 2-eigenspace is given by

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Finally, put all the eigenspace bases together and see if we have as many vectors as the size of A. In this case, $2+1=3$, and we have a basis of eigenvectors. To write $A = PDP^{-1}$ with P invertible and D diagonal, use the basis of eigenvectors for the columns of P, and put their corresponding eigenvalues on the diagonal of D:

$$P = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

- **Topic 3:**

1) (a) Write $\mathbf{x}$ as a linear combination of the basis vectors in $\mathcal{E}$. This one is easy since we can see that $\mathbf{x} = 2\mathbf{e}_1 - 2\mathbf{e}_2 + 2\mathbf{e}_3$. The coefficients give the coordinates:

$$[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}.$$

(b) Write $\mathbf{x}$ as a linear combination of the basis vectors in $\mathcal{B}$. This time, we can see that $\mathbf{x} = 2\mathbf{b}_3 = 0\mathbf{b}_1 + 0\mathbf{b}_2 + 2\mathbf{b}_3$. (You could also solve the vector equation $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{x}$.) The coefficients give the coordinates:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

2) (a) The change of coordinates matrix is

$$P_{\mathcal{B}\leftarrow\mathcal{E}} = \begin{bmatrix} | & | & | \\ [\mathbf{e}_1]_{\mathcal{B}} & [\mathbf{e}_2]_{\mathcal{B}} & [\mathbf{e}_3]_{\mathcal{B}} \\ | & | & | \end{bmatrix}.$$

We have to find all the $\mathcal{B}$-coordinates for the standard basis vectors. It amounts to solving three vector equations $c_{1j}\mathbf{b}_1 + c_{2j}\mathbf{b}_2 + c_{3j}\mathbf{b}_3 = \mathbf{e}_j$ for $j = 1, 2, 3$. Make an augmented matrix with three augmented columns, and row reduce:

$$\left[\begin{array}{ccc|ccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R_2=R_2+R_3}{\sim} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{R_1=R_1-R_3}{\sim} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R_1=R_1-R_2}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{R_1=\frac{1}{2}R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

The columns of the matrix on the right give the solutions:

$$\begin{aligned} \tfrac{1}{2}\mathbf{b}_1 + 0\mathbf{b}_2 + 0\mathbf{b}_3 &= \mathbf{e}_1 \\ -\tfrac{1}{2}\mathbf{b}_1 + \mathbf{b}_2 + 0\mathbf{b}_3 &= \mathbf{e}_2 \\ -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 &= \mathbf{e}_3. \end{aligned}$$

The coefficients give coordinates, so

$$P_{\mathcal{B}\leftarrow\mathcal{E}} = \begin{bmatrix} | & | & | \\ [\mathbf{e}_1]_{\mathcal{B}} & [\mathbf{e}_2]_{\mathcal{B}} & [\mathbf{e}_3]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Notice it is the same as the matrix on the right when we finished row reducing.)

(b) To verify that $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{E}}[\mathbf{x}]_{\mathcal{E}}$, just multiply our answers for $P_{\mathcal{B}\leftarrow\mathcal{E}}$ and $[\mathbf{x}]_{\mathcal{E}}$:

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

That is the same as our answer for $[\mathbf{x}]_{\mathcal{B}}$. It checks out.

3) The answer is

$$[T]_{\mathcal{E}} = \begin{bmatrix} | & | & | \\ [T(\mathbf{e}_1)]_{\mathcal{E}} & [T(\mathbf{e}_2)]_{\mathcal{E}} & [T(\mathbf{e}_3)]_{\mathcal{E}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The columns say what T does to the basis vectors. Our answer means that, for any $\mathbf{x} \in \mathbb{R}^3$, $[T(\mathbf{x})]_{\mathcal{E}} = [T]_{\mathcal{E}}[\mathbf{x}]_{\mathcal{E}}$.

4) The matrix is

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & [T(\mathbf{b}_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}.$$

Use $P_{\mathcal{B}\leftarrow\mathcal{E}}$ and $[T]_{\mathcal{E}}$ to find the columns:

$$[T(\mathbf{b}_1)]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{E}}[T(\mathbf{b}_1)]_{\mathcal{E}} = P_{\mathcal{B}\leftarrow\mathcal{E}}[T]_{\mathcal{E}}\,[(\mathbf{b}_1)]_{\mathcal{E}},$$

i.e.,

$$[T(\mathbf{b}_1)]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

Repeat to find $[T(\mathbf{b}_2)]_{\mathcal{B}}$ and $[T(\mathbf{b}_3)]_{\mathcal{B}}$:

$$[T(\mathbf{b}_2)]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix},$$

$$[T(\mathbf{b}_3)]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -1 \end{bmatrix}.$$

The answer is

$$[T]_{\mathcal{B}} = \begin{bmatrix} -1 & -\frac{3}{2} & \frac{1}{2} \\ 2 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

- **Topic 4:**

(a) The closest vector to $\mathbf{b}$ in W is given by its orthogonal projection $\hat{\mathbf{b}}$. In order to find that, we first need an orthogonal basis for W. We can see that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, so they form a basis for W. Use the Gram–Schmidt algorithm. Start by setting

$$\mathbf{u}_1 := \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Let $\mathbf{p}_2$ be the orthogonal projection of $\mathbf{v}_2$ onto $\operatorname{span}\{u_1\}$:

$$\mathbf{p}_2 = \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \frac{2}{2}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Then define

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Now $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthogonal basis for W, and we can use them to find our projection:

$$\hat{\mathbf{b}} = \frac{\mathbf{b} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{b} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \frac{-2}{2}\mathbf{u}_1 + \frac{2}{1}\mathbf{u}_2 = -\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

This is the closest approximation to $\mathbf{b}$ in W.

(b) The distance from $\mathbf{b}$ to W is $\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|$. We have

$$\mathbf{b} - \hat{\mathbf{b}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix},$$

so the distance is

$$\left\|\mathbf{b} - \hat{\mathbf{b}}\right\| = \sqrt{2^2 + 2^2} = 2\sqrt{2}.$$

- **Topic 5:**

  Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the columns of A:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We want an orthonormal basis for $W := \operatorname{Col} A$, and a regular (not even orthogonal) basis is given by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Use the Gram–Schmidt algorithm to find an orthogonal basis. In every step, we find an orthogonal projection and then rip it out.

First set

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Let $\mathbf{p}_2$ be the orthogonal projection of $\mathbf{v}_2$ onto $\operatorname{span}\{\mathbf{u}_1\}$:

$$\mathbf{p}_2 = \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \frac{1}{3}\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}.$$

Then define

$$\mathbf{u}_2 := \mathbf{v}_2 - \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \\ -\frac{1}{3} \end{bmatrix}.$$

Now $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthogonal basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. We can get a nicer basis by rescaling $\mathbf{u}_2$ to clear the fractions:

$$\mathbf{u}_2' := 3\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

Now $\mathbf{u}_1$ and $\mathbf{u}_2'$ also form an orthogonal basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Go again. Let $\mathbf{p}_3$ be the orthogonal projection of $\mathbf{v}_3$ onto $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$:

$$\mathbf{p}_3 = \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{v}_3 \cdot \mathbf{u}_2'}{\mathbf{u}_2' \cdot \mathbf{u}_2'}\mathbf{u}_2' = \frac{0}{3}\mathbf{u}_1 + \frac{3}{15}\mathbf{u}_2' = \frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

Then define

$$\mathbf{u}_3 := \mathbf{v}_3 - \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{5}\left(\begin{bmatrix} 0 \\ 5 \\ 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -2 \\ 4 \\ 2 \\ 6 \end{bmatrix}.$$

Now $\mathbf{u}_1$, $\mathbf{u}_2'$, and $\mathbf{u}_3$ form an orthogonal basis for $W = \operatorname{Col} A$. We can get a nicer-looking orthogonal basis by rescaling $\mathbf{u}_3$ to clear the fractions:

$$\mathbf{u}_3' := \frac{5}{2}\mathbf{u}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}.$$

Finally, normalize the orthogonal basis to get an orthonormal basis $\left\{\frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1, \frac{1}{\|\mathbf{u}_2'\|}\mathbf{u}_2', \frac{1}{\|\mathbf{u}_3'\|}\mathbf{u}_3'\right\}$. We have

$$\|\mathbf{u}_1\| = \sqrt{3}, \quad \|\mathbf{u}_2'\| = \sqrt{15}, \quad \|\mathbf{u}_3'\| = \sqrt{15},$$

so our answer is

$$\left\{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{15}}\begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}\right\}.$$

- **Topic 6:**
  (a) Find the characteristic polynomial and set it equal to zero:

  $$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 20 \\ -1 & 4-\lambda \end{vmatrix} = (-\lambda)(4-\lambda) - (20)(-1) = \lambda^2 - 4\lambda + 20.$$

  Use the quadratic equation to find the zeros:

  $$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(20)}}{2} = \frac{4 \pm \sqrt{-64}}{2} = \frac{4 \pm 8i}{2} = 2 \pm 4i.$$

  (b) The matrices P and C exist since A has a complex eigenvalue $\lambda = a - bi$. It doesn't matter which one we use, but let's just pick $\lambda = 2 - 4i$. Then $a = 2$ and $b = 4$, so that

  $$C = \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix}.$$

  To find P, we just need to get a complex eigenvector $\mathbf{v} \in \mathbb{C}^2$ with eigenvalue $\lambda = 2 - 4i$, and then we set $P = \begin{bmatrix} \operatorname{Re}\mathbf{v} & \operatorname{Im}\mathbf{v} \end{bmatrix}$. We want to solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$, so we row reduce the augmented matrix:

  $$\begin{bmatrix} A - \lambda I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -2+4i & 20 & 0 \\ -1 & 2+4i & 0 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} -1 & 2+4i & 0 \\ -2+4i & 20 & 0 \end{bmatrix}.$$

This matrix is guaranteed to have a nontrivial null space since λ is an eigenvalue, so the rows cannot be linearly independent: there has to be some constant c such that $R_2 = cR_1$. (In fact, $c = 2 - 4i$, but the exact value isn't important.) Replace R_2 with $R_2 - cR_1$ to get

$$\begin{bmatrix} -1 & 2+4i & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{-R_1}{\sim} \begin{bmatrix} 1 & -2-4i & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It says that $v_1 - (2+4i)v_2 = 0$, or $v_1 = (2+4i)v_2$. Then

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (2+4i)v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 2+4i \\ 1 \end{bmatrix}.$$

We can set the free variable to $v_2 = 1$ to get

$$\mathbf{v} = \begin{bmatrix} 2+4i \\ 1 \end{bmatrix} = \begin{bmatrix} 2+4i \\ 1+0i \end{bmatrix}$$

Then

$$P = \begin{bmatrix} \operatorname{Re}\mathbf{v} & \operatorname{Im}\mathbf{v} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}.$$