Collocation Design Document and Notes

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1 Transcription Method

1.1 Runge-Kutta Methods

To begin, one-step methods called *S-stage Runge-Kutta* can be defined as follow:

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h_k \sum_{j=1}^{S} \beta_j \mathbf{f}_{kj} \tag{1}$$

where for $1 \leq j \leq S$

$$\mathbf{y}_{kj} = \mathbf{y}_k + h_k \sum_{l=1}^{S} \alpha_{jl} \mathbf{f}_{kl} \tag{2}$$

$$\mathbf{f}_{kj} = \mathbf{f}\left[\mathbf{y}_{kj}, \mathbf{u}_{kj}, t_{kj}\right] \tag{3}$$

$$\mathbf{u}_{kj} = \mathbf{u}(t_{kj}) \tag{4}$$

$$t_{kj} = t_k + h_k \rho_j \tag{5}$$

$$h_k = t_{k+1} - t_k \tag{6}$$

S is referred to as the "stage", and the intermediate values of $\mathbf{y}_{kj}$ called *internal stages*. In these expressions, $\{\rho_j, \beta_j, \alpha_{jl}\}$ are known constants with $0 \leq \rho_1 \leq \rho_2 \leq \cdots \leq \rho_S \leq 1$. A common way to define the coefficients in to use the Butcher array

$$\begin{array}{c|ccc}
\rho_1 & \alpha_{11} & \cdots & \alpha_{1S} \\
\vdots & \vdots & & \vdots \\
\rho_S & \alpha_{S1} & \cdots & \alpha_{SS} \\
\hline
 & \beta_1 & \cdots & \beta_S
\end{array}$$

These schemes are called *explicit* if $\alpha_{jl} = 0$ for $l \geq j$ and *implicit* otherwise.

1.2 Variable Phase Length

For many optimal control problems, it's convenient to break the problem into *phases* either for numerical purposes or to describe different physical processes. In general, the length of a phase is defined by t_I and t_F. Therefore, define a time transformation

$$t = t_I + \tau(t_F - t_I) = t_I + \tau\sigma \tag{7}$$

where the phase length $\sigma = t_F - t_I$ and $0 \leq \tau \leq 1$. Thus for $\Delta\tau_k = (\tau_{k+1} - \tau_k)$ we have

$$h_k = (\tau_{k+1} - \tau_k)(t_F - t_I) = \Delta\tau_k \sigma \tag{8}$$

With this transformation

$$\mathbf{y}' = \frac{d\mathbf{y}}{d\tau} = \frac{d\mathbf{y}}{dt}\frac{dt}{d\tau} = \sigma\dot{\mathbf{y}} \tag{9}$$

and the original ODE becomes

$$\mathbf{y}' = \sigma\mathbf{f}\left[\mathbf{y}(\tau), \mathbf{u}(\tau), \tau\right] \tag{10}$$

1.3 Collocation Methods

Suppose we consider approximating the solution of the ODE by a function $\mathbf{z}(t)$, with components $z(t)$. As an approximation, let us use a polynomial of degree S (order $S+1$) over each step $t_k \leq t \leq t_{k+1}$:

$$z(t) = a_0 + a_1(t - t_k) + \cdots + a_S(t - t_k)^S \tag{11}$$

The coefficients $(a_0, a_1, \ldots, a_S)$ are chosen such that the approximation matches at the beginning of the step t_k, i.e.,

$$z(t_k) = y_k \tag{12}$$

and has derivatives that match at the internal stage points

$$\frac{dz(t_{kj})}{dt} = f\left[\mathbf{y}_{kj}, \mathbf{u}_{kj}, t_{jk}\right] = f_{kj} \tag{13}$$

Observe that within a particular step $t_k \leq t \leq t_{k+1}$ the parameter $0 \leq \rho \leq 1$ defines the *local* time parameterization $t = t_k + h_k\rho$ and so it follows that

$$z(t) = a_0 + a_1 h_k \rho_j + \cdots + a_S h_k^S \rho_j^S \tag{14}$$

and similarly from

$$\frac{dz(t)}{dt} = a_1 + \cdots + a_{S-1}(S-1)(t - t_k)^{S-2} + a_S S(t - t_k)^{S-1} \tag{15}$$

substitution gives

$$f_{kj} = a_1 + \cdots + a_{S-1}(S-1)h_k^{S-2}\rho_j^{S-2} + a_S S h_k^{S-1}\rho_j^{S-1} \tag{16}$$

The conditions shown in Eq. (13) are called *collocation* conditions and the resulting method is referred to as a *collocation method.*

The focus of a collocation method is on a polynomial representation for the different state variables. When the state is a polynomial of degree S over each step $t_k \leq t \leq t_{k+1}$ it is natural to use a polynomial approximation of degree $S-1$ for the algebraic variables $u(t)$, i.e.,

$$\nu(t) = b_0 + b_1(t - t_k) + \cdots + b_{S-1}(t - t_k)^{S-1} \tag{17}$$

for $j = 0, \ldots, S-1$ and the coefficients $(b_0, b_1, \ldots, b_{S-1})$ are determined such that the approximation matches at the intermediate points for $j = 1, \ldots, S$

$$\nu(t_{kj}) = \mathbf{u}_{kj} \tag{18}$$

1.3.1 Lobatto IIIA, S = 2

The simplest Lobatto IIIA method has two stages and is of order $\eta = 2$. It is commonly referred to as the *trapezoidal method.* The nonlinear programming constraints, called defects, and the corresponding NLP variables are as follows:

Defect Constraints:

$$\mathbf{0} = \boldsymbol{\zeta}_k = \mathbf{y}_{k+1} - \mathbf{y}_k - \frac{\Delta\tau_k}{2}\left[\sigma\mathbf{f}_k + \sigma\mathbf{f}_{k+1}\right] \tag{19}$$

variables

$$\mathbf{x} = \begin{pmatrix} \vdots \\ \mathbf{y}_k \\ \mathbf{u}_k \\ \mathbf{y}_{k+1} \\ \mathbf{u}_{k+1} \\ \vdots \\ \mathbf{p} \\ t_I \\ t_F \\ \vdots \end{pmatrix} \tag{20}$$

1.3.2 Lobatto IIIA, S = 3

There are three common forms when there are three stages all having order $\eta = 4$. We abbreviate the primary form LA3.

Primary Form
Defect Constraints

$$\mathbf{0} = \mathbf{y}_{k+1} - \mathbf{y}_k - \Delta\tau_k \left[\beta_1 \sigma \mathbf{f}_k + \beta_2 \sigma \mathbf{f}_{k2} + \beta_3 \sigma \mathbf{f}_{k+1}\right] \tag{21}$$

$$\mathbf{0} = \mathbf{y}_{k2} - \mathbf{y}_k - \Delta\tau_k \left[\alpha_{21} \sigma \mathbf{f}_k + \alpha_{22} \sigma \mathbf{f}_{k2} + \alpha_{23} \sigma \mathbf{f}_{k+1}\right] \tag{22}$$

where

$$\mathbf{f}_{k2} = \mathbf{f}\left[\mathbf{y}_{k2}, \mathbf{u}_{k2}, t_{k2}\right] \tag{23}$$

$$t_{k2} = t_k + h_k \rho_2 = t_k + \frac{1}{2} h_k \tag{24}$$

$$\mathbf{u}_{k2} = \mathbf{u}(t_{k2}) \tag{25}$$

Variables

$$\mathbf{x} = \begin{pmatrix} \vdots \\ \mathbf{y}_k \\ \mathbf{u}_k \\ \mathbf{y}_{k2} \\ \mathbf{u}_{k2} \\ \mathbf{y}_{k+1} \\ \mathbf{u}_{k+1} \\ \vdots \\ \mathbf{p} \\ t_I \\ t_F \\ \vdots \end{pmatrix} \tag{26}$$

Hermite-Simpson (Separated):
This method is referred to as *Hermite-Simpson (Separated)* or *Separated Simpson* and abbreviated HSS.

Defect Constraints

$$\mathbf{0} = \mathbf{y}_{k+1} - \mathbf{y}_k - \Delta\tau_k \left[\beta_1 \sigma \mathbf{f}_k + \beta_2 \sigma \mathbf{f}_{k2} + \beta_3 \sigma \mathbf{f}_{k+1}\right] \tag{27}$$

$$\mathbf{0} = \mathbf{y}_{k2} - \frac{1}{2}(\mathbf{y}_k + \mathbf{y}_{k+1}) - \frac{\Delta\tau_k}{8}(\sigma \mathbf{f}_k - \sigma \mathbf{f}_{k+1}) \tag{28}$$

Hermite-Simpson (Compressed):
This method is referred to as *Hermite-Simpson (Compressed)* or *Compressed Simpson* and is abbreviated HSC.

Defect Constraints

$$\mathbf{0} = \mathbf{y}_{k+1} - \mathbf{y}_k - \Delta\tau_k \left[\beta_1 \sigma \mathbf{f}_k + \beta_2 \sigma \mathbf{f}_{k2} + \beta_3 \sigma \mathbf{f}_{k+1}\right] \tag{29}$$

where

$$\mathbf{y}_{k2} = \frac{1}{2}(\mathbf{y}_k - \mathbf{y}_{k+1}) + \frac{h_k}{8}(\mathbf{f}_k - \mathbf{f}_{k+1}) \tag{30}$$

$$\mathbf{f}_{k2} = \mathbf{f}\left[\mathbf{y}_{k2}, \mathbf{u}_{k2}, t_{k2}\right] \tag{31}$$

$$t_{k2} = t_k + h_k \rho_2 = t_k + \frac{1}{2} h_k \tag{32}$$

$$\mathbf{u}_{k2} = \mathbf{u}(t_{k2}) \tag{33}$$

Variables

$$\mathbf{x} = \begin{pmatrix} \vdots \\ \mathbf{y}_k \\ \mathbf{u}_k \\ \mathbf{u}_{k2} \\ \mathbf{y}_{k+1} \\ \mathbf{u}_{k+1} \\ \vdots \\ \mathbf{p} \\ t_I \\ t_F \\ \vdots \end{pmatrix} \tag{34}$$

1.3.3 Lobatto IIIA, S = 4

This sixth order scheme is abbreviated LA4.

Defect Constraints

$$\mathbf{0} = \mathbf{y}_{k+1} - \mathbf{y}_k - \Delta\tau_k \left[\beta_1 \sigma \mathbf{f}_k + \beta_2 \sigma \mathbf{f}_{k2} + \beta_3 \sigma \mathbf{f}_{k3} + \beta_4 \sigma \mathbf{f}_{k+1}\right] \tag{35}$$

$$\mathbf{0} = \mathbf{y}_{k2} - \mathbf{y}_k - \Delta\tau_k \left[\alpha_{21} \sigma \mathbf{f}_k + \alpha_{22} \sigma \mathbf{f}_{k2} + \alpha_{23} \sigma \mathbf{f}_{k3} + \alpha_{24} \sigma \mathbf{f}_{k+1}\right] \tag{36}$$

$$\mathbf{0} = \mathbf{y}_{k3} - \mathbf{y}_k - \Delta\tau_k \left[\alpha_{31} \sigma \mathbf{f}_k + \alpha_{32} \sigma \mathbf{f}_{k2} + \alpha_{33} \sigma \mathbf{f}_{k3} + \alpha_{34} \sigma \mathbf{f}_{k+1}\right] \tag{37}$$

where

$$\mathbf{f}_{k2} = \mathbf{f}\left[\mathbf{y}_{k2}, \mathbf{u}_{k2}, t_{k2}\right] \tag{38}$$

$$t_{k2} = t_k + h_k \rho_2 \tag{39}$$

$$\mathbf{u}_{k2} = \mathbf{u}(t_{k2}) \tag{40}$$

$$\mathbf{f}_{k3} = \mathbf{f}\left[\mathbf{y}_{k3}, \mathbf{u}_{k3}, t_{k3}\right] \tag{41}$$

$$t_{k3} = t_k + h_k \rho_3 \tag{42}$$

$$\mathbf{u}_{k3} = \mathbf{u}(t_{k3}) \tag{43}$$

Variables

$$\mathbf{x} = \begin{pmatrix} \vdots \\ \mathbf{y}_k \\ \mathbf{u}_k \\ \mathbf{y}_{k2} \\ \mathbf{u}_{k2} \\ \mathbf{y}_{k3} \\ \mathbf{u}_{k3} \\ \mathbf{y}_{k+1} \\ \mathbf{u}_{k+1} \\ \vdots \\ \mathbf{p} \\ t_I \\ t_F \\ \vdots \end{pmatrix} \tag{44}$$

1.3.4 Lobatto IIIA, S = 5

This eighth order scheme is abbreviated LA5

Defect Constraints

$$\mathbf{0} = \mathbf{y}_{k+1} - \mathbf{y}_k - \Delta\tau_k \left[\beta_1 \sigma \mathbf{f}_k + \beta_2 \sigma \mathbf{f}_{k2} + \beta_3 \sigma \mathbf{f}_{k3} + \beta_4 \sigma \mathbf{f}_{k4} + \beta_5 \sigma \mathbf{f}_{k+1}\right] \tag{45}$$
$$\mathbf{0} = \mathbf{y}_{k2} - \mathbf{y}_k - \Delta\tau_k \left[\alpha_{21} \sigma \mathbf{f}_k + \alpha_{22} \sigma \mathbf{f}_{k2} + \alpha_{23} \sigma \mathbf{f}_{k3} + \alpha_{24} \sigma \mathbf{f}_{k4} + \alpha_{25} \sigma \mathbf{f}_{k+1}\right] \tag{46}$$
$$\mathbf{0} = \mathbf{y}_{k3} - \mathbf{y}_k - \Delta\tau_k \left[\alpha_{31} \sigma \mathbf{f}_k + \alpha_{32} \sigma \mathbf{f}_{k2} + \alpha_{33} \sigma \mathbf{f}_{k3} + \alpha_{34} \sigma \mathbf{f}_{k4} + \alpha_{35} \sigma \mathbf{f}_{k+1}\right] \tag{47}$$
$$\mathbf{0} = \mathbf{y}_{k4} - \mathbf{y}_k - \Delta\tau_k \left[\alpha_{41} \sigma \mathbf{f}_k + \alpha_{42} \sigma \mathbf{f}_{k2} + \alpha_{43} \sigma \mathbf{f}_{k3} + \alpha_{44} \sigma \mathbf{f}_{k4} + \alpha_{45} \sigma \mathbf{f}_{k+1}\right] \tag{48}$$

where

$$\mathbf{f}_{k2} = \mathbf{f}\left[\mathbf{y}_{k2}, \mathbf{u}_{k2}, t_{k2}\right] \tag{49}$$
$$t_{k2} = t_k + h_k \rho_2 \tag{50}$$
$$\mathbf{u}_{k2} = \mathbf{u}(t_{k2}) \tag{51}$$
$$\mathbf{f}_{k3} = \mathbf{f}\left[\mathbf{y}_{k3}, \mathbf{u}_{k3}, t_{k3}\right] \tag{52}$$
$$t_{k3} = t_k + h_k \rho_3 \tag{53}$$
$$\mathbf{u}_{k3} = \mathbf{u}(t_{k3}) \tag{54}$$
$$\mathbf{f}_{k4} = \mathbf{f}\left[\mathbf{y}_{k4}, \mathbf{u}_{k4}, t_{k4}\right] \tag{55}$$
$$t_{k4} = t_k + h_k \rho_4 \tag{56}$$
$$\mathbf{u}_{k4} = \mathbf{u}(t_{k4}) \tag{57}$$

Variables

$$\mathbf{x} = \begin{pmatrix} \vdots \\ \mathbf{y}_k \\ \mathbf{u}_k \\ \mathbf{y}_{k2} \\ \mathbf{u}_{k2} \\ \mathbf{y}_{k3} \\ \mathbf{u}_{k3} \\ \mathbf{y}_{k4} \\ \mathbf{u}_{k4} \\ \mathbf{y}_{k+1} \\ \mathbf{u}_{k+1} \\ \vdots \\ \mathbf{p} \\ t_I \\ t_F \\ \vdots \end{pmatrix} \tag{58}$$

1.3.5 Quadrature Equations

The IRK methods provide a way to solve ODE's. When dealing with problems involving integral expressions such as

$$\mathcal{I} \int_{t_I}^{t_F} \mathbf{w}\left[\mathbf{y}(t), \mathbf{u}(t), t\right] dt \tag{59}$$

it's common to introduce new dynamic variables $\mathbf{r}(t)$ and then solve the following augmented system:

$$\dot{\mathbf{y}} = \mathbf{f}\left[\mathbf{y}(t), \mathbf{u}(t), t\right] \tag{60}$$
$$\dot{\mathbf{r}} = \mathbf{w}\left[\mathbf{y}(t), \mathbf{u}(t), t\right] \tag{61}$$

in conjunction with the initial condition $\mathbf{r}(t_I) = 0$. It then follows that

$$\mathbf{r}(t_F) = \mathcal{I} \tag{62}$$

If we apply a recursive scheme to the augmented system we can write

$$\mathbf{r}(t_F) = \mathbf{r}_M = \sum_{k=1}^{M-1} \left(\mathbf{r}_{k+1} - \mathbf{r}_k\right) \tag{63}$$

It then follows that

$$\mathbf{r}_{k+1} - \mathbf{r}_k = \begin{cases} \Delta\tau_k \left[\beta_1 \sigma \mathbf{w}_k + \beta_2 \sigma \mathbf{w}_{k+1}\right] & S = 2 \\ \Delta\tau_k \left[\beta_1 \sigma \mathbf{w}_k + \beta_2 \sigma \mathbf{w}_{k2} + \beta_3 \sigma \mathbf{w}_{k+1}\right] & S = 3 \\ \Delta\tau_k \left[\beta_1 \sigma \mathbf{w}_k + \beta_2 \sigma \mathbf{w}_{k2} + \beta_3 \sigma \mathbf{w}_{k3} + \beta_4 \sigma \mathbf{w}_{k+1}\right] & S = 4 \\ \Delta\tau_k \left[\beta_1 \sigma \mathbf{w}_k + \beta_2 \sigma \mathbf{w}_{k2} + \beta_3 \sigma \mathbf{w}_{k3} + \beta_4 \sigma \mathbf{w}_{k4} + \beta_5 \sigma \mathbf{w}_{k+1}\right] & S = 5 \end{cases} \tag{64}$$

2 Nonlinear Programming

The general nonlinear programming (NLP) problem can be stated as follows: Find the n-vector $x^T = (x_1, \ldots, x_n)$ to minimize the scalar objective function

$$F(\mathbf{x}) \tag{65}$$

subject to the m constraints

$$\mathbf{c}_L \leq \mathbf{c}(\mathbf{x}) \leq \mathbf{c}_U \tag{66}$$

and simple bounds

$$\mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U \tag{67}$$