An Analysis of Hare and Lynx Population Dynamics

Master of Data Science - Strategic Leadership

Summative Report

Year of Study: 2023/24

INTRODUCTION

Predator-prey relationships often involve multiple predators and prey, presenting a difficult approach when modelling the life cycles between them. However, Canadian lynx and Snowshoe hares are one of the most tightly linked relationships observed, as the lynx are the main predators of hares, and hares represent the main prey for lynx (1). Up to 96% of the Canadian lynx's diet is Snowshoe hares (2). Whilst lynx do eat alternative prey when hares are scarce (such as mice, squirrels and voles), these food sources are insufficient in providing them with the nutrients they need. As such, the populations of both species are hugely interdependent, with changes in either population directly affecting the population of the other (3). This relationship allows us to analyse the population dynamics between these two species.

To help us quantitatively analyse this relationship, we will use data on the number of lynx and hare pelts collected by the Hudson Bay Company in our project. This is a fur trade company, in which the recorded pelts can be used to approximate the hare and lynx populations (4). An initial analysis of the pelts collected reveals that the populations of these species follow a cycle of around 8-11 years.

The peak of this cycle is reached when the hares reach a density of around 1500 hares per km squared, which is too large of a capacity for the environment to support. This leads to insufficient resources for the hares, resulting in a decline in the population, which is also exacerbated by predation by lynxes. The hare population stabilises for several years when it reaches a low level, and as vegetation and food sources replenish, the population begins to increase again. Hares breed at a fast rate as they produce several litters each year, so the population increases at a faster rate as breeding occurs, reaching higher densities once more. The population remains high for a year or two before the cycle repeats itself (3).

The lynx population follows this cycle closely. As the hare population approaches the carrying capacity and faces limitations such as lack of space and increased competition for resources, the population of lynxes remains stable as they continue to thrive with the abundant food supply of hares. However, this is not sustainable for long, as the decreasing number of hares causes the lynx population to decline as their main food source becomes limited. Without the nutrients they need, breeding amongst the lynxes becomes drastically reduced, as their weakened condition means that fewer females breed and any litter that is produced tends to be smaller. For a period of three to five years, the lynx population remains low. This decline in the lynx population alleviates the predation pressure on hares, allowing the hare population to recover as their resources replenish. This cycle continues as the recovering population provides a new opportunity for the lynx population to grow, causing a recurring oscillation between high and low numbers of hares and lynxes, in which the lynx oscillations are dependent on the hare population (3).

This project will attempt to create a model to represent this approximate 10-year cycle for hare and lynx populations so that the effects of different variables on this

population cycle can be modelled and analysed. For instance, this model will be able to predict how the occurrence of human hunting of lynxes, as well as the presence of a competitive predator for the hares, affects the population numbers.

1. MODELLING ASSUMPTIONS

We propose our model to include the following variables (we use these in section 2.3):

a = Lynx/Hare Predation Rate , b = Lynx/Hare Conversion , c = Lynx/Hare Michaelis Constant , d = Lynx Death Rate , k = Hare Carrying Capacity and r = Hare Reproduction Rate
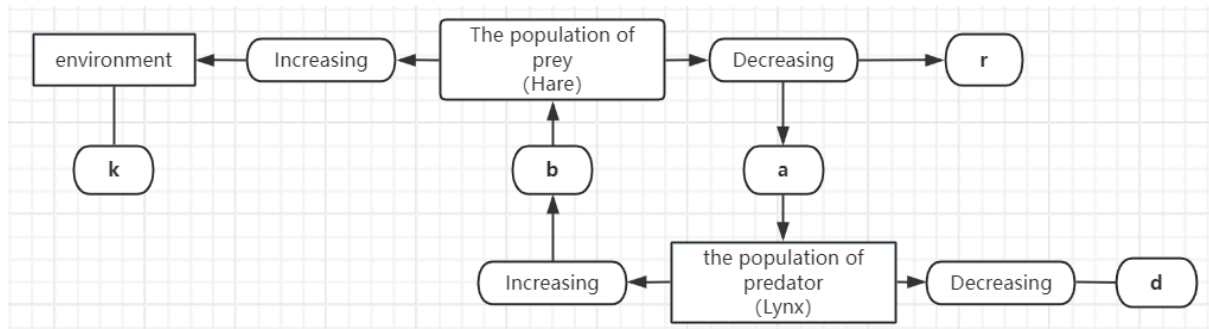

Figure 1: Flow chart explaining the relationship

Figure 2 saw the relatively regular periodic changes, Therefore, we will further amend this model to produce stable limit cycles in order to better fit reality.
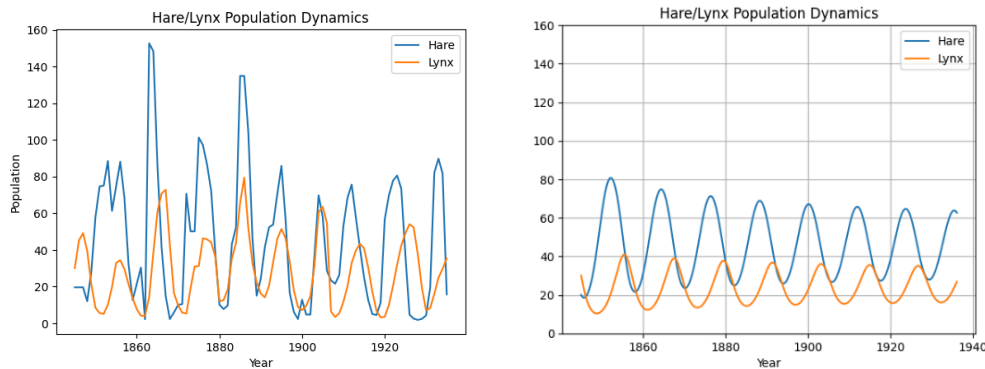

Figure 2: The changes in species numbers between 1845 and 1935 and our fitted model

Before we start to formulate our model (see section 2.3) we make various assumptions about our model.

Firstly, we assume that the parameters {a, b, c, d, k, r} are constant over time:

$$\frac{dr}{dt} = \frac{da}{dt} = \frac{db}{dt} = \frac{dc}{dt} = \frac{dk}{dt} = \frac{dd}{dt} = 0$$

This assumption simplifies the analysis of the model as it allows us to derive more straightforward and stable results. However, in reality, parameters may change over time due to various factors - for instance, carrying capacity k, can change with changing environments such as resource availability and damage to the habitat.

Secondly, we assume that there is no emigration/immigration of lynx or hares. Since the rate of change of the hare population (dH/dt) and the rate of change of the lynx population (dL/dt) only include death, birth, predation rate, and carrying capacity, our model assumes that no additional input or output will occur which isn't entirely realistic. To go further, we assume that the members of the predator and prey population randomly mix through their habitat - are homogeneous.

Thirdly, we assume that our function H(t) for the population of hares over time and function L(t) for a population of lynxes over time are continuous. Therefore, we lose stochasticity which can be seen in Figure 2, instead, we have perfect curves and symmetrical structure which is unrealistic.

We also assume that predators are entirely dependent on the prey for their food. Without prey, predators starve and their population declines.

Finally, we assume that the growth rate of hares is logistic and slows down as it approaches carrying capacity k but is not limited by food and other resources that are in ample supply, this approach is often used, however, it oversimplifies the complexities of real ecosystems where various factors influence population growth and interactions. Consequently, in the absence of predators, the prey population will grow logistically until it reaches carrying capacity k.

2. SIMULATE
2.1 Traditional models
If we consider only the growth of hares as a time-dependent growth of a species' function $x(t)$. If we assume the food supply is unlimited then it would be reasonable to assume the growth of hares is proportional to the population size of hares minus the death rate of hares by natural causes which is proportional to the hare's population i.e.:

$$\frac{dx}{dt} = \alpha x - \beta x \Rightarrow x(t) = Ae^{(\alpha-\beta)t}.$$

Given the growth rate of hares is given by a, if we assume a > 0, the death rate of hares β should be less than α and A is the population at x(0) and these coefficients are constant. We would see the population of hares grow exponentially without bound over time.

In reality, this is not the case as the population of hares is inhibited by predation by lynxes. Therefore, we model a new differential equation y(t) to represent the population of lynxes. In this case, the hare population will decrease proportionally to the number of lynxes y multiplied by the number of hares x. Additionally, the lynxes eat hares, hence, the growth rate of hares α is proportional to the population of lynxes y multiplied by the population of hares x. We assume the death rate of hares is simply due to natural causes therefore the death rate δ is simply proportional to the population of lynxes.

Hence, we get equilibria: $\frac{dx}{dt} = ax - \beta xy, \frac{dy}{dt} = \epsilon\beta xy - \delta y$

$(x^*, y^*) = (0, 0), \left(\frac{\delta}{\epsilon\beta}, \frac{a}{\delta}\right)$ (i.e.when dx/dt = 0, dy/dt = 0 and dx/dy = 0, see Appendix C)

$$J = \begin{bmatrix} a - \beta y & -\beta x \\ \epsilon\beta y & \epsilon\beta x - \delta \end{bmatrix}$$

$$J_{(0,\,0)} = \begin{bmatrix} a & 0 \\ 0 & -\delta \end{bmatrix} \implies \lambda_1 = a > 0 \, , \lambda_2 = -\delta < 0$$

$$J_{\left(\frac{\delta}{c\beta},\frac{a}{\delta}\right)} = \begin{bmatrix} 0 & \frac{-\delta}{\epsilon} \\ \epsilon a & 0 \end{bmatrix} \implies \det(J - \lambda I) = 0 \implies \lambda = \pm i\sqrt{(a\delta)} \, , Re(\lambda) = 0 \implies centre$$

trace(J) = 0 + 0 = 0 $\quad \det(J) = ad > 0 \implies centre$

With zero real parts at our non-trivial eigenvalue, the equilibrium system is neutrally stable. This means that small perturbations from the equilibrium cause the system not to converge or diverge but instead lead to periodic orbits

2.2 More advanced model

Let us propose a new system extending the classic predator-prey model:

$$\frac{dH}{dT} = r \cdot H \left(1 - \frac{H}{k}\right) - \frac{a \cdot H \cdot L}{c + H} \quad , \quad \frac{dL}{dt} = \frac{b \cdot a \cdot H \cdot L}{c + H} - d \cdot L$$

incorporate logistic growth of Hares by introducing a carrying capacity k to account for limited resources that may inhibit the growth rate of Hares. Additionally, we introduce a saturation effect on predation (c + H) reflecting environmental limits on their growth. Incorporating logistic growth for the prey carrying capacity and saturated decline for the predators introduces additional nonlinear terms that can help improve the stability of the system. The system has the following differential equations: (See B:3 for parameters)

Let us prove this system is more stable. The nullclines for Hares are where:

$$\frac{dH}{dt} = r \cdot H \cdot \left(1 - \frac{H}{k}\right) - \frac{a \cdot H \cdot L}{c + H} = 0 \rightarrow \begin{cases} H^* = 0 \\ L^* = \frac{r}{a} \cdot (c + H) \cdot \left(1 - \frac{H}{k}\right) \end{cases}$$

The nullclines for Lynx are where:

$$\frac{dL}{dt} = b\frac{a \cdot H \cdot L}{c + H} - d \cdot L = 0 \rightarrow \begin{cases} L^* = 0 \\ H^* = \frac{c \cdot d}{a \cdot b - d} \end{cases}$$

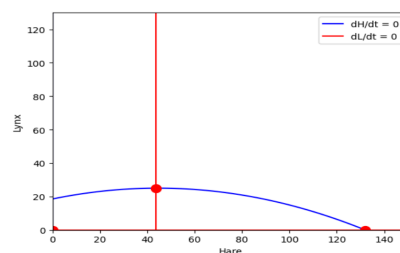We obtained three equilibrium points through the equations.

Figure 5: Nullclines

$$J = \begin{vmatrix} f'_H & f'_L \\ g'_H & g'_L \end{vmatrix} = \begin{vmatrix} r - \frac{2 \cdot r \cdot H}{k} + \frac{L \cdot a \cdot (c+H) - H \cdot L \cdot a}{(c+H)^2} & -\frac{a \cdot H}{c+H} \\ \frac{L \cdot a \cdot b \cdot (c+H) - H \cdot L \cdot a \cdot b}{(c+H)^2} & \frac{a \cdot b \cdot H}{c+H} - d \end{vmatrix}$$

$$J_{(0,0)} = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix} \implies \lambda_1 = r > 0, \lambda_2 = -d < 0$$

The equilibrium at the point (0,0) is unstable; for the point $(\frac{c \cdot d}{a \cdot b - d}, \frac{r}{a}$

$(c + H) \bullet (1 - \frac{H}{k}))$, it lies precisely at a critical point, making it challenging to determine stability. We will analyse this problem using the phase trajectories below. Dividing the first quadrant into four parts using the straight lines $H = \frac{c \cdot d}{a \cdot b - d}$ and

$L = \frac{r}{a}(c + H) \bullet (1 - \frac{H}{k})$, the signs of $\frac{dH}{dt}$ and $\frac{dL}{dt}$ within each section become clear:
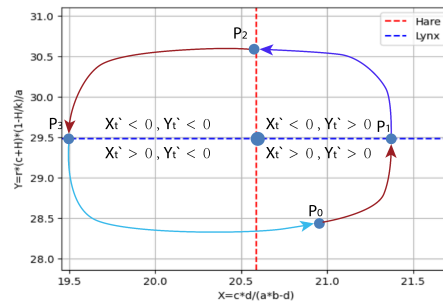

Figure 6: H-L Phase Trajectory Analysis

2.3 Find the stability of the coexistence equilibrium
For better calculation, we make the variables dimensionless (see Appendix C for full details):

$$\frac{dH}{dt} = r * H * \left(1 - \frac{H}{k}\right) - \frac{a * H * L}{c + H} \qquad \frac{dL}{dt} = \frac{b * a * H * L}{c + H} - d * L$$

Let $x = \frac{H}{k}$, $y = \frac{L}{c}$, $t' = r*t$,

Substituting these into the given equation, we get:

$$\frac{dx}{dt'} = x(1 - x) - \frac{a \bullet x \bullet y}{1+x} \qquad \frac{dy}{dt'} = \frac{b \bullet a \bullet x \bullet y}{1+x} - d \bullet y$$

To continue simplifying the equation, we let $f(x) = \frac{x}{1+x}$ and $g(x) = (1+x) \bullet (1-x)$, so we get.

$$\frac{dx}{dt'} = f(x)[g(x) - ay] \qquad \frac{dy}{dt'} = (b \bullet a \bullet f(x) - d) \bullet y$$

There are three equilibria for this system: (0,0), (k,0), $(\frac{d}{b \bullet a - d}, \frac{(1+x)(1-x)}{a})$

Therefore, at the coexistence equilibrium, $(x^*, g(x^*))$, the Jacobian is:

$$J = \begin{vmatrix} f(x) \cdot g'(x) & -a \cdot f(x) \\ b \cdot a \cdot y \cdot f'(x) & 0 \end{vmatrix} \rightarrow \begin{vmatrix} \frac{d}{b \cdot a} \cdot g'(x) & -\frac{d}{b} \\ b \cdot a \cdot g(x) \cdot f'(x) & 0 \end{vmatrix}$$

From the Routh-Hurwitz criteria, we know that the coexistence equilibrium is stable only if trace<0 and det(J)>0.

$$\text{Tr}(J) = \frac{d}{b \cdot a} \cdot g'(x) \qquad \Delta = d \cdot a \cdot g(x) \cdot f'(x)$$

since d, a>0 and $f'(x) = \frac{1}{(1+x)^2} > 0$, the second condition holds whenever $y^* > 0$.

so det(J)>0. The coexistence equilibrium is stable if $g'(x\ *) < 0$, and is unstable if $g'(x\ *) > 0$. Moreover, if $g'(x\ *) = 0$ then tau=0 and $\triangle$>0, we have imaginary eigenvalues, and so we expect complex eigenvalues when $g'(x\ *) \approx 0$

Therefore, we have a stable cyclic dynamic between the hares and lynx populations and no indication of chaotic behaviour as categorised with the classic predator-prey equation (see Figure 10B in Appendix B) which shows the phase plot with limit cycles indicating a stable solution.

3. Identify a problem in the world
3.1. Visualising the data
We approximate the population of hares and lynxes at time t as continuous functions H(t) and P(t) using Python's numerical integration technique "odeint" or other methods such as the Euler Method. After the "odeint" function is called it returns an array "sol" that contains the numerical solutions of the ODE system. We use this approximation to plot a 2D diagram of the dynamic between the population of hares and lynxes.

3.2 Fit model to real-life data
In the context of the differential equations modelling of predator and prey dynamics, fitting the parameters to our context is a crucial step if we want to get meaningful insight from our data. Consequently, fitting our model to the observed data of Hares and lynxes in the Hudson Bay region, we can more accurately represent the dynamics of the actual predator-prey system that we are studying and hopefully capture the nuances of the interaction of hares and lynxes and their effect on population sizes.

The fitting process involves finding parameters (r, k, a, b, c, d) that minimise the sum of the squared residuals e_i^2, calculated as: $\text{minimize} \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$

where n is the number of data points, y_i is the observed population at the time i and $\hat{y}_i$ is the corresponding simulated population. This is our objective function.

We use Python's "curve_fit" function to minimise the objective function e_i^2 specified above. By default, it uses the Lavenberg-Marquadt algorithm, a popular method for solving nonlinear least-squares problems. We estimate the parameters (r, k, a, b, c, d) to be (0.88, 131.98, 2.055, 0.99, 43, 1.03) respectively.

We obtain a fairly accurate fit with the fitted values vs residuals for both hares and lynxes centring around 0 and appear to be normally distributed. We also find the initial_params, final_params = (148.793.56, 46,523.73), therefore, we have a substantial improvement in the fit of the model to our data. The p-value of the

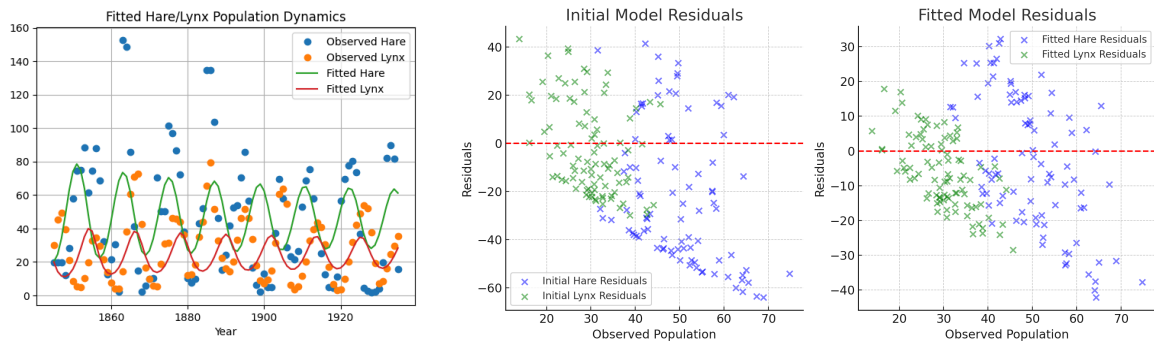Shapiro test is greater for our final_params suggesting better fit.

Figure 7: Parameter comparison between past and present

Now that we have modelled the relationship between this predator-prey relationship, and figured out how they keep stable long-term oscillatory dynamics. We are interested to see the external effects on the predator, such as hunting lynx. Then, we try to adjust the already established model to predict the possible changes.

3.2 Get insight from our model

$$\frac{dH}{dT} = r \cdot H \left(1 - \frac{H}{k}\right) - \frac{a \cdot H \cdot L}{c + H} \quad , \quad \frac{dL}{dt} = \frac{b \cdot a \cdot H \cdot L}{c + H} - d \cdot L$$

To get insight into the behaviour of lynx and hare population dynamics we will alter our fitted parameters { r, k, a, c, b, a, d} (see figure B:3) to stimulate new situations. This way we can try to model what effect changes in population dynamics of hares and lynxes will have on their numbers in a real environment (e.g. Hudson Bay). We have also used this model to predict future populations of hares and lynxes over 10 years (see Appendix A:2).

We will model the following situations:
1. Increase the death rate d to simulate a situation where the number of lynxes dying increases due to the lynxes being hunted for fur.
2. Set the initial population of lynxes to 0 i.e. (L(0)) to determine what happens when there is no predation of hares.
3. Change the carrying capacity to observe how alteration in carrying capacity affects the dynamics of the predator-prey system.

3.3 Analysis

Situation 1
The parameter d is a constant that represents the death rate of lynxes. By altering d, we can model the increased death rate of lynxes. If we assume the increased death rate of lynxes is due to events such as humans hunting lynxes for fur, then we can determine how many lynxes can be hunted before it causes their extinction.
Let us increase the death rate of lynxes dramatically, say the new death rate is double the original death rate. We model the result in Figure 8, and then downward pressure on the lynxes causes their population to decrease exponentially. The

decrease in lynx population results in less predation of hares by lynxes and as a result the hares grow exponentially slowing down as they reach the carrying capacity – the maximum number of hares the environment can support without predation. Therefore, we can deduce that if we hunt lynxes at twice the rate then the population will die out.

When we decrease the population slightly, we observe Figure B:12. We see an oscillation behaviour typical of predator-prey models, but with Lynxes trending downwards and eventually stabilising. The trend of the lynx downwards and their stabilisation suggests the system is reaching a new equilibrium where the reduced number of lynxes can no longer effectively control the hare population. However, this is likely not representative of reality as we make a few flawed assumptions, and a more well-suited model would be needed to monitor the dynamics of hares with human intervention. This is because we assume that the death of lynxes by human intervention is constant, however, this is not the case, the hunting rate of Lynxes would be dependent on several factors, including the human density in the area. In our future research, we propose adding a new differential equation to model the population dynamics of hares so that we can have a more complex model that includes additional data that more accurately models the population dynamics.

Situation 2

Figure 9 shows the result of our model when the initial population of lynxes is zero. It should be noticed that the hares increase exponentially and stabilise at our carrying capacity k. This is expected with our background research and given the assumptions instated with our model. The hares don't suffer from predation and as a result, can grow without a bound until they reach a point where they are inhibited by other resources e.g. the amount of space.

Situation 3

The carrying capacity (k) is the maximum amount of prey that can be supported without predation (at Hudson Bay). In Figure 11, with a lower k of 100, the hare population peaks at lower levels, which insufficiently supports the lynx population and causes them as there is increased competition for hares by lynxes. Conversely, in Figure 10, an increased k of 200 allows for higher peaks in the hare population, this can sustain a larger lynx population. Thus, a higher carrying capacity allows both the predator and prey populations to attain greater levels. However, the actual carrying capacity of an environment for hares involves many factors such as availability of food, which we assume is unlimited in our model. Furthermore, in real-life situations, the carrying capacity would also not be fixed between 100 and 130 with changes in environmental conditions and other ecological factors affecting the carrying capacity.
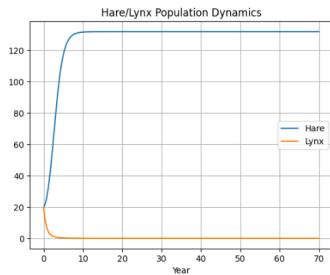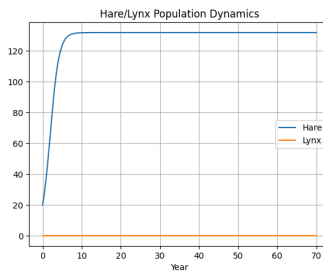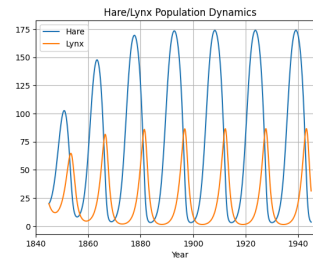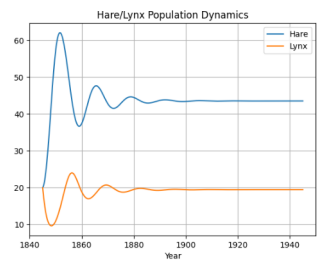
Figure 8

Figure 9

Figure 10

Figure 11

CONCLUSIONS
Results

In our analysis of the population dynamics of Hares and Lynx we successfully devised a predator-prey model and deduced by analysis of its equilibria points that it has structurally stable limit cycles. Structurally stable limit cycles help ensure our model has robust and predictable long-term behaviour. We successfully determined the model's parameters by finding parameters that minimise the squared difference between our historical data and our model's predicted value (See Appendix B:3 for our parameters). This allowed us to obtain parameters specific to the focus of our research, the population dynamic of hares and lynxes in the Hudson Bay region.

Using these parameters, we predicted the future population of lynxes and hares over ten-year periods (their oscillation cycles). These projections can be seen in Appendix A:2. These projections had the expected cyclical oscillation that aligns with our research in section 1.

In scenario simulations, we explored the impact of modifying our fitted parameters to model the effect of various ecological changes on the populations of hares and lynxes in the Hunson Bay area.

First, we discovered that doubling the lynx death rate resulted in significant shifts in the population dynamics of hares and lynxes. We found that It led to the population of the lynxes decaying exponentially, consequently, the hare populations increased logistically to reach their carrying capacity (see Figure 8 for our results).

Our simulations also identified a carrying capacity for hares between 100 and 130 (Situation 3, see Figures 10 and 11). When the carrying capacity is less than 100, hare populations peak at levels insufficient to sustain lynxes, potentially causing periodic decline in the lynx population. Alternatively, when the carrying capacity was above 130, hare populations peaked at higher levels and as a result, the lynxes also experienced higher population peaks.

Furthermore, we identified equilibrium points in the model where the populations of both species remained stable (see section 2.3), this insight is key in our future efforts to determine sustainable population levels with varying ecosystems.

Importance of the Model and Future Strategies

Our success at implementing a model that has structurally stable limit cycles lends itself well to the success of our model as it indicates that the predicted populations of hares and lynxes exhibit oscillations that have a predictable pattern. This feature is essential for the success of our model at predicting future population dynamics of hares and lynxes that align and are consistent with the actual population of hares and lynxes in the Hudson Bay region.

The outcomes derived from our model highlight the substantial dependence of lynxes on a substantial environmental carrying capacity to ensure an abundant prey supply. The practical significance of this observation emphasises the need for a conservation-centric approach when assessing the implications for lynxes. Our model indicates that alterations in carrying capacity, whether induced by human activities or influenced by climate change, may precipitate significant shifts in the population cycles between these two species. This shows the importance of proactive conservation measures to maximise the carrying capacity for hares, as this allows both populations to become stable.

Limitations

It must be noted that our model is an extremely simplified mathematical model that assumes that the environment and interactions in it are all not influenced by other species or external factors. In real life, the interactions can prove to be a lot more complex, thus, the result of our model should be taken with a grain of salt

This model assumes that the parameters are constant. This is troublesome in specific contexts, for example, the carrying capacity k can change when there is a change in the environment such as climate change. If we extrapolate our data into the future, our model does not account for the change in carrying capacity due to the environment. Therefore, we would expect significant variation of our predicted result from reality.

Furthermore, we model the impact of human intervention in situation 1, however, we fail to account for the varying population size of humans and as a result the varying rate of lynxes being hunted for fur. Instead, we assume that the hunting rate of lynxes for fur is constant which is an extreme oversimplification on our end. This may contribute to the further variation between reality and our predicted result.

These limitations highlight the importance of caution when interpreting the model results.

Further Applications of the Model

We believe our model has the potential to give huge insight into the population dynamics if we explore the nullclines and eigenvectors to see the behaviour of our model for our fitted model with varying parameters to model specific situations (see section 2.3). In the future, we aim to use our model and the identified equilibrium points where the populations of both species are constant to get an insight into the predator-prey dynamics for specific parameters. More analysis of the equilibrium can help determine if certain scenarios cause chaotic behaviour and how stable the model is at specific parameters. Further analysis of the eigenvectors can help determine the local stability of equilibrium points, e.g. if small disturbances will die over time or grow indefinitely.

Our model can be used to assess ecological balance by calling historical population numbers to assess current populations, such as whether the lynx are endangered. However, these estimates are based on more sophisticated models.

Therefore, in the future, we will further consider the impact of climate change, human activities, and the migration of species. For example, global warming has a great impact on winter snowfall in Canada. Lynx may therefore lose their hunting advantage, and white-furred hares will also be more vulnerable to prey in winters with less snowfall. We can simulate this situation and predict the population changes of lynx and hares, to carry out targeted protection.

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APPENDIX A

(Figure 1:Actual Result)

Year	Hare	Lynx
1845	19.58	30.09
1846	19.60	45.15
1847	19.61	49.15
1848	11.99	39.52
1849	28.04	21.23
1850	58.00	8.42
1851	74.60	5.56
1852	75.09	5.08
1853	88.48	10.17
1854	61.28	19.60
1855	74.67	32.91
1856	88.06	34.38
1857	68.51	29.59
1858	32.19	21.30
1859	12.64	13.69
1860	21.49	7.65
1861	30.35	4.08
1862	2.18	4.09
1863	152.65	14.33
1864	148.36	38.22
1865	85.81	60.78
1866	41.41	70.77
1867	14.75	72.77
1868	2.28	42.68
1869	5.91	16.39
1870	9.95	9.83
1871	10.44	5.80
1872	70.64	5.26

1873	50.12	18.91
1874	50.13	30.95
1875	101.25	31.18
1876	97.12	46.34
1877	86.51	45.77
1878	72.17	44.15
1879	38.32	36.33
1880	10.11	12.03
1881	7.74	12.60
1882	9.67	18.34
1883	43.12	35.14
1884	52.21	43.77
1885	134.85	65.69
1886	134.86	79.35
1887	103.79	51.65
1888	46.10	32.59
1889	15.03	22.45
1890	24.20	16.16
1891	41.65	14.12
1892	52.34	20.38
1893	53.78	33.33
1894	70.40	46.00
1895	85.81	51.41
1896	56.69	46.43
1897	16.59	33.68
1898	6.16	18.01
1899	2.30	8.86
1900	12.82	7.13
1901	4.72	9.47
1902	4.73	14.86
1903	37.22	31.47
1904	69.72	60.57

1905	57.78	63.51
1906	28.68	54.70
1907	23.37	6.30
1908	21.54	3.41
1909	26.34	5.44
1910	53.10	11.65
1911	68.48	20.35
1912	75.58	32.88
1913	57.92	39.55
1914	40.97	43.36
1915	24.95	40.83
1916	12.59	30.36
1917	4.97	17.18
1918	4.50	6.82
1919	11.21	3.19
1920	56.60	3.52
1921	69.63	9.94
1922	77.74	20.30
1923	80.53	31.99
1924	73.38	42.36
1925	36.93	49.08
1926	4.64	53.99
1927	2.54	52.25
1928	1.80	37.70
1929	2.39	19.14
1930	4.23	6.98
1931	19.52	8.31
1932	82.11	16.01
1933	89.76	24.82
1934	81.66	29.70
1935	15.76	35.40

(Figure 2:Predicted Result)

Year	Hare Population	Lynx Population
1850	62.34877661	11.69132563
1860	31.06784659	13.4923238
1870	23.65479526	24.75405234
1880	37.22733084	36.85117928
1890	59.42444547	32.79430382
1900	66.97289234	22.81995548
1910	56.092062	16.77081864
1920	39.00634376	16.01686844
1930	28.96575013	20.79446569
1940	30.77964712	29.74659157
1950	44.07072304	34.38740564
1960	58.43420146	29.60748247
1970	61.46140677	22.28913602
1980	51.97992717	17.78758477
1990	38.56457861	17.46919885
2000	30.57162096	21.72043767
2010	32.19206102	29.18427495
2020	43.33876687	33.30603965
2030	56.05718251	29.68029808

APPENDIX B

Parameter	Value
Hare Reproduction Rate (r)	0.8861
Hare Carrying Capacity (k)	131.9805
Lynx/Hare Predation Rate (a)	2.0543
Lynx/Hare Conversion (b)	0.9905
Lynx/Hare Michaelis Constant (c)	42.9626
Lynx Death Rate (d)	1.0246

Table 1: Parameter values for the model.

(Figure 3:Parameter values)
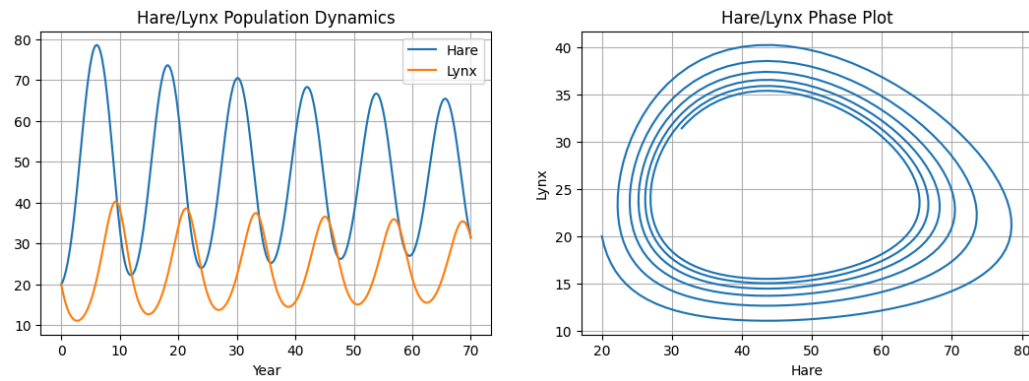
(Figure 10: Time series plot and Phase Plot)
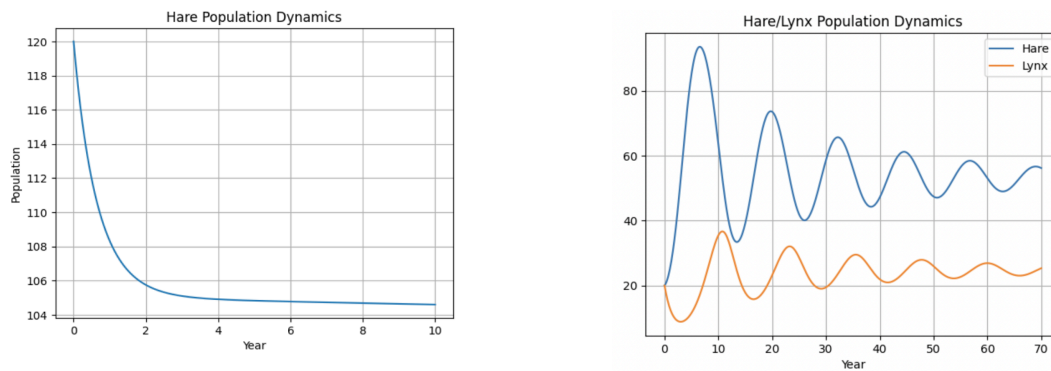
(Figure 11)

(Figure 12)

The link to our Python code:

https://colab.research.google.com/drive/149F4cFUvI8-D-C6sGe88V8AqRdtF_vvF

APPENDIX C

(a)

$$\frac{dx}{dt} = 0 \implies x = 0 \implies a - \beta y = 0 \implies y = \frac{a}{\beta}$$

$$y = \frac{a}{\beta} > 0, \ \frac{dy}{dt} = 0 \rightarrow \epsilon\beta xy - \delta y = \epsilon\beta x - \delta \implies x = \frac{\delta}{\epsilon\beta}$$

Hence, we get equilibria (x, y) = $(0, 0)$ or $\left(\frac{\delta}{\epsilon\beta}, \frac{a}{\delta}\right)$

(b) the full detail of find the stability of the coexistence equilibrium

For better calculation, we make the variables dimensionless:

$$\frac{dH}{dt} = r * H * \left(1 - \frac{H}{k}\right) - \frac{a * H * L}{c + H}$$

$$\frac{dL}{dt} = \frac{b * a * H * L}{c + H} - d * L$$

Let x=H/k, y=L/c, t`=r*t

Substituting these into the given equation, we get:

$$\frac{dx}{dt'} = x(1 - x) - \frac{a \bullet x \bullet y}{1+x} \qquad \frac{dy}{dt'} = \frac{b \bullet \alpha \bullet x \bullet y}{1+x} - d \bullet y$$

Here, α=a/k and β=b/c, this is the dimensionless equation, where t` is the new dimensionless time.

$$\frac{dx}{dt'} = \frac{x}{1+x} \bullet ((1 + x) \bullet (1 - x) - a \bullet y$$

$$\frac{dy}{dt'} = \frac{x}{1+x} \bullet (b \bullet \alpha \bullet y) - d \bullet y$$

To continue simplifying the equation, we let f(x)=x/(1+x) and g(x)=(1+x)·(1-x),so we get.

$$\frac{dx}{dt'} = f(x)[g(x) - ay]$$

$$\frac{dy}{dt'} = (b \bullet a \bullet f(x) - d) \bullet y$$

The nullclines for hare:

$$f(x) = 0 \rightarrow \frac{x}{1+x} = 0 \rightarrow x = 0 \qquad g(x) = ay \rightarrow y = \frac{(1+x)(1-x)}{a}$$

The nullclines for lynx:

$$y = 0 \qquad x = \frac{d}{b \cdot a - d}$$

There are three equilibria for this system: (0,0), (k,0), (d/(b·a-d), ((1+x)(1-x))/a)

The Jacobian J, of the system is:

$$J = \begin{vmatrix} f(x) \cdot g'(x) + f'(x) \cdot (g(x) - ay) & -a \cdot f(x) \\ b \cdot a \cdot y \cdot f'(x) & b \cdot a \cdot f(x) - d \end{vmatrix}$$

Like we mention before,g(x)-a·y=0 and b·a·f(x)-d=0

Therefore, at the coexistence equilibrium,(x^*,g(x^*)), the Jacobian is:

$$J = \begin{vmatrix} f(x) \cdot g'(x) & -a \cdot f(x) \\ b \cdot a \cdot y \cdot f'(x) & 0 \end{vmatrix} \rightarrow \begin{vmatrix} \frac{d}{b \cdot a} \cdot g'(x) & -\frac{d}{b} \\ b \cdot a \cdot g(x) \cdot f'(x) & 0 \end{vmatrix}$$

From the Routh-Hurwitz criteria, we know that the coexistence equilibrium is stable only if trace<0 and det(J)>0.

$$\text{Tr}(J) = \frac{d}{b \cdot a} \cdot g'(x) \quad \Delta = d \cdot a \cdot g(x) \cdot f'(x)$$

since d, a>0 and f'(x)=1/ $[(1+x)]$ ^2 >0, the second condition holds whenever y^*>0. so det(J)>0.

The the coexistence equilibrium is stable if g'(x*)<0, and is unstable if g'(x*)>0. Moreover,if g'(x*)=0 then tau=0 and △>0, so we have imaginary eigenvalues, and so we expect complex eigenvalues when g'(x*)≈0