# A Friendly Introduction to Abstract Nonsense

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Spring 2025

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# 1 Introduction

Category theory was first introduced by MacLane and Eilenberg to give a precise definition to what it meant for something to be "natural" [1]. The definition of naturality requires natural transformations, which in turn require functors, which reqiure categories. And thus, category theory was born. In this sense, category theory itself can be thought of as "natural".

Category theory has evolved over the near century since it's inception, and nowadays is of fundamental importance to many branches of mathematics, especially algberaic ones such as algebraic topology, algebraic geometry, and of course, algebra itself. This makes motivating it with algebraic and topological definitions very easy, and I will make use of these fields at an undergraduate level as motivation and examples throughout these notes. However, I don't want to assume extensive prerequisite knowledge in these fields, and I would prefer these notes to be accessible to anyone with only basic knowledge of what sets and functions are. Whether it's prudent or not to introduce students to categories before algebra is a debate[1] for another day. My goal is not to teach a course on this material, but rather to give a friendly introduction, and as such I believe it fit to keep these notes as accessible as possible.

These notes will be structured into a few sections. I'll start with motivation, then will define a category and give some examples. After that I will continue with more important categorical definitions, along with a brief intermission for the Yoneda lemma. I'll finish up with some more advanced topics that I either think are important, or am personally interested in.

## Part I

# The Basics of Categories

# 2 Motivation

## 2.1 The Cartesian Product

Now I want to introduce a construction we are all familiar with, even if we don't realize it. The cartesian product $A \times B$ of two sets $A, B$ is defined as the set of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$. For example, I'm sure you've all drawn the coordinate plane before. Every point on the coordinate plane is a pair $(x, y)$, where $x$ and $y$ are both real numbers, This means that the coordinate plane is really the cartesian product $\mathbb{R} \times \mathbb{R}$.

Now let's consider an arbitrary cartesian product $A \times B$. There are two important functions I want to consider. Let $\pi_A : A \times B \to A$ be the function $\pi_A(a, b) = a$, and let $\pi_B : A \times B \to B$ be the function $\pi_B(a, b) = b$. We call these the *natural projections* or *canonical projections* onto $A$ and $B$. We will see that these are extremely important functions in this section. In fact, as we will see later, these functions characterize cartesian products "up to isomorpism."

**Proposition 1.** *Let $X, Y, Z$ be sets, and let $\pi_X, \pi_Y$ be the natural projections of $X \times Y$ onto $X$ and $Y$, respectively. Let $f : Z \to X$ and $g : Z \to Y$ be any functions. Then there exists a unique function $\varphi : Z \to X \times Y$ such that*

$$f = \pi_X \circ \varphi \qquad \textit{and} \qquad g = \pi_Y \circ \varphi.$$

*In other words, the diagram*

[1] It's not.

$$\begin{tikzcd}
 & X \\
Z \arrow[ur, "f"] \arrow[r, dashed, "\varphi"] \arrow[dr, "g"'] & X \times Y \arrow[u, "\pi_X"'] \arrow[d, "\pi_Y"] \\
 & Y
\end{tikzcd}$$

Diagram 1: Universality of Product

*commutes.*

To understand this statement, we need to understand what it means for a diagram to "commute". Consider the much simpler diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
 & C
\end{tikzcd}$$

Diagram 2: Simple Diagram

A diagram is a drawing like this, with letters and arrows. The letters represent sets, and the arrows represent functions between these sets. So the arrow labeled $f$ in the smaller diagram represents a function $f : A \to B$. We then say a diagram *commutes* if any possible way of getting from one point to another are the same. For example, in this smaller diagram, there are two ways to go from $A$ to $C$: we can take the function $f$ from $A$ to $B$, then take the function $g$ from $B$ to $C$. Alternatively, we can take the function $h$ straight from $A$ to $C$. In this case, the diagram will commute if $h = g \circ f$, meaning both paths are the same.

Now back to our bigger diagram (diagram 1). It's a little less apparent what it means for this to commute, since there are now two endpoints. It's often easier to break a bigger diagram up into smaller diagrams. So for example, consider the smaller "subdiagram"

$$\begin{tikzcd}
 & X \\
Z \arrow[ur, "f"] \arrow[r, dashed, "\varphi"'] & X \times Y \arrow[u, "\pi_X"']
\end{tikzcd}$$

This will commute if

$$\pi_X \circ \varphi = f.$$

This means that for our main diagram 1 to commute, we must have

$$\pi_X \circ \varphi = f \qquad \text{and} \qquad \pi_Y \circ \varphi = g.$$

This is precisely the statement of the proposition.

The proof of the proposition is very simple, and I leave it as an exercise. The map $\varphi$ is defined as $\varphi(z) = (f(z), g(z))$. Uniqueness is manifestly satisfied, so simply showing that the diagram commutes is sufficient.

Even after all this, the statement of the proposition may be fairly difficult to parse. However, I want to focus on the following idea. We are able to view the cartesian product purely in terms of functions, specifically using the natural projections, and how other functions behave with them. Later, we will see that this viewpoint actually *defines* the cartesian product *up to isomorphism*, meaning any two sets that satisfy this property behave the exact same way (at least, as far as functions are concerned).

## 2.2 Fundamental Symmetries

Now for some higher level motivation. For those who have taken algebra, you have seen the *first isomorphism theorem.* And you likely know that this theorem applies not only to groups, but also to *rings.* However, there's even more than that. Given a vector space $V$ with a subspace $U \subseteq V$, you can define a *quotient space* $V/U$. Since a vector space is nothing more than an abelian group with some extra structure, which we call *scalar multiplication*, you can just consider the quotient *group* $V/U$, and define scalar multiplication on this new abelian group in the most obvious way. I leave rigorizing this definition as an exercise to the reader, but it should be clear that it is possible.

Additionally, note that if $T : V \to W$ is any linear transformation, then $\ker T$ is a subspace of $V$. This means that we can consider the quotient $V/\ker T$. The natural question is whether the first isomorphism theorem applies, and indeed, it does.

**Theorem 1** (First Isomorphism Theorem)**.** *Let $T : V \to W$ be a linear transformation. Then $V/\ker T \cong \operatorname{im} T$. In particular, there is a unique* isomorphism *of vector spaces $\phi$ making the diagram*

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\pi"'] & \operatorname{im} T \\
V/\ker T \arrow[ur, dashed, "\phi"'] &
\end{tikzcd}$$

*commute.*

In this case, the linear transformation $\pi$ is the natural projection onto the quotient, defined by $v \mapsto v + \ker T$. Why does the first isomorphism theorem apply in so many areas of algebra?

We're actually not done. There is a "first isomorphism theorem" for regular functions as well. Another way to rephrase the first isomorphism theorem is that for any linear transformation $T$, we can break it down into a surjection $\pi : V \to V/\ker T$, an isomorphism $V/\ker T \to \operatorname{im} T$, and then an inclusion map $\operatorname{im} T \hookrightarrow W$. An inclusion map is where you take a set $\operatorname{im} T$ which is a subset of a bigger set $W$, and define a map $\operatorname{im} T \to W$ by mapping $x \mapsto x$. You "inject" $\operatorname{im} T$ into the bigger set $W$. You will not be surpised to hear inclusion maps are injective, and thus we can break any linear transformation into a surjection, isomorphism, and then an injection:

$$\begin{tikzcd}
V \arrow[r, two heads, "\pi"] \arrow[rrr, bend left, "T"] & V/\ker T \arrow[r, dashed, "\phi"'] & \operatorname{im} T \arrow[r, hook, "i"'] & W.
\end{tikzcd}$$

However, we can actually do this for any function. Given a set $X$ with an equivalence rlation $\sim$, we can define the quotient set $X/\sim$ as the set of all equivalence classes $[x]_\sim$ in $X$ under $\sim$. If $f : X \to Y$ is a function, then consider the following equivalence relation in $X$:

$$x \sim y \iff f(x) = f(y).$$

I leave it as an exercise to prove that this is an equivalence relation. This means that for each equivalence class $[x]_\sim$, we must have $f(x) = f(y)$ for all $x, y \in [x]_\sim$, and thus $f[x]_\sim = f(x)$. In other words, the entire equivalence class corresponds to one element of the image of $f$. It is then natural to think that $X/\sim$ may be in correspondence with $\operatorname{im} f$, and indeed this is the case; there is a bijection $X/\sim \to \operatorname{im} f$ defined as $[x]_\sim \mapsto f(x)$. This is well-defined by the definition of $\sim$.

There then exists a quotient function $\pi$ defined as $x \mapsto [x]_\sim$, which is manifestly surjective, and then we indeed have that $f$ decomposes as

$$\begin{tikzcd} X \arrow[r, two heads, "\pi"] \arrow[rrr, bend left, "f"] & X/\sim \arrow[r, dashed, "\phi"'] & \operatorname{im} f \arrow[r, hook, "i"'] & Y. \end{tikzcd}$$

This is often called the *canonical factorization* of a function.

And there are more symmetries than just this. The definition I gave of a product in section 2.1 generalizes as well. If you replace "function" with "group homomorphism" and "set" with "group", then I claim the exact same definition gives you a definition of direct product of groups, which is well-defined up to group isomorphism. Similarly, if you use "ring" and "ring homomorphism", you get a direct product of rings. If you use "continuous map" and "topological space", you get a product topology. If you use "vector space" and "linear transformation", you get a product of vector spaces. And there are other, similar constructions for the *kernel*, the *image of a function*, the *quotient*, and more. In a sense, these constructions are "universal", they are always defined the same way, and if they exist, they are always defined up to isomorphism.

Given the uncanny symmmetry between vastly different areas of mathematics, we would expect that there should be some "unifying language" of mathematics, that would explain these symmetries in a natural way. This is *precisely* what category theory offers us.

# 3 Definition

## 3.1 Viewpoint Shift

One common theme so far in this talk has been functions. I started off by discussing some properties of functions, and I continued by showing that the Cartesian Product can be characterized pretty much only using functions. I then worked with the first isomorphism theorem and constructions like the cartesian product in vastly different areas of math, all of which heavily utilize *functions*.

Many of you may be wondering why I'm putting so much emphasis on functions, but to that I would respond with "why not?" Functions are some of the most important things in mathematics. We spend multiple years in high school studying *functions*. Calculus is nothing more than the study of the derivative and integral of *functions*. When we get to higher level math, we want to study groups and rings and fields and topologies and vector spaces and modules and manifolds and all these different constructions using *functions*.

However, this viewpoint can only take us so far. When we study more advanced objects, or even when we study calculus, we aren't concerned with *all* functions, we only care about differentiable and integrable functions. When we study groups, we are only concerned with group homomorphisms. What we *need* if we are to take this viewpoint seriously is an *abstraction* of the idea of a function.

Abstractions are common in advanced mathematics. We take an important object, figure out all the important properties of that object, then ask "what happens if we look at *all objects* that satisfy these important properties? Can we find anything new?"

Category theory is this idea applied to mathematics itself. Much of math is taking a type of object, and a type of function, and studying it. Category theory abstracts the very idea of an object and a function.

### 3.2 Definition

So how do we abstract a function? We first need to identify the important properties of a function. I claim the most important thing functions give us is function composition. If you have a function $f : X \to Y$ and $g : Y \to Z$, you can always compose them into a function $g \circ f : X \to Z$. So we want there to be composition. We will also see later that the idea of inverse functions is indeed very important, but for there to be an inverse function, we must first have the identity function. So we also want an identity function to exist. Finally, we will ask that function composition be associative, since that is a very reasonable request to make. These are the three properties we will require our abstraction of functions to have. We call our abstraction of a function a *morphism*. This leads us to the definition of a category.

**Definition 1** (Category)**.** A category $\mathcal{C}$ consists of two types of information:

- A class $\mathrm{Obj}(\mathcal{C})$ of *objects*;
- For any two objects $X$ and $Y$ in $\mathcal{C}$, there is a class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms*; we say a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is *a morphism from $X$ to $Y$*, and write $f : X \to Y$. We call $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ a hom-class.

The objects and morphisms are required to satisfy the following properties:

1. For all $f : X \to Y$ and $g : Y \to Z$, we can compose $f$ and $g$ into a morphism $g \circ f : X \to Z$, called the *composite* of $f$ and $g$;
2. The operation $\circ$ of morphism composition is associative, meaning $h \circ (g \circ f) = (h \circ g) \circ f$;
3. For all objects $X \in \mathrm{Obj}(\mathcal{C})$, there is an identity morphism $1_X : X \to X$;
4. The identity morphism is an identity with respect to composition, meaning for any $f : Z \to X$ and $g : X \to Z$,
$$1_X \circ f = f \qquad \text{and} \qquad g \circ 1_X = g.$$

It is common to simply write $X \in \mathcal{C}$ to indicate that $X$ is an object of $\mathcal{C}$, rather than writing $X \in \mathrm{Obj}(\mathcal{C})$. I will use this convention. It is not uncommon to write $\mathrm{Hom}(X, Y)$, omitting the subscript $\mathcal{C}$ when the category is understood. Another common notation for $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is $\mathcal{C}(X, Y)$. I will be using the notation $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Another common notation for identity morphisms is $\mathrm{id}_X$. I will use the slightly less common notation $1_X$ instead.

The reason I use the word “class” instead of “set” is because we want to be able to discuss the *category of all sets*, or the *category of all groups*, and similar contructions. However, the collection of all sets, for example, is too big to be a set. If we allow it to be a set, we inevitably get horrible contradictions. The same is true for groups. Classes are things that are like sets, but are allowed to grow bigger than a set is. All sets are also classes, but smaller. A class that is too big to be a set is called a proper class. In order to talk about many of the categories we *want* to talk about, we have to allow $\mathcal{C}$ to form a proper class of objects. We call a category whose class of objects is not a set a *large* category. A category whose collection of objects is indeed a set is a *small* category.

Similarly, the hom-classes can indeed be proper classes. However, in most important categories, the hom-classes form sets, whereas the overall collection of objects forms a proper class. We call a category whose hom-classes form sets a *locally small* category, and we instead use the terminology *hom-set*.

# 4 Examples

## 4.1 Simple Examples

This definition is not easy to parse. Note that in definition 1, we spent most of our time discussing morphisms, and how morphisms behave. There is clearly an emphasis on morphisms, which is not unexpected given our extensive discussion of functions. However, this makes understanding morphisms crucial to continued understanding of category theory. To this end, we ask the question:

What is a morphism?

You will not like the answer. Before we can see why this is, let's explore some tame examples of categories. Consider the category $\mathcal{S}\text{et}$, where $\text{Obj}(\mathcal{S}\text{et})$ is the class of all sets, and $\text{Hom}_{\mathcal{S}\text{et}}(X, Y)$ is the set of all functions $f : X \to Y$. Verifying that $\mathcal{S}\text{et}$ is a category is straightforward, as we explicitly built categories as an abstraction of the idea of $\mathcal{S}\text{et}$. Other common categories include

- $\mathcal{G}\text{rp}$: objects are groups and morphisms are group homomorphisms;
- $\mathcal{A}\text{b}$: objects are abelian groups and morphisms are group homomorphisms;
- $\mathcal{R}\text{ing}$: objects are (usually unital) rings and morphisms are (usually unital) ring homomorphisms;
- $k$-$\mathcal{V}\text{ect}$: objects are $k$-vector spaces and morphisms are $k$-linear transformations;
- $R$-$\mathcal{M}\text{od}$: objects are $R$-modules and morphisms are $R$-linear maps;
- $\mathcal{T}\text{op}$: objects are topological spaces and morphisms are continuous maps;
- $\mathcal{M}\text{an}$: objects are smooth manifolds and morphisms are smooth maps;
- $\mathcal{C}\text{h}(R)$: objects are chain complexes in $R$-$\mathcal{M}\text{od}$ and morphisms are chain complex morphisms.

However, the definition of a category is exceedingly general, so there are many more categories we can define. The following category shows just how general these definitions are.

## 4.2 The Poset $\mathbb{Z}$

Let $\mathcal{C}$ be the category where $\text{Obj}(\mathcal{C}) = \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. Morphisms will be defined as follows: for any $n, m \in \mathcal{C}$, if $n \le m$, then there is *exactly one morphism* $n \to m$; if $n > m$, then there is *no morphism* $n \to m$. This is an exceedingly strange and abstract definition, and we will parse it fully after we show this construction satisfies the category axioms.

First, note that if $f : n \to m$ and $g : m \to p$ are morphisms, then $n \le m \le p$, and thus there isa morphism $h : n \to p$. We can define $g \circ f := h$, and this gives us composition. This composition is clearly associative. Moreover, since $n \le n$, there is precisely one morphism $1_n : n \to n$. Consider the morphism $f : n \to m$. This can be composed into a morphism $f \circ 1_n : n \to m$. However, the only morphism $n \to m$ is $f$, so we must have $f \circ 1_n = f$. The morphism $1_n$ is shown to be a left-identity in the same way. Therefore, $\mathcal{C}$ is a category.

It may not be clear what is happening in this example. We have defined morphisms to be the exact same thing as the $\leq$ relation. However, we want to intuitively think of morphisms as functions, but $\leq$ is definitely not a function. When we write $n \to m$, we think of something going from $n$ to $m$, but this morphism absolutely does not do that. All a morphism $n \to m$ does is say $n \leq m$. It's unclear why we even write these morphisms as arrows at all. These morphisms are absolutely nothing like functions.

However, this definition, as confusing as it may be, does indeed form a category. And this is the uncomfortable truth of category theory. Some people say "objects don't have to have elements", but I feel this is a misnomer. It's difficult to think of objects that don't have elements. Instead, think about it like this:

In category theory, morphisms need not act on elements. Objects are understood only via their relationships (morphisms/arrows) to one another.

It is with this in mind that we must part ways with our set-theoretic roots in some senses. In classical set theory, we learn about functions and sets by considering their elements, and their action on elements. In category theory, we cannot rely on morphisms acting on elements. All we can do is rely on the relationships between morphisms with *other morphisms*. This may seem very convoluded at the moment, but after we develop functors and natural transformations, we will see that although it's convoluded, it's also natural in a sense.

# 5 Functors

## 5.1 Covariance

Pausing our quest to understand morphisms for a short while, we come to a very natural question. In set theory, there are functions between sets. In group theory, there are group homomorphisms between groups. In topology, there are continuous maps between topological spaces. How, then, do we go from one category to another? The answer is with *functors*.

**Definition 2** (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ consists of

- A function $\mathcal{F} : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$;
- For every $X, Y \in \mathcal{C}$, a function $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$. In other words, for every morphism $f : X \to Y$, a morphism $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$.

These functions must satisfy the following axioms.

1. For every identity morphism $1_X$, we must have $\mathcal{F}(1_X)$ be the identity morphism of $\mathcal{F}(X)$. In other words,
$$\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}.$$
2. For every $f : X \to Y$ and $g : Y \to Z$, we must have
$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Functors are how we choose to define morphisms between categories. They carry meaningful information about the internal contents of a category, so they are a very natural definition.

For example, consider the functor $\mathcal{U} : \mathcal{G}\mathrm{rp} \to \mathcal{S}\mathrm{et}$. Since every group is just a set with some extra structure, and every group homomorphism is just a function with some special properties, we can send a group to it's underlying set and a function to it's underlying function, and this relation is clearly functorial. We "forget" the extra structure of $\mathcal{G}\mathrm{rp}$, and for this reason $\mathcal{U}$ is called an *forgetful functor*. Similarly, we have forgetful functors for vector spaces, modules rings, topological spaces, etc. Forgetful functors will show up later due to their presence in adjunction relations.

For those who have taken topology, the first fundamental group $\pi_1$ is a functor from the category $\mathcal{T}\mathrm{op}_*$ of *pointed topological spaces*, meaning a topological space $X$ plus a distinguished base point $x_0$. to the category $\mathcal{G}\mathrm{rp}$.

A classical example of a functor is the functor Hom. Let $\mathcal{C}$ be a category, and let $X \in \mathcal{C}$. Define the functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ as

$$A \mapsto \mathrm{Hom}_{\mathcal{C}}(X, A).$$

That is, we take an object $A \in \mathcal{C}$, and map it to the set $\mathrm{Hom}_{\mathcal{C}}(X, A)$. In this way, Hom defines a functor $\mathcal{C} \to \mathcal{S}\mathrm{et}$, called the hom-functor.

But wait, how does Hom act on morphisms? Let $\varphi : A \to B$ be a morphism. What is $\mathrm{Hom}_{\mathcal{C}}(X, \varphi)$ defined as? For notational purposes, denote the induced morphism as $\varphi^*$. Since $\varphi^*$ is a function between *sets*, we can define it's action on elements. Let $f : X \to A$ be a morphism. We need to send it to a morphism $X \to B$. Since $\varphi$ is a morphism from $A \to B$, we can just consider the composition

$$X \xrightarrow{f} A \xrightarrow{\varphi} B.$$

We thus define $\varphi^*(f) = \varphi \circ f$. Showing that this relation is functorial is a great exercise, and I encourage you all to do it. In this case, a diagram means the same thing as it did before, except letters now represent objects, and arrows represent morphisms. Commutative diagrams are also interpreted the same way.

## 5.2 Contravariance

A natural question at this point is "what about $\mathrm{Hom}_{\mathcal{C}}(-, X)$?" Indeed, this also forms a functor, but not in the way we have discussed. The functors I have described herein are also known as *covariant functors*. However, there's another "type"[2] of functor, known as a *contravariant* functor.

If $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a covariant functor and $f : A \to B$ is a morphism in $\mathcal{C}$, then we saw that $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ is a morphism in $\mathcal{D}$.

Contravariant functors satisfy all the usual axioms of a functor, except they *flip* the arrows. So if $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ is a contravariant functor and $f : A \to B$ in $\mathcal{C}$, then $\mathcal{G}(f) : \mathcal{G}(B) \to \mathcal{G}(A)$ is a morphism in $\mathcal{D}$. This is more easily seen in a diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \Big\downarrow \mathcal{G} & \\ \mathcal{G}(A) & \xleftarrow[\mathcal{G}(f)]{} & \mathcal{G}(B) \end{array}$$

It turns out that although $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is covariant, the functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$ is actually *contravariant*. To show this, consider the action it has on morphisms. Let $\varphi : A \to B$ be a morphism in $\mathcal{C}$, and let $\varphi_*$ be the induced morphism under this contravariant flavor of Hom. If we define it in the same way, then we would need to compose on the *right* this time, to make sure the codomain of the composition is $X$. Which means we're going to have a morphism

$$A \xrightarrow{\varphi} B \longrightarrow X.$$

[2] We will see later in the section on duality that all functors are indeed covariant.

This means that in order for $\varphi_*$ to be defined in a similar way, we must necessarily view it as a function

$$\varphi_* : \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X).$$

In other words, the arrow has *flipped*. So indeed, $\mathrm{Hom}_{\mathcal{C}}(-, X)$ is contravariant.

# 6 Natural Transformations

## 6.1 Definition

The next step may feel unmotivated at the moment, but the motivation will become clear at the end of the section.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, we can contemplate a *functor category*, where objects are functors $\mathcal{C} \to \mathcal{D}$. We often denote this as $\mathcal{D}^{\mathcal{C}}$ or $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. I prefer the former, but I will use the latter for these notes, as I think it's clearer. In order for this to be a category, we need a way of defining morphisms between functors. This means given two functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$, we need a way of transforming $\mathcal{F}$ into $\mathcal{G}$, or vice versa. We call this a *natural transformation*.

**Definition 3** (Natural Transformation)**.** Let $\mathcal{F}, \mathcal{G} \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. A natural transformation $\eta$ from $\mathcal{F}$ to $\mathcal{G}$, denoted $\eta : \mathcal{F} \Rightarrow \mathcal{G}$, consists of a collection of morphisms $\eta_X : \mathcal{F}(X) \to \mathcal{G}(X)$ *for every* $X \in \mathcal{C}$, such that for all $f : X \to Y$ in $\mathcal{C}$,

$$\eta_Y \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_X.$$

In other words, the diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_X"'] & \mathcal{F}(Y) \arrow[d, "\eta_Y"] \\ \mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(Y) \end{tikzcd}$$

commutes.

We call $\eta_X$ the *component* of $\eta$ at $X$, and we say $\eta_X$ is *natural* in $X$. We often denote the set of natural transformations from $\mathcal{F}$ to $\mathcal{G}$ as $\mathrm{Nat}(\mathcal{F}, \mathcal{G})$.

## 6.2 Yoneda

Now that we have introduced natural transformations, we can introduce one of the most important results in category theory, even if it's also one of the simplest. This result is known as the *Yoneda lemma*.

**Lemma 1** (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category, and let $\mathcal{F} : \mathcal{C} \to \mathcal{S}\mathrm{et}$ be any covariant functor, and let $C \in \mathcal{C}$. Then there is a* natural isomorphism of sets *(a bijection)*

$$\mathrm{Nat}\left(\mathrm{Hom}_{\mathcal{C}}(C, -), \mathcal{F}\right) \cong \mathcal{F}(C).$$

*Similarly, if $\mathcal{G} : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is any contravariant functor, and $C \in \mathcal{C}$, then there is a natural isomorphism*

$$\mathrm{Nat}\left(\mathrm{Hom}_{\mathcal{C}}(-, C), \mathcal{G}\right) \cong \mathcal{G}(C).$$

The proof of the Yoneda lemma is beyond these notes, but it's a good idea to unpack the implications of it. Let $C, D \in \mathcal{C}$, and note that $\mathrm{Hom}_{\mathcal{C}}(D, -)$ is covariant. This means we can plug in $\mathrm{Hom}_{\mathcal{C}}(D, -)$ for $\mathcal{F}$ in the Yoneda lemma to get

$$\mathrm{Nat}\left(\mathrm{Hom}_{\mathcal{C}}(C, -), \mathrm{Hom}_{\mathcal{C}}(D, -)\right) \cong \mathrm{Hom}_{\mathcal{C}}(D, C).$$

This means that the natural transformations between $\mathrm{Hom}_{\mathcal{C}}(C, -)$ and $\mathrm{Hom}_{\mathcal{C}}(D, -)$ are exactly the same as the morphisms from $D$ to $C$. Although this is extrmely interesting, it turns out that the contravariant flavor of the lemma is more important. Since $\mathrm{Hom}_{\mathcal{C}}(-, D)$ is contravariant, the Yoneda lemma also gives us

$$\mathrm{Nat}\left(\mathrm{Hom}_{\mathcal{C}}(-, C), \mathrm{Hom}_{\mathcal{C}}(-, D)\right) \cong \mathrm{Hom}_{\mathcal{C}}(C, D).$$

What we have seen is that although this is the contravariant flavor of the lemma, it seems more "covariant", because now our natural transformations are the same as the morphisms $C \to D$. The arrows are no longer being flipped. This result allows us to define an important *new* functor $\mathcal{Y}$[3], known as the *Yoneda Embedding.*

For simplicity of notation, let $h_X$ denote the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$. Let

$$y_{C,D} : \mathrm{Nat}\left(h_C, h_D\right) \to \mathrm{Hom}_{\mathcal{C}}(C, D)$$

be the natural bijection given by the Yoneda lemma, and let $y^{-1}_{C,D}$ be it's inverse. Let

$$\mathcal{Y} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$$

be the functor $X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X) = h_X$. Since the morphisms in $\mathcal{C}$ from $C$ to $D$ are *exactly the same* as the morphisms from $h_C$ to $h_D$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$, for any morphism $f : C \to D$ in $\mathcal{C}$, we can define

$$\mathcal{Y}(f) = y^{-1}_{C,D}(f).$$

Since the Yoneda lemma tells us that this is a natural isomorphism, this is clearly a functorial assignment. Moreover, the hom-sets in this new functor category are *exactly the same* as the hom-sets in $\mathcal{C}$, because $\mathcal{Y}$ is defined by an *isomorphism*. This means that $\mathcal{Y}$ in a sense "embeds" $\mathcal{C}$ into the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$, but this embedding *preserves all structure in* $\mathcal{C}$ by preserving all morphisms. We call such an embedding *fully faithful.*

This means that if $C$ is an object in a category, we can identity it with it's hom-functor $\mathrm{Hom}_{\mathcal{C}}(-, C)$ in such a way that *no structure is lost.* The identification $C \mapsto \mathrm{Hom}_{\mathcal{C}}(-, C)$ gives the *exact same category.* Thus, the true implications of the Yoneda lemma are as such:

> Any object in any (locally small) category can be fully represented by how it interacts with every other object in the category. In other words, everything about objects can be fully explained by their relations with other objects.

We will use these philosophical implications in full in the next section. In the next section, we will also clarify a slight bit of machinery I skipped over in this section. I only defined natural transformations between *covariant functors*, but the Yoneda embedding involves *contravariant* functors. Reconciling this is not difficult, but it involves viewing contravariant functors as covariant functors from the *opposite category*, a topic that will be explored in the next section.

One of the most fascinating things is that the Yoneda embedding arises in a natural way. Recall that the bijection of hom-set to natural transformations was *natural*, meaning it can be viewed as a natural transformation itself. Doing this is more complex, but I encourage the reader to try and figure it out on their own, or to have fun going down the relevant Wikipedia/ChatGPT rabbit hole.

[3] The Yoneda embedding is sometimes denoted よ. I've never actually seen this used but I just think it's interesting.

# 7 Universality

## 7.1 Duality

After this lengthy detour into the Yoneda lemma, we are ready to come back to our original question: how exactly do we study objects and morphisms in arbitrary categories? We already know the answer will be "through relationships with other morphisms", but what exactly does this mean? The first answer to this question will come via universal properties. In order to define these, we need to cover a few final definitions in category theory, which are of fundamental importance to the field. The first definition will be that of the opposite category, and the notion of duality.

**Definition 4** (Opposite Category)**.** Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{\mathrm{op}}$ is the category where

$$\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C}),$$

and for all $A, B \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A).$$

In other words, if $f : A \to B$ in $\mathcal{C}$, then $f : B \to A$ in $\mathcal{C}^{\mathrm{op}}$. All we do is "flip the arrows", in the same exact way we flipped the arrows in a contravariant functor. In fact, any contravariant functor $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ is actually the same as a *covariant* functor $\mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. I leave it to the reader to make this notion precise, and to use this notion to precisely recover the Yoneda Embedding from this definition.

I want to note here that there appears to be a sort of correspondence between covariant and contravariant functors. They are really the same thing, except one goes through $\mathcal{C}$, and one goes through the opposite category. This notion of a corespondence between $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ is called *duality*, and as we will see in this section, it generalizes much further than to just functors.

## 7.2 Isomorhisms, Initial & Terminal Objects

Back when we defined a category, we required that identity morphisms exist, that way we might have a meaningful notion of an inverse morphism. It's finally time to use this axiom to it's fullest, and define one of the most fundamental concepts in category theory.

**Definition 5** (Isomorphism)**.** Let $f : A \to B$ be a morphism. We say $f$ is an *isomorphism* if there exists an *inverse morphism* $f^{-1} : B \to A$ such that

$$f \circ f^{-1} = 1_B \qquad \text{and} \qquad f^{-1} \circ f = 1_A.$$

If there is an isomorphism $f : A \to B$, we say $A$ is *isomorphic* to $B$, and write $A \cong B$.

This is the same as our definition of an isomorphism of *sets* as an invertible function. Isomorphisms actually answer an important question that I haven't yet addressed. When we define a category, we have to choose what our morphisms are going to be. This choice feels very arbitrary, and it's not necessarily always clear why we might choose one type of morphisms over another. The arbitrariness of this choice vanishes once we introduce isomorphisms.

If $A \cong B$ are isomorphic, then they are *exactly the same* as far as morphisms are concerned. To make this precise, let $\varphi$ be an isomorphism $A \to B$. It's not difficult to show that if $\mathcal{F}$ is any functor, and $\varphi$ is an isomorphism, then $\mathcal{F}(\varphi)$ is an isomorphism. Consider the functor $\mathrm{Hom}_{\mathcal{C}}(-, Z)$, where $Z$ is any object in $\mathcal{C}$. Then we have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(A, Z) \xrightarrow{\varphi} \mathrm{Hom}_{\mathcal{C}}(B, Z)$$

This is an isomorphism of sets, meaning that $\mathrm{Hom}_{\mathcal{C}}(A, Z)$ and $\mathrm{Hom}_{\mathcal{C}}(B, Z)$ are essentially the same. Every morphism $f : A \to Z$ can be turned into a morphism $f' : B \to Z$ and vice versa, and this can all be done without effecting any other morphisms in $\mathcal{C}$. The reverse direction concerning morphisms $Z \to A, B$ is also easily shown using the covariant flavor of Hom.

In this sense, we can now make the choice of morphisms less arbitrary. Let's say we have a certain type of object we want to study. For example, a group. When we study groups, we care about the *structure* of a group. We want isomorphic groups to be "essentially the same group", so what we need is a definition of morphisms in such a way that isomorphic groups have the exact same group structure. Group homomorphisms are the proper choice for this. A group homomorphism $\varphi : G \to H$ basically identifies the group structure of $G$ with $H$ in some way. If two groups are *isomorphic*, then their structure must identify with *every other group* in *exactly the same way.* We can immediately conclude that they must then have the same structure.

And this is usually how we choose our morphisms. We figure out what it is that we want to study, and we choose morphisms in such a way that isomorphic objects have *exactly the same properties*, as far as we are concerned.[4]

**Definition 6** (Initial/Terminal Objects)**.** Let $\mathcal{C}$ be a category. An object $I \in \mathcal{C}$ is *initial* if for all $X \in \mathcal{C}$, there is exactly *one morphism* $I \to X$. Equivalently, $\mathrm{Hom}_{\mathcal{C}}(I, X) = \{*\}$ for all $X$. Similarly, an object $T \in \mathcal{C}$ is *terminal* if $\mathrm{Hom}_{\mathcal{C}}(X, T) = \{*\}$ for all $X \in \mathcal{C}$, meaning there is a unique morphism $X \to T$.

Initial and terminal objects are of fundamental importance, almost due to the following basic, yet important result.

**Proposition 2.** *Let $I, J \in \mathcal{C}$ be initial objects. Then $I \cong J$. Similarly, if $S, T \in \mathcal{C}$ are terminal, then $S \cong T$.*

> **Proof.** Since $I$ is initial, there is a unique morphism $f : I \to J$. Since $J$ is initial, there is a unique morphism $g : J \to I$. These can be composed into a morphism $g \circ f : I \to I$. However, since $I$ is initial, there is only one morphism $I \to I$, and by the category axioms, it must be the identity morphism $1_I$. Therefore, $g \circ f = 1_I$. By the same logic, $f \circ g = 1_J$. The proof for final objects is done the exact same way. ■

Interestingly, note that if an object is initial in $\mathcal{C}$, it must be terminal in $\mathcal{C}^{\mathrm{op}}$, and vice versa. This is one way that duality arises. If we have an initial object in $\mathcal{C}$, then we can usually define a *dual* version of it by flipping the arrows, and it will be initial in $\mathcal{C}^{\mathrm{op}}$, and thus terminal in $\mathcal{C}$. This idea of duality is of fundamental importance, and we will see as much in this next section.

### 7.3 Universal Properties

Universal properties summarize everything we have done so far in a beautiful way, and they are an extremely important concept in category theory, algebra, topology, and beyond. Universal properties bleed unequivically into adjunction relations, which are a special topic I will cover at the end. For now, I will introduce them in a less precise manner, although this will make it more clear what a universal property actually is.

Put simply, an object $X$ is *universal* if it is initial or terminal in some category. The issue is, categories like $\mathcal{S}\mathrm{et}$ and $\mathcal{G}\mathrm{rp}$ may indeed have interesting intitial and terminal objects, but they are

[4] There are other ways we may choose morphisms, for example as an intermediary step in a proof we may need to define a new category in a certain way. These definitions are often natural and do not require arbitrary choices, so they are not of concern here.

not interesting in the sense that we are currently looking for. Universal properties often require creating an entire new category on the spot, just so that you can view $X$ as initial or final in that category.

Usually, the objects of this new category all satisfy a specific property that we are concerned with. We then say any initial or final object of that category is *universal* with respect to that property. Before we unpack the significance of such a construction, it's prudent to consider an example.

Back in the beginning of these notes, we said that the cartesian product $A \times B$ of two sets comes with natural projections, $\pi_A$ and $\pi_B$. The cartesian product thus satisfies the following property: it has a function going to $A$, and a function going to $B$. Let's create a new category $\mathcal{C}$, where objects are simply sets $X$, together with functions $f : X \to A$ and $g : X \to B$. In other words, objects are things that satisfy the property we are concerned about. Objects thus look like the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

The way we define morphisms in $\mathcal{C}$ is as follows: consider two objects, $(X, f, g)$ and $(Y, \phi, \psi)$:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & & & Y \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ A & & B & A & & B \end{tikzcd}$$

A morphism $(X, f, g) \to (Y, \phi, \psi)$ looks like a function $\alpha : X \to Y$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "\alpha"] \arrow[dr, "g"] & \\ A & Y \arrow[l, "\phi"] \arrow[r, "\psi"'] & B \end{tikzcd}$$

commutes.

For those who remember proposition 1, this diagram may look familiar. We said that if we put the cartesian product $A \times B$, together with the natural projections $\pi_A, \pi_B$ into the diagram, then the function $\alpha$ becomes unique:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, dashed, "\alpha"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

In other words, for all objects $(X, f, g)$ in $\mathcal{C}$, there is *exactly one morphism* $\alpha : (X, f, g) \to (A \times B, \pi_A, \pi_B)$ in $\mathcal{C}$. In other words, the cartesian product is *final* in $\mathcal{C}$.

Since the category was built out of objects satisfying a specific property, we say that the cartesian product is *universal* with respect to this property. That is, $A \times B$ is *universal* with respect to the property of having a function going to $A$, and another going to $B$. In other words, the cartesian product *satisfies a universal property.*

Universal properties are more significant than you likely think. If two objects satisfy the same universal property, then they must both be either initial or terminal in the same category, and thus they are *isomorphic*. This means that we can actually *define* the cartesian product of two sets as any set $X$ together with natural projections, which satisfy the universal property of the cartesian product. This definition is *well-defined up to isomorphism*, meaning if two sets satisfy the universal property (and thus would be cartesian products in this new definition), they would have to be isomorphic.

Although having multiple objects satisfying the same definition may seem strange, the isomorphism condition means that the objects behave in the *exact same way*, as far as our category is concerned. The universal property an object satisfies is also usually more useful than any precise definition of the object, since the universal property usually tells us *exactly* how important morphisms work, and how they are related to each other. Since the categorical perspective is to use morphisms rather than elements to study objects, once we start viewing things from the lens of category theory, it's perfectly fine to define things based on the universal property they satisfy. In fact, the Yoneda lemma seems to tell us that this is fine as well, since objects are fully described by their morphisms. This notion can be made precise, and we will do that in the section on adjunction.

For now, I want to wrap up universal properties by giving some powerful examples. The universal property of the cartesian product can actually be extended to *any category*. If we have a category $\mathcal{C}$, with objects $X, Y \in \mathcal{C}$, we can just define the object $X \times Y$ as the object, together with morphisms $\pi_X, \pi_Y$ satisfying the same universal property as the cartesian product:

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dashed, "\alpha"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

This is indeed how we define a product of two elements in any arbitrary category. It's important to note that the product doesn't necessarily exist, as categories don't necessarily have to have initial or terminal objects.

Another useful universal property is the kernel. This shows up oftentimes in algebraic contexts, and it requires two things. First, we need to have what's called a zero-morphism. The intuitive way to think about it is if you have two functions $f, g$, then $f(x) + g(x) = f(x)$. In other words, $g(x) = 0$ for all $x$. In group theory, 0 should be interpreted as the identity of the group. The precise definition is as follows: a zero-morphism is a morphism $0 : X \to Y$ such that for all $f, g : Z \to X$, $0 \circ f = 0 \circ g$; and for all $f, g : Y \to Z$, $f \circ 0 = g \circ 0$.

This allows us to define kernels. Let $\varphi : X \to Y$ be any morphism in a category with zero-morphisms. The kernel is an object $\operatorname{Ker}\varphi$, together with a morphism $\ker\varphi$ (I use lowercase k for the morphism to distinguish them), such that for any morphism $f : Z \to X$ such that $\varphi \circ f = 0$, there exists a unique morphism $\overline{\varphi} : Z \to \operatorname{Ker}\varphi$ such that the diagram

$$\begin{tikzcd}
Z \arrow[r, "f"'] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\overline{\varphi}"'] & X \arrow[r, "\varphi"'] & Y \\
 & \operatorname{Ker}\varphi \arrow[u, "\ker\varphi"'] &
\end{tikzcd}$$

commutes. This is typically phrased as "any morphism $f$ such that $\varphi \circ f = 0$ factors uniquely through the kernel." I leave it as an exercise to show that the usual definitions of kernel in

$\mathcal{G}$rp, $\mathcal{R}$ing, $\mathcal{A}$b, $k$-$\mathcal{V}$ect satisfy this universal property. Alternatively, the kernel is the equalizer of $\varphi$ and 0, but I don't have time to get into this definition unfortunately.

There are many more important universal properties, more than I have time to show here. However, for each intial object, there is a terminal object in the opposite category, and vice versa. This means that if we have a universal property, we can flip all the arrows and obtain a *new* universal property, called the *dual construction*. These dual properties are usually given the prefix "co". For example, the *coproduct* $X \amalg Y$ of $X$ and $Y$ is the object, together with natural *injections* $i_X : X \to X \amalg Y$ and $i_Y : Y \to X \amalg Y$, which is *initial* with respect to this property. That is, for any set $Z$ with functions $f : X \to Z$ and $g : X \to Z$, there exists a unique morphism $\varphi$ making the diagram

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] \arrow[r, "i_X"'] & X \amalg Y \arrow[u, dashed, "\varphi"'] & Y \arrow[ul, "g"'] \arrow[l, "i_Y"] \end{tikzcd}$$

commute. In $\mathcal{S}$et, the coproduct is the *disjoint union*. Aka, you take two sets $X, Y$, and forcibly make them disjoint. For example, you do $X \times \{0\}$ and $Y \times \{1\}$, making new sets which are isomorphic to the originals, but disjoint. You then take the union of these *new* sets.

One interesting remark is that in $\mathcal{G}$rp, the coproduct is the *free product* of two groups. However, in the subcategory $\mathcal{A}$b of $\mathcal{G}$rp, the (finite) coproduct is the *same as the product*. When (finite) products and coproducts coencide, we call them the *direct sum*, and denote them by $\oplus$. This shows that subcategories need not have the same universal objects as their parent categories, since the product does *not* satisfy the universal property of coproducts in $\mathcal{G}$rp.

Another example is the *cokernel*. This is obtained by reversing the diagram for kernels:

$$\begin{tikzcd} & \operatorname{Coker}\varphi \arrow[dr, dashed, "\overline{\varphi}"] & \\ X \arrow[r, "\varphi"] \arrow[rr, bend right, "0"'] & Y \arrow[u, "\operatorname{coker}\varphi"] \arrow[r, "f"] & Z \end{tikzcd}$$

I leave it as an exercise to make precise the universal property the cokernel satisfies. In the category $\mathcal{G}$rp, the cokernel can be realized explicitly as the group $\operatorname{Coker}\varphi \cong Y/\operatorname{im}\varphi$, and the morphism $\operatorname{coker}\varphi$ can be seen as the quotient morphism $\pi : Y \to Y/\operatorname{im}\varphi$. Quotients indeed satisfy their own universal property, and it involves the quotient morphism, and I encourage the reader to dig into this.

Universal properties are kind of like blueprints for building important objects. The important thing is that they are indeed "universal" in the truest sense of the word. No matter what category you are working in, or what you're doing, you can always look for something that satisfies a certain universal property. It may not exist, but you can always look.

# Part II
# Special Topics

## 8 Adjunction

### 8.1 Universal Arrows

This notion of a "universal property" is useful, but we haven't given a *precise definition* of what it is. We just said "view an object as universal in some category". How do we make this notion precise? One way to do this is to first make precise the notion of a universal arrow.

**Definition 7** (Universal Arrow)**.** Let $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ be a functor, and let $D \in \mathcal{D}$ be an object. A universal arrow (also called a universal morphism) from $D$ to $\mathcal{U}$ is an object $C \in \mathcal{C}$, together with a morphism $\eta : D \to \mathcal{U}(C)$, such that for every $Z \in \mathcal{C}$ and every morphism $f : D \to \mathcal{U}(Z)$, there is a unique morphism $f^* : C \to Z$ such that

$$\mathcal{U}(f^*) \circ \eta = f.$$

In other words, the diagram

$$\begin{tikzcd} D \arrow[r, "\eta"] \arrow[dr, "f"'] & \mathcal{U}(C) \arrow[d, dashed, "\mathcal{U}(f^*)"] & C \arrow[d, dashed, "f^*"] \\ & \mathcal{U}(Z) & Z \end{tikzcd}$$

commutes.

It is important to recap that in this definition, the universal arrow $D \to \mathcal{U}$ is the pair $(C, \eta)$. We can also contemplate the *dual* notion: a universal arrow $\mathcal{U} \to D$ can also be a pair $(C, \eta : \mathcal{U}(C) \to D)$ such that

$$\begin{tikzcd} D & \mathcal{U}(C) \arrow[l, "\eta"'] & C \\ & \mathcal{U}(Z) \arrow[ul, "f"] \arrow[u, dashed, "\mathcal{U}(f^*)"'] & Z \arrow[u, dashed, "f^*"'] \end{tikzcd}$$

commutes, with notation as above.

### 8.2 Products as Universal Arrows

Let's recover universal properties from this notion of a universal arrow. We will start with the universal property of product, which we have seen extensively thus far. Let $\mathcal{C}$ be a category where for any $X, Y \in \mathcal{C}$, we also have $X \times Y \in \mathcal{C}$ (we say such a category *has finite products*). Given a category, we can define an *product category* $\mathcal{C} \times \mathcal{C}$, where objects are pairs $(X, Y)$ with $X, Y \in \mathcal{C}$, and morphisms $(X_1, X_2) \to (Y_1, Y_2)$ are also pairs $(f_1, f_2)$, where

$$f_1 : X_1 \to Y_1, \qquad f_2 : X_2 \to Y_2.$$

Morphism composition is defined componentwise, meaning

$$(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1).$$

Identity morphisms are $1_{(A,B)} = (1_A, 1_B)$. It's a good exercise to show this satisfies the category axioms.

Let $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the functor $X \mapsto (X, X)$ and $f \mapsto (f, f)$, for $X \in \mathcal{C}$ and $f$ any morphism in $\mathcal{C}$. This functor is known as the (binary) *diagonal functor*.

Let $X, Y \in \mathcal{C}$. Since $\mathcal{C}$ has products, we also have the product $X \times Y$, together with natural projections $\pi_X, \pi_Y$. Let's construct a universal arrow using the product. We will be using the dual construction, since the product is terminal and the dual construction defines terminal universal objects.

Let's construct the universal arrow $\Delta \to (X, Y)$. To do this, we need an object $C \in \mathcal{C}$ and a morphism $\eta : \Delta(C) \to D$ in $\mathcal{C} \times \mathcal{C}$ satisfying the above requiremenets. The natural definition is to simply set

$$C = X \times Y, \qquad \eta = (\pi_X, \pi_Y).$$

This can be seen using the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\Delta"] & (X \times Y, X \times Y) \arrow[r, shift left, "\pi_X"] \arrow[r, shift right, "\pi_Y"'] & (X, Y). \end{tikzcd}$$

Is this indeed a univeral arrow? Let $Z \in \mathcal{C}$, and consider any morphism $(f, g) : (Z, Z) \to (X, Y)$ in $\mathcal{C} \times \mathcal{C}$. Since $f, g$ are morphisms in $\mathcal{C}$, and we must have $f : Z \to X$ and $g : Z \to Y$, by the universal property there exists a unique morphism $\varphi$ making the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

commute. We can then consider $\Delta(\varphi) : \Delta(Z) \to \Delta(X \times Y)$ to construct the diagram

$$\begin{tikzcd} (X, Y) & \Delta(X \times Y) \arrow[l, "(\pi_X, \pi_Y)"'] & & X \times Y \\ & \Delta(Z) \arrow[ul, "(f,g)"] \arrow[u, dashed, "\Delta(\varphi)"'] & & Z \arrow[u, dashed, "\varphi"'] \end{tikzcd}$$

Indeed, if we look back at the diagram for universal arrows, we will see that this satisfies the property of universal arrows, and thus $(X \times Y, (\pi_X, \pi_Y))$ defines a universal arrow $\Delta \to (X, Y)$.

Therefore, the universal property of products induces a universal arrow, but it is simple to show the converse is also true. When we defined universal properties, we found that universal objects are always isomorphic. The same is true here: if $(A, \mu)$ and $(B, \nu)$ are both universal arrows $D \to \mathcal{U}$ (or $\mathcal{U} \to D$) for some functor $\mathcal{U}$ and object $D$, then we necessarily have $A \cong B$. This is a great exercise to do on your own. Therefore, each universal arrow induces a universal property, and the challenge is simply to show that each universal property can be encoded as a universal arrow. This is indeed possible, as we saw in the product example, but is more difficult to do in generality.

## 8.3 Adjoint Functors

Universal properties arise as special cases, or are characterized by, many different constructions. For example, one can view universal properties as initial or terminal objects, which is the most simple definition. Additionally, they can be seen as limits or colimits of functors[5], representable functors, Kan extensions, universal arrows, and many other generalizations of categorical notions. One of the most important things, however, is *adjoint functors*.

We discussed earlier the idea of a *forgetful functor* $\mathcal{G}\text{rp} \to \mathcal{S}\text{et}$, which "forgets" information about a group. For those without an algebra background, consider the category $\mathcal{S}\text{et}_*$ of *pointed sets*. Objects are pairs $(S, s)$, where $s$ is just an element of the set $S$. Morphisms $(S, s) \to (T, t)$ are functions $f : S \to T$ that *preserve base points*, meaning $f(s) = t$. Basically, we have added some "extra structure" to the category $\mathcal{S}\text{et}$. We can define what's called a *forgetful functor* $\mathcal{U} : \mathcal{S}\text{et}_* \to \mathcal{S}\text{et}$ by *forgetting* this structure. The extra structure here is the "base point", so we just forget about the base point. $\mathcal{U}$ is then defined as $(S, s) \mapsto S$, and it's action on morphisms is just $f \mapsto f$.

Forgetful functors are generally notated with $U$ or $\mathcal{U}$, as I have done here. However, when I introduced universal arrows, I also used $\mathcal{U}$ for the functor in that definition. This is not a coincidence; the notion of *adjunction* arises naturally from universal arrows, and as we will see, forgetful functors usually arise as part of an adjunction relation.

Let $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ be a functor. Let $(A_1, u_1)$ be a universal morphism $X_1 \to \mathcal{U}$, and let $(A_2, u_2)$ be a universal morphism $X_2 \to \mathcal{U}$. This means $X_1, X_2 \in \mathcal{D}$ and $A_1, A_2 \in \mathcal{C}$. By the universal property of universal morphisms, any morphism $f : X_1 \to X_2$ induces a unique morphism $g : A_1 \to A_2$ making the diagram

$$\begin{tikzcd} X_1 \arrow[r, "u_1"] \arrow[d, "f"'] & \mathcal{U}(A_1) \arrow[d, dashed, "\mathcal{U}(g)"] & A_1 \arrow[d, dashed, "g"] \\ X_2 \arrow[r, "u_2"'] & \mathcal{U}(A_2) & A_2 \end{tikzcd}$$

commute. This follows due to the fact that $u_2 \circ f$ is a morphism $X_1 \to \mathcal{U}(A_2)$, so there is an induced morphism by the universal property.

Suppose that *every* object $X_i$ in $\mathcal{D}$ admits a universal arrow $(A_i, u_i) : D_i \to \mathcal{U}$. This means that we can assign every object $D_i$ to it's corresponding arrow $A_i$, which is an assignment $\mathcal{D} \to \mathcal{C}$. We want to turn this into a functor, but to do so we must define action on morphisms. As seen in the diagram above, for every morphism $f : X_i \to X_j$, by the universal property of universal arrows, there is a unique induced morphism $g : A_i \to A_j$. We can then simply map $f \mapsto g$, and this defines a functor. Denote this functor by $\mathcal{F} : \mathcal{D} \to \mathcal{C}$. We now have a pair of functors:

$$\begin{aligned} \mathcal{F} : \quad & \mathcal{D} \longrightarrow \mathcal{C} \\ \mathcal{U} : \quad & \mathcal{C} \longrightarrow \mathcal{D} \end{aligned}$$

These are very special functors. Specifically, they are *adjoint* functors. To proprly define adjoint functors, we need to first explore some of the properties of the functors $\mathcal{F}$ and $\mathcal{U}$.

Let $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Since $D$ admits a universal arrow $A_i$ to $\mathcal{U}$, we know that every morphism $D \to \mathcal{U}(C)$ induces a unique morphism $A_i \to C$, and moreover $A_i = \mathcal{F}(D)$. So we have that every morphism $D \to \mathcal{U}(C)$ induces a morphism $\mathcal{F}(D) \to C$. In other words, we have a function

$$\text{Hom}_{\mathcal{D}}(D, \mathcal{U}(C)) \to \text{Hom}_{\mathcal{C}}(\mathcal{F}(D), C).$$

Since the arrow is unique, this function is injective. However, we can show that this function is also *surjective*. Let $f : \mathcal{F}(D) \to C$. Recall that $\mathcal{F}(D) = A_i$, so we have

$$(\mathcal{U} \circ \mathcal{F})(D) = \mathcal{U}(A_i).$$

[5]Discussed later.

In other words, this raises to a morphism

$$\mathcal{U}(f) : \mathcal{U}(A_i) \to \mathcal{U}(C).$$

Since there is a morphism $u : D \to \mathcal{U}(A_i)$, there is a composition of morphisms

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[dr, "\mathcal{U}(f)\circ u"'] & \mathcal{U}(A_i) \arrow[d, "\mathcal{U}(f)"] \\ & \mathcal{U}(C) \end{tikzcd}$$

Therefore, for each $f : \mathcal{F}(D) \to C$, there is a morphism $\mathcal{U}(f)\circ u : D \to \mathcal{U}(C)$. The universal property of universal arrows says it's unique, and thus we have that the morphism $\mathcal{U}(f) \circ u : D \to \mathcal{U}(C)$ induces the morphism $f : \mathcal{F}(D) \to C$. Since this is precisely how our function was defined, we have that the function

$$\mathrm{Hom}_{\mathcal{D}}(D, \mathcal{U}(C)) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(D), C)$$

is bijective, and thus

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(D), C) \cong \mathrm{Hom}_{\mathcal{D}}(D, \mathcal{U}(C)).$$

This is called an *adjunction*.

**Definition 8** (Adjunction)**.** Let $\mathcal{C}, \mathcal{D}$ be categories, let $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ and $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ be functors. If for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$, there is an isomorphism

$$\varphi_{D,C} : \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(D), C) \cong \mathrm{Hom}_{\mathcal{D}}(D, \mathcal{U}(C))$$

which is natural in $C$ and $D$, then we say $\mathcal{F}$ and $\mathcal{U}$ are *adjoint functors*.

Since $\mathcal{F}$ appears on the *left* side of Hom, we say it is *left-adjoint* to $\mathcal{U}$, and call it a *left-adjoint functor*. Similarly, since $\mathcal{U}$ appears on the *righ* slot of Hom, we say it is *right-adjoint* to $\mathcal{F}$, and call it a *right-adjoint functor*. We write $\mathcal{F} \dashv \mathcal{U}$ to say that $\mathcal{F}$ is left-adjoint to $\mathcal{U}$, and $\mathcal{U}$ is right-adjoint to $\mathcal{F}$.

Adjoint functors are extremely important. MacLane, one of the founders of category theory, famously said "adjoint functors arise everywhere." One example for those with some algebra knowledge is Hom and tensor. Given a commutative ring $R$, we can define the functor $X \otimes_R - : R\text{-}\mathcal{M}\mathrm{od} \to R\text{-}\mathcal{M}\mathrm{od}$ for any $R$-module $X$. It turns out that for any $R$-modules $Y, Z$, there is an isomorphism

$$\mathrm{Hom}_R(X \otimes_R Y, Z) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_R(X, Z)),$$

where $\mathrm{Hom}_R(Y, Z)$ is viewed as an $R$-module by taking addition to be pointwise addition, and defining scalar multiplication in the natural way. In other words,

$$X \otimes_R - \dashv \mathrm{Hom}_R(X, -).$$

In general, forgetful functors are right-adjoint functors, and their left-adjoints are usually called *free functors*. For example, the forgetful functor $\mathcal{U} : \mathcal{G}\mathrm{rp} \to \mathcal{S}\mathrm{et}$ has a left-adjoint $\mathcal{F} : \mathcal{S}\mathrm{et} \to \mathcal{G}\mathrm{rp}$ called the *free* functor, which assigns each set $X$ to the *free group* $\mathcal{F}(X)$ on $X$. For those who know linear algebra, the forgetful functor $\mathcal{U} : k\text{-}\mathcal{V}\mathrm{ect} \to \mathcal{S}\mathrm{et}$ is right-adjoint, and it's left-adjoint $\mathcal{F} : \mathcal{S}\mathrm{et} \to k\text{-}\mathcal{V}\mathrm{ect}$ is also a free functor. The vector space $\mathcal{F}(X)$ for $X$ a set is defined as *all possible $k$-linear combinations in $X$*. This is called the *free* vector space on $X$, and it has $X$ as it's basis. Interestingly, all vector spaces are free, since being free simply means a space has a basis. This free functor is more interesting when we consider modules over an arbitrary ring, since modules need not be free.

**Remark.** Right-adjoint functors here arose as a family of universal morphisms $X_i \to \mathcal{U}$ for all $X_i \in \mathcal{D}$. Similarly, left-adjoint functors arise from the dual construction; meaning there is a family of universal morphisms $\mathcal{F} \to A_i$ for all $A_i \in \mathcal{C}$. Left- and right-adjoints can in fact be *defined* in this way, but we generally opt for the more traditional definition 8.

## 8.4 Naturality

I said that the isomorphism in adjunction relations is *natural* in both elements. Unpacking this means that if $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ are adjoint, and $Y_1, Y_2 \in \mathcal{C}$, $X_1, X_2 \in \mathcal{D}$, then the isomorphisms

$$\varphi_{X_1,Y_1} : \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(X_1), Y_1) \cong \mathrm{Hom}_{\mathcal{D}}(X_1, \mathcal{U}(Y_1)),$$

and similarly for $\varphi_{X_2,Y_2}$, must be such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(X_1), Y_1) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(f),g)"] \arrow[d, "\varphi_{X_1,Y_1}"'] & \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(X_2), Y_2) \arrow[d, "\varphi_{X_2,Y_2}"] \\
\mathrm{Hom}_{\mathcal{D}}(X_1, \mathcal{U}(Y_1)) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(f,\mathcal{U}(g))"'] & \mathrm{Hom}_{\mathcal{D}}(X_2, \mathcal{U}(Y_2))
\end{tikzcd}$$

commutes for all morphisms

$$\begin{array}{c} X_1 \xrightarrow{f} Y_1 \\ Y_1 \xrightarrow{g} Y_2 \end{array},$$

in $\mathcal{D}$ and $\mathcal{C}$, respectively.

This is a terrible way of seeing naturality, since it requires understanding $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(-), -)$, and similarly $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{U}(-))$ as *bifunctors*

$$\begin{array}{c} \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathcal{F}^{\mathrm{op}} \times 1_{\mathcal{C}}} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,-)} \mathcal{S}\mathrm{et} \\ \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \xrightarrow[1_{\mathcal{D}} \times \mathcal{U}^{\mathrm{op}}]{} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow[\mathrm{Hom}_{\mathcal{D}}(-,-)]{} \mathcal{S}\mathrm{et} \end{array}.$$

It is much better to see the naturality of adjunction via units and counits. Unfortunately, these notes are long enough and we do not have sufficient time to get into the weeds. I encourage the reader to pick up a text and explore this further on their own, as adjunction is extremely important. I will however present the unit and counit definition, but will not show how this definition flows from the adjunction relation.

Given an adjunction $\mathcal{F} \dashv \mathcal{U}$, one can define two natural transformations. The first is the *unit*, often denoted $\eta : 1_{\mathcal{D}} \Rightarrow \mathcal{U} \circ \mathcal{F}$, in which all components $\eta_X$ are *universal arrows.* The second natural transformation is the *counit*, often denoted $\varepsilon : \mathcal{F} \circ \mathcal{U} \Rightarrow 1_{\mathcal{C}}$, in which once again all components $\varepsilon_X$ are *universal arrows.* These natural transformations also satisfy the following property: for all $X \in \mathcal{C}$, we have that

$$1_{\mathcal{F}(X)} = \varepsilon_{\mathcal{F}(X)} \circ \mathcal{F}(\eta_X),$$
$$1_{\mathcal{U}(X)} = \mathcal{U}(\varepsilon_X) \circ \eta_{\mathcal{U}(X)}.$$

This is usually summarized as

$$1_{\mathcal{F}} = \varepsilon\mathcal{F} \circ \mathcal{F}\eta,$$

$$1_{\mathcal{U}} = \mathcal{U}\varepsilon \circ \eta\mathcal{U}.$$

We interpret the notation $\varepsilon\mathcal{F}$ as the natural transformation whose components look like $\varepsilon_{\mathcal{F}(X)}$, and similarly $\mathcal{F}\eta$ is intepreted as the natural transformation whose components look like $\mathcal{F}(\eta_X)$. Diagramatically, we write these as the commutative diagrams

$$\begin{tikzcd}
\mathcal{F} \arrow[r, Rightarrow, "\mathcal{F}\eta"'] \arrow[rr, Rightarrow, bend left, "1_{\mathcal{F}}"] & \mathcal{F} \circ \mathcal{U} \circ \mathcal{F} \arrow[r, Rightarrow, "\varepsilon\mathcal{F}"'] & \mathcal{F} \\
\mathcal{U} \arrow[r, Rightarrow, "\eta\mathcal{U}"] \arrow[rr, Rightarrow, bend right, "1_{\mathcal{U}}"'] & \mathcal{U} \circ \mathcal{F} \circ \mathcal{F} \arrow[r, Rightarrow, "\mathcal{U}\varepsilon"] & \mathcal{U}
\end{tikzcd}$$

Most authors use single arrows $\to$ instead of double arrows $\Rightarrow$, as natural transformations in this diagram are 1-morphisms and double arrows generally denote 2-moprhisms. I wanted to clarify that the arrows were natural transformations, so I opted fo that notation instead.

Adjunction is an extremely important phenomenon in category theory. Fully understanding it and it's implications is *not* a simple task, and I encourage the reader to explore further on their own.

# 9 Limits and Colimits

## 9.1 Definition, Cones

Another useful notion, and indeed yet *another* way of characterizing universal properties, is with the notion of the *limit* or *colimit* of a functor. These admit simple descriptions using universal arrows, but we will spell out their definitions in full after.

Earlier, we defined the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$. If $\mathcal{J}$ is also a category, we can also define a diagonal functor $\Delta : \mathcal{C} \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$. For all $C \in \mathcal{C}$, the image $\Delta(C)$ is a functor $\mathcal{J} \to \mathcal{C}$. Specifically, $\Delta(C)$ is the *constant functor*, which maps $J \mapsto C$ for all $J \in \mathcal{J}$, and each morphism to $1_C$. Each morphism $f : C \to D$ in $\mathcal{C}$ is mapped to the natural transformation $\Delta(f) : \Delta(C) \Rightarrow \Delta(D)$, where each component $\Delta(f)_J : C \to D$ is just defined to be $f$ for all $J \in \mathcal{J}$. I leave it as an exercise to reconcile this definition with our earlier definition of a diagonal functor.

**Definition 9** (Limit/Colimit)**.** Let $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ be a functor. With notation as above, the *limit* of $\mathcal{F}$, denoted $\varprojlim \mathcal{F}$, is a *universal arrow* from $\Delta$ to $\mathcal{F}$. The *colimit* of $\mathcal{F}$ is a universal arrow from $\mathcal{F}$ to $\Delta$.

Let's unpack this definition. Typically the category $\mathcal{J}$ is a indexing category, and usually $\mathcal{J}$ is finite, or at least small. Let's consider a *limit* of the functor $\mathcal{F} : \mathcal{J} \to \mathcal{C}$. Per our definition, this would be a universal arrow $\Delta \to \mathcal{F}$. Since $\Delta$ is a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$, a universal arrow would be an object $L \in \mathcal{C}$ and a *natural transformation* $\eta : \Delta(L) \Rightarrow \mathcal{F}$. The universality of the arrow tells us that for any $Z \in \mathcal{C}$ and any natural transformation $\mu : \Delta(Z) \Rightarrow \mathcal{F}$, there is a unique morphism $f : Z \to L$ making the diagram

$$\begin{tikzcd}
\mathcal{F} & \Delta(L) \arrow[l, "\eta"'] & L \\
& \Delta(Z) \arrow[ul, "\mu"] \arrow[u, dashed, "\Delta(f)"'] & Z \arrow[u, dashed, "f"]
\end{tikzcd}$$

commute.

Recall that a natural transformation $\eta : \Delta(L) \Rightarrow \mathcal{F}$ is just a family of morphisms $\eta_J : \Delta(L)(J) \to \mathcal{F}(J)$ for each $J \in \mathcal{J}$, satisfying a certain property. However, $\Delta(L)$ is the *constant* functor, meanining $\Delta(L)(J)$ is always $L$. This means that $\eta$ defines a family of morphisms $\eta_J : L \to \mathcal{F}(J)$. Additionally, we defined $\Delta(L)(f) = 1_L$ for any morphism $f$ in $\mathcal{J}$.

Since $\eta$ is a natural transformation, we must have that for any $\varphi : J_1 \to J_2$ in $\mathcal{J}$, the diagram

$$\begin{tikzcd}
L \arrow[r, "1_L"] \arrow[d, "\eta_1"'] & L \arrow[d, "\eta_2"] \\
\mathcal{F}(J_1) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(J_2)
\end{tikzcd}$$

commutes. Recall here that $\Delta(L)(J) = L$ and $\Delta(L)(\varphi) = 1_L$. This can be simplified to just saying that the diagram

$$\begin{tikzcd}
& L \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] & \\
\mathcal{F}(J_1) \arrow[rr, "\mathcal{F}(\varphi)"'] & & \mathcal{F}(J_2)
\end{tikzcd}$$

commutes for all $J_1, J_2 \in \mathcal{J}$ and all $\varphi : J_1 \to J_2$. We say an object $L$, together with a family of morphisms $\eta_i$ for each $J_i \in \mathcal{J}$ is a *cone* of $\mathcal{F}$.

The universality condition says that if we have another natural transformation $\mu : \Delta(Z) \Rightarrow \mathcal{F}$; that is, another *cone* for $\mathcal{F}$:

$$\begin{tikzcd}
& Z \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & \\
\mathcal{F}(J_1) \arrow[rr, "\mathcal{F}(\varphi)"'] & & \mathcal{F}(J_2)
\end{tikzcd}$$

then there is a unique morphism $f : Z \to L$ such that each $\mu_i = \eta_i \circ f$. In other words, making the diagram

$$\begin{tikzcd}
& Z \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\mu_1"'] \arrow[ddr, bend left, "\mu_2"] & \\
& L \arrow[dl, "\eta_1"] \arrow[dr, "\eta_2"'] & \\
\mathcal{F}(J_1) \arrow[rr, "\mathcal{F}(\varphi)"'] & & \mathcal{F}(J_2)
\end{tikzcd}$$

commute for any $\varphi : J_1 \to J_2$ in $\mathcal{J}$.

This means that the criteria for being a universal arrow $\Delta \to \mathcal{F}$, and thus a limit of a functor, is to be *terminal with respect to cones*. In this example, $\varprojlim \mathcal{F} = L$ is the limit of $\mathcal{F}$. Colimits arise analogously as initial objects with respect to *co-cones*. I encourage the reader to make precise what a "co-cone" is, and make precise this statement.

## 9.2 Products as Limits

We now return to our universal property of products to give an example of a statement made previously, which is that *all universal properties are limits or colimits*. Let $\mathcal{J}$ be the category with

2 objects $\{\mathbf{1}, \mathbf{2}\}$, and two morphisms, which are the identity morphisms:

$$1_{\mathbf{1}} \circlearrowright \mathbf{1} \qquad \mathbf{2} \circlearrowleft 1_{\mathbf{2}}$$

Since any functor $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ must preserve identity morphisms, defining $\mathcal{F}$ simply amounts to choosing two elements $X_1, X_2 \in \mathcal{C}$ such that $\mathcal{F}(\mathbf{1}) = X_1$, $\mathcal{F}(\mathbf{2}) = X_2$.

A limit $\varprojlim \mathcal{F}$, if it exists, would then come with two morphisms, $\eta_1 : \varprojlim \mathcal{F} \to X_1$ and $\eta_2 : \varprojlim \mathcal{F} \to X_2$. As a diagram, this looks like

$$\begin{tikzcd}
& \varprojlim \mathcal{F} \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] & \\
X_1 & & X_2
\end{tikzcd}$$

Since there is no morphism $\mathbf{1} \to \mathbf{2}$, there is no *relevant* morphism $X_1 \to X_2$. The fact that the limit is terminal simply means that if $Z, \mu_1, \mu_2$ is any other cone, then the diagram

$$\begin{tikzcd}
& Z \arrow[d, dashed, "\varphi"] \arrow[ddl, bend right, "\mu_1"'] \arrow[ddr, bend left, "\mu_2"] & \\
& \varprojlim \mathcal{F} \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] & \\
X_1 & & X_2
\end{tikzcd}$$

commutes. We can rewrite this as

$$\begin{tikzcd}
& Z \arrow[dl, "\mu_1"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "\mu_2"] & \\
X_1 & \varprojlim \mathcal{F} \arrow[l, "\eta_1"] \arrow[r, "\eta_2"'] & X_2
\end{tikzcd}$$

Note that this is exactly the same as the universal property of products:

$$\begin{tikzcd}
& Z \arrow[dl, "\mu_1"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "\mu_2"] & \\
X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2
\end{tikzcd}$$

This means that

$$\varprojlim \mathcal{F} \cong X_1 \times X_2.$$

And thus, the universal property of products is simply the limit of a functor $\mathcal{F} : \mathcal{J} \to \mathcal{C}$.

## 9.3 Preservation of Limits

**Theorem 2.** *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a left-adjoint functor. Then $\mathcal{F}$ preserves colimits, meaning for any functor $\mathcal{G}$, one has*

$$\mathcal{F}\left(\varinjlim \mathcal{G}\right) \cong \varinjlim \left(\mathcal{F} \circ \mathcal{G}\right).$$

*Similarly, a right-adjoint functor preserves limits, meaning if $\mathcal{U}$ is right-adjoint functor, then*

$$\mathcal{U}\left(\varprojlim \mathcal{G}\right) \cong \varprojlim \left(\mathcal{U} \circ \mathcal{G}\right).$$

Theorem 2 is a classical theorem in category theory, and it's one of the many reasons adjoint functors are important, and also one of the reasons limits are useful. For example, here's a fun application of this theorem. In the category $k$-$\mathcal{V}$ect of $k$-vector spaces, and more generally the category $R$-$\mathcal{M}$od of $R$-modules, the product $A \times B$ and coproduct $A \amalg B$ are isomorphic. Because of this, we instead call it the *direct sum*, and denote it $A \oplus B$. Since the product is a limit, and the coproduct is a *colimit*, any adjoint functor will preserve it. Let's work with vector spaces, that way any finite-dimensional $k$-vector space $V$ is isomorphic to $k^n$ for some $n$.

Recall that I previously stated that the tensor product $X \otimes_k -$ is left-adjoint to Hom, and is thus a left-adjoint functor. By theorem 2, if $U, V, W$ are vector spaces, then

$$(V \oplus W) \otimes U \cong (V \otimes U) \oplus (W \otimes U).$$

Induction then gives us

$$\left(\bigoplus_{i \in I} V_i\right) \otimes U \cong \bigoplus_{i \in I} (V_i \otimes U).$$

Question: what is the dimension of $U \otimes V$? Since $U, V$ are vector spaces, we can write them as $k^n$ and $k^m$, respectively. We then have

$$k^n \otimes k^m = \left(\bigoplus_{i=1}^{n} k\right) \otimes \left(\bigoplus_{j=1}^{m} k\right) \cong \bigoplus_{i=1}^{n} \left(k \otimes \bigoplus_{j=1}^{m} k\right) \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} (k \otimes k).$$

It's easily shown that $k \otimes k \cong k$, and thus we have

$$\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} (k \otimes k) \cong \bigoplus_{i=1}^{mn} k = k^{mn}.$$

Therefore,

$$k^n \otimes k^m \cong k^{mn},$$

and thus

$$\dim\left(k^n \otimes k^m\right) = mn.$$

**Remark.** Just like universal products, a category need not have a (co)limit for every functor. If the category has all *small* limits, meaning all limits where the indexing category $\mathcal{J}$ is small, then we say it is *complete*. Similarly, if the category has all small *colimits*, we say it is *cocomplete*.

# 10 Monads

## 10.1 A Monad is a Monoid in the Category of Endofunctors

The first sentence on the Wikipedia page for monads used to say (roughly) "A monad is a monoid in the category of endofunctors". Although technically true, this is not really what a person trying to learn about monads wants to be the first definition they read, and as such the sentence became a minor meme for some time. In this penultimate section, we will unpack what a monad is, and in the final section, we will learn what a monoid is, and will show that, indeed, a monad is a monoid in the category of endofunctors.

Let's begin with some preliminaries. An endomorphism is a morphism $X \to X$; that is, a morphism from an object to itself. In the same vein, we call a functor $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ from a category to itself an *endofunctor*. We can then consider the composition

$$\mathcal{T}^2 = \mathcal{T} \circ \mathcal{T} : \mathcal{C} \to \mathcal{C}.$$

Similarly, we can consider the composition

$$\mathcal{T}^3 = \mathcal{T}^2 \circ \mathcal{T} : \mathcal{C} \to \mathcal{C}.$$

If $\mu : \mathcal{T}^2 \Rightarrow \mathcal{T}$ is a natural transformation with components $\mu_X : \mathcal{T}^2(X) \to \mathcal{T}(X)$ for any $X \in \mathcal{C}$, then we can consider the two *new* natural transformations $\mathcal{T}\mu$ and $\mu\mathcal{T}$, both of which go $\mathcal{T}^3 \Rightarrow \mathcal{T}^2$. We define these natural transformations as having components

$$(\mathcal{T}\mu)_X : \mathcal{T}^3(x) \to \mathcal{T}^2(x),$$

and

$$(\mu\mathcal{T})_X = \mu_{\mathcal{T}(X)}.$$

Now that we understand this notation, we can present the definition of a monad.

**Definition 10** (Monad)**.** A monad is a category $\mathcal{C}$, together with an endofunctor $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ and two natural transformations

$$\eta : 1_{\mathcal{C}} \Rightarrow \mathcal{T}, \qquad \mu : \mathcal{T}^2 \Rightarrow \mathcal{T},$$

such that

$$\mu \circ \mathcal{T}\mu = \mu \circ \mu\mathcal{T},$$
$$\mu \circ \mathcal{T}\eta = \mu \circ \eta\mathcal{T} = 1_{\mathcal{T}}.$$

Equivalently, the diagrams

$$\begin{tikzcd}
\mathcal{T}^3 \arrow[r, "\mathcal{T}\mu"] \arrow[d, "\mu\mathcal{T}"'] & \mathcal{T}^2 \arrow[d, "\mu"] \\
\mathcal{T}^2 \arrow[r, "\mu"'] & \mathcal{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T} \arrow[r, "\eta\mathcal{T}"] \arrow[dr, "1_{\mathcal{T}}"'] & \mathcal{T}^2 \arrow[d, "\mu"] & \mathcal{T} \arrow[l, "\mathcal{T}\eta"'] \arrow[dl, "1_{\mathcal{T}}"] \\
& \mathcal{T} &
\end{tikzcd}$$

commute.

Let's unpack this definition. To do this, we need to introduce the *classical* notion of a monoid. We will revisit monoids from a categorical perspective soon, but for now the following definition will suffice. Let $M$ be a set, and let $\cdot : M \times M \to M$ be an operation known as *multiplication*; if $m, n \in M$, we denote their product as $m \cdot n$ or $mn$. Multiplication must be associative:

$$(mn)p = m(np)$$

and there must exist $1 \in M$ such that for all $m \in M$,

$$m1 = 1m = m.$$

That is, 1 is an identity element. For the algebraists here, a monoid is a group that need not have inverses.

We can view a monad as a type of monoid, but with a key distinction that makes this definition much more abstract. In a regular monoid, we have multiplication defined between two elements $x, y \in M$. This means that multiplication takes a pair $(x, y)$, and maps it to the element $xy$. Hence, $\cdot$ is a function $M \times M \to M$. The key thing to note here is that we are using the *cartesian product* of $M$ to define multiplication. However, we could also try defining multiplication as a function from the *tensor product* of two modules $M \otimes M \to M$, or from the *direct sum* of two abelian groups $M \oplus M \to M$, or other such "product-like" operations. These multiplication definitions would be very different from what we're used to, since there may be no meaningful way to multiply *any* arbitrary elements $x, y \in M$, depending on what the structure of the product.

In a monad, instead of having a set $M$, we instead have an *endofunctor* $\mathcal{T} : \mathcal{C} \to \mathcal{C}$, which means that we probably need to try one of these new products. We use *morphism composition* between functors as the product, meaning that instead of having something like $\mathcal{T} \times \mathcal{T}$, we instead use $\mathcal{T} \circ \mathcal{T}$. How is it then possible to define multiplication of elements in a monad?

First, recall that multiplication is given by a morphism. Since our object is a functor, we need a morphism $\mu : \mathcal{T} \circ \mathcal{T} \to \mathcal{T}$ to give multiplication. In other words, $\mu$ is a *natural transformation*. Also recall that general categories need not have a meaniful notion of elements, and thus instead of describing how to multiply elements of $M$, it suffices to describe the action of multiplication on the object $M$; or in this case, $\mathcal{T}$. This is done by the first commutative diagram, which essentially states that multiplication should be associative.

Finally, a monoid has a unit element. In a classical monoid, this can be described by finding an object $I \in \mathcal{S}\mathrm{et}$ which is an *identity* with respect to the cartesian product $\times$ (a singleton), and considering a morphism $I \to M$. Since $I$ is a singleton, it's image in $M$ defines the identity element. This means the information in the identity can *also* be encoded by a morphism, so long as we have an identity $I$ for the arbitrary product $\otimes$ we are working with. In this case, the product is composition, so the *identity functor* $1_{\mathcal{C}}$ is an identity by definition. Therefore, an identity element of the *monad* is simply a morphism $\eta : 1_{\mathcal{C}} \to \mathcal{T}$; once again, $\eta$ must be a natural transformation. The final diagram just defines left- and right-unital laws.

## 10.2 Monads via Adjunction

Every adjunction gives rise to a monad. To see this, let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{U} : \mathcal{D} \to \mathcal{C}$ have $\mathcal{F} \dashv \mathcal{U}$. For now, we omit the $\circ$ in function composition for simplicity. The adjunction induces an endomorphism $\mathcal{UF} : \mathcal{C} \to \mathcal{C}$, and they also induce the unit $\eta : 1_{\mathcal{C}} \to \mathcal{UF}$, and the counit $\varepsilon : \mathcal{FU} \to 1_{\mathcal{D}}$. We can clearly view the unit $\eta$ as the identity in the monad $\mathcal{UF}$, but defining multiplication is less straightforward.

Consider the natural transformation $\mu = \mathcal{U}\varepsilon\mathcal{F}$. Since $\varepsilon : \mathcal{FU} \to 1_{\mathcal{D}}$, we have

$$\mu : \mathcal{UFUF} \to \mathcal{UF}.$$

This can be seen explicitly by constructing each component $\varepsilon_X$ as $\mathcal{U}(\varepsilon_{\mathcal{F}(X)})$, noting that $\mathcal{F}(X) \in \mathcal{D}$, and thus $\mathcal{U}(\varepsilon_{\mathcal{F}(X)}) \in \mathcal{C}$. This suffices as a definition for multiplication, since associativity is equivalent to commutivity of the absurd looking diagram

$$\begin{tikzcd}
\mathcal{UFUFUF} \arrow[r, "\mathcal{UFU}\varepsilon\mathcal{F}"] \arrow[d, "\mathcal{U}\varepsilon\mathcal{FUF}"'] & \mathcal{UFUF} \arrow[d, "\mathcal{U}\varepsilon\mathcal{F}"] \\
\mathcal{UFUF} \arrow[r, "\mathcal{U}\varepsilon\mathcal{F}"'] & \mathcal{UF}
\end{tikzcd}$$

This diagram is induced by the slightly more general diagram

$$\begin{tikzcd} \mathcal{FUFU} \arrow[r, "\mathcal{FU}\varepsilon"] \arrow[d, "\varepsilon\mathcal{FU}"'] & \mathcal{FU} \arrow[d, "\varepsilon"] \\ \mathcal{FU} \arrow[r, "\varepsilon"'] & 1_{\mathcal{D}} \end{tikzcd}$$

We have not yet built the machinery to precisely formulate *why* this diagram commutes. If you are already familiar with *horizontal composition*, try showing that the horizontal composite $\varepsilon\varepsilon = \varepsilon(\mathcal{FU}\varepsilon) = (\varepsilon\mathcal{FU})\varepsilon$.

The left and right unit laws

$$\begin{tikzcd} \mathcal{UF} \arrow[r, "\eta\mathcal{UF}"] \arrow[d, "\mathcal{UF}\eta"'] \arrow[dr, "1_{\mathcal{UF}}"] & \mathcal{UFUF} \arrow[d, "\mathcal{U}\varepsilon\mathcal{F}"] \\ \mathcal{UFUF} \arrow[r, "\mathcal{U}\varepsilon\mathcal{F}"'] & \mathcal{UF} \end{tikzcd}$$

amount to the triangular identities $\mathcal{U}\varepsilon \circ \eta\mathcal{U} = 1$ and $\varepsilon\mathcal{F} \circ \mathcal{F}\eta = 1$. I leave rigorizing this as an exercise. Every adjunction thus induces a monad. The converse is also true, and I encourage the enterprising reader to explore it on their own.

### 10.3 Comonads

As is always the case in categorical constructions, there is indeed a dual notion to a monad, being that of a comonad. As usual, this is a monad in the opposite category, and thus flipping all the arrows suffices. Given a category $\mathcal{C}$ and an endofunctor $\mathcal{L}$ on $\mathcal{C}$, a natural transformation $\varepsilon : \mathcal{L} \Rightarrow 1_{\mathcal{C}}$, and another natural transforation $\delta : \mathcal{L} \Rightarrow \mathcal{L}^2$, we say that $(\mathcal{L}, \varepsilon, \delta)$ forms a *comonad* if the diagrams

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathcal{L}^2 \arrow[d, "\mathcal{L}\delta"] \\ \mathcal{L}^2 \arrow[r, "\delta\mathcal{L}"'] & \mathcal{L}^3 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, "1_{\mathcal{L}}"] & \mathcal{L}^2 \arrow[d, "\mathcal{L}\varepsilon"] \\ \mathcal{L}^2 \arrow[r, "\varepsilon\mathcal{L}"'] & \mathcal{L} \end{tikzcd}$$

commute. Any adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\mathcal{F}"] & \mathcal{D} \arrow[l, shift left, "\mathcal{U}"] \end{tikzcd}$$

with $\mathcal{F} \dashv \mathcal{U}$ defines a comonad in $\mathcal{D}$.

## 11 Monoids

### 11.1 Monoidal Categories

We saw in the last section that monads arise from adjunction. In fact, monads can be used to study adjunctions, and vice versa. Monads also served as a sort of abstraction of monoids, so a more thorough abstraction of monoids is in order to provide some extra framework for the study of monads.

In order to define monoids, we first need a suitable category for them to live in. Recall when we defined monads, we replaced $\times$ with $\circ$ for our definition of multiplication. We thus need a category

$\mathcal{C}$, equipped with a suitable product $\otimes$ often called the *tensor product*, that serves as a generalization of the cartesian product construction.

We would expect that a monoidal category be a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is *associative*, meaning we have an equality of functors

$$\otimes(-, \otimes(-,-)) = \otimes(\otimes(-,-),-) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

In more conventional notation, this implies that for any $X, Y, Z \in \mathcal{C}$, we have an equality

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z).$$

Since monoids in the classical sense have identity elements, we would also expect that there exists $I \in \mathcal{C}$ such that $I \otimes X = X$ and $X \otimes I = X$ for all $X \in \mathcal{C}$.

Although this is the intuitive definition, it is also fairly naive. As has likely become apparent by now, in category theory we rarely want to ask that two objects be preciely equals to each other. This is an extremely strong requirement, and we are also often more concerned with when they are *isomorphic*. This is especially true for categorical constructions such as functors. In our definition, we required an *equality* of functors

$$\otimes(-, \otimes(-,-)) = \otimes(\otimes(-,-),-),$$

and an *equality*

$$I \otimes X = X \otimes I = X.$$

It is much more natural to ask that these *not* be equal, but instead be *naturally isomorphic* by a given natural isomorphism. More explicitly, there is a natural isomorphism $\alpha : \otimes(1_{\mathcal{C}} \times \otimes) \Rightarrow \otimes(\otimes \times 1_{\mathcal{C}})$, which induces isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

for all $A, B, C \in \mathcal{C}$. We also expect that there be two more natural isomorphisms $\lambda : I \otimes - \Rightarrow 1_{\mathcal{C}}$ and $\rho : - \otimes I \Rightarrow 1_{\mathcal{C}}$, meaning that for any $A \in \mathcal{C}$, there are induced morphisms

$$\lambda_A : I \otimes A \cong A, \qquad \rho_A : A \times I \cong A.$$

In this sense, $\otimes$ is associative *up to natural isomorphism*, and $I$ is a unit *up to natural isomorphism.* This is indeed the definition of a monoidal category.

**Definition 11** (Monoidal Category)**.** Let $\mathcal{C}$ be a category, let $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a bifunctor, let $I \in \mathcal{C}$ be an object, and let there exist natural transformations

$$\alpha : \otimes(-, \otimes(-,-)) \Rightarrow \otimes(\otimes(-,-),-),$$

$$\lambda : - \otimes I \Rightarrow 1_{\mathcal{C}}, \qquad \rho : I \otimes - \Rightarrow 1_{\mathcal{C}}.$$

Then $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is a *monoidal category* if for all $A, B, C, D \in \mathcal{C}$, the *pentagon diagram*

$$\begin{tikzcd}
 & & (A \otimes B) \otimes (C \otimes D) \arrow[drr, "\alpha"] & & \\
A \otimes (B \otimes (C \otimes D)) \arrow[urr, "\alpha"] \arrow[dr, "1 \otimes \alpha"'] & & & & ((A \otimes B) \otimes C) \otimes D \\
 & A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[ur, "\alpha \otimes 1"'] &
\end{tikzcd}$$

commutes, and the triangular diagram

$$\begin{tikzcd}
A \otimes (I \otimes C) \arrow[r, "\alpha"] \arrow[dr, "1\otimes\lambda"'] & (A \otimes I) \otimes C \arrow[d, "\rho\otimes 1"] \\
& A \otimes C
\end{tikzcd}$$

commutes, with components of the natural transformations as is evident.

The pentagon and triangle diagrams are *coherence conditions*, which simply ensure that although associativity and unity are only defined up to isomorphism, things still behave as we would expect.

**Remark.** We generally call $\alpha$ the associator, and we call $\lambda$ and $\rho$ the left- and right-unitors, respectively.

## 11.2 Monoids

Monoidal categories can be thought of as a "good place to define monoids in". They show up in other important areas of category theory; for example one can define an "enriched" category $\mathcal{C}$ over a monoidal category, wherein hom-sets in $\mathcal{C}$ are actually *objects* in the monoidal category. Although this is an interesting aside and illustrates that monoidal categories have widespead applications, it is indeed a topic for another discussion. Instead, we turn our attention to monoids themselves.

**Definition 12** (Monoid)**.** Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. A *monoid $M$* in $\mathcal{C}$ is an object $C \in \mathcal{C}$ and two morphisms

$$\mu : C \otimes C \to C, \qquad \eta : I \to C$$

such that the diagrams

$$\begin{tikzcd}
C \otimes (C \otimes C) \arrow[r, "\alpha"] \arrow[d, "1\otimes\mu"'] & (C \otimes C) \otimes C \arrow[r, "\mu\otimes 1"] & C \otimes C \arrow[d, "\mu"] \\
C \otimes C \arrow[rr, "\mu"'] & & C
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes C \arrow[r, "\eta\otimes 1"] \arrow[dr, "\lambda"'] & C \otimes C \arrow[d, "\mu"] & C \otimes I \arrow[l, "1\otimes\eta"'] \arrow[dl, "\rho"] \\
& C &
\end{tikzcd}$$

commute.

## 11.3 A Monad is a Monoid in the Category of Endofunctors

**Theorem 3** (A Monad is a Monoid in the Category of Endofunctors)**.** *A monad is a monoid in the category of endofunctors.*

**Proof.** First, we must show that the category $\mathrm{End}(\mathcal{C})$ of endofunctors $\mathcal{C} \to \mathcal{C}$ is indeed a monoidal category, as defined for monads. We take the tensor product to be composition of functors $\circ$, which is associative by definition of a morphism. Thus, we can define the associator $\alpha$ as the identity natura transformation 1, and the pentagon diagram commutes manifestly.

Moreover, note that since identity morphisms are identities with respect to morphism composition, we have $\mathcal{T} \circ 1_{\mathcal{C}} = 1_{\mathcal{C}} \circ \mathcal{T} = \mathcal{T}$ for any $\mathcal{T} \in \mathrm{End}(\mathcal{C})$. Thus, the left and right unitors can be defined as the identity natural transformations, and the triangular diagram commutes manifestly once again. Thus, $(\mathrm{End}(\mathcal{C}), \circ, 1_{\mathcal{C}})$ is a monoidal category.

Now let $(\mathrm{End}(\mathcal{C}), \mathcal{T}, \mu, \eta)$ be a monad. We want to show it is a monoid. Since the alternator and unitors in $\mathrm{End}(\mathcal{C})$ are the identities, we can reduce the diagrams in definition 12 to

$$\begin{tikzcd}
\mathcal{T} \circ \mathcal{T} \circ \mathcal{T} \arrow[r, "\mu \circ 1_{\mathcal{T}}"] \arrow[d, "1_{\mathcal{T}} \circ \mu"'] & \mathcal{T} \circ \mathcal{T} \arrow[d, "\mu"] \\
\mathcal{T} \circ \mathcal{T} \arrow[r, "\mu"'] & \mathcal{T}
\end{tikzcd}$$

and

$$\begin{tikzcd}
1_{\mathcal{C}} \circ \mathcal{T} \arrow[r, "\eta \circ 1_{\mathcal{T}}"] \arrow[dr, "1_{\mathcal{T}}"'] & \mathcal{T} \circ \mathcal{T} \arrow[d, "\mu"] & \mathcal{T} \circ 1_{\mathcal{C}} \arrow[l, "1_{\mathcal{T}} \circ \eta"'] \arrow[dl, "1_{\mathcal{T}}"] \\
 & \mathcal{T} &
\end{tikzcd}$$

In this case, the action of $\circ$ on morphisms of $\mathrm{End}(\mathcal{C})$ (aka composition of natural transformations) is *horizontal composition* ■

# A Glossary of Notation

$f : X \to Y$: $f$ is a morphism from $X$ to $Y$
$\eta : \mathcal{F} \Rightarrow \mathcal{G}$: $\eta$ is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$. $\eta_X$ then denotes the component of $\eta$ at $X$, i.e. the morphism $\eta_X : \mathcal{F}(X) \to \mathcal{G}(X)$
$\mathrm{Hom}_{\mathcal{C}}(X, Y)$: class of all morphisms $X \to Y$ in a category $\mathcal{C}$
$\Delta$: the diagonal functor
$\eta$ (in monoids or adjunction): unit in adjunctions and monoids
$\varepsilon$ (in adjunction): counit in adjunctions
$\mu$ (in monoids): multiplication in monoids
$\otimes$: tensor product or monoidal product
$\alpha$: the associator
$\lambda$: the left-unitor
$\rho$: the right-unitor
$\varprojlim \mathcal{F}$: limit of a functor $\mathcal{F}$
$\varinjlim \mathcal{F}$: colimit of a functor $\mathcal{F}$
$\mathrm{End}_{\mathcal{C}}(X)$: morphisms $X \to X$ in a category $\mathcal{C}$
$\mathrm{Fun}(\mathcal{C}, \mathcal{D})$: class of all functors $\mathcal{C} \to \mathcal{D}$
$\mathrm{Nat}(\mathcal{F}, \mathcal{G})$: class of all natural transformations $\mathcal{F} \Rightarrow \mathcal{G}$
$\times$: product
$\amalg$: coproduct
$\oplus$: direct sum
$\cong$: isomorphism
$\mathrm{Hom}_{\mathcal{C}}(X, -)$: the functor $A \mapsto \mathrm{Hom}_{\mathcal{C}}(X, A)$
$h_X$: $\mathrm{Hom}(-, X)$
$X \otimes_R -$: the functor $M \mapsto X \otimes_R M$
$\mathcal{C}^{\mathrm{op}}$: the opposite category
$f : X \hookrightarrow Y$: $f$ is an injective function from $X$ to $Y$
$f : X \twoheadrightarrow Y$: $f$ is a surjective function from $X$ to $Y$

# References


[1] Emily Riehl. Category theory in context. https://math.jhu.edu/~eriehl/context.pdf, 2014. Johns Hopkins University lecture notes.