

Contents

1	Preliminaries	2
1.1	Sets and Equivalence Relations	2
2	The Integers	7
2.1	Mathematical Induction	7
2.2	The Division Algorithm	8
3	Groups	10
3.1	Integer Equivalence Classes and Symmetries	10
3.2	Definitions and Examples	11
3.3	Subgroups	15
4	Cyclic Groups	16
4.1	Cyclic Subgroups	16
4.2	Multiplicative Group of Complex Numbers	17
4.3	The Method of Repeated Squares	19
5	Permutation Groups	20
5.1	Definitions and Notation	20
5.2	Dihedral Groups	26
6	Cosets and Lagrange's Theorem	27
6.1	Cosets	27
6.2	Lagrange's Theorem	28
6.3	Fermat's and Euler's Theorems	31
7	Isomorphisms	32
7.1	Definition and Examples	32
7.2	Direct Products	36
7.2.1	External Direct Products	36

Chapter 1

Preliminaries

1.1 Sets and Equivalence Relations

Set Theory

Definition 1.1 (Set). A set is any well-defined collection of objects; that is, it is defined in such a manner that we can tell whether x belongs to the set or not.

We denote sets by capital letters, and we refer to the objects that belong to them as their elements. We have two notations for defining sets. Sometimes we write all their elements within curly brackets:

$$X = \{x_1, x_2, \dots, x_n\}$$

and sometimes we use set-builder notation:

$$X = \{x \mid x \text{ satisfies } \mathcal{P}\}$$

where all elements x of the set X must satisfy some property \mathcal{P} . For example, consider the set of even positive integers:

$$E = \{x \in \mathbb{N} \mid \frac{x}{2} \in \mathbb{N}\}$$

We write $2 \in E$ to denote set membership, and $-3 \notin E$ to denote set exclusion. The following are shorthands for rather important sets:

- \mathbb{N} is the set of all naturals.
- \mathbb{Z} is the set of all integers.
- \mathbb{Q} is the set of all rationals.
- \mathbb{R} is the set of all reals.
- \mathbb{C} is the set of all complex numbers.

We can define various relations and operations on sets.

Definition 1.2 (Subset). if A and B are sets, we say that $A \subset B$, or A is a subset of B , if every element of A is also in B .

For example,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Trivially, all sets are their own subsets. We also find that two sets are equal if $A \subseteq B$ and $B \subseteq A$.

Definition 1.3 (Empty Set). The empty set, denoted \emptyset , is the set $\{\}$.

We can define two operations on sets to construct new sets from two sets:

Definition 1.4 (Union). The union $A \cup B$ of two sets A and B is defined as

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

In other words, all elements in either A , B , or both.

Definition 1.5 (Intersection). The intersection $A \cap B$ of two sets A and B is defined as

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

In other words, all elements in both A and B .

We can consider the union and intersection of more than just two sets with the big operators. The union is given as

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

and the intersection is

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Definition 1.6 (Disjoint). Two sets A and B are disjoint if

$$A \cap B = \emptyset$$

Oftentimes, we will work within one fixed set U , called the *Universal Set*.

Definition 1.7 (Compliment of a Set). From the previous definition of a Universal Set, we define the compliment of a set A as

$$A' = \{x \mid x \in U \wedge x \notin A\}$$

Definition 1.8 (Set Difference). We define the difference between two sets A and B as

$$A \setminus B = A \cap B' = \{x \mid x \in A \wedge x \notin B\}$$

Proposition 1.9. Let A , B , and C be sets.

- $A \cup A = A$, $A \cap A = A$, and $A \setminus A = \emptyset$
- $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup B = B \cup A$ and $A \cap B = B \cap A$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Cartesian Products and Mappings

Given sets A and B , we define the **Cartesian Product** $A \times B$ as a set of ordered pairs.

Definition 1.10 (Cartesian Product). The cartesian product of two sets $A \times B$ is given as

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Proposition 1.11 (Generalization of Cartesian Product). The cartesian product generalizes to n sets:

$$A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) \mid a_1 \in A_1 \wedge \cdots \wedge a_n \in A_n\}$$

We will often use the shorthand $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$.

Definition 1.12 (Function/Mapping). Subsets of $A \times B$ are known as **relations**. We define a **mapping** or **function** $f \subset A \times B$ from set A to B as a special type of relation where each element $a \in A$ has a unique element $b \in B$ such that $(a, b) \in f$. Often, we write $f : A \rightarrow B$ and $f : a \mapsto b$.

If $f : A \rightarrow B$, then A is the **domain** of f , and

$$f(A) = \{f(a) \mid a \in A\} \subseteq B$$

is the **range** or **image** of f .

Definition 1.13 (Surjective). A map $f : A \rightarrow B$ is **onto** or **surjective** if

$$\forall(a \in A)(\exists(b \in B)(f(a) = b))$$

In other words, every element of the domain is mapped to an element in the image, **and** every element in the image is mapped from an element in the domain. The entirety of both sets is used.

Definition 1.14 (Injective). A map $f : A \rightarrow B$ is **one-to-one** or **injective** if

$$a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

Equivalently, a function is injective if $a_1 = a_2 \implies f(a_1) = f(a_2)$.

In other words, each element of the domain is mapped to a **unique** element of the image.

Definition 1.15 (Bijective). A map is called **bijective** if it is both injective and surjective.

Definition 1.16 (Function Compositions). If we have the maps $f : A \rightarrow B$ and $g : B \rightarrow C$, we can define the **composition** of f and g from A to C as

$$(g \circ f)(x) = g(f(x))$$

where we have

$$g \circ f : A \rightarrow C$$

A map from \mathbb{R}^n to \mathbb{R}^m given by a matrix is known as a **linear map**, or **linear transformation**.

Definition 1.17 (Permutation). For some set S , a bijection from S to S , formally $\pi : S \rightarrow S$, is known as a **permutation** of S .

Theorem 1.18 (Properties of Compositions). Let $f : A \rightarrow B$, $g : B \rightarrow C$, and $h : C \rightarrow D$.

- The composition of mappings is associative: $(h \circ g) \circ f = h \circ (g \circ f)$
- If f and g are both injective, then $g \circ f$ is injective.
- If f and g are both surjective, then $g \circ f$ is surjective.
- If f and g are both bijective, then $g \circ f$ is bijective.

Definition 1.19 (Identity Mapping). For some set S , we will use id_S or id as the **identity mapping** from S to itself.

$$id(s) = s \forall s \in S$$

Definition 1.20 (Inverse Mappings). From 1.19, we can formally define an **inverse mapping** $g : B \rightarrow A$ of some map $f : A \rightarrow B$, where

$$g \circ f = id_A \text{ and } f \circ g = id_B$$

A map is said to be **invertible** if it has an inverse. We often give f^{-1} as the inverse of f .

Theorem 1.21. A Map is Invertible if and only if it is Bijective.

Proof. Suppose $f : A \rightarrow B$ is invertible with inverse $g : B \rightarrow A$. Then $g \circ f = id_A$ is the identity map, meaning $g(f(a)) = a$. If $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$, then

$$a_1 = g(f(a_1)) = g(f(a_2)) = a_2$$

Thus, f is injective. Now suppose $b \in B$. Let $a \in A$ be the number such that $f(a) = b$. Recall that $f(g(b)) = b$. Thus, we have $a = g(b)$. This means each element of the domain will have some corresponding element in the image, and thus f is surjective. Let f be bijective and let $b \in B$. Since f is surjective, there exists $a \in A$ where $f(a) = b$. Since f is injective, a is unique. Define $g(b) = a$. This is the inverse of f . ■

Definition 1.22 (Equivalent Relation). An equivalence relation on a set X is a relation $R \subset X \times X$ satisfying the properties

- $(x, x) \in R \forall x \in X$ **reflexive property**
- $(x, y) \in R \implies (y, x) \in R$ **symmetric property**
- $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ **transitive property**

For an equivalence relation on X with $(x, y) \in R$, we often write $x \sim y$. If the relation has an associated notation such as $=, \equiv, \cong$, then that will be used.

Example. Suppose f and g are differentiable functions on \mathbb{R} . We can define an equivalence relation $f(x) \sim g(x)$ if $f'(x) = g'(x)$.

Definition 1.23 (Partition). A partition \mathcal{P} of a set X is a collection of sets $X_1, X_2, \dots \neq \emptyset$ such that

$$X_i \cap X_j = \emptyset \text{ for } i \neq j$$

and

$$\bigcup_k X_k = X$$

Definition 1.24 (Equivalence Class). Let \sim be an equivalence relation on a set X with $x \in X$. Then the **equivalence class** of x is given as

$$[x] = \{y \in X \mid y \sim x\}$$

Theorem 1.25. Given an equivalence relation \sim on a set X , the equivalence classes of X form a partition of X . Conversely, if $\mathcal{P} = X_i$ is a partition of a set X , then there is an equivalence relation on X with equivalence classes X_i .

Corollary 1.26. Two equivalence classes of an equivalence relation are either disjoint or equal.

Chapter 2

The Integers

2.1 Mathematical Induction

Definition 2.1 (First Principle of Induction). Mathematical induction is a method of proof wherein a statement about integers with $n \in \mathbb{N}$. We prove the statement for a specific integer n_0 , and then we show that the statement being true for some n implies the statement is true for $n + 1$. This implies this statement holds for all integers $n \geq n_0$.

A standard example of induction follows.

Example. Show that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for any $n \in \mathbb{N}$.

We can show this for $n = 1$:

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Now we can assume that we have verified the first n cases.

$$\begin{aligned} 1 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{n^{2+3n+2}}{2} \\ &= \frac{(n+1)((n+1)+1)}{2} \end{aligned}$$

This is the formula for the $n + 1$ th case, so we are done.

Definition 2.2 (Second Principle of Induction). Let $S(n)$ be a statement about the integers for $n \in \mathbb{N}$. Suppose $S(n_0)$ is true. If $S(n_0), S(n_0 + 1), \dots, S(k)$ implies $S(k + 1)$ for $k \geq n_0$, then $S(n)$ is true for all integers $n \geq n_0$.

This is more complicated, so an example has not been given. Obviously.

Definition 2.3 (Principle of Well-Ordering). Every nonempty subset of the natural numbers is well-ordered.

Lemma 2.4. The principle of mathematical induction implies 1 is the least positive natural.

Theorem 2.5 (Principle of Well-Ordering). The principle of well ordering can follow from the principle of mathematical induction. Furthermore, every nonempty subset of \mathbb{N} contains at least one element.

2.2 The Division Algorithm

Theorem 2.6 (Division Algorithm). Let a and b be integers with $b > 0$. Then there exists *unique* integers q and r such that

$$a = bq + r$$

where $0 \leq r < b$.

Let $a, b \in \mathbb{Z}$. If $b = ak, k \in \mathbb{Z}$, then we write $a|b$. We say an integer d is a **common divisor** of a and b if $d|a$ and $d|b$. The **greatest common divisor** of integers a, b is a positive integer d such that d is a common divisor of a and b and if d' is any other common divisor of a and b , then $d'|d$. We write $\gcd(a, b)$.

Definition 2.7 (Relatively Prime Numbers). Two numbers $a, b \in \mathbb{Z}$ are relatively prime if $\gcd(a, b) = 1$.

Theorem 2.8. Let $a, b \in \mathbb{Z} \setminus \{0\}$. Then there exist $r, s \in \mathbb{Z}$ such that

$$\gcd(a, b) = ar + bs$$

Corollary 2.9. For any two $a, b \in \mathbb{Z}$ that are relatively prime, there exist $r, s \in \mathbb{Z}$ such that $ar + bs = 1$.

The Euclidean Algorithm

2.8 allows us to compute the gcd of two integers. This will be illustrated with an example.

Example. Let's compute $\gcd(945, 2415)$. First, observe:

$$2415 = 945 \cdot 2 + 525$$

$$945 = 525 \cdot 1 + 420$$

$$525 = 420 \cdot 1 + 105$$

$$420 = 105 \cdot 4 + 0$$

Reversing our steps, we find that 105 divides 420, 105 divides 525, 105 divides 945, and 105 divides 2415. Hence, 105 divides both numbers. If d were another common divisor of 945 and 2415, then d would have to divide 105. Hence, $\gcd(2415, 945) = 105$.

To compute $\gcd(a, b) = d$, we are dividing multiple times to obtain a decreasing sequence of integers. We use the algorithm

$$\begin{aligned} b &= aq_1 + r_1 \\ a &= r_1q_2 + r_2 \\ r_1 &= r_2q_3 + r_3 \\ &\vdots \\ r_{n-2} &= r_{n-1}q_n + r_n \\ r_{n-1} &= r_nq_{n+1} \end{aligned}$$

where q_i is the highest integer number of times r_i can divide r_{i-1} , and r_{i+1} is the remainder.

Prime Numbers

Some $p \in \mathbb{N} \setminus \{1\}$ is a **prime number** if the only numbers that divide p are 1 and itself. Non-prime integers > 1 are known as **composite**.

Lemma 2.10 (Euclid). Let $a, b \in \mathbb{Z}$ and $p \in \mathbb{P}$, where \mathbb{P} is the set of primes. Since $\gcd(a, p) = 1$, then there exists $r, s \in \mathbb{Z}$ such that $ar + ps = 1$.

Theorem 2.11 (Euclid). There exist an infinite number of primes.

Proof. We will do this proof via contradiction. Suppose there are a finite number of primes, say $\{p_1, \dots, p_n\} = \mathbb{P}$.

$$\text{Let } P = 1 + \prod_{i \in \mathbb{P}} i$$

It follows that P is divisible by some p_i for $i \in [1, n]$. In this case, p_i must divide $P - p_1p_2 \cdots p_n = 1$. However, it does not, so there must either exist some $p_{n+1} \neq p_i$, or $P \in \mathbb{P}$. ■

Theorem 2.12 (Fundamental Theorem of Arithmetic). Let $n \in \mathbb{N}$ where $n > 1$. Then

$$n = p_1p_2 \cdots p_k$$

where $p_1, \dots, p_k \in \mathbb{P}$, and are not necessarily distinct. Furthermore, this factorization is unique.

Chapter 3

Groups

3.1 Integer Equivalence Classes and Symmetries

Two integers $a, b \in \mathbb{Z}$ are equivalent mod n if n divides $a - b$ with no remainder. The integers mod n partition \mathbb{Z} into n different equivalence classes, the set of which is denoted as \mathbb{Z}_n . For example, consider \mathbb{Z}_{12} , the set of the following equivalence classes:

$$\begin{aligned}[0] &= \{\dots, -12, 0, 12, 24, \dots\} \\ [1] &= \{\dots, -11, 1, 13, 25, \dots\} \\ &\vdots \\ [11] &= \{\dots, -1, 11, 23, 35, \dots\}\end{aligned}$$

We can define arithmetic over \mathbb{Z}_n , where we say addition modulo n is given as $(a + b) \bmod n$.

Example. Operations modulo n :

$7 + 4 \equiv 1 \pmod{5}$	$7 \cdot 3 \equiv 1 \pmod{5}$
$3 + 5 \equiv 0 \pmod{8}$	$3 \cdot 5 \equiv 7 \pmod{8}$
$3 + 4 \equiv 7 \pmod{12}$	$3 \cdot 4 \equiv 0 \pmod{12}$

Proposition 3.1. Let \mathbb{Z}_n be the set of equivalence classes of the integers mod n and $a, b, c \in \mathbb{Z}_n$.

1. Addition and multiplication are commutative:

$$a + b \equiv b + a \pmod{n}$$

$$ab \equiv ba \pmod{n}$$

2. Addition and multiplication are associative:

$$(a + b) + c \equiv a + (b + c) \pmod{n}$$

$$(ab)c \equiv a(bc) \pmod{n}$$

3. There are both additive and multiplicative identities:

$$a + 0 \equiv a \pmod{n}$$

$$a \cdot 1 \equiv a \pmod{n}$$

4. Multiplication distributes over addition:

$$a(b + c) \equiv ab + ac \pmod{n}$$

5. For every integer there exists an additive inverse:

$$a + (-a) \equiv 0 \pmod{n}$$

6. If $a \neq 0$, then $\gcd(a, n) = 1$ iff there exists a multiplicative inverse b for $a \pmod{n}$; that is, a $b \neq 0$ such that

$$ab \equiv 1 \pmod{n}$$

3.2 Definitions and Examples

Definition 3.2 (Group). A group (G, \circ) is a set G together with a **binary operation** (or **law of composition**) $G \times G \rightarrow G$ given as $(a, b) \mapsto a \circ b$ that satisfies the following axioms.

- The binary operation is **associative**. That is,

$$(a \circ b) \circ c = a \circ (b \circ c)$$

Formally,

$$\forall(a, b, c \in G)((a \circ b) \circ c = a \circ (b \circ c))$$

- There exists an **identity element** $e \in G$, such that

$$e \circ a = a \circ e = a$$

Formally,

$$\exists(e \in G)(\forall a \in G(a \circ e = e \circ a = a))$$

- For all $a \in G$, there exists an **inverse element** $a^{-1} \in G$, such that

$$a \circ a^{-1} = a^{-1} \circ a = e$$

Formally,

$$\forall(a \in G)(\exists(a^{-1} \in G)(a \circ a^{-1} = a^{-1} \circ a = e))$$

The \circ operator can be thought of as simply a placeholder for a binary operation such as addition, multiplication, or even an inner product-like operation.

Definition 3.3 (Abelian Group). A group (G, \circ) with the property $a \circ b = b \circ a \forall a, b \in G$ is called **abelian** or **commutative**. A group not satisfying this property is known as **nonabelian** or **noncommutative**.

Notation. We will use ab to denote a binary operation associated with a set rather than $a \circ b$, partially for brevity but also to distinguish between group operations and compositions. If a standard symbol is associated with the operation, such as $+$, that will be used instead.

For example, the set of integers \mathbb{Z} forms a group under addition, because for all $a, b \in \mathbb{Z}$, we have $(a + b) \in \mathbb{Z}$, $a + 0 = a$, $a + (-a) = 0$, and $-a \in \mathbb{Z}$. We also have $a + b = b + a$, so the integers over addition $(\mathbb{Z}, +)$ form an abelian group.

The integers mod n also form a group under addition modulo n . Addition can be shown with a **Cayley table**, or a table describing addition/multiplication. Consider \mathbb{Z}_5 :

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

It's also to note that not every set with a binary operation defines a group. For example, the integers mod n are not necessarily closed under multiplication, so they cannot necessarily constitute

a group. However, we can recall from 3.1 that $ab \equiv 1 \pmod n$ iff a, b are relatively prime; that is $\gcd(a, b) = 1$. Thus, we can define the multiplicative group of nonzero relatively prime numbers in \mathbb{Z}_n as $U(n)$. For example, consider the Cayley table for $U(8)$:

\cdot	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Example. Let

$$\begin{aligned} 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & i &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ j &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & k &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \end{aligned}$$

Notice the following.

$$\begin{aligned} \hat{\mathbf{i}}^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \\ \hat{\mathbf{j}}^2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \\ \hat{\mathbf{k}}^2 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \\ \implies \hat{\mathbf{i}}^2 &= \hat{\mathbf{j}}^2 = \hat{\mathbf{k}}^2 = -1 \end{aligned}$$

The set $Q_8 = \{\pm 1, \pm \hat{\mathbf{i}}, \pm \hat{\mathbf{j}}, \pm \hat{\mathbf{k}}\}$ forms a group (Q_8, \cdot) known as the **quaternion group**, which is nonabelian.

One final important example is the **general linear group**. This is the group of all invertible matrices over the multiplication operator, and is denoted $GL(n, S)$ where the matrices are of order n with elements $a_{ij} \in S$.

Definition 3.4 (Order of a Group). The order of a group G , denoted $|G|$, is the number of elements it has. If the group has **finite order**, we write $|G| = n$, where $n \in \mathbb{N}$. If a group has **infinite order**, we write $|G| = \infty$.

Basic Properties of Groups

Proposition 3.5. The identity element of a group is unique.

Proof. Suppose e and e' are both identities in G . Then $eg = ge = g$ and $e'g = ge' = g$ for all $g \in G$. However, $e, e' \in G$, so we can say

$$\begin{aligned} ee' &= e' \\ ee' &= e \\ \implies e &= ee' = e' \\ \therefore e &= e' \end{aligned}$$

■

Proposition 3.6. If G is a group, then for any $g \in G$, its inverse g^{-1} is unique.

Proposition 3.7. Let G be a group. If $a, b \in G$, then $(ab)^{-1} = b^{-1}a^{-1}$.

Proof. Let $a, b \in G$. Then $abb^{-1}a^{-1} = aea^{-1} = aa^{-1} = e$. Similarly, $b^{-1}a^{-1}ab = e$. By the previous proposition, we know that inverses are unique, and by 3.2 we know group operations are associative. Thus,

$$\begin{aligned} (ab)(b^{-1}a^{-1}) &= e \\ \therefore (ab)^{-1} &= b^{-1}a^{-1} \end{aligned}$$

■

Proposition 3.8. If G is a group, then $\forall a \in G$, it will follow that $(a^{-1})^{-1} = a$.

Proposition 3.9. If G is a group and $a, b, c \in G$, then $ba = ca \implies b = c$ and $ab = ac \implies b = c$.

The previous proposition shows that **right and left cancellation laws** hold in groups. This is due to the existence of inverses over the associated binary operation.

Notation. Oftentimes, exponential notation will be used in a group to denote repeated applications of the operation:

$$g^n = \underbrace{g \circ g \circ \cdots \circ g}_{n \text{ times}}$$

And

$$g^{-n} = \underbrace{g^{-1} \circ g^{-1} \circ \cdots \circ g^{-1}}_{n \text{ times}}$$

Proposition 3.10. In a group G , the following properties of exponentiation hold for all $g, h \in G$:

- $g^m g^n = g^{m+n} \quad \forall m, n \in \mathbb{Z}$
- $(g^m)^n = g^{mn} \quad \forall m, n \in \mathbb{Z}$
- $(gh)^n = (h^{-1}g^{-1})^{-n} \quad \forall n \in \mathbb{Z}$

Remark. In the above proposition, $(gh)^n \neq g^n h^n$ in general, since the group may not be abelian. If the group is abelian, this property holds.

Remark. If the group has an addition operator, especially if the group is \mathbb{Z} or \mathbb{Z}_n , we write the operation of exponentiation as multiplication, that is an rather than n^a for $n \in G$, $a \in \mathbb{Z}$.

3.3 Subgroups

Definitions and Examples

Definition 3.11 (Subgroup). A **subgroup** H of a group G is a subset $H \subseteq G$ such that when the group operation of G is restricted to H , then H is still a group. That is, if $H \subseteq G$ and (G, \circ) is a group, then H is a subgroup if (H, \circ) is a group.

Remark. The subgroup $H = \{e\}$ is called the **trivial subgroup**, and a proper subset that forms a group $H \subsetneq G$ is known as a **proper subgroup**.

Example. Consider the set of nonzero real numbers \mathbb{R}^* over the operation of multiplication. The identity is 1 and the inverse of any $a \in \mathbb{R}^*$ is just a^{-1} . We can show that \mathbb{Q}^* is a proper subgroup of \mathbb{R}^* . Trivially, $1 \in \mathbb{Q}^*$. Also, the inverse of any $\frac{p}{q} \in \mathbb{Q}^*$ must also be in \mathbb{Q}^* , since $\left(\frac{p}{q}\right)^{-1} = \frac{q}{p} \in \mathbb{Q}^*$. Since multiplication in \mathbb{R}^* is associative, so is multiplication in \mathbb{Q}^* . Therefore, \mathbb{Q}^* is a proper subgroup of \mathbb{R}^* .

Remark. A subset of some set G can be a group without being a *subgroup* of G , if it is a group under a different operation. For example, the general linear is not a subgroup of the additive group of $n \times n$ matrices, since it uses a different operation.

Example. Let $\text{SL}(2, \mathbb{R}) \subset \text{GL}(2, \mathbb{R})$ be the set of matrices $A \in \text{SL}(2, \mathbb{R})$ satisfying $\det(A) = 1$. Multiplication is closed since $\det(A)\det(B) = \det(AB)$, and the other properties of groups follow directly from the general linear. This group is known as the **special linear group**.

Some Subgroup Theorems

Proposition 3.12. A subset H of G is a subgroup iff it satisfies the following conditions:

1. The identity $e \in G$ is in H .
2. If $h_1, h_2 \in H$, then $h_1 h_2 \in H$.
3. If $h \in H$, then $h^{-1} \in H$.

Proof. Trivial. ■

Proposition 3.13. Let H be a subset of G . Then H is a subgroup of G iff $H \neq \emptyset$, and whenever $g, h \in H$ then $gh^{-1} \in H$.

Chapter 4

Cyclic Groups

4.1 Cyclic Subgroups

Oftentimes, a subgroup can be constructed from a single element of a group. For example, consider $3 \in \mathbb{Z}$. now consider the set of all multiples of 3, that is

$$3\mathbb{Z} = \{\dots, -3, 0, 3, 6, \dots\}$$

Obviously, $3\mathbb{Z}$ is a subgroup of \mathbb{Z} . Specifically, it's a **cyclic subgroup** "generated" by 3.

Theorem 4.1 (Cyclic Subgroups). If G is a group and $a \in G$, then the set

$$\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$$

is a subgroup of G . Furthermore, $\langle a \rangle$ is the smallest subgroup of G containing a .

Remark. If we use the $+$ notation, then we write

$$\langle a \rangle = \{na \mid n \in \mathbb{Z}\}$$

Definition 4.2 (Cyclic Groups and Subgroups). If G is a group with $a \in G$, then we say $\langle a \rangle$ is the **cyclic subgroup** generated by a . If there exists some $b \in G$ where $\langle b \rangle = G$, then G is a **cyclic group** generated by b . We call a and b the **generators** of their respective groups.

Definition 4.3 (Order of a Cyclic Group). The **order** of a cyclic group $\langle a \rangle$ is the smallest **positive** integer $n \in \mathbb{Z}^+$ such that $a^n = e$, where $e \in \langle a \rangle$ is the identity element. If there does not exist some n , we say $|a| = \infty$.

Remark. When considering the above definition, it's important to note that $0 \notin \mathbb{Z}^+$.

Example. Notice that both 1 and 5 generate $(\mathbb{Z}_6, +)$. Also, not every element necessarily generates the group. The element $2 \in \mathbb{Z}_6$ generates $\langle 2 \rangle = \{0, 2, 4\}$.

Theorem 4.4. Every cyclic subgroup is abelian.

Proof. Let $\langle a \rangle = G$ and $a \in G$. If $g, h \in G$, then they can be written as powers of a . Let $r, s \in \mathbb{Z}$ such that $g = a^r, h = a^s$.

$$gh = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = hg$$

Therefore G is abelian. ■

Subgroups of Cyclic Groups

Theorem 4.5. Every subgroup of a cyclic group is cyclic.

Corollary 4.6. The subgroups of \mathbb{Z} are exactly $n\mathbb{Z}$ for $n \in \mathbb{N} \cup \{0\}$.

Proposition 4.7. Let G be a cyclic subgroup of order n and suppose $\langle a \rangle = G$. Then $a^k = e$ iff n divides k with no remainder.

Theorem 4.8. Let G be a cyclic group of order n with $\langle a \rangle = G$. If $b = a^k$, then the order of b is $\frac{n}{d}$, where $d = \gcd(k, n)$.

Corollary 4.9. The generators of \mathbb{Z}_n are the integers $r \in [1, n)$ where $\gcd(r, n) = 1$.

4.2 Multiplicative Group of Complex Numbers

Definition 4.10 (Complex Numbers). The **complex numbers** are defined as

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

where $i = \sqrt{-1}$

Definition 4.11 (Arithmetic in \mathbb{C}). If $z = a + bi$ and $w = c + di$ with $z, w \in \mathbb{C}$, then we have

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$$

$$zw = (a + bi)(c + di) = ac + bdi^2 + adi + bci = (ac - bd) + (ad + bc)i$$

Every nonzero complex number $z = a + bi$ has a multiplicative inverse, that is $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = z^{-1}z = 1$.

$$z^{-1} = \frac{a - bi}{a^2 + b^2}$$

Definition 4.12 (Complex Conjugate). If $z = a + bi$, then the complex conjugate $\bar{z} = a - bi$.

Complex numbers can also be represented in **polar coordinates** of the form

$$z = a + bi = r(\cos \theta + i \sin \theta)$$

and

$$r = |z| = \sqrt{a^2 + b^2}$$

and

$$a = r \cos \theta \quad b = r \sin \theta$$

and

$$\theta = \arctan\left(\frac{b}{a}\right)$$

Notation. We can abbreviate $r(\cos \theta + i \sin \theta) =: r \operatorname{cis} \theta$.

Proposition 4.13. Let $z = r \operatorname{cis} \theta$ and $w = s \operatorname{cis} \phi$. Then

$$zw = rs \operatorname{cis}(\theta + \phi)$$

Theorem 4.14 (DeMoivre). Let $z = r \operatorname{cis} \theta \in \mathbb{C} \setminus \{0\}$. Then

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta)$$

The Circle Group and the Roots of Unity

The multiplicative group of complex numbers $(\mathbb{C}, \cdot) \triangleq \mathbb{C}^*$ has interesting subgroups. Specifically, \mathbb{R}^* and \mathbb{Q}^* have no subgroups of finite order, but \mathbb{C}^* has many.

Definition 4.15 (Circle Group). The circle group is the multiplicative group $\mathbb{T} \subsetneq \mathbb{C}$ defined as

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$$

Remark. This group forms a circle on the complex grid with radius 1.

Proposition 4.16. The circle group \mathbb{T} is a subgroup of \mathbb{C}^*

The circle group has many interesting subgroups, such as $\{1, -1, i, -i\}$. These numbers are the numbers satisfying $z^4 = 1$. These are known as the **4th roots of unity**.

Definition 4.17 (Roots of Unity). The set of complex numbers z satisfying $z^n = 1$ for some $n \in \mathbb{N}$ are known as the **n th roots of unity**.

Theorem 4.18 (Roots of Unity Definition). If $z^n = 1$, then the n th roots of unity are

$$z = \operatorname{cis}\left(\frac{2k\pi}{n}\right)$$

where $k \in \{0, 1, \dots, n-1\}$. Furthermore, the n th roots of unity are a subgroup of \mathbb{T} of order n .

Definition 4.19 (Primitive n th root of unity). A primitive n th root of unity is some z that generates the n th roots of unity.

For example, the 8th roots of unity have 4 generators:

$$\begin{aligned}w &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\w^3 &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\w^5 &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \\w^7 &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\end{aligned}$$

4.3 The Method of Repeated Squares

looked boring ill come back to it later

Chapter 5

Permutation Groups

Consider an equilateral triangle $\triangle ABC$. The symmetries of this triangle actually consist of permutations of the three vertices, where a **permutation** of the set $S = \{A, B, C\}$ is a bijective map $\pi : S \rightarrow S$. The three vertices have the following six permutations:

$$\begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix} \quad \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} \quad \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$

$$\begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \quad \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix} \quad \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

In this case, the array

$$\begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$

denotes the permutation that sends $A \rightarrow B$, $B \rightarrow C$, and $C \rightarrow A$:

$$A \mapsto B$$

$$B \mapsto C$$

$$C \mapsto A$$

The symmetries of a triangle form a group. In this chapter we study groups of this type.

5.1 Definitions and Notation

The permutations of a set X form a group S_X . If X is finite, then we assume $X = \{1, 2, \dots, n\}$. In this case we write S_n instead of S_X . We call this the **symmetric group** on n letters.

Theorem 5.1 (Symmetric Group on n Letters). The symmetric group on n letters S_n is a group with $n!$ elements, wherein the binary operation is the composition of maps.

Proof. The identity of S_n is the identity map. If $f : S_n \rightarrow S_n$ is a permutation, then f^{-1} exists by 1.20 since f is bijective. Thus, every map has an inverse under the composition operation. By 1.18, compositions are associative. Thus, (S_n, \circ) is a group. There are $n!$ ways to arrange sets of order n , so $|S_n| = n!$. ■

Definition 5.2 (Permutation Group). A subgroup of S_n is a **permutation group**.

Example. Consider the subgroup G of S_5 consisting of the identity permutation id and the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$$

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

We can construct this table defining how to operate with elements in this permutation group.

\circ	id	σ	τ	μ
id	id	σ	τ	μ
σ	σ	id	μ	τ
τ	τ	μ	id	σ
μ	μ	τ	σ	id

Remark. Functions are composed from right to left. That is, $(\sigma \circ \tau)(x) = \sigma(\tau(x))$. We can thus adopt the convention of operating from right to left on permutations, that is given $\sigma\tau$ we will simply do τ first.

Example. Permutation groups are usually not abelian. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Then

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

and

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

Cycle Notation

Definition 5.3 (Permutations as Cycles). A permutation $\sigma \in S_X$ is a **cycle of length k** if there exists $a_1, \dots, a_k \in X$ such that

$$\sigma(a_1) = a_2$$

$$\sigma(a_2) = a_3$$

$$\vdots$$

$$\sigma(a_k) = a_1$$

and $\sigma(x) = x$ for all other $x \in X$. We write (a_1, \dots, a_k) to denote the cycle σ .

Remark. Cycles are the building blocks of all permutations.

For example, the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 5 & 1 & 4 & 2 & 7 \end{pmatrix} = (162354)$$

is a cycle of length 6, and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 5 & 6 \end{pmatrix} = (243)$$

is a cycle of length 3. Not all permutations are cycles. For example,

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} = (1243)(56)$$

consists of both a cycle of length 2 and of length 4.

Example. It's easy to compute compositions of cycles. Let

$$\sigma = (1352) \text{ and } \tau = (256)$$

We can think of σ as

$$\sigma \equiv \begin{cases} 1 \mapsto 3 \\ 3 \mapsto 5 \\ 5 \mapsto 2 \\ 2 \mapsto 1 \end{cases}$$

and we can think of τ as

$$\tau \equiv \begin{cases} 2 \mapsto 5 \\ 5 \mapsto 6 \\ 6 \mapsto 2 \end{cases}$$

Thus, we can find

$$(\sigma \circ \tau) \equiv \begin{cases} 2 \mapsto 5 \mapsto 2 \\ 5 \mapsto 6 \\ 6 \mapsto 2 \mapsto 1 \\ 1 \mapsto 3 \\ 3 \mapsto 5 \\ 5 \mapsto 2 \mapsto 5 \end{cases} \equiv \begin{cases} 5 \mapsto 6 \\ 6 \mapsto 1 \\ 1 \mapsto 3 \\ 3 \mapsto 5 \end{cases} \equiv (1356)$$

Similarly, if $\mu = (1634)$, then $\sigma\mu = (1652)(34)$

Definition 5.4 (Disjoint Cycles). If $\sigma, \tau \in S_X$ are cycles such that $\sigma = (a_1, \dots, a_k)$ and $\tau = (b_1, \dots, b_l)$, then σ and τ are **disjoint** if $a_i \neq b_j$ for all i, j .

The products of disjoint cycles cannot be simplified.

Proposition 5.5. Let σ and τ be disjoint cycles in S_X . $\sigma\tau = \tau\sigma$.

Proof. Let $\sigma = (a_1, \dots, a_k)$ and $\tau = (b_1, \dots, b_l)$. We must show $\sigma\tau(x) = \tau\sigma(x)$ for all $x \in X$. If $x \notin \{a_1, \dots, a_k\}$ and $x \notin \{b_1, \dots, b_l\}$, then both σ and τ **fix** x , meaning $\sigma(x) = x$ and $\tau(x) = x$. Thus,

$$\sigma\tau(x) = \sigma(\tau(x)) = \sigma(x) = x = \tau(x) = \tau(\sigma(x)) = \tau\sigma(x)$$

Now, suppose $x \in a_1, \dots, a_k$. Thus, $\sigma(a_i) = a_{(i \bmod k)+1}$, meaning

$$\begin{aligned} a_1 &\mapsto a_2 \\ a_2 &\mapsto a_3 \\ &\vdots \\ a_{k-1} &\mapsto a_k \\ a_k &\mapsto a_1 \end{aligned}$$

Since σ and τ are disjoint, then $\tau(a_i) = a_i$. Thus,

$$\begin{aligned} \sigma\tau(a_i) &= \sigma(\tau(a_i)) \\ &= \sigma(a_i) \\ &= a_{(i \bmod k)+1} \\ &= \tau(a_{(i \bmod k)+1}) \\ &= \tau(\sigma(a_i)) \\ &= \tau\sigma(a_i) \end{aligned}$$

The same logic holds for $x \in \{b_1, \dots, b_l\}$. Since σ and τ are disjoint,

$$\{a_1, \dots, a_k\} \cap \{b_1, \dots, b_l\} = \emptyset$$

■

Theorem 5.6 (Permutations as Cycles). Every permutation $\sigma \in S_n$ can be written as the product of disjoint cycles.

Proof. Assume $X = \{1, 2, \dots, n\}$. Let $\sigma \in S_n$ and define $X_1 = \{\sigma(1), \sigma^2(1), \dots\}$. The set X_1 must be finite, since the set X is finite. Let $i \in \mathbb{Z}$ be the first integer in X that is not in X_1 , and let $X_2 = \{\sigma(i), \sigma^2(i), \dots\}$. Again, this set is finite. Now we can define X_3, X_4, \dots in the same manner. Since X is finite, there will be a finite number of these **disjoint** sets. Let this number be r . If σ_i is given as

$$\begin{cases} \sigma(x) & x \in X_i \\ x & x \notin X_i \end{cases}$$

Then $\sigma = \sigma_1\sigma_2 \cdots \sigma_r$. Since the sets X_1, \dots, X_r are disjoint, then the cycles $\sigma_1, \dots, \sigma_r$ are also disjoint. ■

Remark. Permutations will generally be represented by cycles for brevity, and the identity permutation will be given as (1).

Transpositions

Definition 5.7 (Transposition). A transposition is a cycle of length 2.

Proposition 5.8. Any permutation of a finite set containing at least two elements can be given as the product of transpositions.

Example. For example, the permutation

$$(16)(253) = (16)(23)(25)$$

There is no unique representation of permutations as a product of transpositions. For example, the above permutation can also be given as

$$(16)(45)(23)(45)(25)$$

Interestingly, no permutation can be written as both an odd *and* an even number of transpositions.

Lemma 5.9. If the identity is written as the product of r transpositions

$$\text{id} = \tau_1 \tau_2 \cdots \tau_r$$

then r is even.

Theorem 5.10. If a permutation σ can be expressed as the product of an even number of transpositions, then all other products of transpositions equaling σ must contain an even number of transpositions. Similarly, if σ can be expressed as an odd number of transpositions, then any set of transpositions equaling σ must be odd.

Proof. Suppose

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_m = \tau_1 \tau_2 \cdots \tau_n$$

where m is even. The inverse of σ is $\sigma_m \cdots \sigma_1$.

$$\text{id} = \sigma \sigma_m \cdots \sigma_1 = \tau_1 \cdots \tau_n \sigma_m \cdots \sigma_1$$

thus n is even by 5.9. ■

We say a permutation is **even** if it can be expressed as an even number of transpositions, and **odd** if it requires an odd number.

The Alternating Groups

An important subgroup of S_n is A_n , the set of all even permutations. We say A_n is the **alternating group on n letters**.

Theorem 5.11. The set A_n is a subgroup of S_n .

Proof. Since the product of any two even transpositions will be even, A_n is closed. By 5.9, $\text{id} \in A_n$. Let $\sigma \in A_n$ be an even permutation. Since it's even, there must exist transpositions $\sigma_1, \dots, \sigma_r$ such that

$$\sigma = \sigma_1 \cdots \sigma_r$$

where r is even. Since the inverse of a transposition is itself, it follows that

$$\sigma^{-1} = \sigma_r \cdots \sigma_1$$

It follows that $\sigma^{-1} \in A_n$. ■

Proposition 5.12. The number of even permutations in S_n with $n \geq 2$ is equal to the number of odd permutations, and thus $|A_n| = \frac{n!}{2}$.

Proof. Let A_n and B_n be the sets of even and odd permutations in S_n , respectively. If there exists some bijection between these sets, then $|A_n| = |B_n|$. Let $\sigma \in S_n$ be a transposition. Such a σ exists because $n \geq 2$. Define

$$\lambda_\sigma : A_n \rightarrow B_n$$

as the mapping

$$\lambda_\sigma(\tau) = \sigma\tau$$

We must show λ_σ is bijective. First we show it's injective. Let $\tau, \mu \in A_n$ with $\lambda_\sigma(\tau) = \lambda_\sigma(\mu)$. We must show this implies $\tau = \mu$. It follows that

$$\begin{aligned} \sigma\tau &= \sigma\mu \\ \implies \tau &= \sigma^{-1}\sigma\tau = \sigma^{-1}\sigma\mu = \mu \\ \therefore \tau &= \mu \end{aligned}$$

We must now show λ_σ is surjective. It suffices to show that for any $\tau \in B_n$, there exists some $\mu \in A_n$ such that $\lambda_\sigma(\mu) = \tau$. Let $\tau \in B_n$. It follows that there exists some set of transpositions $\{\tau_1, \dots, \tau_m\}$ with m being odd, such that

$$\tau = \tau_1 \cdots \tau_m$$

Let $\mu := \sigma^{-1}\tau_1 \cdots \tau_m = \sigma^{-1}\tau$. This is still an element of S_n since S_n is a group, but is no longer an element of B_n since it is obviously composed of an even number of transpositions. Thus, $\mu \in A_n$. As such, we can consider

$$\lambda_\sigma(\mu) = \sigma\sigma^{-1}\tau = \tau$$

Therefore, such a μ always exists since S_n is closed under function composition. Therefore,

$$|A_n| = |B_n|$$

■

5.2 Dihedral Groups

A special type of permutation group is the dihedral group. We define the **n th dihedral group** to be the group of rigid motions of a regular n -gon, denoted D_n .

Theorem 5.13. The dihedral group, D_n , is a subgroup of S_n of order $2n$.

This theorem comes from the fact that since the n -gon is rigid, if we denote the vertices of the n -gon in order by $1, 2, \dots, n$, we will find that if the vertex 1 is replaced with some k , then the next vertex can only be $k - 1$ or $k + 1$. This gives rise to $2n$ different orientations of the n -gon, which can be thought of as two sets of order n but with a negative sign flipped.

Theorem 5.14. The group D_n for $n \geq 3$ consists of all products of two elements r and s , where r has order n and s has order 2, and $sr s = r^{-1}$.

Proof. The possible motions of some n -gon consist of rotations and reflections. There are obviously exactly n possible rotations, which can be generated by a rotation of only one vertex. We denote this rotation r as the permutation that rotates by

$$\frac{360^\circ}{n}$$

and find that all other rotations r^k can be represented as k repeated applications of r . Thus, r has order n . Now define s_1, \dots, s_n where s_k is the reflection that fixes vertex k . Two repeated rotations will fix all vertexes, so the order of any s_k is two. Let $s = s_1$. We have $s^2 = 1$ and $r^n = 1$. Any rigid motion $1 \mapsto k$ must replace the next vertex by either $k + 1$ or $k - 1$. If it's replaced by $k + 1$, then the motion is given by r^k . If it's replaced by $k - 1$, it's given by $r^k s$. Therefore, $\{r, s\}$ generates D_n . ■

Chapter 6

Cosets and Lagrange's Theorem

Lagrange's theorem is important in finite group theory, but understanding it relies heavily on the notion of a coset.

6.1 Cosets

Definition 6.1 (Cosets). Let G be a group with a subgroup H and some $g \in G$. The **left coset** of H with representative g is the set

$$gH = \{gh \mid h \in H\}$$

Similarly, the **right coset** of H with representative g is the set

$$Hg = \{hg \mid h \in H\}$$

If left and right cosets are the same or it's obvious which coset is being referenced, often these sets are referenced simply as cosets without specification as to the side.

Proposition 6.2. Left and right cosets in an abelian group are always identical.

Lemma 6.3. Let H be a subgroup of G and let $g_1, g_2 \in G$. The following conditions are equivalent.

- $g_1H = g_2H$
- $Hg_1^{-1} = Hg_2^{-1}$
- $g_1H \subset g_2H$
- $g_2 \in g_1H$
- $g_1^{-1}g_2 \in H$

Proof. Proof not provided, I will do this when im not exhausted i think

■

Theorem 6.4. Let H be a subgroup of some group G . The left cosets of H in G partition G .

Proof. Let $g_1, g_2 \in G$ and g_1H, g_2H be left cosets of H in G . We must show that $g_1H \cap g_2H = \emptyset$ or $g_1H = g_2H$. Suppose $g_1H \cap g_2H \neq \emptyset$. It follows that there exists some $a \in g_1H \cap g_2H$. By definition of a left coset, $a = g_1h_1 = g_2h_2$ for some $h_1, h_2 \in H$. Thus, $g_1 = g_2h_2h_1^{-1}$. Since H is a subgroup, $h_1^{-1} \in H$ and $h_2h_1^{-1} \in H$. Therefore, $g_2h_2h_1^{-1} \in g_2H$. This implies $g_1 \in g_2H$. By lemma 6.3, $g_1H = g_2H$. Therefore, $g_1H \cap g_2H \neq \emptyset \implies g_1H = g_2H$. ■

There is nothing special in this proof for left cosets, so we can define an equivalent statement for right cosets of H in G .

Definition 6.5 (Index). Let G be a group with subgroup H . Define the **index** of H in G to be the number of left cosets of H in G , denoted $[G : H]$.

Theorem 6.6. Let H be a subgroup in G . The number of left cosets of H in G is the same as the number of right cosets of H in G .

Proof. Let \mathcal{R}_H and \mathcal{L}_H be the sets of right and left cosets of H , respectively. We must define a bijection $\phi : \mathcal{L}_H \rightarrow \mathcal{R}_H$. Let $gH \in \mathcal{L}_H$, and define $\phi(gH) = Hg^{-1}$. Lemma 6.3 states that if $g_1H = g_2H$, then $Hg_1^{-1} = Hg_2^{-1}$. This shows that no $\phi(gH)$ can map to multiple values, and thus ϕ is well defined. Furthermore, for any $Hg \in \mathcal{R}_H$, there exists some $g^{-1}H \in \mathcal{L}_H$ such that $\phi(g^{-1}H) = H(g^{-1})^{-1} = Hg$. Such an inverse exists because $g \in G$ and G is a group. Therefore ϕ is surjective. Now show that ϕ is injective. Suppose $\phi(g_1H) = \phi(g_2H)$. By definition, $Hg_1^{-1} = Hg_2^{-1}$. From lemma 6.3, this implies that $g_1H = g_2H$. Therefore ϕ is bijective, and

$$|\mathcal{L}_H| = |\mathcal{R}_H|$$

■

6.2 Lagrange's Theorem

Proposition 6.7. Let H be a subgroup of G with $g \in G$ and define the map $\phi : H \rightarrow gH$ by $\phi(h) = gh$. ϕ is bijective, and thus $|H| = |gH|$.

Proof. To show ϕ is injective, let $h_1, h_2 \in H$ with $h_1 = h_2$. We must show that this implies $\phi(h_1) = \phi(h_2)$. Notice that $\phi(h_1) = gh_1$ and $\phi(h_2) = gh_2$. Thus we have

$$\phi(h_1)h_1^{-1} = gh_1h_1^{-1} = g$$

$$\phi(h_2)h_2^{-1} = gh_2h_2^{-1} = g$$

However, inverses are unique, so $h_1 = h_2 \implies h_1^{-1} = h_2^{-1}$. Therefore, we have

$$\begin{aligned}\phi(h_2)h_2^{-1} &= \phi(h_2)h_1^{-1} \\ \implies \phi(h_1)h_1^{-1} &= \phi(h_2)h_1^{-1} \\ \implies \phi(h_1) &= \phi(h_2)\end{aligned}$$

Therefore, ϕ is injective. Now we must show that ϕ is surjective by showing that for any $\alpha \in gH$, there exists a corresponding $h \in H$ such that $\phi(h) = \alpha$. To do this, start by recognizing that α can be written as some gh , and that ϕ by definition maps $h \mapsto gh$. Therefore, for any $gh \in gH$, simply notice that $h \in H$ has $\phi(h) = gh$. Therefore, ϕ is bijective. ■

Theorem 6.8 (Lagrange's Theorem). Let G be a finite group with subgroup H . Then $|G|/|H| = [G : H]$ is the number of distinct left cosets of H in G . The number of elements in H must divide the number of elements in G .

Proof. By 6.4 and via 6.5, G is partitioned into $[G : H]$ distinct left cosets. By 6.7, each coset has $|H|$ elements. Since each coset is distinct from the rest, the total number of cosets times the order of each coset will give the order of G . Thus, $|G| = [G : H]|H|$. ■

Corollary 6.9. Suppose G is a finite group with $g \in G$. Then the order of g must divide $|G|$.

Corollary 6.10. Let $|G| = p$ with p as a prime number. Then G is cyclic and any $g \in G$ with $g \neq e$ will be a generator.

Proof. Let $g \in G$ with $g \neq e$. Recall that the order of an element is the number of operations required to return to that original element. Since identities are unique, $|\langle g \rangle| \neq 1$, and thus $|\langle g \rangle| = p$ since p is prime and $|\langle g \rangle|$ must divide p by 6.9. Thus, $\langle g \rangle = G$. ■

Remark. The converse of Lagrange's theorem is false. Just because a group can have subgroups of some order does not mean those subgroups actually exist. Proof of this is noted by the next proposition.

Proposition 6.11. The group A_4 has no subgroups of order 6.

Proof. Recall from Chapter 5 that A_4 is the set of all even permutations in S_4 , and

$$|A_4| = \frac{4!}{2} = 12$$

Let $|H| = 6$ and note that $[G : H] = |G|/|H| = 2$. Since one of the cosets is formed by $eH = H$, there must exist some other element $g \in A_4$ with $gH \neq H$. Specifically, $gH = G \setminus H$, since gH is what's "left over" from the other coset. However, $He = H$ is also a right coset, and the same logic shows that $Hg = G \setminus H$. Therefore, $gH = Hg$. This implies that $gHg^{-1} = Hgg^{-1} = H$. This means that for any $g \in G$, $gH = Hg$, since this holds for both $gH = H$ or $gH \neq H$. This shows that for any $g \in H$, $ghg^{-1} \in H$ for some $h \in H$ by the previous logic. Since there are eight cycles of order 3 in A_4 , there will exist at least one in H . WLOG, let $(123) \in H$. Then $(123)^{-1} = (132) \in H$. Since $(124), (243) \in A_4$ with inverses $(142), (234) \in A_4$, respectively, we have

$$(124)(123)(124)^{-1} = (124)(123)(142) = (324) = (243)$$

Therefore $(243) \in H$, and thus $(243)^{-1} = (234) \in H$. We now have defined 4 elements of H .

$$(243)(123)(243)^{-1} = (243)(123)(234) = (142)$$

Therefore $(142) \in H$, and thus $(142)^{-1} = (124) \in H$. We have now defined 6 elements of H , which is the maximum our assumption allowed for. Now notice that $(1) \in H$, which brings the total to 7. Therefore there exists no such H with 6 elements, and A_4 has no subgroup of order 6. ■

Theorem 6.12. Two cycles $\tau, \mu \in S_n$ have the same order if and only if there exists some $\sigma \in S_n$ such that $\mu = \sigma\tau\sigma^{-1}$.

Proof. Let

$$\tau = (a_1, \dots, a_k)$$

$$\mu = (b_1, \dots, b_k)$$

Define $\sigma(a_i) = b_i$ for some index $i \in \mathbb{N}$. For any b_i , observe that

$$\sigma^{-1}(b_i) = a_i$$

$$\mu a_i = a_{i+1}$$

$$\sigma(a_{i+1}) = b_{i+1}$$

$$\mu(b_i) = b_{i+1}$$

where addition in the index is defined to be $(i \bmod k) + 1$. Therefore, $\mu = \sigma\tau\sigma^{-1}$.

For the reverse case, let τ retain the above definition and let $\sigma \in S_n$ be arbitrarily chosen such that $\sigma\tau\sigma^{-1} = \mu$. If $\sigma(a_i) = b$ and $\sigma(a_{(i \bmod k)+1}) = b'$, then $\mu(b) = b'$. Since σ is bijective, the cycle

$$\mu = (\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))$$

is the same length as τ . ■

6.3 Fermat's and Euler's Theorems

The Euler ϕ -Function is the map $\phi : \mathbb{N} \rightarrow \mathbb{N}$ defined as $\phi(1) = 1$ and $\phi(n)$ as the number of integers that are coprime to n ($\gcd(m, n) = 1$ for $1 \leq m < n$), for $n > 1$.

Theorem 6.13. Let $U(n)$ be the group of units in \mathbb{Z}_n . Then $|U_n| = \phi(n)$.

This gives rise to an important theorem in number theory.

Theorem 6.14 (Euler's Theorem (wow great name guys)). Let $a, n \in \mathbb{Z}$ with $n > 0$ and $\gcd(a, n) = 1$. Then $a^{\phi(n)} \equiv 1 \pmod{n}$.

Proof. By 6.13, $|U(n)| = \phi(n)$. Therefore, either $a^{\phi(n)} = 1$ for all $a \in U(n)$, or $a^{\phi(n)} - 1$ is divisible by n . Therefore, $a^{\phi(n)} \equiv 1 \pmod{n}$. ■

Notice that, trivially, $\phi(p) = p - 1$ for any prime number. This leads to the special case known as Fermat's Little Theorem, not to be confused with Fermat's Last Theorem.

Theorem 6.15 (Fermat's Little Theorem). Let p be prime and suppose $p \nmid a$. Then

$$a^{p-1} \equiv 1 \pmod{p}$$

Furthermore, for any $b \in \mathbb{Z}$, $b^p \equiv b \pmod{p}$.

Chapter 7

Isomorphisms

(skipping chapters 7 and 8 because i dont find those applications of algebra very interesting, and since I have to take a class on algebra in college anyways id prefer to read what I find **enjoyable** rather than everything in the textbook).

7.1 Definition and Examples

Definition 7.1 (Group Isomorphism). Two groups (G, \cdot) and (H, \circ) are **isomorphic** if there exists a bijective map $\phi : G \rightarrow H$ that preserves the group operation. That is,

$$\phi(a \cdot b) = \phi(a) \circ \phi(b)$$

for all $a, b \in G$. If G is isomorphic to H , we write $G \cong H$ and call ϕ an isomorphism, specifically a **group isomorphism**.

Example. We can show that $\mathbb{Z}_4 \cong \langle i \rangle$. Define $\phi : \mathbb{Z}_4 \rightarrow \langle i \rangle$ as the map $\phi(n) = i^n$. The reasons for such a definition should be self-evident. Brute force can show that ϕ is bijective:

$$\begin{array}{ll} \phi(0) = 1 & \phi(1) = i \\ \phi(2) = -1 & \phi(3) = -i \end{array}$$

To show that ϕ preserves the group operation, let $m, n \in \mathbb{Z}_4$. We have

$$\phi(m + n) = i^{m+n} = i^m i^n = \phi(m) \phi(n)$$

Therefore the group operation is preserved, and $\mathbb{Z}_4 \cong \langle i \rangle$.

Example. The groups \mathbb{Z}_8 and \mathbb{Z}_{12} cannot be isomorphic, since $|\mathbb{Z}_8| \neq |\mathbb{Z}_{12}|$.

Theorem 7.2. Let $\phi : H \rightarrow G$ be a group isomorphism. Then the following is true.

- $\phi^{-1} : H \rightarrow G$ is an isomorphism.
- $|G| = |H|$
- If G is abelian, then H is abelian.
- If G is cyclic, then H is cyclic.
- If G has a subgroup of order n , then H has a subgroup of order n .

Proof. The first two follow from the fact that ϕ is a bijection. The proofs of 4 and 5 are left as an exercise, so this should be fun.

Let $h_1, h_2 \in H$. Since ϕ is surjective, there exist $g_1, g_2 \in G$ such that $\phi(g_1) = h_1$ and $\phi(g_2) = h_2$. Let G be abelian. Therefore,

$$\begin{aligned} h_1 h_2 &= \phi(g_1) \phi(g_2) \\ &= \phi(g_1 g_2) \\ &= \phi(g_2 g_1) \\ &= \phi(g_2) \phi(g_1) \\ &= h_2 h_1 \end{aligned}$$

To prove the fourth statement, notice that

$$\phi(n^k) = \phi(n \cdots n) = \phi(n) \cdots \phi(n) = \phi(n)^k$$

for some $k \in \mathbb{Z}$. Let G be cyclic with $\langle g \rangle = G$ for some generator $g \in G$. Let $\phi(g) = h$. Since g generates G , any $g' \in G$ can be written as

$$g^k = g'$$

for some $k \in \mathbb{Z}$. Therefore, for all $g' \in G$, we have

$$\phi(g') = \phi(g^k) = \phi(g)^k = h^k$$

Since ϕ is bijective, any h' can be written as some $\phi(g')$, implying that any h' can thus be written as some $\phi(g^k) = h^k$. Therefore, H is cyclic and $\langle h \rangle = H$.

Now we show the fifth condition. Let S be a subgroup of G with $|S| = n$. We can define the set L as the set

$$L := \{\phi(s) \mid s \in S\}$$

If we let $s_1, s_2 \in S$, then $s_1 s_2 \in S$. Furthermore, $\phi(s_1 s_2) = \phi(s_1) \phi(s_2)$, and $\phi(s_1), \phi(s_2) \in L$. However, since $s_1 s_2 \in S$, $\phi(s_1 s_2) \in L$, and thus $\phi(s_1) \phi(s_2) \in L$. Therefore, L is closed under the group operation, and defines a subgroup of H . Because ϕ is a bijection and we can abuse notation to concisely say $\phi(S) = L$, $|S| = |L| = n$. ■

Theorem 7.3. All cyclic groups of infinite order are isomorphic to \mathbb{Z} .

Proof. Let G be a cyclic group with infinite order and let $\langle a \rangle = G$ be a generator for G . Define the map $\phi : \mathbb{Z} \rightarrow G$ by $n \mapsto a^n$. Then

$$\phi(m+n) = a^{m+n} = a^m a^n = \phi(m)\phi(n)$$

Now we must show ϕ is a bijection. Let $m, n \in \mathbb{Z}$ with $m \neq n$, specifically WLOG $m > n$. Suppose for purpose of contradiction that $a^m = a^n$. It follows that

$$a^{m-n} = e \implies \Leftarrow$$

because this contradicts the assumption that G has infinite order, and $m-n > 0$ is finite. Since a is a generator for G , ϕ is onto since for any $\alpha \in G$, there exists some integer n such that $\alpha = a^n$, and $\phi(n) = a^n = \alpha$. Therefore, ϕ is an isomorphism. \blacksquare

Theorem 7.4. Let G be cyclic with $|G| = n$. Then $G \cong \mathbb{Z}_n$.

Proof. Let G be a cyclic group with order n and generator $a \in G$. Define the map $\phi : \mathbb{Z}_n \rightarrow G$ by $k \mapsto a^k$, with $0 \leq k < n$. The proof that ϕ is an isomorphism is left as an exercise
woooooooooo

First, show that ϕ preserves the group operation. Let $m, r \in \mathbb{Z}_n$. Notice that $m, r < n$ in \mathbb{Z}_n . Thus,

$$\phi(m+r) = a^{m+r} = a^m a^r = \phi(m)\phi(r)$$

Now it remains to be shown that ϕ is a bijection. Suppose $m \neq r$. WLOG, let $m > r$. Suppose for purpose of contradiction that $\phi(m) = \phi(r)$. We have

$$\phi(m) = \phi(r) \implies a^m = a^r \implies a^{m-r} = e$$

$$m > r \implies m-r \neq 0 \nexists \alpha \in \mathbb{Z} \implies \Leftarrow$$

Therefore $m \neq r \implies \phi(m) \neq \phi(r)$. It remains only to be shown that ϕ is surjective. Any $g \in G$ can be written as a^k for some $0 \leq k < n$. Notice that

$$g = a^k = \phi(k)$$

Therefore ϕ is surjective, and $\therefore G \cong \mathbb{Z}_n$. \blacksquare

Corollary 7.5. If G is a group of order p with p as a prime number, then $G \cong \mathbb{Z}_p$.

Proof. By 6.10, any $|G| = p$ is cyclic. Thus, by 7.4, $G \cong \mathbb{Z}_p$. \blacksquare

Theorem 7.6. Isomorphisms of groups defines an equivalence relation on the class of all groups.

Proof. Proof is left as an exercise. Luckily I thought of this about a month ago and already proved it!!!

Any group is trivially isomorphic to itself by the identity function. Thus, $G \cong G$ for some group G . Let G and H be groups with isomorphism $\phi : G \rightarrow H$. By 7.2, $\phi^{-1} : H \rightarrow G$ is also an isomorphism, so $G \cong H \implies H \cong G$. Now let $\phi : G \rightarrow H$ define the isomorphism $G \cong H$ and let $\psi : H \rightarrow I$ define the isomorphism $H \cong I$, where I is a group. Consider the function $\psi \circ \phi : G \rightarrow I$. Let $g_1, g_2 \in G$. Since ϕ and ψ are isomorphisms, we have

$$(\psi \circ \phi)(g_1 g_2) = \psi(\phi(g_1 g_2)) = \psi(\phi(g_1) \phi(g_2)) = \psi(\phi(g_1)) \psi(\phi(g_2)) = (\psi \circ \phi)(g_1) (\psi \circ \phi)(g_2)$$

Therefore $\psi \circ \phi$ is an isomorphism, and $G \cong H \wedge H \cong I \implies G \cong I$. Therefore, isomorphisms define an equivalence relation on the class of groups. ■

The result of this theorem gives a goal of group theory: to classify all groups up to isomorphism with groups that are well studied, since we can treat isomorphic groups as equivalent. This idea gives rise to representation theory, where we represent groups with isomorphic groups we know a lot about. The following theorem is a representation theorem, and a very strong one at that.

Theorem 7.7 (Cayley). Every group is isomorphic to a group of permutations.

Proof. Let G be a group. The goal is to find some group of permutations \overline{G} that is isomorphic to G . For any $g \in G$, define a function $\lambda_g : G \rightarrow G$ as $\lambda_g(a) = ga$. We must show each λ_g defines a permutation on G . To show that it is one to one, let $\lambda_g(a) = \lambda_g(b)$ and notice that

$$\begin{aligned} \lambda_g(a) &= ga \\ \lambda_g(b) &= gb \\ \implies gb &= ga \implies g = a \end{aligned}$$

Therefore λ_g is injective. To show that it is surjective, we must show for any $a \in G$, there exists some $b \in G$ such that $\lambda_g(b) = a$. Let $b = g^{-1}a$. We have $\lambda_g(b) = a$.

Now we can define the group \overline{G} :

$$\overline{G} := \{\lambda_g \mid g \in G\}$$

It still remains to be shown that \overline{G} is both a group and that it is isomorphic to G . To show this first thing, since it is more important, we suppose that (\overline{G}, \circ) is a group. Let $g, h, k, a \in G$ and consider closure under addition.

$$(\lambda_g \circ \lambda_h)(a) = \lambda_g(\lambda_h(a)) = \lambda_g(ha) = gha = \lambda_{gh}(a)$$

Since G is a group, $gh \in G$, and thus $\lambda_{gh} \in \overline{G}$. Next, we show associativity.

$$\begin{aligned} (\lambda_g \circ (\lambda_h \circ \lambda_k))(a) &= \lambda_g((\lambda_h \circ \lambda_k)(a)) = \lambda_g(\lambda_h(\lambda_k(a))) = (\lambda_g \circ \lambda_h)(\lambda_k(a)) \\ &= ((\lambda_g \circ \lambda_h) \circ \lambda_k)(a) \end{aligned}$$

Therefore \overline{G} is associative. Notice that $\lambda_e(a) = ea = a$, meaning $\lambda_e = \text{id}_G$. As seen previously, this is an additive identity under function composition. Define $\lambda_g^{-1} = \lambda_{g^{-1}}$. We have

$$(\lambda_{g^{-1}} \circ \lambda_g)(a) = g^{-1}ga = a = \lambda_e(a)$$

Therefore \overline{G} is a group under function composition. Now define the isomorphism $\phi : G \rightarrow \overline{G}$ as the map $g \mapsto \lambda_g$. Trivially, ϕ is surjective. We now show that ϕ is injective. Suppose $g, h \in G$. Then it follows that

$$\phi(g) = \lambda_g$$

and

$$\phi(h) = \lambda_h$$

Suppose $\lambda_h = \lambda_g$. For any $a \in G$,

$$ga = \lambda_g(a) = \lambda_h(a) = ha$$

Thus, ϕ is injective. Finally, we show that ϕ defines an isomorphism.

$$\phi(gh) = \lambda_{gh} = \lambda_g \lambda_h = \phi(g)\phi(h)$$

Therefore, $G \cong \overline{G}$. ■

Remark. The isomorphism in the above proof gives the **left regular representation** of G .

7.2 Direct Products

Direct products allow the study of groups via Cartesian products of groups.

7.2.1 External Direct Products

Definition 7.8 (External Direct Product). Let $(G, \cdot), (H, \circ)$ be groups. We define the **external direct product** of G and H as the Cartesian product $G \times H$, where for any $(g_1, h_1), (g_2, h_2) \in G \times H$, we define the binary operation as

$$(g_1, h_1)(g_2, h_2) = (g_1 \cdot g_2, h_1 \circ h_2)$$

Remark. The external direct product can be repeated an arbitrary number of times:

$$\prod_{i=1}^n G_i = G_1 \times \cdots \times G_n$$

If all these groups are equivalent, we often simply write G^n .

Proposition 7.9. Let G and H be groups. The external direct product $G \times H$ is a group.

Proof. Let $(g_1, h_1), (g_2, h_2) \in G \times H$. We have

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$$

Since G and H are closed under their respective group operations, $g_1g_2 \in G$ and $h_1h_2 \in H$, and thus $(g_1g_2, h_1h_2) \in G \times H$.

Let e_1 be the identity of G and e_2 the identity of H , and note that $(e_1, e_2) \in G \times H$. Then we have

$$(g_1, h_1)(e_1, e_2) = (g_1e_1, h_1e_2) = (g_1, h_1)$$

since each is an identity under their own group. Therefore, $G \times H$ has an identity.

For any $(g, h) \in G \times H$, we can define the identity $(g^{-1}, h^{-1}) \in G \times H$:

$$(g, h)(g^{-1}, h^{-1}) = (gg^{-1}, hh^{-1}) = (e_1, e_2)$$

Associativity is effort to prove but similarly to the first three it follows directly from the fact that G and H are groups. ■

Theorem 7.10. Let $(g, h) \in G \times H$. If g and h have finite orders r and s respectively, then the order of (g, h) in $G \times H$ is the least common multiple of r and s .

Proof. This should be blatantly obvious I don't know why this proof is so long, but there must be some reason. Anyways if $(g, h)^m = (g^m, h^m) = (e_G, e_H)$ then m must be some multiple of both r and s in order for this to occur. Thus, the least common multiple must be the order. ■

Corollary 7.11. Let $(g_1, \dots, g_n) \in \prod G_i$. If g_i has finite order r_i in G_i , then the order of $(g_1, \dots, g_n) \in \prod G_i$ is the least common multiple of r_1, \dots, r_n .