Written Assignment 1
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I have never formally learned how to write proofs and I'm like 100% self taught so I'm sorry if I made any mistakes with notation or stuff, I'm still trying to get better.

1) Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 8 \\ -3x & - & 6y & - & 3z & = & -21 \end{array}$$

Part A:

$$\begin{bmatrix} 1 & 2 & 1 & 8 \\ -3 & -6 & -3 & -21 \end{bmatrix} \xrightarrow{3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 8 \\ -3+3 & -6+6 & -3+3 & -21+24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\therefore 0x + 0y + 0z = 3$$

$$0x + 0y + 0z = 0 \ \forall x, y, z \in \mathbb{R}$$

$$\Rightarrow 0 = 3$$

$$\text{However, } 0 \neq 3 \Rightarrow\Leftarrow$$

Since we have a contradiction, our assumption that the system was consistent must be wrong.

$$\therefore \text{ this system must be inconsistent and have no solution.}$$

Part B:
The fact that this system is inconsistent means there does not exist any point (x, y, z) such that the planes intersect. Since these planes are continuous for all $(x, y, z) \in \mathbb{R}^3$, this necessarily implies that the planes are parallel to each other.

2)

For a 3×5 matrix, if the fifth column is a pivot column, this implies that exactly one row has a leading 1 in the 5th column. A leading 1 must have all entries to the left of it in its column be 0. Thus, at least one column in our matrix will look like this:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This column implies the following statement:

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 1$$

where x_i denotes a variable in $\mathbb{R}$. We know that for all $x_i \in \mathbb{R},\ 0x_i = 0$. Thus, this statement simplifies to

$$0 = 1$$

This is a contradiction, as $0 \neq 1$, so we find that this matrix represents an inconsistent system.

3)

Since we're asked for the sales in the fourth year, it's reasonable to assume the given values start from the first year. We give x as the year, and y as the sales in millions. Thus, we must find an equation of the form

$$y = ax^2 + bx + c$$

that contains the points $(1, 50)$, $(2, 60)$, and $(3, 75)$. By plugging in our point values, we can have

$$\begin{cases} a + b + c = 50 \\ 4a + 2b + c = 60 \\ 9a + 3b + 1 = 75 \end{cases}$$

This system of equations has the matrix representation

$$\begin{bmatrix} 1 & 1 & 1 & 50 \\ 4 & 2 & 1 & 60 \\ 9 & 3 & 1 & 75 \end{bmatrix}$$

We use the elementary row operations allowed under Gaussian elimination to solve this system.

$$\begin{bmatrix} 1 & 1 & 1 & 50 \\ 4 & 2 & 1 & 60 \\ 9 & 3 & 1 & 75 \end{bmatrix} \xrightarrow{-4R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 50 \\ 0 & -2 & -3 & -140 \\ 9 & 3 & 1 & 75 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 50 \\ 0 & 1 & \frac{3}{2} & 70 \\ 9 & 3 & 1 & 75 \end{bmatrix}$$

$$\xrightarrow{-9R_1+R_3 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 50 \\ 0 & 1 & \frac{3}{2} & 70 \\ 0 & -6 & -8 & -375 \end{bmatrix} \xrightarrow{R_3+6R_2 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 50 \\ 0 & 1 & \frac{3}{2} & 70 \\ 0 & 0 & 1 & 45 \end{bmatrix} \xrightarrow{-R_3+R_1 \to R_1} \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & \frac{3}{2} & 70 \\ 0 & 0 & 1 & 45 \end{bmatrix}$$

$$\xrightarrow{-\frac{3}{2}R_3+R_2 \to R_2} \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 45 \end{bmatrix} \xrightarrow{-R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 45 \end{bmatrix}$$

The matrix, now given in reduced row-echelon form, indicates the following equation will satisfy this problem:

$$y = \frac{5}{2}x^2 + \frac{5}{2}x + 45$$

To predict the sales in the fourth year, we can solve for y given $x = 4$:

$$\begin{aligned} y =& \frac{5}{2}4^2 + \frac{5}{2}4 + 45 \\ =& \frac{5 \cdot 16}{2} + \frac{5 \cdot 4}{2} + 45 \\ =& 40 + 10 + 45 \\ =& 95 \end{aligned}$$

This model predicts the sales in year 4 will reach \$95 million.

4)

$$\begin{aligned} x - y + 2z &= 0 \\ -x + y - z &= 0 \\ x + ky + z &= 0 \end{aligned}$$

This problem was very interesting. First we give the matrix representation of this system:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & k & 1 & 0 \end{bmatrix}$$

We perform the following operation:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & k & 1 & 0 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_1} \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & k & 1 & 0 \end{bmatrix}$$

Looking at the first row, we see that for all x, y, it must be true that $z = 0$. We cannot solve for x and y from here, since we do not know what k is. However, we will be using the fact that $\forall x, y (z = 0)$ later in the problem.
This system of equations is homogeneous, and thus has the trivial solution $(x, y, z) = (0, 0, 0)$. By a theorem from section 1.1, we know that the solution set (which I have denoted as $\mathcal{S}$ because I wanted it to look cool) of a system of equations must satisfy exactly one of these statements:

1. $\mathcal{S}$ has exactly 1 element.
2. $\mathcal{S}$ has an infinite number of elements.
3. $\mathcal{S} = \varnothing$

Since $(0, 0, 0) \in \mathcal{S}\ \forall k \in \mathbb{R}$, and thus $\mathcal{S} \neq \varnothing$, we only need to find values of k that would allow for an infinite number of solutions.
We know that $x - y + 2z = 0$ and $x + ky + z = 0$. $\forall (x, y, z) \in \mathcal{S}$, the following statements hold:

$$\begin{aligned} & x + ky + z = x - y + 2z \\ \iff & (x - x) + (ky + y) + (z - 2z) = 0 \\ \iff & (ky + y) - z = 0 \\ \iff & ky + y = z \\ \iff & y(k + 1) = z \end{aligned}$$

We are going to now isolate k, which entails dividing by the variable y. This is generally discouraged since division by 0 is undefined over the reals. However, there is a rather trivial proof that the only solution with $y = 0$ is the solution $(0, 0, 0)$, and we're trying to find the values of k that allow for solutions beyond the aforementioned solution. To prove this, suppose we have $y = 0$ and recall that $z = 0\ \forall x, y$. Recalling the equations from our original system, we observe

$$\begin{aligned}
&x - y + 2z = 0 \\
\Rightarrow &x - 0 + 0 = 0 \\
\Rightarrow &x = 0 \\
&-x + y - z = 0 \\
\Rightarrow &-x + 0 - 0 = 0 \\
\Rightarrow &x = 0 \\
&x + ky + z = 0 \\
\Rightarrow &x + k(0) + 0 = 0 \\
\Rightarrow &x = 0
\end{aligned}$$

Thus the only solution with $y = 0$ is $(0, 0, 0)$. For any k allowing for solutions beyond this solution, there must exist a corresponding $y \neq 0$ that satisfies the system of equations. We thus allow division by y in this scenario.

$$\text{Recall } y(k+1) = z$$
$$\Rightarrow k + 1 = \frac{z}{y}$$
$$\text{Recall } z = 0$$
$$\Rightarrow k + 1 = 0$$
$$\Rightarrow k = -1$$

If this statement is true, that is $k = -1$, it will imply that there are an infinite number of solutions. This becomes more obvious when we realize that $k = -1$ turns the third equation in our system into the negative of the second equation, effectively turning the system into a system of two equations with three variables, which cannot have only one solution since the number of variables is greater than the number of equations, and by a theorem from 1.1 this implies either no solutions or infinite solutions exist. Since this system is homogeneous, and thus $\mathcal{S} \neq \varnothing$, there must be infinite solutions.
If it is not true, that is $k \neq -1$, then it is necessary that $y = 0$, otherwise there would be no other way to arrive at this contradiction. To illustrate this, suppose $k \neq -1$. It follows that $k + 1 \neq 0$. We previously saw the statement

$$y(k+1) = z$$

Remember that z is always equal to 0. Thus, we have

$$y(k+1) = 0$$

If $k + 1 \neq 0$, then we can divide both sides by $k + 1$ and see the following:

$$\frac{y(k+1)}{k+1} = \frac{0}{k+1}$$
$$\Leftrightarrow y = 0$$

Thus, $k \neq -1$ necessarily implies $y = 0$. We previously saw that $y = 0 \Leftrightarrow \mathcal{S} = \{(0,0,0)\}$. Thus, any $k \neq -1$ will give this system of equations exactly one solution.

$$\therefore k \in \mathbb{R} \setminus \{-1\}$$

5)

Consider the following matrices:

$$X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad W = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Part A:

$$\text{Let } a, b \in \mathbb{R}$$

$$\text{Let } Z = aX + bY$$

$$\begin{aligned}
\Rightarrow \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} &= a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} a+b \\ b \\ a \end{bmatrix}
\end{aligned}$$

$$\iff \begin{cases} a + b = 2 \\ b = -1 \\ a = 3 \end{cases}$$

$$a + b = 3 - 1 = 2$$

$$\therefore a = 3 \text{ and } b = -1$$

Part B:

$$\text{Suppose for contradiction that } \exists a, b \in \mathbb{R} \text{ such that } aX + bY = W$$

$$\implies a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\iff \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\iff \begin{bmatrix} a+b \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore a = 1 \text{ and } b = 1 \text{ and } a + b = 1$$

$$\text{However, } a = 1 \text{ and } b = 1 \text{ implies } a + b = 1 + 1 = 2 \neq 1 \Rightarrow\Leftarrow$$

$$\text{Therefore, no such scalars } a \text{ and } b \text{ exist.}$$

$$\therefore \nexists a, b \in \mathbb{R}(aX + bY = W)$$

Part C:

$$\text{Let } aX + bY + cW + \vec{0}$$

$$\Rightarrow \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \\ + \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+b+c \\ b+c \\ a+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow a + b + c = b + c = 0$$

$$\Rightarrow a = 0$$

$$\Rightarrow \begin{bmatrix} b+c \\ b+c \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow c = 0$$

$$\Rightarrow \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow b = 0$$

$$\therefore aX + bY + cW = \vec{0} \Leftrightarrow a = 0,\ b = 0,\ c = 0$$

Part D:

$$aX + bY + cZ = \vec{0}$$

$$\Rightarrow a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+b+2c \\ b-c \\ a+3c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow b - c = 0 \Rightarrow b = c$$

$$Rightarrow \begin{bmatrix} a+3c \\ c-c \\ a+3c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow a+3c=0 \Rightarrow a=-3c$$

$$\text{Let } c=1$$

$$\Rightarrow b=1, \quad a=-3$$

$$\Rightarrow \begin{bmatrix} -3+1+2 \\ 1-1 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore the scalars $a=-3$, $b=1$, and $c=1$ satisfy the statements $aX+bY+cZ=\vec{0}$ and $\{a,b,c\} \setminus \{0\} \neq \varnothing$.