Written Assignment 4
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Note - I often use the notation $\exists(x)(P(x))$ to mean "there exists x such that P(x). Basically I encase each argument in parenthases to be 100% clear about which part of the statement is which, and I leave the "such that" implicit in my notation between the parenthases because I'm too lazy to write it most of the time. I often use similar notation for $\forall(x)(P(x))$, meaning "for all x, P(x) is true." I think I've seen similar notation used before so I would assume it's not extremely nonstandard, but none of my profs have written like that so I like to clarify my notation.

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1) Let W be the set of all vectors in $\mathbb{R}^2$ whose components are integers. Either show that W is a subspace of $\mathbb{R}^2$, or provide a specific example that violates the test for a subspace (Theorem 4.5).

Let $\mathbb{R}^2$ be defined over $\mathbb{R}$ with $W \coloneqq \mathbb{Z}^2$. Trivially, $W \subset \mathbb{R}^2$. Equally as trivially, W is not a subspace of $\mathbb{R}^2$ due to not being closed under the standard definition of scalar multiplication. Consider $0.5 \in \mathbb{R}$. For any odd number $\alpha \in \mathbb{Z}$, we have

$$0.5\alpha \notin \mathbb{Z}$$

Thus, for any $\mathbf{v} \in W$ with a component α, it follows that $0.5\mathbf{v} \notin W$. For example, $0.5(3, 1) = (1.5, 0.5) \notin W$. This can be taken a step further.

$$\pi \in \mathbb{R} \setminus \mathbb{Q}$$
$$\implies \nexists(\beta, \gamma \in \mathbb{Z}) \left(\pi = \frac{\beta}{\gamma}\right)$$

The second step is true by definition of an irrational number. For any $\beta, \gamma \in \mathbb{Z}$, we thus have

$$\pi \neq \frac{\beta}{\gamma}$$
$$\implies \gamma\pi \neq \beta$$
$$\therefore \forall\,(\beta \in \mathbb{Z})\,(\beta\pi \notin \mathbb{Z})$$

Therefore, we can confidently say that for all $\mathbf{v} \in W$, it follows that $\pi\mathbf{v} \notin W$, and the same will follow for any irrational number. Therefore W is not closed under scalar multiplication. ■

This problem actually reminds me of when I tried to prove $\mathbb{Z}$ couldn't form a vector space a little while ago, but looking back on it my proof was not very correct. Like I said $\{0, 1\}$ wasn't a field since $1+1 = 2 \notin \{0, 1\}$, but $\mathbb{Z}/2\mathbb{Z}$ actually does form a field with $1+1 = 0$. I think my proof works however, I just need to clean up some of the arguments in it.

2) Let A be a fixed $m \times n$ matrix. Use properties of matrix addition and multiplication to show that the set

$$S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

is a subspace of $\mathbb{R}^n$.

For any $A \in M_{m,n}$, the trivial solution $\mathbf{x} = \mathbf{0}$ will be a solution to the equation $A\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{0} \in S$. By the definition of S, we also have

$$\forall(\mathbf{x} \in S)\,(\mathbf{x} \in \mathbb{R}^n)$$

Thus, $S \subseteq \mathbb{R}^n$.
Let $\mathbf{x}, \mathbf{y} \in S$. It follows that

$$A\mathbf{x} = \mathbf{0}$$
$$A\mathbf{y} = \mathbf{0}$$

Consider $\mathbf{x} + \mathbf{y}$. By the properties of matrix multiplication and the fact that $\mathbb{R}^n$ is a vector space, we have

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = 0 + 0 = 0$$
$$\therefore \mathbf{x} + \mathbf{y} \in S$$

Thus S is closed under vector addition. Let $\alpha \in \mathbb{R}$. By the associative property defined over matrices, real numbers, and the vector space $\mathbb{R}^n$, we have

$$A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha(0) = 0$$
$$\therefore \alpha\mathbf{x} \in S$$

Thus, S is closed under scalar multiplication. By one of the theorems, since $S \subseteq \mathbb{R}^n$, $S \neq \varnothing \because \mathbf{0} \in S$, and S is closed under vector addition and scalar multiplication as defined in $\mathbb{R}^n$, it follows that S is a subspace of $\mathbb{R}^n$. ■

I'm just realizing this as I'm checking over my work before submi0tting this - if $A : \mathbb{R}^n \to \mathbb{R}^m$ were a linear map, wouldn't the nullspace just be $\ker(A)$? Which would be $\{\mathbf{0}\}$ if A is an isomorphism, but the general case is just a homomorphism so that's not necessarily true. Wait if an isomorphism is a bijective homomorphism, and all linear maps are homomorphic, then any linear map given by an invertible matrix gives an isomorphism since a function is bijective iff it is invertible. And for $A\mathbf{x} = \mathbf{0}$, the nullspace is $\{\mathbf{0}\}$ iff A^{-1} exists. Wow why does every single part of linear algebra connect together so well. And why am I thinking out loud into a document.

3) Let A be a fixed $m \times n$ matrix. Determine whether the set

$$T = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{b} \neq \mathbf{0}\}$$

is a subspace of $\mathbb{R}^n$? Justify your answer.

Let $A \in M_{m,n}$ be an arbitrary matrix with real entries. T is a subspace iff $\mathbf{0} \in T$. However, for $\mathbf{x} = \mathbf{0}$,

$$A\mathbf{0} = \mathbf{0}$$

However, T requires $A\mathbf{x} \neq \mathbf{0}$, meaning $\mathbf{0} \notin T$. Thus T is not a vector space, and cannot be a subspace of $\mathbb{R}^n$. $\blacksquare$

4) Let $S = \{(1,2,-3),(-1,2,2)\}$

1. Are the vectors in S linearly independent? Justify your answer.
2. Do the vectors in S span $\mathbb{R}^3$? Justify your answer.
3. Find a basis for the span(S). Justify your answer.

Suppose there exists some $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \beta \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

If this statement holds for some $(\alpha, \beta) \neq (0,0)$, then S is linearly dependent. From the above statement, we have

$$\begin{bmatrix} \alpha \\ 2\alpha \\ -3\alpha \end{bmatrix} = \begin{bmatrix} -\beta \\ 2\beta \\ 2\beta \end{bmatrix}$$

Since matrix equality is defined component-wise, from the first entry we have $\alpha = -\beta$. However, from the second entry, we have $2\alpha = 2\beta$, but if $\beta = -\alpha$, we have

$$2\alpha = -2\alpha \iff \alpha = -\alpha \iff \alpha = 0$$

If $\alpha = 0$, then $\beta = 0$ as well, meaning the only solution $\alpha, \beta \in \mathbb{R}$ is the trivial solution. Therefore, S is linearly independent.
We know that $\dim(\mathbb{R}^3) = 3$ due to the standard basis, $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ having $|B| = 3$. Thus, all bases for $\mathbb{R}^3$ have 3 elements. It follows from the definition of a basis that there are no sets with less elements than a basis that span the space. However, $|S| = 2 \neq 3$, meaning $\text{span}(S) \subsetneq \mathbb{R}^3$. Therefore, S does not span $\mathbb{R}^3$.
Suppose for purpose of contradiction that there exists some set $B_S \subset \mathbb{R}^3$ that forms a basis for $\text{span}(S)$ with $|B_S| < |S|$. Since $|S| = 2$, it follows that $|B_S| = 1$. Let $\mathbf{e} \in B_S$ be the element of this basis. Notice that $S \subset \text{span}(S)$. From this,

$$\exists(\alpha, \beta \in \mathbb{R}) \left(\alpha\mathbf{e} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \wedge \beta\mathbf{e} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right)$$

However, S is linearly independent, meaning that there exists no set $|R| < |S|$ where the elements of S can be written as a linear combination of the elements of R. Thus, no such B_S exists, since the elements of S cannot be written as a linear combination of B_S, and by contradiction it follows that S must be its own basis for $\text{span}(S)$. ■

5) Let $T = \{(1,2,-3),(2,-1,2),(5,0,1),(3,0,3)\}$

1. Are the vectors in T linearly independent? Justify your answer.
2. Do the vectors in S span $\mathbb{R}^3$? Justify your answer.
3. Find a basis for the span(T). Justify your answer.

We know that $T \subset \mathbb{R}^3$, and from the above problem $\dim(\mathbb{R}^3) = 3$. However, $|T| = 4 > \dim(\mathbb{R}^3)$, so T can't be linearly independent since it has more elements than the basis of $\mathbb{R}^3$.
Since the rowspace of any two row-equivalent matrices is equal, we can find a basis for span(T) as follows.

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 5 & 0 & 1 \\ 3 & 0 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\operatorname{span}(T) = \operatorname{span}\{(1,0,0),(0,1,0),(0,0,1)\}$. This set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is the standard basis for $\mathbb{R}^3$, and as such $\operatorname{span}\{(1,0,0),(0,1,0),(0,0,1)\} = \mathbb{R}^3$. Thus, $\operatorname{span}(T) = \mathbb{R}^3$. We also find that a basis for span(T) is the standard basis for $\mathbb{R}^3$:

$$\{(1,0,0),(0,1,0),(0,0,1)\}$$

$\blacksquare$