Calculus 3 Notes

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Abstract

Calculus 3 extends the derivative, limit, and integral to functions of multiple variables. It introduces vectors, vector functions, partial derivatives, the gradient vector, iterated integration, line and surface integrals, directional derivatives and tangent planes, optimization, Lagrange multipliers, and the classical theorems of Green, Gauss, and Stokes. This class was taught my Dr. Kassebaum at AACC.

Contents

Chapter 1

Vectors and the Geometry of Space

1.1 Vectors

Theorem 1.1 (Properties of Vectors). Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$ be vectors, and let $c, d \in \mathbb{R}$ be scalars. The following properties are satisfied:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$$

$$\vec{a} + \vec{0} = \vec{a}$$

$$\vec{a} + (-\vec{a}) = \vec{0}$$

$$c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$$

$$(cd)\vec{a} = c(d\vec{a})$$

$$1\vec{a} = \vec{a}$$

Definition 1.2 (Unit Vector). A unit vector is any vector $\vec{v}$ satisfying $|\vec{v}| = 1$. In physics, we give $\hat{v}$ to denote the unit vector in the direction of $\vec{v}$.

The vectors $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$, and $\langle 0, 0, 1\rangle$ are the **standard basis vectors** of $\mathbb{R}^3$. We denote them $\hat{\textbf{\i}}$, $\hat{\textbf{\j}}$, and $\hat{\mathbf{k}}$, respectively.
For some vector $\vec{v}$, the unit vector with the same direction as $\vec{v}$ is given as

$$\frac{\vec{v}}{|\vec{v}|}$$

This notation is slightly abusive, as multiplication and division are not defined for vectors.
One use of vectors is to denote forces. For example, given some $\vec{F_1} = \sqrt{2}\langle -5, 5\rangle$ and $\vec{F_2} = \langle 9\sqrt{3}, 9\rangle$, we can find the resultant force vector and its magnitude:

$$\vec{F_1} + \vec{F_2} = (9\sqrt{3} - 5\sqrt{2})\hat{\textbf{\i}} + (9 + 5\sqrt{2})\hat{\textbf{\j}}$$

$$\Rightarrow |\vec{F_1} + \vec{F_2}| = \sqrt{(9\sqrt{3} - 5\sqrt{2})^2 + (9 + 5\sqrt{2})^2}$$

$$= \sqrt{424 + 90(\sqrt{2} - \sqrt{6})}$$

1.2 The Dot Product

Definition 1.3 (The Dot Product). For $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, we define the dot product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

This is an example of an inner product with the mapping $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The extention from $\mathbb{R}^3$ to $\mathbb{R}^n$ is given as

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$$

where a_i, b_i represent components of the vectors $\vec{a}$ and $\vec{b}$.

Definition 1.4 (Orthogonal Vectors). $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\langle \vec{a}, \vec{b} \rangle = 0$, where $\langle \cdot, \cdot \rangle$ gives an inner product. In calculus 3, this inner product is assumed to always be the dot product, and is generally denoted $\vec{a} \cdot \vec{b}$.

Definition 1.5 (Direction Angles). For some $\vec{a} = \langle a_1, a_2, a_3 \rangle$, we give α, β, γ as the angles formed with the x, y, and z-axes. We can define them as

$$\cos\alpha = \frac{a_1}{|\vec{a}|} \qquad \cos\beta = \frac{a_2}{|\vec{a}|} \qquad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

We also find that

$$\frac{\vec{a}}{|\vec{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

As an example, consider $\vec{a} = \langle 1, 2, 3 \rangle$ and $\vec{b} = -4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$. From 1.17, we have

$$\vec{a} \cdot \vec{b} = -4 + 4 - 3 = -3$$

$$\vec{b} \cdot \vec{a} = -4 + 4 - 3 = -3$$

$$\vec{a} = 1 + 4 + 9 = 14$$

$$\vec{b} = 16 + 4 + 1 = 21$$

We see there are some properties of the dot product.

Theorem 1.6 (Properties of the Dot Product). Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$ be vectors, and let $c, d \in \mathbb{R}$ be scalars. The following properties are satisfied:

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$$

$$\vec{0} \cdot \vec{a} = 0$$

Theorem 1.7 (Alternate Definition of the Dot Product)**.** Let $\vec{a}, \vec{b}$ be vectors and let θ be the angle between them.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

We can now define new things using the dot product.

Definition 1.8 (Vector Projection)**.** The projecion of a vector $\vec{v}$ onto another vector $\vec{u}$ is the component of the vector $\vec{v}$ in the direction of $\vec{u}$. One can imagine two vectors coming from the origin. Forming a right triangle using the top vector as the hypotenuse, the length of the leg that lies along the other vector is the projection of the first vector onto the second vector. The projection of $\vec{u}$ onto $\vec{v}$ is

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{v} \cdot \vec{u}}{|\vec{v}|^2}\vec{v}$$

Definition 1.9 (Scalar Projection)**.** We can define the scalar projection, which is simply the signed magnitude of the vector projection, as seen in 1.8. The scalar projection of $\vec{u}$ onto $\vec{v}$ is given as

$$\text{comp}_{\vec{v}}\vec{u} = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|}$$

As an example, we can use the scalar projection to define work. If a force $\vec{F}$ moves an object from P_1 to P_2, the displacement vector $\vec{d} = \overrightarrow{P_1P_2}$. The work done by the force over this distance is

$$W = \text{comp}_{\vec{d}}\vec{F}|\vec{d}|$$

which is the product of the component of the force over the distance, times the total distance moved. This simplifies to

$$W = \vec{F} \cdot \vec{d}$$

1.3 The Cross Product

We begin by defining the determinant of a 2×2 and 3×3 matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Definition 1.10 (The Cross Product). The cross product is a vector operation defined with the mapping $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, where for some $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, we give

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

If we recall the way to evaluate a 3×3 determinant from above, we have the useful memorization method:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This isn't really a valid determinant to evaluate, but if we treat it as such then it's easier to remember.

The cross product will always give a vector orthogonal to the two starting vectors. We also find that $\vec{a} \times \vec{a} \equiv \vec{0}$. If for some $\vec{a} \times \vec{b}$, the angle between the two vectors $\theta \in [0, \pi]$ goes counter-clockwise, then the resulting vector will point in the positive direction. If the angle goes clockwise, it will point in the negative direction (the negative of its magnitude).

Theorem 1.11 (Sinusoidal Definition of the Cross Product).

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \vartheta$$

Theorem 1.12 (Properties of the Cross Product). Take $|\varphi\rangle$, $|\psi\rangle$, and $|\vartheta\rangle$ as vectors in $\mathbb{R}^3$, because $\vec{a}$ and $\vec{b}$ have different heights and it's annoying. Let $\alpha \in \mathbb{R}$. The following statements will hold:

$$|\psi\rangle \times |\varphi\rangle = -|\varphi\rangle \times |\psi\rangle$$

$$(\alpha|\psi\rangle) \times |\varphi\rangle = |\psi\rangle \times (\alpha|\varphi\rangle) = \alpha(|\psi\rangle \times |\varphi\rangle)$$
$$|\psi\rangle \times (|\varphi\rangle + |\vartheta\rangle) = |\psi\rangle \times |\varphi\rangle + |\psi\rangle \times |\vartheta\rangle$$
$$|\psi\rangle \cdot (|\varphi\rangle \times |\vartheta\rangle) = (|\psi\rangle \times |\varphi\rangle) \cdot |\vartheta\rangle$$
$$|\psi\rangle \times (|\varphi\rangle \times |\vartheta\rangle) = (|\psi\rangle \cdot |\vartheta\rangle)|\varphi\rangle - (|\psi\rangle \cdot |\varphi\rangle)|\vartheta\rangle$$

Remark. using this notation for calculus was a mistake.

1.4 Equations of Lines and Planes

Theorem 1.13 (Vector Parametrization of Lines). Given a point P and a direction vector $\vec{v}$, we define a line as the set of terminal points of position vectors $\vec{r}$. If $P = (p_1, \ldots, p_n)$, we give $\vec{r_0} = \langle p_1, \ldots, p_n \rangle$. We give the vector equation of a line as

$$\vec{r} = \vec{r_0} + t\vec{v}, t \in \mathbb{R}$$

For example, take the point $P = (2, 3)$ and $\vec{v} = \langle 4, 1 \rangle$. We have $\vec{r} = \langle 2, 3 \rangle + \langle 4t, t \rangle$. Since we're defining a line as the set of terminal points of $\vec{r}$, we can write $\langle x, y \rangle = \langle 2 + 4t, 3 + t \rangle$. It follows from the definition of matrix addition that we have $x = 2 + 4t$ and $y = 3 + t$. These are the parametric

equations of this line. We solve for t and have $t = \frac{x-2}{4} = y - 3$. These are the symmetric equations of this line.
Sidenote: if $f : \langle x_1, \ldots, x_n \rangle \mapsto (x_1, \ldots, x_n)$, then we can give our line in set-builder notation as

$$\{f(\vec{r}) \mid \vec{r} = \vec{r_0} + t\vec{v}, t \in \mathbb{R}\}$$

We will be using $P \sim \vec{r}$ and $\vec{r} \sim P$ to notate the position vector $\vec{r}$ for a point P.

Definition 1.14 (Direction Numbers). If we take $\vec{v} = \langle \alpha, \beta, \gamma \rangle$ in the above example, then the direction numbers are α, β, γ since these are the denominators of our symmetric equations.

Theorem 1.15 (Defining Line Segments with Position Vectors). The line segment between points P_0 and P_1 with corresponding position vectors $P_0 \sim \vec{r}_0$ and $P_1 \sim \vec{r}_1$ is given by

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, t \in [0, 1] \subset \mathbb{R}$$

We can verify the above theorem trivially. We have the parametric variable t spanning the interval $[0, 1]$, so we must have $t = 0 \Rightarrow \vec{r}(0) = \vec{r}_0$ and $t = 1 \Rightarrow \vec{r}(1) = \vec{r}_1$. Consider the following:

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$$

$$\begin{aligned} \vec{r}(0) &= (1-0)\vec{r}_0 + 0\vec{r}_1 \\ &\Rightarrow \vec{r}(0) = \vec{r}_0 \\ \vec{r}(1) &= (1-1)\vec{r}_0 + 1\vec{r}_1 \\ &\Rightarrow \vec{r}(1) = \vec{r}_1 \end{aligned}$$

Definition 1.16 (Planes). A plane in $\mathbb{R}^3$ is determined by some point $P_0 = (x_0, y_0, z_0)$ and a normal vector $\vec{n}$ which is orthogonal to the plane.

For example, consider the point $P_0 = (0, 2, 3)$ and vector $\vec{n} = \langle 2, 0, 0 \rangle$. We find $\vec{n} \sim \hat{\mathbf{i}}$, and $P_0 \in \mathbb{R}_y \times \mathbb{R}_z$, thus this vector is orthogonal to the yz plane, which the point happens to be in. Thus this vector and point describe the yz-plane.
Now consider the point $P_0 = (0, 2, 3)$ but the vector $\vec{n} = \langle 2, 0, 1 \rangle$. If we define the plane described herein as the set of points ψ, then we can define $P = (x, y, z) \neq P_0 \wedge P \in \psi$. Let $\vec{r}_0 \sim P_0$ be the position vector of point P_0, and let $\vec{r} \sim P$ be the position vector of point P. This means

$$\vec{r}_0 = \langle 0, 2, 3 \rangle$$

$$\vec{r} = \langle x, y, z \rangle$$

We define a new vector $\vec{r} - \vec{r}_0 = \overrightarrow{PP_0}$ which is parallel to the plane. Thus,

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

is the vector equation of the plane. We also find

$$\Rightarrow \langle 2, 0, 1 \rangle \cdot \langle x - 0, y - 2, z - 3 \rangle = 0$$

$$\Rightarrow 2x + z - 3 = 0$$

This is the scalar equation of the plane. Notice this is a linear equation in x, y, z. In fact, any linear equation x, y, z such that $ax + by + cz + d = 0$ has a plane with normal vector $\vec{n} = \langle a, b, c \rangle$. Another thing we can do is find a vector normal to the plane given points. For example, consider $P = (1, 3, 0)$, $Q = (2, -1, 5)$, and $R = (4, 0, 2)$. We can give the vectors

$$\overrightarrow{PR} = \langle -3, 3, 2 \rangle, \qquad \overrightarrow{QP} = \langle 1, -4, 5 \rangle$$

We compute $\overrightarrow{PR} \times \overrightarrow{QP}$ to find a vector normal to the plane, since we know $\overrightarrow{QP}$ and $\overrightarrow{PR}$ are both in the plane.

$$\overrightarrow{PR} \times \overrightarrow{QP} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 3 & -2 \\ 1 & -4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -2 \\ -4 & 5 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} -3 & -2 \\ 1 & 5 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} -3 & 3 \\ 1 & -4 \end{vmatrix} \hat{\mathbf{k}}$$

$$= (15 - 8)\hat{\mathbf{i}} - (-15 + 2)\hat{\mathbf{j}} + (12 - 3)\hat{\mathbf{k}}$$

Definition 1.17 (Angle Between Planes). For planes ψ and ϕ, the angle between the planes $\theta \in [0, \pi]$ is given by the angle θ between the normal vectors $\vec{n}_\psi$ and $\vec{n}_\phi$.

Definition 1.18 (Parallel and Orthogonal Planes). For planes ψ and ϕ, they are parallel if and only if their normal vectors $\vec{n}_\psi$ and $\vec{n}_\phi$ are parallel. Furthermore, $\phi \perp \psi$ if and only if $\vec{n}_\phi \cdot \vec{n}_\psi = 0$.

1.5 Cylinders and Quadric Surfaces

Definition 1.19 (Traces). A trace of a surface is the curve of its intersection with a coordinate plane.

Definition 1.20 (Cylinder). A surface consisting of all lines parallel to it that pass through a given plane curve.

Definition 1.21 (Quadric Surface). A surface defined by a polynomial equation of degree 2, that is an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0, \; A, B, C, D, E, F, G, H, I, J \in \mathbb{R}$$

Basic quadric surfaces can be defined of the forms

- $z = Ax^2 + By^2$ describes a paraboloid
- $z^2 = Ax^2 + By^2$ describes a double cone
- $\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 1$ describes a hyperboloid of one sheet

- $-\frac{x^2}{A^2} - \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$ describes a hyperboloid of two sheets
- $\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$ descibes an elipsoid

Chapter 2

Vector Functions

2.1 Vector Functions and Space Curves

Definition 2.1 (Vector-Valued Function). A vector-valued function $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ has a domain of real numbers and a range of vectors.

Definition 2.2 (Component Functions). Component functions f, g, h of a vector function $\vec{r}$ are given as

$$\vec{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$$

Definition 2.3 (Space Curve). A space curve is the set C of points (x, y, z) where

$$C = \{(x, y, z) \mid t \in I \subseteq \mathbb{R}, x = f(t), y = g(t), z = h(t)\}$$

Let $\vec{r}(t)\langle 1 + 3t, 2 - 4t, 7 + t\rangle$. It follows that

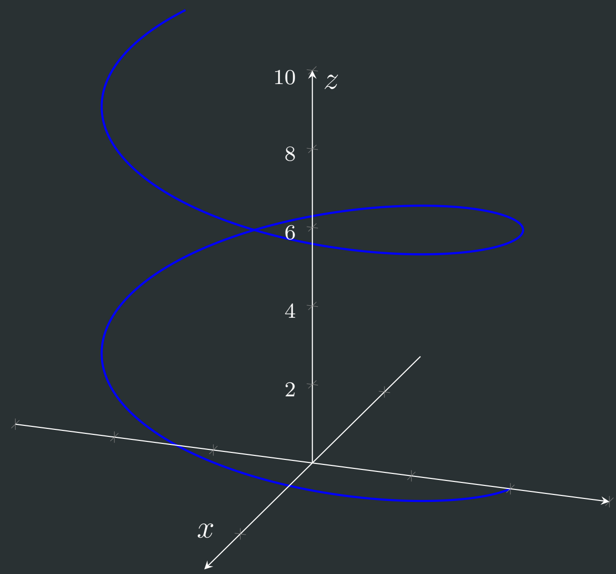
$$\langle x, y, z\rangle = \langle 1 + 3t, 2 - 4t, 7 + t\rangle$$

$$\Rightarrow x = 1 + 3t, \quad y = 2 - 4t, \quad z = 7 + t$$

These are the **parametric equations** of this vector function. We can then also show that

$$t = \frac{x - 1}{3} = \frac{y - 2}{-4} = z - 7$$

These are the **symmetric equations** of this vector function.



$$\vec{r}(t) = \sin(t)\hat{\mathbf{i}} + \cos(t)\hat{\mathbf{j}} + t\hat{\mathbf{k}},\ t \in \mathbb{R}^+ \cup \{0\}$$

Definition 2.4 (Limit of a Vector Function)**.** If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then we will have

$$\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} \langle f(t), g(t), h(t)\rangle = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$$

provided the limits of the component functions exist.

2.2 Derivatives and Integrals of Vector Functions

Definition 2.5 (Derivative of a Vectoe Function)**.** For a vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle, t \in \mathbb{R}$, we define

$$\vec{r}\,' = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

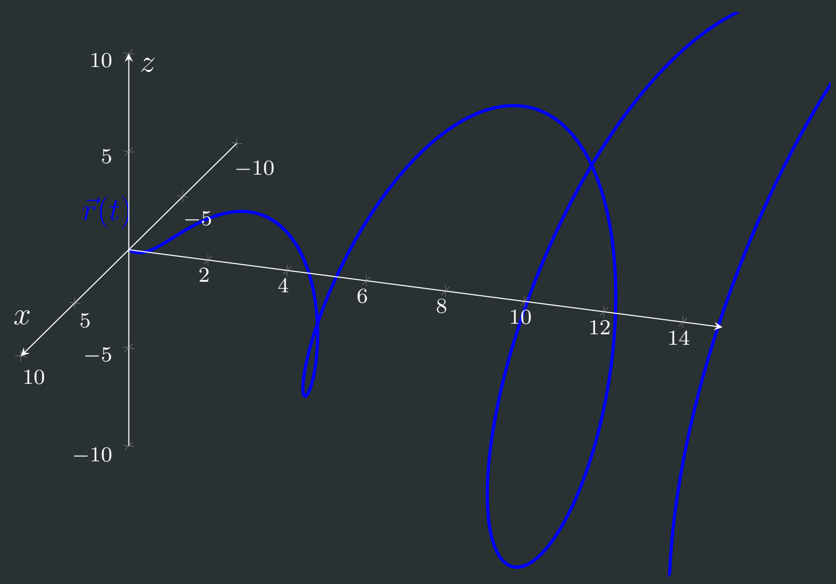
provided this limit exists.

From our earlier definition of the limit of a vector function, we have

$$\lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \left\langle \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}, \lim_{h\to 0} \frac{g(t+h) - g(t)}{h}, \lim_{h\to 0} \frac{h(t+h) - h(t)}{h} \right\rangle$$

We call this vector the **tangent vector**.
For example, consider the following space curve:

$$\vec{r}(t) = \langle t\cos(t), t, t\sin(t)\rangle$$



We give

$$\vec{r}'(t) = \langle \cos(t) - t\sin(t), 1, \sin(t) + t\cos(t) \rangle$$

Definition 2.6 (Tangent Line). The tangent line to the curve traced by $\vec{r}(t)$ at point $P = (x_0, y_0, z_0)$ is the line parallel to the tangent vector and through point P. We give the following definition:

$$\frac{x - x_0}{f'(t)} = \frac{y - y_0}{g'(t)} = \frac{z - z_0}{h'(t)}$$

Definition 2.7 (Unit Tangent Vector). We often use the **Unit Tangent Vector**, $\vec{T}(t)$, where

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Example: find the unit tangent vector at $t = 0$ for

$$\vec{r}(t) = t^3 e^t \hat{\mathbf{i}} - 4t^3 \hat{\mathbf{j}} + \sin(3t)\hat{\mathbf{k}}$$

$$\vec{r}'(t) = (3t^2 e^{t + t^3 e^t})\hat{\mathbf{i}} - 12t^2 \hat{\mathbf{j}} + 3\cos(3t)\hat{\mathbf{k}}$$
$$\implies \vec{r}'(0) = 3\hat{\mathbf{k}}$$

If we changed the $\hat{\mathbf{k}}$ component to cos instead of sin, we would have $\vec{r}'(0) = \vec{0}$, meaning the derivative is undefined for $t = 0$.

Definition 2.8 (Differentiation Rules for Vector-Valued Functions). Let $\vec{v}(t), \vec{u}(t)$ be differentiable vector functions, $c \in \mathbb{R}$, and f is a real-valued function.

- $\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$
- $\frac{d}{dt}(c\vec{u}(t)) = c\vec{u}\,'(t)$
- $\frac{d}{dt}(f(t)\vec{u}(t)) = f(t)\vec{u}\,'(t) + f'(t)\vec{u}(t)$
- $\frac{d}{dt}(\vec{v}(t) \cdot \vec{u}(t)) = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{v}\,'(t) \cdot \vec{u}(t)$
- $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$
- $\frac{d}{dt}\vec{u}(f(t)) = f'(t)\vec{u}\,'(f(t))$

For all of these, order of operation will only matter for the cross product.

Definition 2.9 (Integrals of Vector Functions). For $t \in I = [\alpha, \beta] \subseteq \mathbb{R}$ we give

$$\int_I \vec{r}(t)dt = \left\langle \int_\alpha^\beta f(t)dt, \int_\alpha^\beta g(t), \int_\alpha^\beta h(t) \right\rangle + \vec{C}$$

2.3 Arc Length and Curvature

Definition 2.10 (Arclength Function s in 3 Dimensions). For $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, t \in [a, b] \subseteq \mathbb{R}$, we have

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

Definition 2.11 (Smooth Parametrization of a Curve). $\vec{r}(t)$ is smooth for an interval $t \in I$ if $\vec{r}\,'$ is continuous and $\vec{r}\,'(t) \neq \vec{0} \forall t \in I$.

Definition 2.12 (Curvature (The intuitive and useless definition)).

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$

Suppose we have $|\vec{r}(t)| = c \in \mathbb{R} \forall t$. We then have

$$\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2$$

$$\implies \frac{d\vec{r}}{dt} = 0 = 2(\vec{r}\,'(t) \cdot \vec{r}(t))$$

$$\left|\vec{T}(t)\right| = 1 \forall t$$

$$\therefore \vec{T}(t) \perp \vec{T}\,'(t)$$

Definition 2.13 (Principal Unit Normal Vector (also Unit Normal)). The unit normal $\vec{N}(t)$ is given as

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\left|\vec{T}\,'(t)\right|}$$

Definition 2.14 (Binormal Vector). The binormal vector $\vec{B}(t)$ is given as

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Definition 2.15 (Torsion of a Curve). Torsion generally indicates how tight a curve bends. We give it as τ.

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}(t) = -\frac{\vec{B}(t) \cdot \vec{N}(t)}{|\vec{r}\,'(t)|}$$

Recall that for some $(x, y) = (f(t), g(t))$, arclength s is given as

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt$$

In three dimensions with a curve parametrized by a vector function where $t \in [a, b]$, we have

$$s(t) = \int_a^t |\vec{r}(u)| \; dt$$

By the fundamental theorem of calculus, we also conclude that

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

From the definition of the magnitude of a vector, we have

$$s(t) = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2} \; dt$$

Since s is a function of t, we can often reparametrize $\vec{r}$ as a function of its length, which can be exceedingly useful, but rather tough. To do this, we compute the integral definition of arc length, solve for t in terms of s, and substitute our solution in for t in the original $\vec{r}$.
An inverse function does not always exist though, and we need the implicit function theorem to tell us if it does. So this is not always possible to do. However, it's possible to *assume* we can reparametrize something that we can't. Recall curvature $\kappa = \left|\frac{d\vec{T}}{dt}\right|$. From our knowledge of the derivatives of parametric functions, and the fact that $\vec{T}$ is a function of s, even though it may not always be possible to write it as one, we give

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$$

An easier way of thinking about this is as the chain rule:

$$\frac{d}{dt}\vec{T}(s(t)) = \vec{T}'(s(t))s'(t)$$

We can now solve for $\frac{d\vec{T}}{ds}$:

$$\frac{d\vec{T}}{dt} = \frac{\vec{T}}{ds}\frac{ds}{dt} \implies \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{T}}{ds}$$

Recall $\kappa = \left|\frac{d\vec{T}}{ds}\right|$ and $\frac{ds}{dt} = |\vec{r}'(t)|$. We have

$$\kappa(t) = \left|\frac{\frac{d\vec{T}}{dt}}{|\vec{r}'(t)|}\right|$$

This re-definition of curvature is sometimes useful.

Definition 2.16 (Curvature (The Useful One)). The useful definition of curvature:

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

For example, consider the curvature of the curve C parameterized by $\vec{r}(t) = \langle t, t^2, t^3\rangle$.

$$\begin{gathered}
\left\langle t, t^2, t^3\right\rangle = \langle 2, 4, 8\rangle \iff t = 2\vec{r}'(t) = \left\langle 1, 2t, 3t^2\right\rangle \\
\vec{r}'(2) = \langle 1, 4, 12\rangle \\
\vec{r}''(t) = \langle 0, 2, 6t\rangle \\
\vec{r}''(2) = \langle 0, 2, 12\rangle \\
\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 4 & 12 \\ 0 & 2 & 12 \end{vmatrix} = \begin{vmatrix} 4 & 12 \\ 2 & 12 \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} 1 & 12 \\ 0 & 12 \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}\hat{\mathbf{k}} = 24\hat{\mathbf{i}} - 12\hat{\mathbf{j}} + 2\hat{\mathbf{k}} \\
\implies \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\sqrt{24^2 + 12^2 + 2^2}}{(1 + 16 + 144)^{\frac{3}{2}}} \approx 0.0131713554 \\
\therefore \kappa(2) \approx 0.0131713554
\end{gathered}$$

Consider a space curve, in this case parametrized by $3\sin(t)\hat{\mathbf{i}} + 3\cos(t)\hat{\mathbf{j}} + 4t\hat{\mathbf{k}}$. Consider the point on the curve $t_0 = \frac{\pi}{2}$.

$$\begin{gathered}
\vec{r}(t_0) = 3\hat{\mathbf{i}} + 2\pi\hat{\mathbf{k}} \\
\vec{r}'(t) = 3\cos(t)\hat{\mathbf{i}} - 3\sin(t)\hat{\mathbf{j}} + \hat{\mathbf{k}} \\
\vec{r}'(t_0) = -3\hat{\mathbf{j}} + 4\hat{\mathbf{k}} \\
\vec{T}(t) = \frac{3\cos(t)\hat{\mathbf{i}} - 3\sin(t)\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{9(\cos^2(t) + \sin^2(t)) + 16}} \\
\vec{T}(t) = \frac{1}{5}\left(3\cos(t)\hat{\mathbf{i}} - 3\sin(t)\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right) \\
\vec{T}(t_0) = \frac{1}{5}\langle 0, -3, 4\rangle
\end{gathered}$$

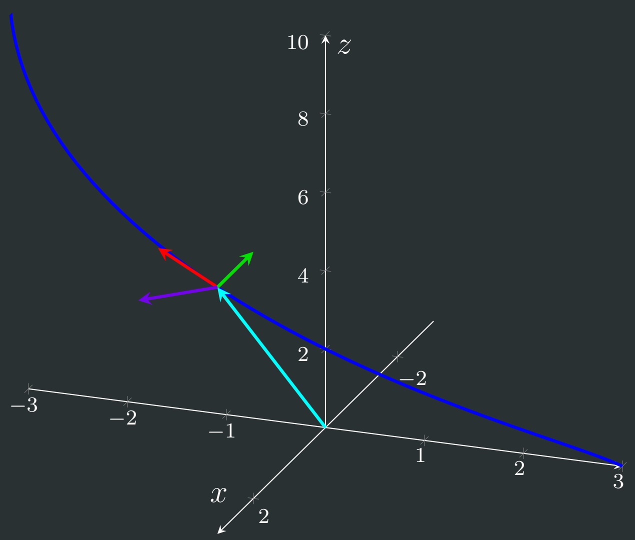
$$\vec{T}'(t) = \frac{1}{5}\left(-3\sin(t)\hat{\mathbf{i}} - 3\cos(t)\hat{\mathbf{j}}\right)$$

$$\vec{T}'(t_0) = \left\langle -\frac{3}{5}, 0, 0 \right\rangle$$

$$\vec{N}(t_0) = -\hat{\mathbf{i}}$$

$$\vec{B}(t_0) \sim \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\frac{3}{5} & \frac{4}{5} \\ -1 & 0 & 0 \end{vmatrix}$$

$$= -\frac{4}{5}\hat{\mathbf{j}} - \frac{3}{5}\hat{\mathbf{k}}$$



We can create a basis for $\mathbb{R}^3$ due to the fact that $\vec{T} \perp \vec{N} \perp \vec{B}$. Computing all these vectors for the general case can be **hard**, so we often dont.

Definition 2.17 (Osculating Plane)**.** The tangent plane to a curve, parallel to the unit tangent and orthogonal to the binormal vector.

Definition 2.18 (Normal Plane)**.** The plane orthogonal to the curve at all points, and thus orthogonal to the unit tangent.

2.4 Velocity and Acceleration

Let $\vec{r}(t)$ be a position function for a particle with respect to the parameter time. Likewise, let $\vec{v}$ and $\vec{a}$ define velocity and acceleration. Let v define speed.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{v}$$

$$\implies \vec{v} = v\vec{T}$$

$$\implies \vec{a} = v'\vec{T} + v\vec{T}\,'$$

$$\text{Recall } \kappa(t) = \frac{\left|\vec{T}\,'\right|}{|\vec{r}\,'|} = \frac{\vec{T}\,'}{v}$$

$$\implies \left|\vec{T}\,'\right| = v\kappa$$

$$\implies \vec{N} = \frac{\vec{T}\,'}{v\kappa} \implies \vec{T}\,' = \vec{N}v\kappa$$

$$\implies v\vec{T}\,' + v^2\kappa\vec{N}$$

We now see that acceleration always occurs in the plane parallel to the unit tangent and unit normal vectors, and parallel to the binormal vector. Thus, acceleration only happens in the osculating plane. This process is known as vector decomposition.

Definition 2.19 (Components of Acceleration)**.** We give $a_T = v'$ and $a_N = v^2\kappa$, thus

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

Also keep in mind that when integrating a vector-valued function representing acceleration or velocity, we can't simply add v_0 or r_0, we need to calculate the value for a constant such that it will come out to that at $t = 0$ EG: if $\vec{a} = e^t$, then $\vec{v} = e^t + c_1$. If $\vec{v}(0) = 0$, then $\vec{v}(0) = e^0 + c_1 = 0 \implies c_1 = -1$.

Chapter 3

Partial Derivatives

3.1 Functions of Several Variables

We've previously seen calculus defined over single variable vector- and scalar-valued functions, but now we define calculus for scalar functions of multiple variables. We focus on

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$f : \mathbb{R}^n \to \mathbb{R}$$

Definition 3.1 (Domain and Range of Multivariate Functions). If f is a function of 2 variables, we have

$$f : D \subseteq \mathbb{R} \to \mathbb{R}$$

where $D = \text{dom}(f) = \{(x, y) \mid (x, y) \mapsto f(x, y)\}$.
We also define $R = \text{rng}(f) = \{f(x, y) \in \mathbb{R} \mid (x, y) \in D\}$.

We often have $z = f(x, y)$ to show that x, y are independent and can vary throughout D, whereas z is dependent on x, y.
Example: The domain of $f(x, y) = 3\ln(x + y^2)$ is:

$$D = \{(x, y) \mid x + y^2 > 0\}$$

because the function is defined as $f : D \to \mathbb{R}$, and $\ln(\cdot) : \mathbb{R}^+ \to \mathbb{R} \wedge \mathbb{R}^- \to \mathbb{C}$, and is undefined for 0. The range will be $\mathbb{R}$, obviously.

Definition 3.2 (Level Curve). For a function $f(x, y) = z$, a level curve is the 2d graph formed by taking $f(x, y = k)$, where $k \in \mathbb{R}$ and is not a variable.

We often overlay many level curves onto one 2D graph to form a contour (topological) map. The 4D analogue of a level curve is a level surface formed by taking $f(x, y, z) = k$ in the same manner.

3.2 Limits and Continuity

Let f be a function defined on an open interval containing x values arbitrarily close to a. Recall that

$$\lim_{x \to a} f(x) = L \iff \forall(\varepsilon > 0)(\exists(\delta > 0)(0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon))$$

We now generalize this to multivariate scalar functions.

Definition 3.3 (Formal Definition of 2D Limit). Let f be a function of two variables with domain D including points arbitrarily close to a, b.

$$\lim_{(x,y) \to (a,b)} f(x,y) = L \iff \forall(\varepsilon > 0)(\exists(\delta > 0)(0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) - L| < \varepsilon))$$

Notice the use of the distance formula, or more generally, the *euclidean norm*. This generalizes to n-dimensions:

Definition 3.4 (Formal Definition of Limits). Let f be a function of n variables with domain D including points arbitrarily close to $\vec{a} \in \mathbb{R}^n$. For the sake of being concise, let $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L \iff \forall(\varepsilon > 0)(\exists(\delta > 0)(0 < \|\vec{x} - \vec{a}\| < \delta \implies |f(\vec{x}) - L| < \varepsilon))$$

We have the same definition of continuity.

Definition 3.5 (Continuity at a Point). If $f(a, b) = \lim_{(x,y) \to (a,b)}$, then f is continuous at (a, b).

Theorem 3.6 (Continuous Functions). All elementary functions (polynomial, logarithmic, radical, rational, exponential, trig, inverse trig) are continuous **on their domains**. Furthermore, compositions of continous functions are continuous on their domains.

When a function is continous at a point, we can evaluate its limit at that point by showing that the function is continous there, and then plugging in values.

Theorem 3.7 (Limit Laws).

$$\lim_{(x,y) \to (a,b)} x = a$$

$$\lim_{(x,y) \to (a,b)} y = b$$

$$\lim_{(x,y) \to (a,b)} c = c$$

Theorem 3.8 (Squeeze Theorem). If $f(x) \leq g(x, y) \leq h(x)$ when x is near a, and

$$\lim_{(x,y) \to (a,b)} f(x) = \lim_{(x,y) \to (a,b)} h(x) = L$$

then

$$\lim_{(x,y) \to (a,b)} g(x) = L$$

Note the use of single variable functions in the previous theorem.
A good checklist of strategies for finding limits of 2D functions is:

1. Is the function continuous?
2. Can the function be simplified algebreicly?
3. Try to show that the limit does *not* exist by showing two different paths $(x, y) \to (a, b)$ give different limit values.
4. Try using polar coordinates to find it. Give $x = r\cos\theta$, $y = r\sin\theta$, and $(x, y) \to (0, 0)$ is equivalent to $r \to 0$. Note this only works for $(a, b) = (0, 0)$.
5. Use the squeeze theorem.
6. Use the epsilon-delta definition of limit.

Example. Evaluate the limit, or show it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2}$$

$\frac{0}{0}$ is indeterminate, so $(0, 0) \notin D$.
Path 1:

$$y = x \text{ and } x \to 0$$

$$\implies \lim_{x\to 0} \frac{x^3}{x^2 + x^4} = \lim_{x\to 0} \frac{x}{1 + x^2} = \frac{0}{1} = 0$$

NOTE: just because we have shown that one path gives one value does not mean this is the value of the limit. The entire point of this method is to show the same limit has multiple values, thus it can only disprove the existence of a limit, not prove it. Thus, we consider more paths.
Path 2:

$$x = y^2 \text{ and } y \to 0$$

$$\lim_{y\to 0} \frac{y^4}{y^4 + y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

$0 \neq \frac{1}{2}$, so the limit does not exist at $(0, 0)$.
A note about the previous example: we cannot choose a definition for x or y where we do not get to the desired point.
Example. Evaluate the limit, or show it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$$

We will use the squeeze theorem.

$$0 \leq \frac{x^2}{x^2 + 2y^2} \leq 1 \;\forall x \in \mathbb{R}$$

$$\implies 0 \leq \frac{x^2 \sin^2 y}{x^2 + 2y^2} \leq \sin^2 y$$

$$\lim_{(x,y)\to(0,0)} 0 = 0$$

$$\lim_{(x,y)\to(0,0)} \sin^2 y = 0$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2} = 0 \text{ by sqeeze theorem. } \square$$

3.3 Partial Derivatives

Definition 3.9 (Partial Derivatives of a Function of 2 Variables). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $z = f(x, y)$. Then the partial derivatives are given as

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

A way to think about the above definition is by having a function $g(x) = f(x, b)$, where b is constant. In other words, we let x vary while holding y constant for some arbitrary $b \in \mathbb{R}$. In this case, $g'(x) = f_x(x, b)$, or g' is the partial derivative of f with respect to x.

Notation. The following notations are equivalent.

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = z_x = D_1 f = D_x f$$

Generally we want to avoid big D notation for partial derivatives. The same statements hold for y.

Example. Let $f(x, y) = 3x^3y^2 - 4xy$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = 9x^2y^2 - 4y$$

$$\frac{\partial f}{\partial y} = 6x^3y - 4x$$

Example. Let $g(x) = \cos\left(\frac{x^2}{1+y}\right)$. Find g_x and g_y.

$$g_x = -\sin\left(\frac{x^2}{1+y}\right)\left(\frac{2x}{1+y}\right)$$

$$g_y = -\sin\left(\frac{x^2}{1+y}\right)\left(\frac{-x^2}{(1+y)^2}\right)$$

Remark. Partial derivatives for functions of n variables exist in the same way.

Notation. Higher order partial derivatives can be given in various ways:

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(f(x,y)\right)\right)$$

When in Leibnitz notation, repeated partials are operated from left to right, similarly to function compositions.

Example. Let $f(x,y) = 3x^3y^2 - 4xy$. From a previous example, we have

$$\frac{\partial f}{\partial x} = 9x^2y^2 - 4y \qquad \frac{\partial f}{\partial y} = 6x^3y - 4x$$

Find the second order partials.

$$f_{xx} = 18xy^2 \qquad f_{yy} = 6x^3$$

$$f_{xy} = 18x^2y - 4 \qquad f_{yx} = 18x^2y - 4$$

Theorem 3.10 (Clairaut's Theorem). If f is defined on a disc D which contains (a,b), and if f_{xy} and f_{yx} are continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Remark. The aforementioned disk is assigned the radius of some arbitrary ε. From this, a better interpretation of the above theorem can be stated as such:
If f is continuous on a closed region D, and its partials are continuous on D, then the order of differentiation doesn't matter for any points not on the border of D.
(A disk of radius ε is also known as an epsilon ball.)

Definition 3.11 (Laplace's Equation). The partial differential equation (PDE) $u_{xx} + u_{yy} = 0$ is known as **Laplace's Equation**. Solutions u to this equation are known as **harmonic functions**.

Example. Does $u(x,y) = e^x \sin(y)$ satisfy Laplace's Equation?

$$u_{xx} = e^x \sin(y)$$

$$u_{yy} = -e^x \sin(y)$$

$$u_{xx} + u_{yy} = e^x \sin(y) - e^x \sin(y) = 0$$

Therefore this equation is a solution to Laplace's Equation.

Definition 3.12 (Wave Equation). The **Wave Equation** is a PDE given as $u_{tt} = a^2 u_{xx}$.

Example. Let $u(x,t) = \sin(x - 5t)$. Show u is a solution to the wave equation.

$$u_t = -5\cos(x-5t) \qquad u_x = \cos(x-5t)$$

$$u_{tt} = -25\sin(x-5t) \qquad u_{xx} = -\sin(x-5t)$$

$$\implies u_{tt} - 25u_{xx} = -25\sin(x-5t) + 25\sin(x-5t) = 0$$

$$\implies u_{tt} = 25u_{xx}$$

3.4 Tangent Planes and Linear Approximations

Definition 3.13 (Tangent Plane). If $z = f(x,y)$ defines a surface with $P = (x_0, y_0, z_0) \in f$, then let C_1 and C_2 be the curves $x = x_0$ and $y = y_0$. The tangent plane izs the plane containing the lines tangent to C_1 and C_2.

Definition 3.14 (Linearization). If $f(x,y)$ defines a surface with point (a,b), then near (a,b) it follows that

$$f(x,y) \approx L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Definition 3.15 (Differentiability of a 2D Function). Let $\Delta z = f(a,b) - f(x,y)$. Let $\varepsilon_1, \varepsilon_2$ be functions of Δx and Δy such that $\varepsilon_1, \varepsilon_2$ go to 0 as $(\Delta x, \Delta y) \to (0,0)$. f is differentiable at (a,b) iff

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Definition 3.16 (Tangent Plane). Suppose a function $z = f(x,y)$ has a point $(x_0, y_0, z_0) \in f$. The tangent plane to f at this point can be written as

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Remark. The variables in this function are only x, y, z.

Corollary 3.17. The normal vector to this plane is

$$\vec{N} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$$

Example. Find the equation of the tangent plane to $z = y \ln x$ at $(e, 4, 4)$.

$$z - 4 = \frac{4}{e}(x - e) + (y - 4)$$

The tangent plane can be used as a **linear approximation** for the function z near the point P.

Definition 3.18 (Linearization). If $z = f(x, y)$, then near $P = (x_0, y_0, z_0) \in f$,

$$f(x, y) \approx L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$$

Theorem 3.19. Let $f : D \to \mathbb{R}$ be a function of two variables with $(a, b) \in D$. If f_x and f_y exist near and at (a, b), and are continuous at (a, b), then f is differentiable at (a, b).

Definition 3.20 (Differentials). For a differentiable function $z = f(x, y)$, let dx and dy be independent variables. Then the **differential** of z (or **total differential**) is given as

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

3.5 The Chain Rule

From Calc 1, we know if f is a function of a function, that is $f(g(t)) = f(x)$, meaning $x = g(t)$, then we give

$$f'(x) = f'(g(t))g'(t)$$

Or in Leibnitz' notation,

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

This is the chain rule. It's important to realize here that variables can be thought of as nested functions of other variables. This will be how we extend the chain rule to multidimensions.

Theorem 3.21 (Chain Rule, Case 1). If $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$, meaning $z = f(g(t), h(t))$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

where z is differentiable.

Example. Let $f(x, y) = x^3y - 4x^2y^2$, where $x = \sin(2t)$ and $y = \cos(3t)$. Find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = (3x^2 - 8xy^2)(2\cos(2t)) + (x^3 - 8x^2y)(-3\sin(3t))$$

Corollary 3.22. In general, for $z = f(x_1, \ldots, x_n)$ where the partial derivatives are continuous, then

$$\frac{dz}{dt} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i}\frac{dx_i}{dt}$$

Theorem 3.23 (Chain Rule, Case 2). Let $f(x, y) = z$, $x = g(s, t)$, and $y = h(s, t)$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Theorem 3.24 (Chain Rule, General Case). If f is a differentiable function in n variables $x_1, \ldots, x_n$, and each x_i is a differentiable function of m variables $t_1, \ldots, t_n$, then

$$\begin{aligned}\frac{\partial f}{\partial t_j} &= \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_j} \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t_j}\end{aligned}$$

We can now do **implicit differentiation** in multiple variables.

Example. Let $w^3 + x^3 + y^3 + z^3 + 6wz = 4y$. Find $\frac{\partial z}{\partial x}$.

$$\frac{\partial}{\partial x}\left(w^3 + x^3 + y^3 + z^3 + 6wz\right) = \frac{\partial}{\partial x}(4y)$$

$$3w^2 w_x + 3x^2 + 3y^2 y_x + 3z^2 z_x + 6wz_x + 6zw_x = 4y_x$$

$$3z^2 z_x + 6wz_x = 4y_x - 3w^2 w_x - 3x^2 - 3y^2 y_x - 6zw_x$$

$$z_x(3z^2 + 6w) = 4y_x - 3w^2 w_x - 3x^2 - 3y^2 y_x - 6zw_x$$

$$z_x = \frac{4y_x - 3w^2 w_x - 3x^2 - 3y^2 y_x - 6zw_x}{3z^2 + 6w}$$

3.6 Directional Derivatives and the Gradient Vector

Definition 3.25 (Gradient Vector). The gradient vector (field) of a function ∇f in n variables is given as

$$\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$$

Definition 3.26 (Directional Derivative). The derivative of a function $f(x, y)$ in the arbitrary direction of some unit vector $\vec{u} = \langle a, b\rangle$ at a point (x_0, y_0) is defined as

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(x_0 - ha, y_0 - hb) - f(x_0, y_0)}{h}$$

Remark. The above definition can be extended to $\mathbb{R}^n$.

Theorem 3.27 (Directional Derivative Thm. 1). The directional derivative of a function f in the direction of unit vector $\vec{u} = \langle a, b \rangle$ is given as

$$D_{\vec{u}} f = f_x(x, y)a + f_y(x, y)b$$

Remark. The above definition can be extended to $\mathbb{R}^n$.

Proposition 3.28 (Directional Derivative as a Dot Product). With the previous definitions, we have

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

Remark. The gradient vector points in the steepest rate of change and is perpendicular to all level curves (contour map).

Theorem 3.29 (Directional Derivative Thm. 2). For a differentiable function f of two variables, the maximum value of $D_{\vec{u}} f(\vec{x})$ is

$$|\nabla f(\vec{x})|$$

and is in the direction of ∇f.

Remark. The above definition can be extended to $\mathbb{R}^n$.

3.7 Maximum and Minimum Values

Theorem 3.30 (First Derivative Test). If f is a function of **two variables** that has a local max or min at (a, b), and the first order partial derivatives exist at (a, b), then

$$\nabla f(a, b) = \vec{0}$$

Theorem 3.31 (Second Derivative Test). Let f be a function of **two variables**. Suppose the partial derivatives of f are continuous on a disk centered at (a, b), and $\nabla f(a, b) = \vec{0}$. Define a new function D, the disciminant function:

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$$

- If $D(a, b) > 0$ and f_{xx} or f_{yy} are greater than 0, then $f(a, b)$ is a local min.
- If $D(a, b) > 0$ and f_{xx} or f_{yy} are less than 0, then $f(a, b)$ is a local max.
- If $D(a, b) < 0$, then $f(a, b)$ is a saddle point.
- If $D(a, b) = 0$, then the test is inconclusive.

Remark. A pneumonic for remembering the discriminant function is

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Note that the partials are defined on a disk so 3.10 applies, and $f_{xy} f_{yx} = (f_{xy})^2$.

Notation. This is beyond calc 3.
If S is a set, then $\max(S)$ gives the maximum element of the set. Likewise, $\min(S)$ gives the min. If a function is defined $f : X \to Y$, then

$$\arg\max_{x \in S} f(x) = \{x \in S \mid \forall s \in X(f(s) \leq f(x))\}$$

$\arg\min$ is defined similarly.

Theorem 3.32 (Extreme Value Theorem for Multivariate Functions)**.** If f is defined on a closed, bounded set $D \subseteq \mathbb{R}^n$, then f has an absolute max and min on D.

To find the extreme values for a function of only **two variables** on a set $D \subseteq \mathbb{R}^2$, set $\nabla f = \vec{0}$, solve for points, evaluate f at the points found, and evaluate f on the boundary of D (notated ∂D).

3.8 Lagrange Multipliers

The method of Lagrange multipliers can optimize a function given some constraint. Suppose we have some function f and a constaint function g, where g has some set value. Find values where $\nabla f \parallel \lambda \nabla g$. Set up and solve a system of equations. Plug in values to the original function.

Example. Let $f(x, y) := (x-2)^2 + (y-2)^2$ and $g(x, y) := x^2 + y^2 = 9$.

$$\nabla f = \langle 2x - 4, 2y - 4 \rangle$$

$$\nabla g = \langle 2x, 2y \rangle$$

$$\nabla = \lambda \nabla g$$

$$\implies \begin{cases} 2x - 4 = \lambda 2x \\ 2y - 4 = \lambda 2y \end{cases}$$

Solve system of equations

$$x, y \in \left\{ 1 + \frac{2\sqrt{2}}{3}, 1 - \frac{2\sqrt{2}}{3} \right\}$$

Plug in values to f to find the maxes and mins.

Chapter 4

Multiple Integrals

4.1 Double Integrals over Rectangles

Let a function $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous on $D \subset \mathbb{R}^2$, where

$$D = [a, b] \times [c, d]$$

Using Riemann sums, we can extend the definition of an integral over rectangular regions such as this. Let $[a, b]$ be partitioned into n disjoint subsets of an equal length Δx, and let $[c, d]$ be similarly paritioned into m disjoint subsets of an equal length Δy. Let $\left(x_i^*, y_j^*\right)$ be the midpoint of the ijth subset of $[a, b] \times [c, d]$. We say that the volume under some function $f(x, y)$ over the region D is approximately

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta y \Delta x$$

We say that $\Delta y \Delta x = \Delta A$ for convenience. This leads to the following definition.

Definition 4.1 (Double Integral over a Rectangular Region). Let f be defined over a rectangular region $R \subset \mathbb{R}^2$. We say

$$\iint_R f(x, y)\, dA = \lim_{(n,m)\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A$$

provided the limit exists.

Remark. When we integrate a function of multiple variables with respect to a single variable, we hold other variables fixed with respect to the variable being integrated with respect to. This is known as **partial integration**.

Theorem 4.2 (Fubini's Theorem). Let f be continuous on $D = \{(x, y) \mid x \in [a, b] \wedge y \in [c, d]\} \subset \mathbb{R}^2$. Then

$$\iint_D f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy$$

Remark. The restriction of continuity can be relaxed in some cases, and it can be required only that f be bounded on D and be discontinuous on a finite number of smooth curves. Proof of this fact most likely follows from utilization of lebesgue integration, but either way it'll likely be shown in an analysis course.

4.2 Double Integrals over General Regions

Definition 4.3 (Type I Region). A type I region is a region bounded by two continuous functions of x.

Definition 4.4 (Type II Region). A type II region is a region bounded by two continuous functions of y.

Definition 4.5 (Integrals over Type I or II Regions). Let D_1 be a type I region with

$$D_1 = \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b] \land y \in [g_1(x), g_2(x)]\}$$

and let f be defined and integrable over D_1. Then

$$\iint\limits_{D_1} f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dydx$$

Furthermore, let D_2 be a type II region with

$$D_2 = \{(x, y) \in \mathbb{R}^2 \mid x \in [h_1(x), h_2(x)] \land y \in [c, d]\}$$

and let f be defined and integrable over D_2. Then

$$\iint\limits_{D_2} f(x, y) dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y)\, dxdy$$

Remark. Notice that when integrating over a type I or type II region, the integral bounded by constants is the last integral evaluated.

The following are properties of double integrals.

1. If $f(x, y) \geq 0\, \forall (x, y) \in D$, then the volume V of the solid region above D and below $z = f(x, y)$ is
$$V = \iint\limits_D f(x, y)\, dA$$
2. The average value of a function $f : \mathbb{R}^2 \to \mathbb{R}$ on $D \subset \mathbb{R}^2$ is
$$f_{avg} = \frac{1}{A(D)} \iint\limits_D f(x, y)\, dA$$
where $A(D)$ is the area of the region D.

3. Integral of a sum:

$$\iint\limits_D f(x,y) + g(x,y)\, dA = \iint\limits_D f(x,y)\, dA + \iint\limits_D g(x,y)\, dA$$

4. Integral of a constant multiple:

$$\iint\limits_D cf(x,y)\, dA = c\iint\limits_D f(x,y)\, dA$$

5. If $f(x,y) \geq g(x,y) \forall (x,y) \in D$, then

$$\iint\limits_D f(x,y)\, dA \geq \iint\limits_D g(x,y)\, dA$$

6. If $D = \bigcup_{i=1}^{n} D_i$ where $\bigcap_{i=1}^{n} D_i = \varnothing$, then

$$\iint\limits_D f(x,y)\, dA = \sum_{i=1}^{n} \iint\limits_{D_i} f(x,y)\, dA$$

7. Integral of the constant function:

$$\iint\limits_D 1 dA = A(D)$$

8. If $m \leq f(x,y) \leq M \, \forall (x,y) \in D$, then

$$mA(D) \leq \iint\limits_D f(x,y)\, dA \leq MA(D)$$

4.3 Double Integrals in Polar Coordinates

Integrals over polar coordinates are defined through a Riemann sum of polar rectangles. It's very convoluded so I will leave the proof as an excercize to myself when I feel like it in the future.

Definition 4.6 (Polar Rectangle). A set of the form $\{(r\cos\theta, r\sin\theta) \mid r \in [a,b] \land \theta \in [\alpha, \beta]\}$. This can be seen as a subset of a circle in both radius and angle.

Theorem 4.7 (Polar Integrals). If f is continuous on a polar rectangle R with $0 \leq a \leq r \leq b$ and $\alpha \leq \theta \leq \beta$, where $\beta - \alpha \leq 2\pi$, then

$$\iint\limits_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f\left(r\cos\theta, r\sin\theta\right) r\, dr d\theta$$

Notice the extra r in the integrand in the above theorem.

Exercise. Evaluate $\iint\limits_R (3x + 4y^2)\, dA$, where R is the region bdd by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$, where $y \geq 0$.

Solution.

$$R = \{(r\cos\theta, r\sin\theta) \mid \theta \in [0,\pi] \wedge r \in [2,3]\}$$

$$\begin{aligned}
\iint\limits_R \left(3x + 4y^2\right) dA &= \int_0^\pi \int_2^3 \left(3r\cos\theta + 4(r\sin\theta)^2\right) r\,drd\theta \\
&= \int_0^\pi \int_2^3 3r^2\cos\theta + 4r^3\sin^2\theta\,drd\theta \\
&= \int_0^\pi r^3\cos\theta + r^4\sin^2\theta\Big|_2^3 d\theta \\
&= \int_0^\pi 19\cos\theta + 65\sin^2\theta\,d\theta \\
&= 10\sin\theta\Big|_0^\pi + \int \frac{65}{2}\left(1 - \cos 2\theta\right) d\theta \\
&= \frac{65}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\Bigg|_0^\pi = \frac{65\pi}{2}
\end{aligned}$$

□

Exercise.

$$\int_{-5}^5 \int_0^{\sqrt{25-x^2}} \left(\frac{1}{2}x^2 + y^2\right) dydx$$

Solution.

$$\begin{aligned}
\int_{-5}^5 \int_0^{\sqrt{25-x^2}} \left(\frac{1}{2}x^2 + y^2\right) dydx &= \int_0^\pi \int_0^5 \frac{1}{2}r^3\cos^2\theta + r^3\sin^2\theta\,drd\theta \\
&= \int_0^\pi \frac{r^4}{8}\cos^2\theta + \frac{r^4}{4}\sin^2\theta\Big|_0^5 d\theta \\
&= \int_0^\pi \frac{625}{8}\cos^2\theta + \frac{625}{4}\sin^2\theta\,d\theta \\
&= \frac{625}{8}\int_0^\pi \cos^2\theta + 2\sin^2\theta\,d\theta \\
&= \frac{625}{8}\int_0^\pi \cos^2\theta + \sin^2\theta + \sin^2\theta\,d\theta \\
&= \frac{625}{8}\int_0^\pi 1 + \sin^2\theta\,d\theta \\
&= \frac{625}{16}\int_0^\pi 1 - \cos 2\theta\,d\theta \\
&= \frac{625\theta}{16} - \frac{625\sin 2\theta}{32}\Bigg|_0^\pi \\
&= \frac{625\pi}{16}
\end{aligned}$$

□

4.4 Applications of Double Integrals

A lamina is basically an extremely thin film, so integrals over regions can approximate properties of a lamina.

Definition 4.8 (Center of Mass of a Lamina). Let $\rho : D \to \mathbb{R}$ be the density function of a lamina defined over some region D. The mass of the lamina is given as

$$m = \iint\limits_D \rho(x,y)\,dA$$

and the center of mass of the lamina is given as

$$(\overline{x}, \overline{y}) = \left(\frac{M_x}{m}, \frac{M_y}{m}\right)$$

where

$$M_x = \iint\limits_D y\rho(x,y)\,dA \qquad M_y = \iint\limits_D x\rho(x,y)\,dA$$

Remark. This definition of mass has applications beyond just a lamina. For example, given a charge density function $\sigma(x,y)$ defined over a region D, then the total charge Q is calculated similarly.

Definition 4.9 (Moment of a Lamina about an Axis). The moment of a lamina defined over the region D with density function $z = \rho(x,y)$ about the x or y axis is given as

$$M_x = \iint\limits_D y\rho(x,y)\,dA \qquad M_y = \iint\limits_D x\rho(x,y)\,dA$$

Notice the use of moment in 4.8

Definition 4.10 (Moment of Inertia). The moment of inertia of a lamina about an axis is given as

$$I_x = \iint\limits_D y^2\rho(x,y)\,dA \qquad I_y = \iint\limits_D x^2\rho(x,y)\,dA$$

The moments about an axis measure the tendency of a lamin ato rotate about an axis, and the moments of inertia measure the difficulty required to start/stop such a rotation.
Another application of double integrals concerns joint probability functions. Let f be probability function for some variable X. Thus, f is a function such that

1. $f(x) \geq 0 \, \forall x \in \mathbb{R}$
2. $\int_{\mathbb{R}} f(x)\,dx = 1$

The probability that X lies between $a, b \in \mathbb{R}$ is given as

$$P(a \leq X \leq b) = \int_a^b f(X)\,dX$$

However, we can use calculus to extend this to two variables.

Definition 4.11 (Joint Density Function). Let $X, Y \in \mathbb{R}$ be probabilistic variables with a probability density $f(x, y)$. The probability that $X \in [a, b]$ and $Y \in [c, d]$ is given as

$$f(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(X, Y)\, dY\, dX$$

4.5 Surface Area

Definition 4.12 (Surface Area). Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and let f_x, f_y be continuous on D. The surface area of the surface $z = f(x, y)$ is given as

$$SA = \iint\limits_D \sqrt{1 + f_x^2 + f_y^2}$$

This was a long section.

4.6 Triple Integrals

Definition 4.13 (Triple Integral). Let $B \subset \mathbb{R}^3$ with $f : B \to \mathbb{R}$. We give

$$\iiint\limits_B f(x, y, z)\, dV := \lim_{\ell, n, m \to \infty} \sum_{i=1}^{\ell} \sum_{j=1}^{n} \sum_{k=1}^{m} f\left(x_i^*, y_j^*, z_k^*\right) \Delta x \Delta y \Delta z$$

4.2 holds for triple integrals, and the integral will always exist if the function is continuous over the region of integration.
Here are triple integral analogues of some of the applications that were seen earlier.

Definition 4.14 (Mass of an Object). Let an object occupy some region $E \subseteq \mathbb{R}^3$ with density function $\rho : E \to \mathbb{R}$. The mass of the object is given as

$$m = \iiint\limits_E \rho(x, y, z)\, dV$$

Definition 4.15 (Center of Mass of an Object). Consider the definitions in 4.14. The center of mass of an object is defined as

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$$

where we give

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV \qquad M_{xz} = \iiint_E y\rho(x,y,z)\,dV \qquad M_{xy} = \iiint_E z\rho(x,y,z)\,dV$$

Exercise. Evaluate $\iiint_B \sqrt{x^2+z^2}\,dV$ where B is bdd by $y = x^2+z^2$ and $y = 4$.

Solution. Recall how to integrate in polar coordinates for this problem. We will convert the xz-axis into polar coordinates.

$$\begin{aligned}
\iiint_B \sqrt{x^2+z^2} &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} dydzdx \\
&= \int_0^{2\pi}\int_0^2\int_{r^2\cos^2\theta+r^2\sin^2\theta}^{4} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\,dy(rdrd\theta) \\
&= \int_0^{2\pi}\int_0^2\int_{r^2}^{4} r^2\,drd\theta \\
&= \int_0^{2\pi}\int_0^2 \left(4-r^2\right) r^2\,drd\theta \\
&= \int_0^{2\pi}\int_0^2 4r^2 - r^4 \\
&= \int_0^{2\pi} \frac{4r^3}{3} - \frac{r^5}{5}\bigg|_0^2 d\theta \\
&= \int_0^{2\pi} \frac{32}{3} - \frac{32}{5}\,d\theta \\
&= 2\pi\left(\frac{32}{3} - \frac{32}{5}\right)
\end{aligned}$$

□

Theorem 4.16. For some arbitrary region $E \subset \mathbb{R}^3$, the volume can be calculated by evaluating

$$\iiint_E dV$$

Exercise. Find the volume of the region B which is bdd by $x^2+y^2 = 9$, $y+z = 5$, and $z = 1$. After integrating wrt z, convert to polar coordinates.

Solution.

$$\begin{aligned} V &= \iiint\limits_B dV \\ &= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} dzdydx \\ &= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4 - y\, dydx \\ &= \int_{0}^{2\pi} \int_{0}^{3} 4r - r^2 \sin\theta\, drd\theta \\ &= \int_{0}^{2\pi} \left[2r^2 - \frac{1}{3} r^3 \sin\theta \right]_0^3 d\theta \\ &= \int_{0}^{2\pi} 18 - 9\sin\theta\, d\theta \\ &= 36\pi \end{aligned}$$

□

We just integrated in cylindrical coordinates!

4.7 Triple Integrals in Cylindrical Coordinates

An intuitive way to think about cylindrical coordinates I've found is to think of polar coordinates along an entire orthogonal axis.
To convert between cylindrical and cartesian coordinates, we give

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

To convert from cartesian to cylindrical coordinates, we give

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Exercise. Express the volume of the object bdd by $z = 4 - x^2 - y^2$, $x^2 + y^2 = 4$, and $z = 9 + x^2 + y^2$ as a triple integral in cartesian coordinates, then as a double integral in cartesian coordinates, then as a triple integral in cylindrical coordinates, then as a double integral in polar coordinates.

Solution. Notice $9 + x^2 + y^2 \geq 4 - x^2 - y^2 \, \forall x, y \in \mathbb{R}$ because $\forall (x \in \mathbb{R}) \, (x^2 \geq 0)$.

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{4-x^2-y^2}^{9+x^2+y^2} dzdydx$$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 5 + 2x^2 + 2y^2 \, dydx$$

$$\int_{0}^{2\pi} \int_{0}^{2} \int_{4-r^2}^{9+r^2} r \, dzdrd\theta$$

$$\int_{0}^{2\pi} \int_{0}^{2} \left(5 + 2r^2\right) r \, drd\theta$$

□

The above question will probably be on the exam.

4.8 Triple Integrals in Spherical Coordinates

Spherical coordinates are given by the ordered triple ρ, θ, ϕ, where

- ρ denotes the distance from the origin
- θ denotes the angle formed with the positive x-axis
- ϕ denotes the angle formed with the positive z-axis

We require that

$$\rho \geq 0 \qquad \phi \in [0, \pi]$$

To convert between spherical to cartesian coordinates, use

$$x = \rho \sin\phi \cos\theta \qquad x = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

and

$$\rho^2 = x^2 + y^2 + z^2$$

Intuition. This is like adding $\rho \sin\phi$ to the polar coordinates for x and y, and this leaves $\cos\phi$ "left over", so we let $\rho \cos\phi$ be z.

Theorem 4.17 (Integrals in Spherical Coordinates)**.** Let $E \subset \mathbb{R}^3$ with $f : E \to \mathbb{R}$. Let E be defined in spherical coordinates as

$$E := \{(\rho, \theta, \phi) \mid \rho \in [a, b] \wedge \theta \in [\alpha, \beta] \wedge \phi \in [c, d]\}$$

where $a \geq 0$, $\beta - \alpha \leq 2\pi$, $d - c \leq \pi$, and $c \geq 0$. We give

$$\iiint\limits_{E} f(x, y, z) \, dV = \int_{c}^{d} \int_{\alpha}^{\beta} \int_{a}^{b} f\left(\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi\right) \rho^2 \sin\phi \, d\rho d\theta d\phi$$

Exercise. Convert the following integral to spherical coordinates:

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x, y, z) \, dz dx dy$$

Solution. Notice that this defines a unit semicircle where $x \geq 0$. Thus, we can give

$$\int_{0}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1} f\left(\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi\right) \rho^2 \sin\phi \, d\rho d\theta d\phi$$

□

4.9 The Jacobian

Definition 4.18 (C^1 Transformation)**.** T is a C^1 transformation from uvw space to xyz space if $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w)$, $y = h(u, v, w)$, $z = k(u, v, w)$, and g, h, and k have continuous first-order partial derivatives.

Definition 4.19 (Jacobian)**.** The **Jacobian** of a C^1 transformation T from uvw space to xyz space is the 3×3 determinant given by

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Transformations can also be defined between n-dimensional spaces, whenever the spaces are of equal dimension. We give

$$\frac{\partial\left(x_1, \ldots, x_n\right)}{\left(u_1, \ldots, u_n\right)} = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

Remark. Online sources make a distinction between the Jacobian matrix $\mathbf{J}$ and its determinant. This text only works with the determinant.

Theorem 4.20 (Change of Variables in Double or Triple Integral)**.** Let $T : S \to R$ be a C^1 transformation from uv space to xy space with a nonzero Jacobian. Suppose f is continuous and T is injective on $S \subset \mathbb{R}^2$, $R \subset \mathbb{R}^2$. Then

$$\iint\limits_R f(x, y) \, dA = \iint\limits_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

The above theorem can be extended to n-dimensions in the exact same manner. For brevity, let

$\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{u} = (u_1, \ldots, u_n)$. With the same definitions except for $S, R \subset \mathbb{R}^n$, we give

$$\int \cdots \int_R f(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_S f(x_1(\mathbf{u}), \ldots, x_n(\mathbf{u})) \left| \frac{\partial(\mathbf{x})}{\partial(\mathbf{u})} \right| d\mathbf{u}$$

Exercise. Prove theorem 4.17 using the Jacobian.

Solution. Let $x = \rho \cos\theta \sin\phi$, $y = \rho \sin\theta \sin\phi$, $z = \rho\cos\phi$. It follows that

$$\begin{aligned}
\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} &= \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi \end{vmatrix} \\
&= \rho^2 \cos\phi \left(-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi\right) \\
&\quad - \rho\sin\phi\left(\rho\cos^2\theta\sin^2\phi + \rho\sin^2\theta\sin^2\phi\right) \\
&= -\rho^2\cos\phi(\sin\phi\cos\phi) - \rho^2\sin\phi\left(\sin^2\phi\right) \\
&= -\rho^2\left(\sin\phi\cos^2\phi + \sin^3\phi\right) \\
&= -\rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right) \\
&= -\rho^2\sin\phi
\end{aligned}$$

We also have $|-\rho^2\sin\phi| = \rho^2|\sin\phi|$, and by taking $\phi \in [0, \pi]$, we have $|\sin\phi| = \sin\phi$. □

Chapter 5

Vector Calculus

5.1 Vector Fields

Definition 5.1 (Vector Field)**.** Let $S \subseteq \mathbb{R}^n$. A vector field is a vector-valued function $\mathbf{F}(x_1, \ldots, x_n)$ where

$$\mathbf{F} : S \to \mathbb{R}^n$$

defined in terms of scalar-valued components.

In calc 3, we work exclusively in terms of $\mathbb{R}^2$ and $\mathbb{R}^3$ in this section, however I have made my above definition a bit more general. Examples and further definitions will all be on $\mathbb{R}^2$ or $\mathbb{R}^3$.
An example of a vector field is the function

$$\mathbf{F}(x, y) = \left\langle \cos(xy), x^2y^2e^{xy\sin x} \right\rangle$$

because for any $(x, y \in \mathbb{R}^2)$, it takes on some value $\mathbf{F}(x, y) \in \mathbb{R}^2$.

Definition 5.2 (Component Functions)**.** For a vector field defined as either

$$\mathbf{F}(x, y) = P(x, y)\hat{\mathbf{\imath}} + Q(x, y)\hat{\mathbf{\jmath}}$$

or

$$\mathbf{F}(x, y, z) = P(x, y, z)\hat{\mathbf{\imath}} + Q(x, y, z)\hat{\mathbf{\jmath}} + R(x, y, z)\hat{\mathbf{k}}$$

where P, Q, R are scalar valued functions, we say that P, Q, R are the component functions of $\mathbf{F}$.

Remark. For some scalar valued $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : \mathbb{R}^3 \to \mathbb{R}$, the gradient vector ∇f is really a vector field, defined as

$$\nabla f = f_x\hat{\mathbf{\imath}} + f_y\hat{\mathbf{\jmath}}$$

or for $f : \mathbb{R}^3 \to \mathbb{R}$,

$$\nabla f = f_x\hat{\mathbf{\imath}} + f_y\hat{\mathbf{\jmath}} + f_z\hat{\mathbf{k}}$$

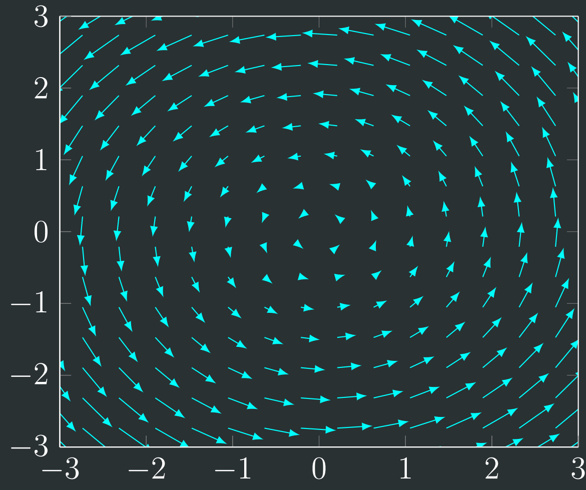
Definition 5.3 (Conservative Vector Fields and Potential Function)**.** Let $\mathbf{F}$ be a vector field. We say that $\mathbf{F}$ is a **conservative vector field** if and only if there exists some scalar valued f such that

$$\mathbf{F} = \nabla f$$

In this case, we say f is the **potential function** of $\mathbf{F}$.

Plotting vector fields by hand is hard, and is best left to computers. Here's an example done with TikZ:

Vector field for $\mathbf{F}(x, y) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$



5.2 Line Integrals

Definition 5.4 (Piecewise Smooth Curve)**.** Any curve C that can be expressed as

$$C = \bigcup_{i=1}^{n} C_i$$

where C_i is a smooth curve and $n \in \mathbb{N}$ is finite.

Definition 5.5 (Orientation of a Parametrized Curve)**.** Let C be a space curve parametrized by t. We say that a given parametrization defines an orientation of C with the positive direction in the direction of increasing t.

Definition 5.6 (Line Integral of a Scalar Function)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a scalar valued function. Let the curve $C \subset \mathbb{R}^2$ be defined by the vector function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $t \in [a, b]$. The line integral of f over C is defined as

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Oftentimes this will be written as $f(\mathbf{r})$ to symbolize that we are plugging in the components of $\mathbf{r}$ as functions of t to f as the variable inputs. This can also be thought of loosely as $f \circ \mathbf{r}$.

Intuition. I found line integrals very odd and tough to remember, so here's how I thought of them. We only want to evaluate the integral of $f(x, y)$ over some $(x, y) \in C$. Since we defined the function $\mathbf{r}(t)$ to take some input t and output all the values $(x, y) \in C$, then to do this we just evaluate $f(\mathbf{r})$ for all of the outputs of $\mathbf{r}$, which are given by t values. Thus we have $f(\mathbf{r}(t))$. The big square root just comes from parametrizing the surface length differential, and the reason for the surface length differential is since dx is just a length along the line $y = 0$, ds is just a length along the line C. This is not asimilar to integrating wrt the measure.

Remark. Oftentimes when we integrate about a closed space curve C, or at least a closed portion of it, we will write

$$\oint_C f(x, y)\, ds$$

Oftentimes when giving a parametrization of a space curve, we write something along the lines of $C : \mathbf{r}(t)$ to show that C is being parametrized by $\mathbf{r}$. This formula also extends to n-dimensions if needed.

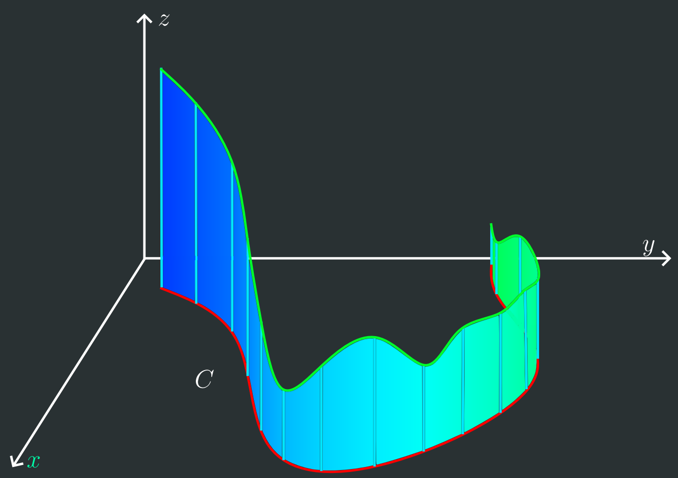


Figure 5.1: A Line Integral Visualized

Definition 5.7 (Line Integral of Vector Field)**.** Let $\mathbf{F}$ be a continuous vector field on a smooth curve C parametrized by $\mathbf{r}(t)$ for $t \in [a, b]$. The line integral of $\mathbf{F}$ over C is given as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

5.3 The Fundamental Theorem for Line Integrals

Definition 5.8 (Path). A path is a curve traced out by a piecewise smooth curve along some orientation.

Definition 5.9 (Independence of Path). Let $\mathbf{f} : D \to \mathbb{R}^3$ be a vector field with $C_1, C_2 \subset D$ as paths with the same initial and terminal points. We say the integral of $\mathbf{F}$ is **independent of path** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Definition 5.10 (Simply-connected Region). A simply-connected region is a connected region D such that every simple closed curve (Jordan curve) in D only encloses points in D.

Intuition. A simply-connected region D has a boundary defined by a curve ∂D that is non self-intersecting. D also has no holes in it.

Theorem 5.11 (Fundamental Theorem of Line Integrals). Let C be a smooth curve parametrized by $\mathbf{r}(t)$ for some $t \in [a, b]$. Let f be a differentiable function of two or three variables with ∇f being continuous over C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Proof. This proof covers the case of $\mathbb{R}^3$. Given the previous definitions and $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, recall the chain rule for some $f(x, y, z)$ with x, y, z as functions of a single parameter t:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

The proof given in class did not use this fact, and there is probably a reason for that, so I have given both my proof using this fact and the proof given in class, starting with mine.

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt \\
&= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\, dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r})\, dt \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a))
\end{aligned}$$

The proof given in class deviates on the escond to last step.

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt \\
&= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\, dt \\
&= \int_a^b \frac{\partial f}{\partial x}\, dx + \int_a^b \frac{\partial f}{\partial y}\, dy + \int_a^b \frac{\partial f}{\partial z}\, dz \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a))
\end{aligned}$$

■

Recall that if there exists some f such that $\mathbf{F} = \nabla f$, then we say that $\mathbf{F}$ is conservative. By 5.11, we can say that for any conservative vector field, the fundamental theorem of calculus applies to line integrals.

Example. Since line integrals are somewhat strange, let's do an example. Let C be a path from $(0,4)$ to $(2,1)$ with an unknown parametrization. Evaluate

$$\int_C \left\langle 6xy, 3x^2 - \frac{2}{\sqrt{y}} \right\rangle \cdot d\mathbf{r}$$

Normally to evaluate a line integral we would evaluate

$$\int_{[a,b]} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

However we don't known $\mathbf{r}$. Thus, we should try to find some f such that $\mathbf{F} =' \nabla f$. Then it will follow that the integral is equal to

$$f(2,1) - f(0,4)$$

since these are the endpoints, and that's what the statement $\mathbf{r}(b)$ and $\mathbf{r}(a)$ gives in 5.11. To find one, we need to undo the del operator via partial integration. Integrating the first component wrt x:

$$\int 6xy\, dx = 3x^2y + C(y)$$

where the constant is now a function of y. We now integrate the y component w.r.t. y:

$$\int 3x^2 - \frac{2}{\sqrt{y}}\, dy = 3x^2y - 4\sqrt{y} + C(x)$$

From this, we know that the function $f(x,y) = 3x^2y - 4\sqrt{y}$ is a potential function for $\mathbf{F}$. By 5.11,

$$\int_C \left\langle 6xy, 3x^2 - \frac{2}{\sqrt{y}} \right\rangle \cdot d\mathbf{r} = 3(2)^2(1) - 4\sqrt{1} - \left(3(0)^2(4) - 4\sqrt{4}\right) = 12 - 4 + 8 = 16$$

Corollary 5.12. Let $\mathbf{F}$ be a conservative vector field that is continuous on D, and let C be a closed curve in D. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

Theorem 5.13. If $\mathbf{F}$ is a continuous vector field on some open, simply connected region $D \subseteq \mathbb{R}^2$, then F is independent of path if and only if it is conservative.

Proof. Let $C_1, C_2 \subset D$ be space curves with initial point a and terminal point b, and let $\mathbf{F} = \nabla f$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(b) - f(a)$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(b) - f(a)$$

$$\therefore \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Proof in the reverse direction was not provided and I'm lazy. ∎

A way to check if some $\mathbf{F}$ is independent of path/conservative employs 3.10. If f is continuous on D, then $f_{xy} = f_{yx}$ on D. Let

$$\mathbf{F} = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

If $\mathbf{F}$ is conservative, then there exists some function such that $\mathbf{F} = \nabla f$. It follows that

$$\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

Assuming 3.10 since $\mathbf{F}$ is conservative, we have

$$P_y = Q_x$$

which is generally stated as

$$Q_x - P_y = 0$$

Theorem 5.14 (Test for Conservative Vector Field)**.** Let $\mathbf{F} = \langle P(x,y), Q(x,y) \rangle$ be a vector field that is continuous on an open, simply connected region D. If the components of $\mathbf{F}$ have continuous first order partial derivatives and

$$Q_x - P_y = 0$$

then $\mathbf{F}$ is conservative.

5.4 Green's Theorem

As Previously Seen. A curve parametrized by some $\mathbf{r}$ is smooth if $\mathbf{r}'$ is continuous and nonzero along the curve.

Definition 5.15 (Positive Orienation of Curves)**.** Let $C \coloneqq \partial D$ be a simple closed curve (Jordan curve) bounding some region D. By convention, the positive orientation of C is defined to be the counterclockwise direction.

Remark. $\oint_C$ or $\oint_{\partial D}$ is often used to denote the integral over a simple closed curve in the direction of positive orientation.

Theorem 5.16 (Green's Theorem). Let $D \subseteq \mathbb{R}^2$ be a closed region with $C := \partial D$ being a positively oriented, piece-wise smooth, simple, closed curve bounding D. If $\mathbf{F} = \langle P(x,y), Q(x,y) \rangle$ is a vector field who's components have continuous first order partial derivatives on D, then

$$\oint_C P\,dx + Q\,dy = \iint\limits_D Q_x - P_y\,dA$$

Proof. Let C be parametrized by $\mathbf{r}(t) = (x(t), y(t))$ for $t \in [a,b]$ with the above definitions. Then

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_a^b \langle P(x,y), Q(x,y) \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt \\ &= \oint_a^b P(x,y)\frac{dx}{dt}dt + Q(x,y)\frac{dy}{dt}\,dt \\ &= \oint_a^b Pdx + Qdy \\ &= \iint\limits_D Q_x - P_y\,dA\end{aligned}$$

■

Corollary 5.17. If $\mathbf{F}$ is conservative, by 5.14,

$$\iint\limits_D Q_x - P_y\,dA = 0$$

Example. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle xy, x \rangle$ and C is defined as $y = \sqrt{4 - x^2}$ from $(2,0)$ to $(-2,0)$.
This curve is a semicircle, and isn't closed. However, it is in the positive orientation. Define C_1 as the line segment from $(-2,0)$ to $(2,0)$ and define

$$\Sigma = C \cup C_1$$

5.16 now applies, and we have

$$\oint_\Sigma \mathbf{F} \cdot d\mathbf{r} = \iint\limits_\Gamma 1 - x\,dA$$

where Γ is the region bdd by : S, and $\partial\Gamma = \Sigma$. We can now evaluate the integral.

$$\begin{aligned}
\iint_{\Gamma} 1 - x\, dA &= \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 1 - x\, dy dx \\
&= \int_0^{\pi}\int_0^2 r - r^2\cos\theta\, dr d\theta \\
&= \int_0^{\pi} \frac{2^2}{2} - \frac{2^3}{3}\cos\theta\, d\theta \\
&= \int_0^{\pi} 2 - \frac{8}{3}\cos\theta\, d\theta \\
&= 2\pi - \frac{8}{3}(\sin\pi - \sin 0) \\
&= 2\pi
\end{aligned}$$

To find the value of the original integral, we take $\oint_{\Sigma} \mathbf{F}\cdot d\mathbf{r} - \int_{C_1} \mathbf{F}\cdot d\mathbf{r}$. To find the second integral, we need to parametrize C_1:

$$C_1 : \mathbf{r}(t) = (1-t)\langle -2, 0\rangle + t\langle 2, 0\rangle = \langle -2+4t, 0\rangle$$

and give

$$\begin{aligned}
\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \langle (-2+4t)(0), (-2+4t)\rangle \cdot \langle 4, 0\rangle\; dt \\
&= \int_0^1 0\, dt \\
&= 0
\end{aligned}$$

$$\therefore \oint_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$$

5.5 Curl and Divergence

Definition 5.18 (Irrotational/Curl-Free). If $\mathbf{F}$ is a vector field, then $\mathbf{F}$ is irrotational at P if $\nabla\times\mathbf{F} = \mathbf{0}$.

Definition 5.19 (Incompressible/Divergence-Free). If $\mathbf{F}$ is a vector field, then $\mathbf{F}$ is incompressible at P if $\nabla\cdot\mathbf{F} = 0$.

Definition 5.20 (Laplace's Operator). We say that

$$\nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and we abbreviate this as $\nabla^2 f$, or sometimes Δf. We call this Laplace's Operator.

In operator theory, we use the del operator as a vector of derivatives:

$$\nabla = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle$$

where ∇ is applied to a function of n variables. This allows us to write

$$\nabla f(x, y, z) = f \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

which is the conventional use of ∇. However, it also givs convenient notations for the curl and divergence operations.

Definition 5.21 (Curl)**.** Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field. Then the **curl** of $\mathbf{F}$, denoted

$$\text{curl}(\mathbf{F})$$

or

$$\nabla \times \mathbf{F}$$

is calculated as

$$\nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Since this is a horrible way to remember this, we can use the notation $\nabla \times \mathbf{F}$ quite literally, and give the memorization pneumonic:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Theorem 5.22 (Test for Conservative Vector Field)**.** If $\mathbf{F}$ has continuous partial derivatives, $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$.

When $\nabla \times \mathbf{F} = \mathbf{0}$, we say that $\mathbf{F}$ is curl-free or irrotational.

Intuition. Curl is the tendency of a vector field to rotate about a point.

Definition 5.23 (Divergence)**.** Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field where P_x, Q_y, R_z exist. The **divergence** of $\mathbf{F}$ is denoted as

$$\text{div}(\mathbf{F})$$

or more commonly

$$\nabla \cdot \mathbf{F}$$

and is defined as

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z$$

Example. Consider $\nabla \cdot (\nabla \times \mathbf{F})$, where $\mathbf{F} = \langle P, Q, R \rangle$. By evaluating the determinant, we find

$$\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

Notice that $\nabla \cdot \nabla \times \mathbf{F} = \mathbf{0}$ if $\mathbf{F}$ is continuous, implying that 3.10 applies.

From the previous example, we know that for a continuous vector field, it's curl is incompressible. There is a theorem that extends this idea into the decomposition of vector fields known as **Helmholtz Decomposition**, also **Helmholtz' Theorem** or the **Fundamental Theorem of Vector Calculus**. The full statement and proof are beyond this class, but the idea can be stated as follows:
Let $\mathbf{F}$ be a vector field defined on a domain bounded by a closed surface $S \subset \mathbb{R}^3$. $\mathbf{F}$ can then be expressed as

$$\mathbf{F} = \mathbf{CF} + \mathbf{DF}$$

where $\mathbf{CF}$ is irrotational and $\mathbf{DF}$ is incompressible.
I tried to understand this and have yet to fully understand it. I self-taught Lebesgue integration and some elementary measure theory, but I need to further understand the dirac delta function and some strange notation to understand it I believe.

Intuition. The curl measures the rate of flow of a vector field towards or away from a point.

Using our knowledge, we can rewrite Green's theorem as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \nabla \times \mathbf{F} \cdot \hat{\mathbf{k}} \, dA$$

5.6 Parametric Surfaces and Their Areas

This section is fascinatingly short. We give the following method for parametrizing surfaces:

Definition 5.24. A surface has a parametrization in terms of two variables. Let $t, u \in \mathbb{R}$.

$$\mathbf{R}(t, u) = \langle x(t, u), y(t, u), z(t, u) \rangle$$

Often it's hard to interpret a surface via a parametrization, so we often give the obvious parametrization. For some surface defined as $f(x, y) = z$, we give the parametrization

$$\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle$$

As Previously Seen. Let $f(x, y) : D \to \mathbb{R}$ define a surface S. Then the area $\mathcal{A}(S)$ is defined as

$$\mathcal{A}(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

5.7 Surface Integrals

Definition 5.25 (Oriented Surface and its Orientation)**.** If there exists some unit normal vector $\mathbf{n}$ to a surface S such that for every $(x, y, z) \in S$, $\mathbf{n}$ varies continuously over S, then S is an oriented surface and the choice of $\mathbf{n}$ gives S an orientation.

Definition 5.26 (Closed Surface)**.** A closed surface is the boundary of some region $E \subseteq \mathbb{R}^3$.

Definition 5.27 (Positive Orientation of Closed Surface). By convention, the positive orientation of a closed surface is outwards.

In regards to the previous definitions, I've elected to bring some of the notes further ahead in the section. Let S be an oriented surface $f(x, y) = z$. We define

$$\mathbf{n}_{up} := \frac{\langle -f_x, -f_y, 1\rangle}{\|\langle -f_x, -f_y, 1\rangle\|}$$

and

$$\mathbf{n}_{down} := \frac{\langle f_x, f_y, -1\rangle}{\|\langle f_x, f_y, -1\rangle\|}$$

by convention, we choose $\mathbf{n}_{up}$ to be the positive orientation of S.

Theorem 5.28 (Surface Integral of a Scalar Function). Let S be a surface defined by the map $g : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ given as $g(x, y) = z$, with parametrization $S : \mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle$ for $(x, y) \in D$. The integral of some function $f(x, y, z)$ over S is given as

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(x, y))\sqrt{1 + g_x^2 + g_y^2}\, dA$$

where $f(\mathbf{r}(x, y)) = f(x, y, g(x, y))$ due to our parametrization of S.

Remark. Surface integrals are tricky to compute, and it is extremely difficult to find ones with "clean" answers. As such, the answers to surface integral questions will not necessarily "look nice".

Theorem 5.29 (Surface Integral of a Scalar Function). Let $\mathbf{F}$ be a continuous vector field defined on the oriented surface S with unit normal $\mathbf{n}$. Then the surface integral of $\mathbf{F}$ over S is given as

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS \text{ or } \iint_S \mathbf{F} \cdot d\mathbf{S}$$

and is calculated as

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_S \mathbf{F}(\mathbf{r}(x, y)) \cdot \mathbf{n}\, dA$$

Remark. Oftentimes the surface integral of a vector field is referred to as the **flux** of $\mathbf{F}$ over a surface S, and often is given the symbol Φ. This can be interpreted as the "amount of stuff passing through a surface", such as magnetic fields.