Modern Algebra 1

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Problem 1. hi
Problem 2. Let $n \geq 2$. Define R as the rotation of the regular n-gon by 360 degrees, S as any reflection of the n-gon, and R_0 as the identity transformation (rotation by 0 degrees). Show

$$D_n = \left\{R_0, R^1, R^2, \dots, R^{n-1}, S, RS, R^2S.., R^{n-1}S\right\}.$$

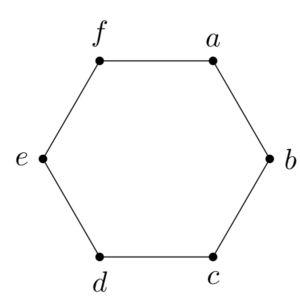
Note that $R^iS = SR^P - i$ for all i.

Solution. Recall the description of the elements of D_n from part (a) of problem 0. The fact that $\left\{R_0, R^1, R^2, \dots, R^{n-1}\right\} \subseteq D_n$ is explained precisely in part (a) of problem 1. It remains to be shown that each reflection $S_1, \dots, S_n$ has the representation R^iS for some i. I have no idea how to show this rigorously, but the problem says an explanation will suffice. Because the n-gon is rigid, it has only two distinct orderings for its angles, which can be called face-up and face-down. $\square$

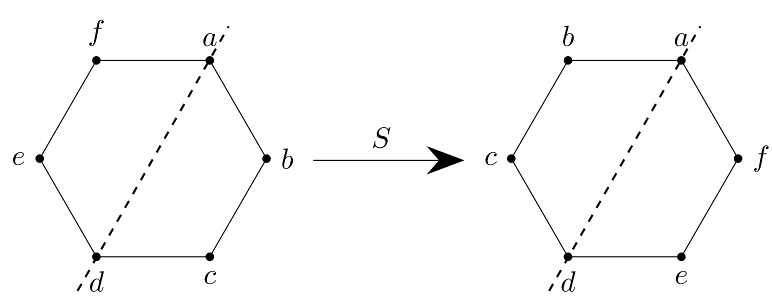
Problem 3. For this problem, recall that D_6 is the dihedral group of order 12, the group of symmetries of the hexagon.

- Find elements $A, B \in D_6$ such that $AB \neq BA$.
- Find elements $A, B, C \in D_6$ such that $AB = BC$ but $A \neq C$.

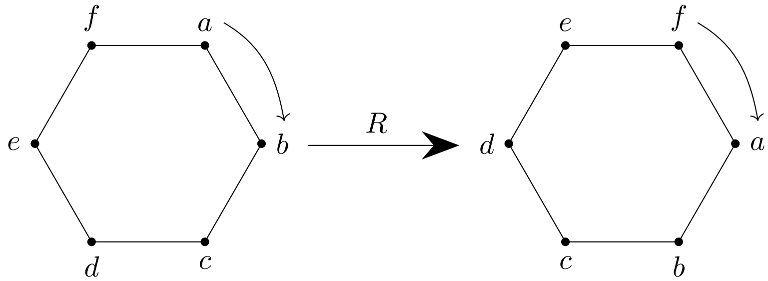
Solution. For simplicity, consider the visualization below.



Take S to be the reflection fixing points a and d.

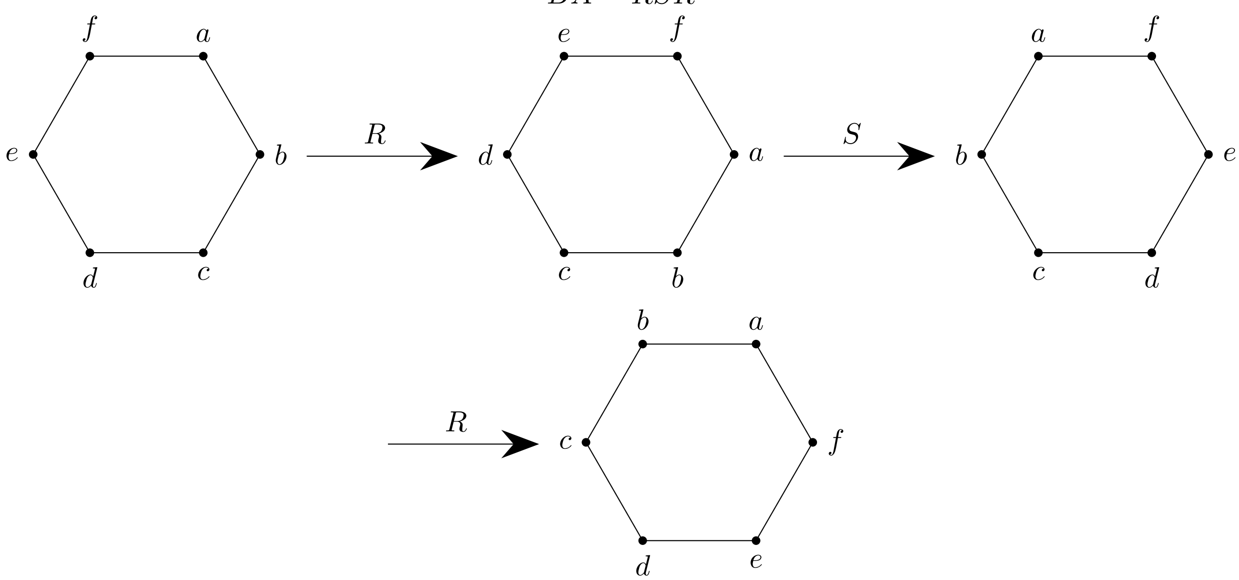


From the visualization, we know S maps b to f, e to c, and vice versa. Now take R to be a clockwise rotation by 60°.

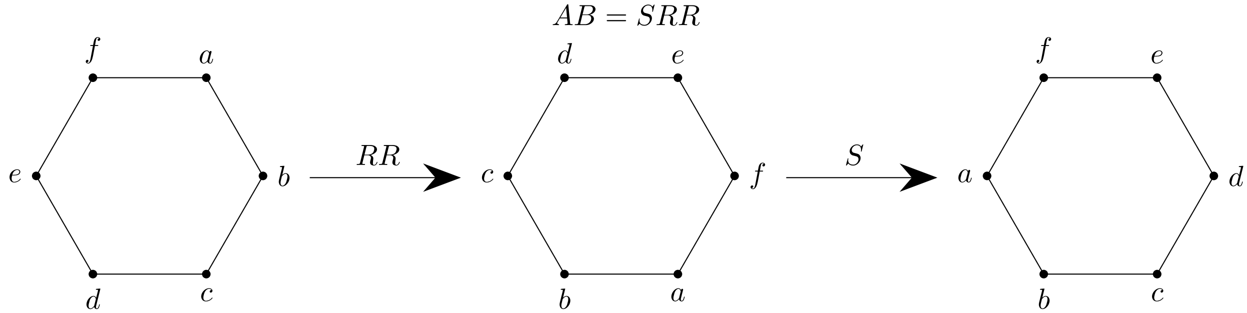


Consider the symmetries $A \coloneqq SR$ and $B \coloneqq R$. We have $AB = SRR$ and $BA = RSR$. Function composition is applied from right to left, so we apply the rightmost transformation first. Rather than tediously explain what point maps to what position, a visual proof has been given.

$BA = RSR$

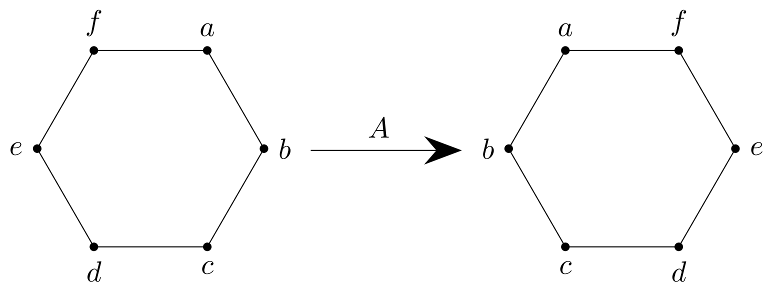


$AB = SRR$

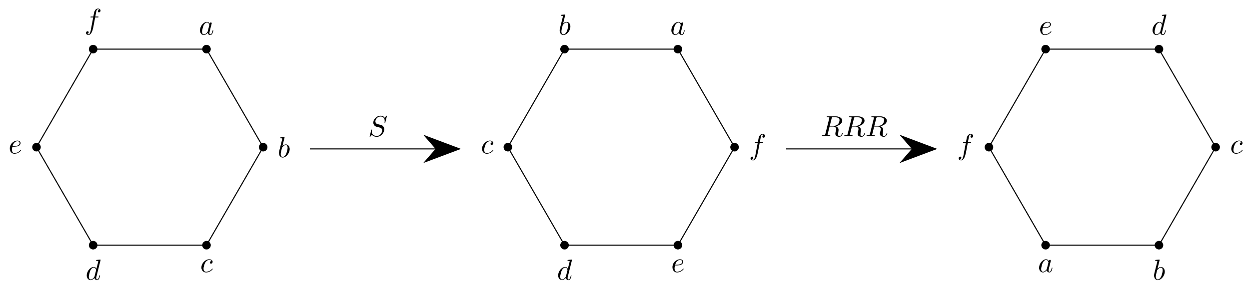


Clearly these hexagons are not in the same orientation, and as such $AB \neq BA$.

For the second part of the problem, let A and B be as previously defined. Notice that the symmetry AB is equivalent to reflecting once and then rotating 4 times. Define $C \coloneqq RRRS$. With this definition, $BC = RRRRS$, one reflection and four rotations. Thus, $AB = BC$. However, it's not necessarily implied that $A \neq C$, since SR and $RRRS$ may simply be different representations of the same symmetry. The result of the transformation A can be seen as the result of the first two transformations present in BA. For simplicity, it has been redrawn below.



Now consider the transformation C:



Clearly, $A \neq C$. As such, we have an example of $AB = BC$ for $A \neq C$. $\square$

Problem 4. Let $G = \{1, 2, 3, 4\}$ with binary operation given by multiplication modulo n. Show that G is a group under this operation.

Solution. Clearly "multiplication modulo n" is a typo, as the value of n matters, and for $n \geq 6$, there are sitiations where for some $a, b \in G$, $ab \mod n \notin G$. For example, take $n = 12$. $3, 2 \in G$, but $3 \cdot 2 = 6 \mod 12 \notin G$. Since the problem requires G is a group, we take $n = 5$.

Due to the ambiguity in the definition of the binary operation, I have chosen to show multiplication modulo 5 is a binary operation on G by brute force in the following Cayley table.

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	1	1

From this table, it can also be seen that 1 is an identity element in G, so there exists an identity element in G. It can also be seen that every element has an inverse; 1 is it's own inverse since $1 \cdot 1 = 1 \mod 5$, but also $2 \cdot 3 = 1 \mod 5$, $3 \cdot 2 = 1 \mod 5$, and $4 \cdot 4 = 1 \mod 5$. So under multiplication modulo 5, the inverse of 2 is 3, the inverse of 3 is 2, and the inverse of 4 is 4, all of which are elements of G. Finally, we show associativity. We know that associativity holds in $\mathbb{Z}$, and $\{1, 2, 3, 4\} \subseteq \mathbb{Z}$. Thus, associativity holds for all elements of G, since all elements of G are also integers, which are known to be associative. Therefore, G is a group under the operation of multiplication modulo 5. $\square$

Problem 5. hi