

# Excursion 1

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## Abstract

This paper explores the Tile Game as presented by Dr. Borkovitz in [Excursion 1](#).

I made very heavy use of theorem and proof environments in some sections, and it negatively impacted readability. As such, I have used nonstandard layouts for my theorem and proof environments to increase readability, while still maintaining some level of professionalism.

## Acknowledgements

Academic Integrity – Please type your name to acknowledge that if you collaborated on this problem, then it was only with classmates from either section of CAS MA 293 (except to have a partner for the game), that you only used allowable resources (notes from class, no Internet searches, AI, etc.), and that you wrote this paper by yourself: Grant Talbert

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## 1 Notes

For a pair  $(a, b)$  with  $a$  tiles color 1 and  $b$  tiles color 2,  $(2, 1)$  is a losing position. It follows that  $(n, n + 1)$  is a losing position by

$$\phi : (n, n + 1) \mapsto (n, n + 1) - (n - 1, n - 1) = (1, 2).$$

## 2 Introduction

**Problem statement.** Given a board position with an arbitrary number of tiles for each color, what is the strategy to consistently win the game, if one exists?

## 3 Mathematical Formalism

**Definition 1** (Board Position). A *board position* is a pair  $(a, b) \in \mathbb{N}^2$ , wherein  $a$  is the number of tiles of color 1, and  $b$  is the number of tiles of color 2. We will often also represent a board position as a column vector

$$\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{N}^2.$$

These representations are equivalent up to isomorphism, and we will use them interchangeably.

**Definition 2** (Moves). This definition can totally be cleaned up.

Let  $\phi : \mathbb{N}^2 \rightarrow \mathbb{N}^2$  be the map<sup>1</sup>

$$\phi(a, b) \equiv \begin{bmatrix} a - 1 \pmod{a} \\ b \end{bmatrix}.$$

Let  $\psi : \mathbb{N}^2 \rightarrow \mathbb{N}^2$  be the map

$$\psi(a, b) \equiv \begin{bmatrix} a \\ b - 1 \pmod{b} \end{bmatrix}.$$

Let  $\xi = (a, b)$  be a board position. Then the set of current valid moves, called *the set of moves on  $\xi$* , denoted as  $\mathcal{M}(\xi)$ , is defined as

$$\mathcal{M}(\xi) := \langle \phi \rangle \cup \langle \psi \rangle \cup \langle \phi\psi \rangle.$$

We may write  $\phi_\xi$  and  $\psi_\xi$  to denote the moves  $\phi$  and  $\psi$  on a board position  $\xi$ , but the subscript will be omitted when possible.

**Proposition 1.** Let  $\xi = (a, b)$  be a board position. Let  $\phi, \psi : \mathbb{N}^2 \rightarrow \mathbb{N}^2$  be as defined in [definition 2](#). Then

$$\phi^k(a, b) \equiv \begin{bmatrix} a - k \pmod{a} \\ b \end{bmatrix}$$

and

$$\psi^k(a, b) \equiv \begin{bmatrix} a \\ b - k \pmod{b} \end{bmatrix}$$

for any  $k \in \mathbb{Z}$ .

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<sup>1</sup>Functions are defined using modular arithmetic to keep the groups they generate finite, as it adds more structure that can be easily explored. This does not violate the rules of the game.

**Proof.** We use the method of proof by induction. We will prove this for  $\phi$ , since the proof for  $\psi$  is identical.

First, let  $k = 1$ . It follows that  $\phi^k(a, b) \equiv \phi^1(a, b) \equiv (a - 1 \pmod{a}, b) \equiv (a - k \pmod{a}, b)$  by definition.

Now, let  $k$  be an arbitrary natural number for which  $\phi^k(a, b) \equiv (a - k \pmod{a}, b)$ . It follows that

$$\begin{aligned}\phi^{k+1}(a, b) &\equiv \phi\left(\phi^k\begin{bmatrix} a \\ b \end{bmatrix}\right) \\ &\equiv \phi\begin{bmatrix} a - k \pmod{a} \\ b \end{bmatrix} \\ &\equiv \begin{bmatrix} a - k - 1 \pmod{a} \\ b \end{bmatrix} \\ &\equiv \begin{bmatrix} a - (k + 1) \pmod{a} \\ b \end{bmatrix}.\end{aligned}$$

By the method of induction, we are done. ■

**Lemma 1.** Let  $\phi, \psi$  be as defined in [definition 2](#). Then  $\phi^k\psi^\ell = \psi^\ell\phi^k$  for any  $k, \ell \in \mathbb{Z}$ .

**Proof.** Let  $(a, b)$  be a board position. By [proposition 1](#), we have

$$\phi^k(a, b) \equiv \begin{bmatrix} a - k \pmod{a} \\ b \end{bmatrix}, \quad \psi^\ell \equiv \begin{bmatrix} a \\ b - \ell \pmod{b} \end{bmatrix}.$$

It follows that

$$\begin{aligned}(\phi^k\psi^\ell)(a, b) &\equiv \phi^k(\psi^\ell(a, b)) \\ &\equiv \phi^k\begin{bmatrix} a \\ b - \ell \pmod{b} \end{bmatrix} \\ &\equiv \begin{bmatrix} a - k \pmod{a} \\ b - \ell \pmod{b} \end{bmatrix} \\ &\equiv \psi^\ell\begin{bmatrix} a - k \pmod{a} \\ b \end{bmatrix} \\ &\equiv \psi^\ell(\phi^k(a, b)) \\ &\equiv (\psi^\ell\phi^k)(a, b).\end{aligned}$$

Therefore,  $\phi^k\psi^\ell = \psi^\ell\phi^k$ . ■

**Lemma 2.** Let  $\xi = (a, b)$  be a board position for which  $\phi$  and  $\psi$  are defined. Then  $\phi^n = \phi^m$  if and only if  $n \equiv m \pmod{a}$ . Likewise,  $\psi^n = \psi^m$  if and only if  $n \equiv m \pmod{b}$ . Additionally,  $|\langle\phi\rangle| = a + 1$  and  $|\langle\psi\rangle| = b + 1$ .

**Proof.** For any cyclic group  $\langle\phi\rangle$ , it follows that  $\phi^\alpha = \phi^\beta$  if and only if  $\alpha \equiv \beta \pmod{|\langle\phi\rangle|}$  [[1](#), [2](#)]. It remains only to be shown that  $|\langle\phi\rangle| = a + 1$  and  $|\langle\psi\rangle| = b + 1$ . ■

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**Lemma 3.** Let  $\xi = (a, b)$  be a board position for which  $\phi$  and  $\psi$  are defined. Then

$$\langle \phi \rangle \cap \langle \psi \rangle = \{ \mathbb{1}_{\mathbb{N}^2} \}.$$

| **Proof.** hi ■

**Theorem 1.** Let  $\xi = (\alpha, \beta)$  be a board position for which  $\phi$  and  $\psi$  are defined. Then  $\Psi := \{ \phi^x \psi^y \mid x, y \in \mathbb{Z} \}$  is an abelian group. Furthermore,  $\Psi \cong \mathbb{Z}_\alpha \times \mathbb{Z}_\beta$ .

**Proof.** The associativity of  $\Psi$  is trivial to prove, but very cumbersome, so we leave it as an exercise to the reader.

First, we must show that  $\Psi$  is closed under function composition. Let  $x, y, n, m \in \mathbb{Z}$ . By the fact that function composition is associative, and by [lemma 1](#),

$$(\phi^x \psi^y) (\phi^n \psi^m) = (\phi^x \phi^n) (\psi^y \psi^m) = \phi^{x+n} \psi^{y+m}.$$

Since  $\mathbb{Z}$  is closed,  $x + n, y + m \in \mathbb{Z}$ , and  $\Psi$  is closed under function composition.

The existence of an identity is trivial.  $\phi^0 \psi^0 = \mathbb{1}_{\mathbb{N}^2} \in \Psi$ .

Finally, we show inverse elements exist. Let  $\phi^x \psi^y \in \Psi$ . Since  $-x, -y \in \mathbb{Z}$ , we have  $\phi^{-x} \psi^{-y} \in \Psi$ . It follows from [lemma 1](#) that

$$(\phi^x \psi^y) (\phi^{-x} \psi^{-y}) = (\phi^x \phi^{-x}) (\psi^y \psi^{-y}) = \phi^{x-x} \psi^{y-y} = \phi^0 \psi^0 = \mathbb{1}_{\mathbb{N}^2}.$$

Therefore,  $\Psi$  is a group. By [lemma 1](#),  $\Psi$  is abelian. We conclude by showing the second part of the theorem. Consider the map

$$\begin{aligned} \tau : \Psi &\rightarrow \langle \phi \rangle \times \langle \psi \rangle \\ &: (\phi^k \psi^\ell) \mapsto (\phi^k, \psi^\ell). \end{aligned}$$

First, we show  $\tau$  is a homomorphism. Let  $\phi^a \psi^b, \phi^k \psi^\ell \in \Psi$ . It follows from [lemma 1](#) that

$$\tau(\phi^a \psi^b) \tau(\phi^k \psi^\ell) = (\phi^a, \psi^b) (\phi^k, \psi^\ell) = (\phi^a \phi^k, \psi^b \psi^\ell) = \tau(\phi^a \phi^k \psi^b \psi^\ell) = \tau((\phi^a \psi^b) (\phi^k \psi^\ell)).$$

Therefore,  $\tau$  is a homomorphism. Now we show  $\tau$  is a bijection.

Let  $(\phi^a, \psi^b) \in \langle \phi \rangle \times \langle \psi \rangle$ . It follows that  $\tau(\phi^a \psi^b) = (\phi^a, \psi^b)$ . Therefore,  $\tau$  is surjective. Now let  $(\phi^k, \psi^\ell) \in \Psi$ . Suppose that  $\tau(\phi^a \psi^b) = \tau(\phi^k \psi^\ell)$ . It follows that

$$\phi^a \psi^b = \phi^k \psi^\ell.$$

Since  $\Psi$  is a group, left and right cancellation properties hold. Thus,

$$\psi^b = \phi^{-a} \phi^k \psi^\ell = \phi^{k-a} \psi^\ell.$$

We also have

$$\phi^{k-a} = \psi^b \psi^{-\ell} = \psi^{b-\ell}.$$

We thus have  $\phi^{k-a} = \psi^{b-\ell}$ . By [lemma 3](#), we have  $\phi^{k-a} = \psi^{b-\ell} = \mathbb{1}_{\mathbb{N}^2}$ , and thus  $k \equiv a \pmod{\alpha}$  and  $b \equiv \ell \pmod{\beta}$ . By [lemma 2](#), we have  $\phi^k = \phi^a$  and  $\psi^\ell = \psi^b$ . Thus,  $\tau$  is bijective, and thus an isomorphism.

Since  $\langle \phi \rangle$  and  $\langle \psi \rangle$  are cyclic, we have  $\langle \phi \rangle \cong \mathbb{Z}_{|\phi|}$  and  $\langle \psi \rangle \cong \mathbb{Z}_{|\psi|}$  [1, 2]. Thus,

$$\langle \phi \rangle \times \langle \psi \rangle \cong \mathbb{Z}_{|\phi|} \times \mathbb{Z}_{|\psi|}.$$

By lemma 2, we have

$$\langle \phi \rangle \times \langle \psi \rangle \cong \mathbb{Z}_{\alpha+1} \times \mathbb{Z}_{\beta+1}.$$

Since isomorphism is an equivalence relation on the class of groups, we have  $\Psi \cong \mathbb{Z}_{\alpha+1} \times \mathbb{Z}_{\beta+1}$ . ■

**Remark.** We may write  $\Phi_\xi$  to denote the group  $\Psi$  on a board position  $\xi$ , but the subscript will be omitted when possible.

**Corollary 1.** *The set  $\langle \phi\psi \rangle \subseteq \langle \phi \rangle \cup \langle \psi \rangle$  is an abelian subgroup.*

**Proof.** whoops ■

**Corollary 2.** *Let  $\xi = (a, b)$  be a board position for which  $\phi$  and  $\psi$  are defined. Then  $\Psi$  is cyclic if and only if  $\gcd(a, b) = 1$ . Specifically,  $\Psi \cong \mathbb{Z}_{(a+1)(b+1)}$ .*

**Proof.** We know that  $\mathbb{Z}_{a+1} \times \mathbb{Z}_{b+1}$  is cyclic if and only if  $\gcd(a+1, b+1) = 1$ , and that if  $\gcd(a+1, b+1) = 1$ ,  $\mathbb{Z}_{a+1} \times \mathbb{Z}_{b+1} \cong \mathbb{Z}_{(a+1)(b+1)}$  [1, 2]. Since  $\Psi \cong \mathbb{Z}_{a+1} \times \mathbb{Z}_{b+1}$ , the same result follows. ■

**Corollary 3.** *Let  $\xi = (a, b)$  be a board position for which  $\phi$  and  $\psi$  are defined. Then  $\phi^a \psi^a = (\phi\psi)^a$ .*

**Proof.** Since  $\phi^a \psi^a \in \Psi$ , and since  $\Psi$  is abelian, it follows that  $\phi^a \psi^a = (\phi\psi)^a$  [1, 2]. ■

**Definition 3** (Win). a move that gives 0,0

**Definition 4** (Primitive Winning Positions). Let  $\eta = (a, b)$  be a board position, often the starting board position. Let

$$PW(\eta) := \{(n, m) \in \mathbb{N}^2 \mid n \leq a \wedge m \leq b \wedge ((n = 0 \wedge m \neq 0) \vee (n \neq 0 \wedge m = 0) \vee (n = m))\}.$$

We call  $PW(\eta)$  the set of *primitive winning positions on  $\eta$* .

**Proposition 2.** *Let  $\xi = (a, b)$  be a board position. If  $\xi \in PW(\eta)$ , then the player who's turn it is to move will win.*

**Proof.** There are three possible cases for  $\xi \in PW(\eta)$ . These are

1.  $\xi = (a, 0)$ ,
2.  $\xi = (0, b)$ , or
3.  $\xi = (a, a)$ .

(1) Since  $a \in \mathbb{N}$ , by definition 2,  $\phi^a \in \mathcal{M}(\xi)$ . By proposition 1,  $\phi^a(a, 0) = (a - a, 0) = (0, 0)$ . By definition 3, this constitutes a win for the player who moved.

(2) Since  $b \in \mathbb{N}$ , by definition 2,  $\psi^b \in \mathcal{M}(\xi)$ . By proposition 1,  $\psi^b(0, b) = (0, b - b) = (0, 0)$ . By definition 3, this constitutes a win for the player who moved.

(3) Since  $a \in \mathbb{N}$ , by definition 2,  $(\phi\psi)^a \in \mathcal{M}(\xi)$ . By proposition 1 and corollary 3,  $(\phi\psi)^a = (a - a, a - a) = (0, 0)$ . By definition 3, this constitutes a win for the player who moved. ■

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**Definition 5** (Primitive Losing Positions). Let  $\eta = (a, b)$  be a board position, often the starting board position. Let

$$PL(\eta) := \{(n, m) \in \mathbb{N}^2 \mid n \leq a \wedge m \leq b \wedge (\forall \pi \in \mathcal{M}(n, m), \pi(n, m) \in PW(\eta))\}.$$

We call  $PL(\eta)$  the set of *primitive losing positions on  $\eta$* .

**Definition 6** (Position Magnitude). Let  $\xi = (a, b)$  be a board position. The *position magnitude* of  $\xi$  is defined as

$$|\xi| := a + b.$$

This is also referred to as *the number of tiles remaining*.

**Theorem 2.** Let  $\eta = (a, b)$  be a board position. For  $|\eta| \leq 2$ ,  $PL(\eta) = \emptyset$ , and for  $|\eta| > 2$ ,  $PL(\eta) = \{(2, 1), (1, 2)\}$ .

| **Proof.** i need to do this too. ■

## 4 Solution

## 5 Explanation and Justification

## 6 Conclusion

## A Appendix

## References

- [1] Joseph A. Gallian. *Contemporary Abstract Algebra*. CRC PRESS, 2021.
- [2] Thomas W. Judson. *Abstract Algebra: Theory and Applications*. 2022.