Honors Differential Equations

Grant Talbert

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BOSTON UNIVERSITY

College of Arts and Sciences
Department of Mathematics and Statistics
CAS MA 231

Contents

Lecture 3: Continuing First Order ODEs

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Note - the previous example was actually

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y - t^2 y}.$$

This makes it way easier:

$$\int y \,\mathrm{d}y = \int \frac{t}{1-t^2}\,\mathrm{d}t.$$

$$\frac{1}{2}y^2 = -\int \frac{1}{u}\,\mathrm{d}u.$$

$$\frac{y^2}{2} = \ln\left|1-t^2\right| + C.$$

Solving from here is trivial.

0.0.1 Mixing Problems

Some container full of some solution, adding more stuff from the top while also draining out some stuff from the bottom. Rates of both can be diff. Stirring all the time to make sure things are mixed.

Example

Tank have 50gal of water. At $t = 0$, the tank contains pure water. Salty brine is poured into the tank at a rate of 3gal/min with a concentration of 1lb/gal of salt. Brine is drained from the tank at a rate of 3gal/min. How much salt is in the tank after 5 minutes?

Solution. Independent variable is time and dependent variable is amount of salt. Alternatively, salt concentration can be used (exercise to reader?)

Let $S(t)$ be the amount, in lbs, of salt at some time t. Since we are adding 1 lb per gallon and 3 gallons per minute, we add 3 lbs per minute. This is a constant positive rate of change of $+3$. We are then removing the amount of water being removed (3 gallons) times the concentration of salt in the tank, which is the amount of salt divided by 50 gallons.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3 - 3\left(\frac{S(t)}{50}\right).$$

The first term is adding salt, and the second term is removing salt. Simplifying, we get

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3\left(1 - \frac{S(t)}{50}\right) = \frac{3(50 - S(t))}{50}.$$

This is separable.

$$\int \frac{1}{50 - S(t)}\,\mathrm{d}S = \int \frac{3}{50}\,\mathrm{d}t.$$

$$-\ln|50 - S(t)| = \frac{3t}{50} + C.$$

$$S(t) = 50 - e^{-C}e^{-\frac{3t}{50}}.$$

$$S(0) = 0 = 50 - e^{-C}.$$

Therefore, $e^{-C} = 50$, and $S(t) = 50 - 50\exp\left(-\frac{3t}{50}\right)$. Plug in the time asked for.

$$S(5) = 50\left(1 - e^{-\frac{15}{50}}\right)\text{ lbs.}$$

□

DETOUR: NUMERICAL ANALYSIS

Recall Taylor Polynomials. We know that an approximation for $f(t)$ about some $a \in \mathbb{R}$ is given by the Taylor Polynomial about a,

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(t-a)^n.$$

A taylor polynomial of order n will be called $T_n(t)$.

As Previously Seen

Euler's Method:

Take an interval split into a grid of points t_j, separated by value Δt. Suppose $y(t_j)$ is known, at least approximately. Obviously $y(t_{j+1}) = y(t_j + \Delta t)$. This is approximately $y(t_j) + y'(t_j)\Delta t$. The derivative is usually given.

Given $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t)$, $y(t_0) = y_0$,

- Pick a grid size Δt.
- Set $Y_0 = y_0$.
- Set $Y_1 = Y_0 + f(Y_0, t_0)\Delta t$.
- Set $Y_2 = Y_1 + f(Y_1, t_1)\Delta t$.
- Set $Y_{n+1} = Y_n + f(Y_n, t_n)\Delta t$

In this case, $t_{n+1} = t_n + \Delta t$.

Theorem 0.1 (Taylor's theorem)

For some $f : \mathbb{R} \to \mathbb{R}$, then
$$f(t) = T_n(t) + R_{n+1}(t).$$
If $f(t)$ is $n+1$ times continuously differentiable on $(a-\delta, a+\delta)$, then there exists a constant C_{n+1} such that $|f(t) - T_n(t)| = |R_{n+1}(t)| \leq C_{n+1} |t-a|^{n+1}$ for all $a - \delta < t < a + \delta$.
$$C_{n+1} \coloneqq \max \left\{ \left| f^{(n+1)}[a-\delta, a+\delta] \right| \right\}.$$

Recall the fundamental theorem of calculus says
$$f(t) - f(a) = \int_a^t f'(s) \, \mathrm{d}s.$$
Let's rewrite this:
$$\int_a^t f'(s) - f'(a) + f'(a) \, \mathrm{d}s = \int_a^t f'(s) - f'(a) \, \mathrm{d}s + \int_a^t f'(a) \, \mathrm{d}s.$$
The second integral is constant wrt s :
$$f(t) - f(a) = \int_a^t f'(s) - f'(a) \, \mathrm{d}s + f'(a)(t-a).$$
Notice the taylor polynomial is appearing.
$$f(t) = f(a) + f'(a)(t-a) + \int_a^t f'(s) - f'(a) \, \mathrm{d}s.$$
So clearly, that final integral is the remainder $R_2(t)$ in $T_1(a)$. We can keep going:
$$f(t) = f(a) + f'(a)(t-a) + \int_a^t \int_a^s f''(q) \, \mathrm{d}q \mathrm{d}s.$$
Now notice that
$$\left| \int_a^b f(x) \, \mathrm{d}x \right| \leq \int_a^b |f(x)| \, \mathrm{d}x.$$
Let $M_2 = \max_{q \in [a,t]} |f''(q)|$. Going back to our double integral, we have
$$|R_2(t)| = \left| \int_a^t \int_a^s f''(q) \, \mathrm{d}q \mathrm{d}s \right| \leq \int_a^t \left| \int_a^s f''(q) \, \mathrm{d}q \right| \mathrm{d}s \leq M_2 \int_a^t \int_a^s \mathrm{d}q \mathrm{d}s = M_2 \int_a^t (s-a) \, \mathrm{d}s = \frac{1}{2} M_2 |t-a|^2.$$
We can do all of this again for R_3.

Lecture 4: Alternative Derivation of Euler's Method

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$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t); \qquad y(t_0) = y_0.$$
Goal: find an approximation to this initial value problem. Since continuous functions cannot be stored in a pc, we store approximations to the values of th efunction at a fixed set of points. On some interval $[t_0, t_k]$, we partition it into subintervals with endpoints $t_i = t_{i-1} + \Delta t$. Euler's method is to set $Y_0 = y_0$, then define $Y_{k+1} \coloneqq Y_k + (\Delta t) \cdot f(Y_k, t_k)$.

Alternatively, we have

$$y_{t_k+1} = y(t_k + \Delta t) = y(t_k) + \Delta t(y'(t_k)) + R_2(\Delta t).$$

Basically, this is just a taylor polynomial at t_k, which is near t_{k+1}. Rearranging, we have

$$y'(t_k)\Delta t = y(t_{k+1}) - y(t_k) - R_2(\Delta t).$$

Thus,

$$y'(t_k) = \frac{y(t_{k+1}) - y(t_k) - R_2(\Delta t)}{\Delta t}.$$

We remove the error to give the approximation

$$y'(t_k) \approx \frac{y(t_{k+1}) - y(t_k)}{\Delta t} \approx f(y(t_k), t_k).$$

The error (called the forward difference) is given as

$$\left| \frac{R_2(\Delta t)}{\Delta t} \right| \leq \frac{C_2 \left|\Delta t\right|^2}{\left|\Delta t\right|} \leq C_2 \left|\Delta t\right|.$$

We also don't use our actual value of $y(t_k)$, we use our value Y_{k+1} and Y_k.

0.1 Picard's Theorem

Differential equations don't always have a solution, and may also have more than one solution. So how do we know if an initial value problem has a **unique solution**?

Theorem 0.2 (Picard)

Consider the intial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t); \qquad y(t_0) = y_0.$$

Suppose there exist constants $t_0 \in (a, b)$ and $y_0 \in (c, d)$ such that $\frac{\partial f}{\partial y}(y, t)$ and $f(y, t)$ are both continuous on $[a, b] \times [c, d]$. Then there exists some $\delta > 0$ such that the initial value problem has a **unique solution** $y(t)$ for all $t \in (t_0 - \delta, t_0 + \delta)$.

Proof.

$$\int_{t_0}^{t} \frac{\mathrm{d}y}{\mathrm{d}t}(s)\,\mathrm{d}s = \int_{t_0}^{t} f(y(s), s)\,\mathrm{d}s.$$

We have

$$\int_{t_0}^{t} \frac{\mathrm{d}y}{\mathrm{d}s}(s)\,\mathrm{d}s = y(t) - y(t_0) = y(t) - y_0.$$

Thus,

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\,\mathrm{d}s.$$

We can inductively define a series of approximations. Our first approximation of y is that $y^{(0)}(t) = y_0$. Our next approximation is

$$y^{(1)}(t) = y_0 + \int_{t_0}^{t} f\left(y^{(0)}(s), s\right)\,\mathrm{d}s.$$

Then we give

$$y^{(2)}(t) = y_0 + \int_{t_0}^{t} f\left(y^{(1)}(s), s\right) \,\mathrm{d}s.$$

We continue to

$$y^{(n+1)}(t) = y_0 + \int_{t_0}^{t} f\left(y^{(n)}(s), s\right) \,\mathrm{d}s.$$

Proof will be finished next time. ■