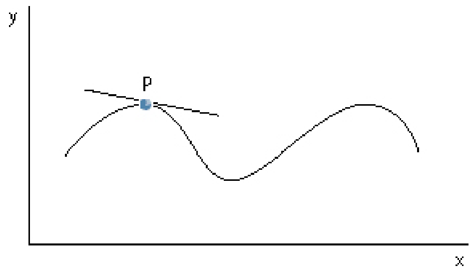
Lecture Notes on Differentiation

A **tangent line** to a function at a point is the line that best approximates the function at that point better than any other line.



The **slope of the function** at a given point is the slope of the tangent line to the function at that point.

The **derivative** of f at $x = a$ is the slope, m, of the function f at the point $x = a$ (if m exists), denoted by $f'(a) = m$. All other notations:

$$y',\ \frac{dy}{dx},\ \frac{df}{dx},\ \frac{d}{dx}f(x),\ D_x y,\ D_x f(x).$$

The function $f(x)$ is **differentiable** at a point x_0 if $f'(x_0)$ exists. If a function is differentiable at all points in its domain (i.e. $f'(x)$ is defined for all x in the domain), then we consider $f'(x)$ as a function and call it the **derivative** of $f(x)$.

The derivative of f that we have been talking about is called the **first derivative**. Now, we define the **second derivative** of a function to be the derivative of f', denoted by $f''(x)$ or $\frac{d^2 f}{dx^2}\left(= \frac{d}{dx}\left(\frac{d}{dx}f\right)\right)$.

Example 1: Given $f(x) = c$ where c is a constant. Then $f'(x) = 0$ because the slope of the function at each point is zero.

Example 2: If $f(x) = 2 - 3x$, then the derivative $f'(x) = 2$ because the slope of the function at each point is 2.

Example 3: Given $f(x) = |x|$. We have

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

However, $f'(0)$ is not defined because there is no unique tangent line to $f(x)$ at $x = 0$.

The following is a table of derivatives of some basic functions:

$f(x)$	$f'(x)$
c	0
$mx + c$	m
x^a	ax^{a-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$

Rules of Differentiation:

1. $(f \pm g)' = f' \pm g'$
2. $(c \cdot f)' = cf'$
3. (Product Rule) $(f \cdot g)' = f'g + fg'$
4. (Quotient Rule) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (where $g(x) \neq 0$)
5. (Chain Rule) $(f \circ g)' = (f(g(x)))' = f'(g(x)) \cdot g'(x)$

The **equation of the tangent line** to the function at point $x = x_0$ is:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Theorem (The Extreme-Value Theorem for Continuous Functions)
If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value value M and an absolute minimum value m somewhere in I.

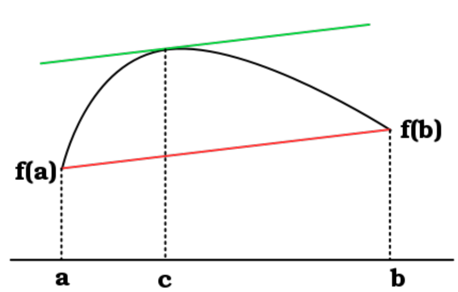
Definition
A point in the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

Theorem
Extreme values (local or global) occur only at critical points and endpoints.

Examples:
1. Find absolute maximum and minimum values of $f(x) = 4 - x^2$ on the interval $[-3, 1]$.
2. Find absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-1, 8]$.
3. Find absolute maximum and minimum values of $f(x) = x^{1/3}$ on the interval $[-1, 1]$.

Theorem (The Mean Value Theorem)
Suppose the $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



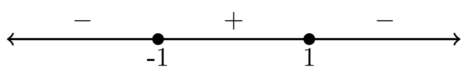
Facts:

- If $f'(x) > 0$ for all x in some interval, then f increases on this interval.
- If $f'(x) < 0$ for all x in some interval, then f decreases on this interval.

Example: Given $f(x) = \dfrac{x}{1+x^2}$.

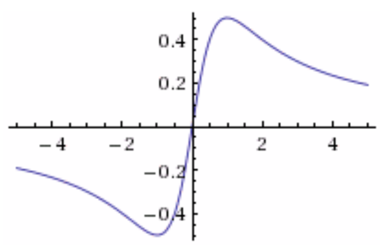
$$f'(x) = \frac{1 \cdot (1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = \frac{(1+x)(1-x)}{(1+x^2)^2}.$$

We can use the **Key Number Method** to test the signs of $f'(x)$.



We know that $f'(x)$ is positive on $(-1, 1)$. Thus, f is increasing on $(-1, 1)$.

Also, $f'(x) < 0$ on $(-\infty, -1)$ and $(1, \infty)$. Thus, f is decreasing on $(-\infty, -1)$ and $(1, \infty)$. The following is the graph of $f(x)$.



The 1st Derivative Test
Suppose f is continuous and differentiable on some open interval containing $x = a$, except possible at $x = a$.
a. If f' changes from $-$ to $+$ at $x = a$, then f has a local minimum at $x = a$.
b. If f' changes from $+$ to $-$ at $x = a$, then f has a local maximum at $x = a$.

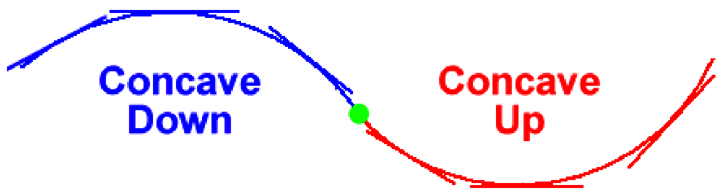
The function $f(x)$ is **concave up** on the interval (a, b) if $f'(x)$ is increasing on (a, b). The function $f(x)$ is **concave down** on the interval (a, b) if $f'(x)$ is increasing on (a, b).

Facts:

- If $f''(x) > 0$ for all x in some interval I, then f' increases on I and thus f is concave up on I.
- If $f''(x) < 0$ for all x in some interval I, then f' decreases on this interval and thus f is concave down on I.

The **inflection point** (or point of inflection) of a function f is defined to be the point at which the concavity changes.

Below is a picture illustrating when a function is concave up or concave down. Notice the tangent lines and their slopes. A point of inflection is also labeled on the picture.



Note: To find the inflection points, we look at the second derivative. Find all the points such that f'' is zero or undefined at those points. Then use the **Key Number Method** to test the sign changes of f'' at those points.

Examples:
1. $f(x) = x^3 - 12x - 5$.
2. $f(x) = x^4 - 4x^3 + 10$.

Examples from Economics

Suppose that

$r(x)$ = the revenue from selling x items
$c(x)$ = the cost of producing the x items
$p(x) = r(x) - c(x)$ = the profit from producing and selling x items.

The **marginal revenue**, **marginal cost**, and **marginal profit** when producing and selling x items are the derivatives

$$\frac{dr}{dx} = \text{marginal revenue,}$$
$$\frac{dc}{dx} = \text{marginal cost,}$$
$$\frac{dp}{dx} = \text{marginal profit.}$$

Let's consider the relationship of p to these derivatives.
If $r(x)$ and $c(x)$ are differentiable for all $x > 0$, and if $p(x) = r(x) - c(x)$ has a maximum value, it occurs at a production level at which $p'(x) = 0$. Since $p'(x) = r'(x) - c'(x)$, $p'(x) = 0$ implies that

$$r'(x) - c'(x) = 0 \text{ or } r'(x) = c'(x).$$

Therefore,

At a production level yielding maximum profit, marginal revenue equals marginal cost.

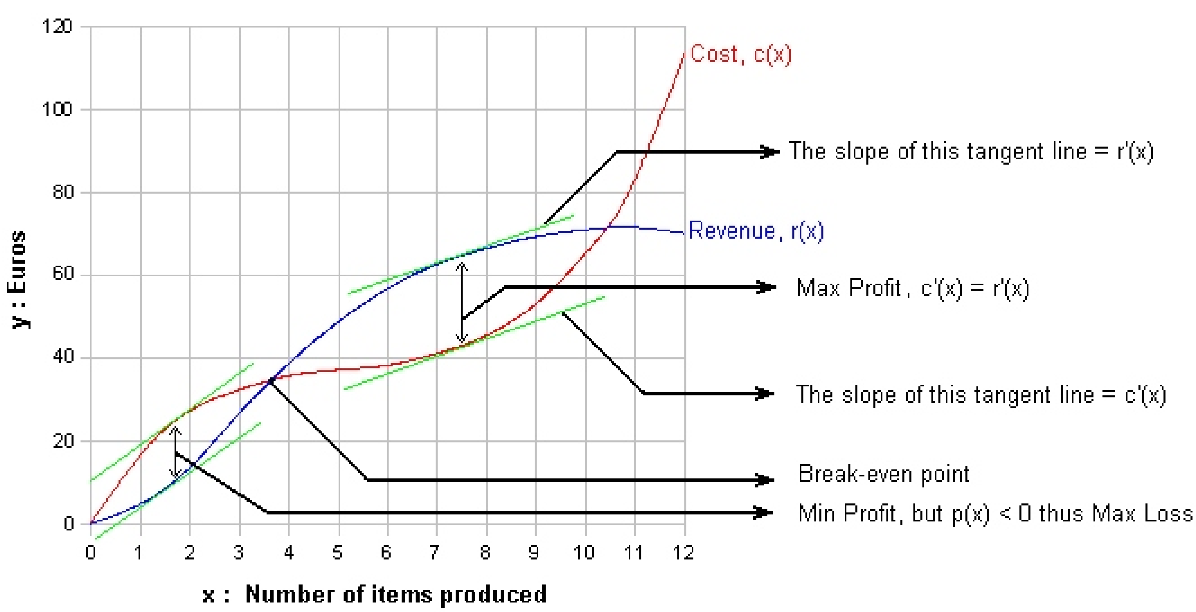


Figure. The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and materials costs and market saturation) and production levels become unprofitable again.

Example Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that maximizes profit? If so, what is it?

Solution Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$. Set $c'(x) = r'(x)$ and get

$$3x^2 - 12x + 15 = 9$$
$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = 2 - \sqrt{2} \text{ and } x_2 = 2 + \sqrt{2}$$

The possible production levels for maximum profit are $x_1 = 2 - \sqrt{2}$ thousand units or $x_2 = 2 + \sqrt{2}$ thousand units. The first derivative of $p(x) = r(x) - c(x)$ is $p'(x) = 9 - 3x^2 + 12x - 15 = -3x^2 - 12x + 6$. By first derivative test, a maximum profit occurs at $x_2 = 2 + \sqrt{2}$ and maximum loss occurs at $x_1 = 2 - \sqrt{2}$.

Example An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution (Outline) Let x be the length of each side of the squares being cut from the corners. Then the volume of the box is $V(x) = x(12 - 2x)^2)$. To maximize the volume, we take the derivative of $V(x)$ and find the critical points. Use 1st derivative test to test for local max. To find the absolute max, compare the local max from the critical points and from the end points of the domain.

Example A manufacturer needs to make a cylindrical can that will hold 1 liter ($= 1000\text{cm}^3$) of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

Solution Hint: It is customary to ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions that make the total surface area as small as possible while satisfying the constraint volume $= 1000\text{cm}^3$.

Let r be the radius of the top circle and h be the high of the can. Then $S = 2\pi r^2 + 2\pi rh$. Since $V = \pi r^2 h = 1000$, we have $h = \frac{1000}{\pi r^2}$. Substitute into S and get $S(r) = 2\pi r^2 + 2\pi r\left(\dfrac{1000}{\pi r^2}\right) = 2\pi r^2 + \dfrac{2000}{r}$. Then take the derivative of $S(r)$ (with respect to r) and find the critical points. Use first derivative test to test for local min.

Lecture Notes on Integration

Mean Value Theorem Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

Corollary 1 If $f'(x) = 0$ at each point of an interval I, then $f(x) = C$ for all x in I, where C is a constant.

Corollary 2 If $f'(x) = g'(x)$ at each point of an interval I, then there exists a constant C such that $f(x) = g(x) + C$ for all x in I.

A function, $F(x)$, is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f.

Example: The function $F(x) = x^2$ is an antiderivative of $f(x) = 2x$. The function $G(x) = x^2 + 4$ is also an antiderivative of $f(x) = 2x$.

The set of all antiderivative of f is the indefinite integral of f with respect to x, denoted by

$$\int f(x)dx$$

The symbol $\int$ is an **integral sign**. The function $f(x)$ is the **integrand** of the integral, and x is the **variable of integration**.

To **verify** $\int xe^x dx = xe^x - e^x + C$, we take the derivative of the right hand side. $\frac{d}{dx} xe^x - e^x + C = e^x + xe^x - e^x = xe^x$. Thus, the integral statement is correct.

Integral formulas
$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
$\int 1dx = \int dx = x + C$
$\int e^x dx = e^x + C$
$\int \frac{1}{x} dx = \ln \lvert x \rvert + C$

Rules for indefinite integrals:

1. $\int kf(x)dx = k \int f(x)dx$

2. $\int -f(x)dx = -\int f(x)dx$

3. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Method of Substitution:

$\int f(g(x))g'(x)dx = \int f(u)du$ where $u = g(x)$ and $du = g'(x)dx$.

Example 1: Find $\int (x^3+2)^5 3x^2 dx$.

Let $u = x^3 + 2$, $du = 3x^2 dx$.
Then

$$\begin{aligned}\int (x^3+2)^5 3x^2 dx &= \int u^5 du \\ &= \frac{u^6}{6} + C \\ &= \frac{(x^3+2)^6}{6} + C\end{aligned}$$

Example 2: Find $\int \sqrt{x^2+1} \cdot 2x dx$.

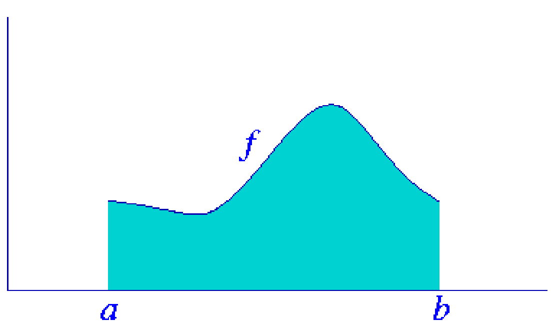
Let $u = x^2 + 1$, $du = 2x dx$.
Then

$$\begin{aligned}\int \sqrt{x^2+1} \cdot 2x dx &= \int \sqrt{u} du \\ &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\ &= \frac{2u^{\frac{3}{2}}}{3} + C \\ &= \frac{2(x^2+1)^{\frac{3}{2}}}{3} + C\end{aligned}$$

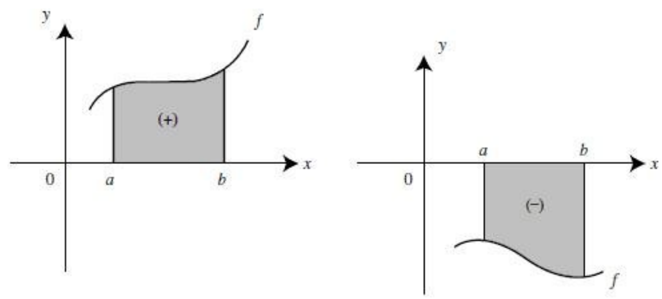
Definition: (Definite Integral)

$$\int_a^b f(x)dx = \text{(signed or net) area between the curve and } x\text{-axis from } a \text{ to } b.$$

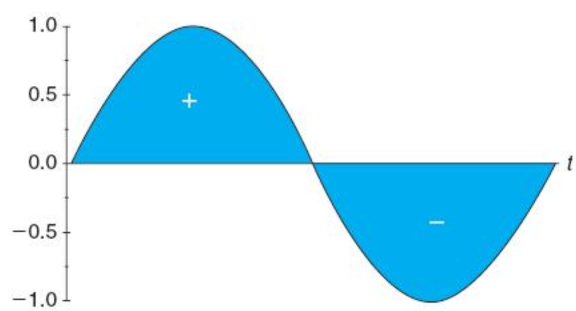
The number a is called the **lower limit** and the number b is called the **upper limit**.



Note: If the curve from a to b is below the x-axis, the definite integral of $f(x)$ from a to b will be negative.

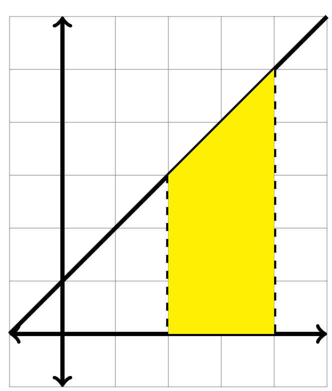


If part of the curve from a to b is below the x-axis and part of it is above the x-axis, the definite integral of $f(x)$ from a to b could be zero.



Example:

$$\int_2^4 (x+1)dx = \frac{(3+5)\cdot 2}{2} = 8$$



Properties for definite integrals:

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a kf(x)dx = 0$

3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

4. $\int_a^b -f(x)dx = -\int_a^b f(x)dx$

5. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

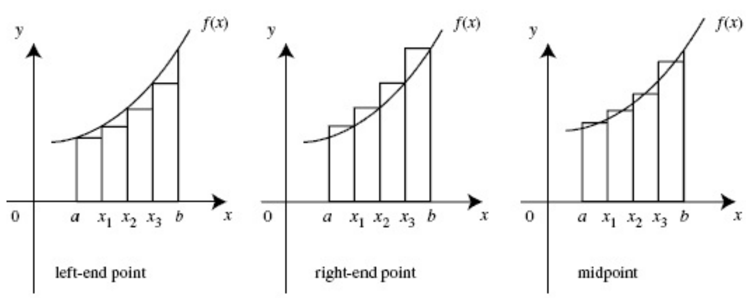
6. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

7. $\min f \cdot (b-a) \leq \int_a^b f(x)dx \leq \max f \cdot (b-a)$

8. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

We can approximate the area under the curve using rectangles.

Left Endpoint Rule: using rectangles with left top corner on the curve
Right Endpoint Rule: using rectangles with right top corner on the curve
Midpoint Rule: using rectangles with top midpoint on the curve



L_n = sum of area of n rectangles using left endpoint rule.

R_n = sum of area of n rectangles using right endpoint rule.

M_n = sum of area of n rectangles using midpoint rule.

The Fundamental Theorem of Calculus
(Part 1) Let f be a continuous function on $[a, b]$. Let F be the function

$$F(x) = \int_a^x f(t)dt.$$

Then, $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

(Part 2) If f is a continuous function on $[a, b]$ and F is any antiderivative of f on [a,b], then

$$\int_a^b f(t)dt = F(b) - F(a) := F(x)\Big|_a^b.$$

Example 1:

$$\frac{d}{dx}\int_1^x \sqrt{t^2+3}dt = \sqrt{x^2+3}$$

Example 2:

$$\frac{d}{dx}\int_3^{x^2} te^t dt = x^2 e^{x^2} \cdot (2x) = 2x^3 e^{x^2}$$

Example 3:

$$\int_1^2 (3x^2+2x)dx = x^3 + x^2\Big|_1^2 = (8+4) - (1+1) = 10$$

Example 4:

$$\int_0^2 2x\sqrt{x^2+1}dx$$

First find an antiderivative of $2x\sqrt{x^2+1}$.

$$\int 2x\sqrt{x^2+1}dx$$

Let $u = x^2+1$, $du = 2xdx$.

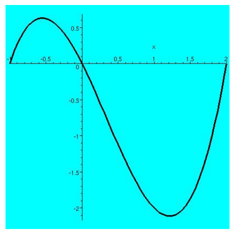
Then $\int 2x\sqrt{x^2+1}dx = \int \sqrt{u}du = \frac{2u^{\frac{3}{2}}}{3} + C = \frac{2(x^2+1)^{\frac{3}{2}}}{3} + C$.

Thus, $\int_0^1 2x\sqrt{x^2+1}dx = \frac{2(x^2+1)^{\frac{3}{2}}}{3}\Big|_0^1 = \frac{2(2)^{\frac{3}{2}}}{3} = \frac{4\sqrt{2}}{3}$.

Note: If a question asks you to find the area of a region, it means the total area, i.e. the (positive) measure of the size of the region.

Example: Find the area of the region between x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

If the graph of the function is not given, you may want to sketch the graph first and see what are the regions. We also want to factor the function and find the x-intercepts. Thus, $f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x+1)(x-2)$. See the following:



Since from $x = -1$ to $x = 0$, the curve is positive and from $x = 0$ to $x = 2$, the curve is negative, we can integrate the function from $x = -1$ to $x = 0$ and from $x = 0$ to $x = 2$ separately.

$$\begin{aligned}\int_{-1}^{0} (x^3 - x^2 - 2x)dx &= \frac{x^4}{4} - \frac{x^3}{3} - x^2 \Big|_{-1}^{0} \\ &= 0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right) = 1 - \frac{1}{4} - \frac{1}{3} = \frac{5}{12}\end{aligned}$$

$$\begin{aligned}\int_{0}^{2} (x^3 - x^2 - 2x)dx &= \frac{x^4}{4} - \frac{x^3}{3} - x^2 \Big|_{0}^{2} \\ &= \frac{2^4}{4} - \frac{2^3}{3} - 2^2 = 4 - \frac{8}{3} - 4 = -\frac{8}{3}\end{aligned}$$

Note that the first integral is positive but the second is negative. Thus, the total area $= \frac{5}{12} + \left| -\frac{8}{3} \right| = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$.

Fact: If $f(x)$ and $g(x)$ are continuous and $f(x) \geq g(x)$ on $[a, b]$, then the area between the two curves is:

$$\int_a^b [f(x) - g(x)]dx$$

Fact: The volume of a solid of known cross-section area $A(x)$ from $x = a$ to $x = b$ is:

$$V = \int_a^b A(x)dx$$

Special Case: The volume of a solid generated by revolving the function $y = f(x)$ about the x-axis from $x = a$ to $x = b$ is:

$$V = \int_a^b \pi[f(x)]^2 dx$$