



DIFFERENTIAL EQUATIONS

Having studied the concept of differentiation and integration, we are now faced with the question where do they find an application.

In fact these are the tools which help us to determine the exact takeoff speed, angle of launch, amount of thrust to be provided and other related technicalities in space launches.

Not only this but also in some problems in Physics and Bio-Sciences, we come across relations which involve derivatives.

One such relation could be $\frac{ds}{dt} = 4.9 t^2$ where s is distance and t is time. Therefore, $\frac{ds}{dt}$ represents velocity (rate of change of distance) at time t .

Equations which involve derivatives as their terms are called differential equations. In this lesson, we are going to learn how to find the solutions and applications of such equations.



OBJECTIVES

After studying this lesson, you will be able to :

- define a differential equation, its order and degree;
- determine the order and degree of a differential equation;
- form differential equation from a given situation;
- illustrate the terms "general solution" and "particular solution" of a differential equation through examples;
- solve differential equations of the following types :

$$(i) \frac{dy}{dx} = f(x)$$

$$(ii) \frac{dy}{dx} = f(x)g(y)$$

$$(iii) \frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$(iv) \frac{dy}{dx} + P(x)y = Q(x)$$

$$(v) \frac{d^2y}{dx^2} = f(x)$$

- find the particular solution of a given differential equation for given conditions.

EXPECTED BACKGROUND KNOWLEDGE

- Integration of algebraic functions, rational functions and trigonometric functions

28.1 DIFFERENTIAL EQUATIONS

As stated in the introduction, many important problems in Physics, Biology and Social Sciences, when formulated in mathematical terms, lead to equations that involve derivatives. Equations

which involve one or more differential coefficients such as $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ (or differentials) etc. and independent and dependent variables are called differential equations.

For example,

$$\begin{array}{lll} \text{(i)} & \frac{dy}{dx} = \cos x & \text{(ii)} \quad \frac{d^2y}{dx^2} + y = 0 \quad \text{(iii)} \quad xdx + ydy = 0 \\ \text{(iv)} & \left(\frac{d^2y}{dx^2} \right)^2 + x^2 \left(\frac{dy}{dx} \right)^3 = 0 & \text{(vi)} \quad y = \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \end{array}$$

28.2 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

Order : It is the order of the highest derivative occurring in the differential equation.

Degree : It is the degree of the highest order derivative in the differential equation after the equation is free from negative and fractional powers of the derivatives. For example,

	Differential Equation	Order	Degree
(i)	$\frac{dy}{dx} = \sin x$	One	One
(ii)	$\left(\frac{dy}{dx} \right)^2 + 3y^2 = 5x$	One	Two
(iii)	$\left(\frac{d^2s}{dt^2} \right)^2 + t^2 \left(\frac{ds}{dt} \right)^4 = 0$	Two	Two
(iv)	$\frac{d^3v}{dr^3} + \frac{2}{r} \frac{dv}{dr} = 0$	Three	One
(v)	$\left(\frac{d^4y}{dx^4} \right)^2 + x^3 \left(\frac{d^3y}{dx^3} \right)^5 = \sin x$	Four	Two

Example 28.1 Find the order and degree of the differential equation :

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}} = 0$$



Solution : The given differential equation is

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}} = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = - \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}$$

The term $\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}$ has fractional index. Therefore, we first square both sides to remove fractional index.

Squaring both sides, we have

$$\left(\frac{d^2y}{dx^2} \right)^2 = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3$$

Hence each of the order of the differential equation is 2 and the degree of the differential equation is also 2.

Note : Before finding the degree of a differential equation, it should be free from radicals and fractions as far as derivatives are concerned.

28.3 LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation in which the dependent variable and all of its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**. A differential equation which is not linear is called non-linear differential equation. For example, the differential equations

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{and} \quad \cos^2 x \frac{d^3y}{dx^3} + x^3 \frac{dy}{dx} + y = 0 \quad \text{are linear.}$$

The differential equation $\left(\frac{dy}{dx} \right)^2 + \frac{y}{x} = \log x$ is non-linear as degree of $\frac{dy}{dx}$ is two.

Further the differential equation $y \frac{d^2y}{dx^2} - 4 = x$ is non-linear because the dependent variable

y and its derivative $\frac{d^2y}{dx^2}$ are multiplied together.

28.4 FORMATION OF A DIFFERENTIAL EQUATION

Consider the family of all straight lines passing through the origin (see Fig. 28.1).

This family of lines can be represented by

$$y = mx \quad \text{.....(1)}$$

Differentiating both sides, we get

$$\frac{dy}{dx} = m \quad \text{.....(2)}$$

From (1) and (2), we get

$$y = x \frac{dy}{dx} \quad \text{.....(3)}$$

So $y = mx$ and $y = x \frac{dy}{dx}$ represent the same family.

Clearly equation (3) is a differential equation.

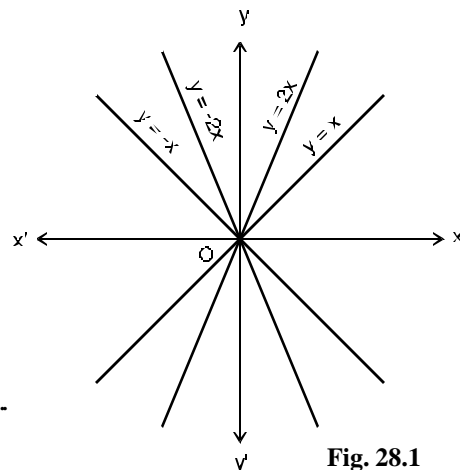


Fig. 28.1

Working Rule : To form the differential equation corresponding to an equation involving two variables, say x and y and some arbitrary constants, say a, b, c , etc.

- (i) Differentiate the equation as many times as the number of arbitrary constants in the equation.
- (ii) Eliminate the arbitrary constants from these equations.

Remark

If an equation contains n arbitrary constants then we will obtain a differential equation of n^{th} order.

Example 28.2 Form the differential equation representing the family of curves.

$$y = ax^2 + bx. \quad \text{.....(1)}$$

Differentiating both sides, we get

$$\frac{dy}{dx} = 2ax + b \quad \text{.....(2)}$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = 2a \quad \text{.....(3)}$$

$$\Rightarrow a = \frac{1}{2} \frac{d^2y}{dx^2} \quad \text{.....(4)}$$

(The equation (1) contains two arbitrary constants. Therefore, we differentiate this equation two times and eliminate 'a' and 'b').

On putting the value of 'a' in equation (2), we get

$$\frac{dy}{dx} = x \frac{d^2y}{dx^2} + b$$

$$\Rightarrow \quad b = \frac{dy}{dx} - x \frac{d^2y}{dx^2} \quad \dots\dots(5)$$

Substituting the values of 'a' and 'b' given in (4) and (5) above in equation (1), we get

$$y = x^2 \left(\frac{1}{2} \frac{d^2y}{dx^2} \right) + x \left(\frac{dy}{dx} - x \frac{d^2y}{dx^2} \right)$$

$$\text{or} \quad y = \frac{x^2}{2} \frac{d^2y}{dx^2} + x \frac{dy}{dx} - x^2 \frac{d^2y}{dx^2}$$

$$\text{or} \quad y = x \frac{dy}{dx} - \frac{x^2}{2} \frac{d^2y}{dx^2}$$

$$\text{or} \quad \frac{x^2}{2} \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0$$

which is the required differential equation.

Example 28.3 Form the differential equation representing the family of curves

$$y = a \cos (x + b).$$

$$\text{Solution :} \quad y = a \cos (x + b) \quad \dots\dots(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = -a \sin(x + b) \quad \dots\dots(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -a \cos (x + b) \quad \dots\dots(3)$$

From (1) and (3), we get

$$\frac{d^2y}{dx^2} = -y \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation.

Example 28.4 Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

Solution : As the centre lies on the x-axis, its coordinates will be (a, 0).

Since each circle passes through the origin, its radius is a.

Then the equation of any circle will be

$$(x - a)^2 + y^2 = a^2 \quad (1)$$

To find the corresponding differential equation, we differentiate equation (1) and get





$$x(x - a)^2 + y \frac{dy}{dx} = 0$$

or

$$x - a + y \frac{dy}{dx} = 0$$

or

$$a = y \frac{dy}{dx} + x$$

Substituting the value of 'a' in equation (1), we get

$$\left(x - y \frac{dy}{dx} - x \right)^2 + y^2 = \left(y \frac{dy}{dx} + x \right)^2$$

or

$$\left(y \frac{dy}{dx} \right)^2 + y^2 = x^2 + \left(y \frac{dy}{dx} \right)^2 + 2xy \frac{dy}{dx}$$

or

$$y^2 = x^2 + 2xy \frac{dy}{dx}$$

which is the required differential equation.

Remark

If an equation contains one arbitrary constant then the corresponding differential equation is of the first order and if an equation contains two arbitrary constants then the corresponding differential equation is of the second order and so on.

Example 28.5

Assuming that a spherical rain drop evaporates at a rate proportional to its surface area, form a differential equation involving the rate of change of the radius of the rain drop.

Solution : Let $r(t)$ denote the radius (in mm) of the rain drop after t minutes. Since the radius is decreasing as t increases, the rate of change of r must be negative.

If V denotes the volume of the rain drop and S its surface area, we have

$$V = \frac{4}{3} \pi r^3 \quad \dots(1)$$

and

$$S = 4\pi r^2 \quad \dots(2)$$

It is also given that

$$\frac{dV}{dt} \propto S$$

or

$$\frac{dV}{dt} = -kS$$

or

$$\frac{dV}{dr} \cdot \frac{dr}{dt} = -kS \quad \dots(3)$$

Using (1), (2) and (3) we have

$$4\pi r^2 \cdot \frac{dr}{dt} = -4k \pi r^2$$

or $\frac{dr}{dt} = k$
which is the required differential equation.



CHECK YOUR PROGRESS 28.1

- Find the order and degree of the differential equation

$$y = x \frac{dy}{dx} + \frac{1}{\frac{dy}{dx}}$$

- Write the order and degree of each of the following differential equations.

$$(a) \left(\frac{ds}{dt} \right)^4 + 3s \frac{d^2s}{dt^2} = 0 \quad (b) y = 2x \frac{dy}{dx} + x \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$(c) \sqrt{1-x^2} dx + \sqrt{1-y^2} dy = 0 \quad (d) \left(\frac{d^2s}{dt^2} \right)^2 + 3 \left(\frac{ds}{dt} \right)^3 + 4 = 0$$

- State whether the following differential equations are linear or non-linear.

$$(a) (xy^2 - x) dx + (y - x^2y) dy = 0 \quad (b) dx + dy = 0$$

$$(c) \frac{dy}{dx} = \cos x \quad (d) \frac{dy}{dx} + \sin^2 y = 0$$

- Form the differential equation corresponding to

$$(x-a)^2 + (y-b)^2 = r^2 \quad \text{by eliminating 'a' and 'b'}$$

- Form the differential equation corresponding to

$$y^2 = m(a^2 - x^2)$$

- Form the differential equation corresponding to

$$y^2 - 2ay + x^2 = a^2, \text{ where } a \text{ is an arbitrary constant.}$$

- Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-3x}$ where A and B are arbitrary constants.
- Find the differential equation of all straight lines passing through the point (3,2).
- Find the differential equation of all the circles which pass through origin and whose centres lie on y-axis.

28.5 GENERAL AND PARTICULAR SOLUTIONS

Finding solution of a differential equation is a reverse process. Here we try to find an equation which gives rise to the given differential equation through the process of differentiations and elimination of constants. The equation so found is called the primitive or the solution of the differential equation.

Remarks

- (1) If we differentiate the primitive, we get the differential equation and if we integrate the differential equation, we get the primitive.
- (2) Solution of a differential equation is one which satisfies the differential equation.



Example 28.6 Show that $y = C_1 \sin x + C_2 \cos x$, where C_1 and C_2 are arbitrary constants, is a solution of the differential equation :

$$\frac{d^2y}{dx^2} + y = 0$$

Solution : We are given that

$$y = C_1 \sin x + C_2 \cos x \quad \dots(1)$$

Differentiating both sides of (1), we get

$$\frac{dy}{dx} = C_1 \cos x - C_2 \sin x \quad \dots(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -C_1 \sin x - C_2 \cos x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

$$\frac{d^2y}{dx^2} + y = C_1 \sin x + C_2 \cos x + (-C_1 \sin x - C_2 \cos x)$$

or
$$\frac{d^2y}{dx^2} + y = 0$$

In integration, the arbitrary constants play important role. For different values of the constants we get the different solutions of the differential equation.

A solution which contains as many as arbitrary constants as the order of the differential equation is called the **General Solution** or complete primitive.

If we give the particular values to the arbitrary constants in the general solution of differential equation, the resulting solution is called a **Particular Solution**.

Remark

General Solution contains as many arbitrary constants as is the order of the differential equation.

Example 28.7 Show that $y = cx + \frac{a}{c}$ (where c is a constant) is a solution of the differential equation.

$$y = x \frac{dy}{dx} + a \frac{dx}{dy}$$

Solution : We have $y = cx + \frac{a}{c}$ (1)

Differentiating (1), we get

$$\frac{dy}{dx} = c \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{c}$$

On substituting the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in R.H.S of the differential equation, we have

$$x(c) + a\left(\frac{1}{c}\right) = cx + \frac{a}{c} = y$$

\Rightarrow R.H.S. = L.H.S.

Hence $y = cx + \frac{a}{c}$ is a solution of the given differential equation.

Example 28.8 If $y = 3x^2 + C$ is the general solution of the differential equation

$\frac{dy}{dx} - 6x = 0$, then find the particular solution when $y = 3$, $x = 2$.

Solution : The general solution of the given differential equation is given as

$$y = 3x^2 + C \quad \text{.....(1)}$$

Now on substituting $y = 3$, $x = 2$ in the above equation, we get

$$3 = 12 + C \quad \text{or} \quad C = -9$$

By substituting the value of C in the general solution (1), we get

$$y = 3x^2 - 9$$

which is the required particular solution.

28.6 TECHNIQUES OF SOLVING A DIFFERENTIAL EQUATION

28.6.1 When Variables are Separable

(i) Differential equation of the type $\frac{dy}{dx} = f(x)$

Consider the differential equation of the type $\frac{dy}{dx} = f(x)$

or $dy = f(x) dx$

On integrating both sides, we get

$$\int dy = \int f(x) dx$$



$$f = \int dx \quad)c +$$

where c is an arbitrary constant. This is the general solution.

Note : It is necessary to write c in the general solution, otherwise it will become a particular solution.

Example 28.9 Solve

$$(x + 2) \frac{dy}{dx} = x^2 + 4x - 5$$

Solution : The given differential equation is $(x + 2) \frac{dy}{dx} = x^2 + 4x - 5$

$$\text{or} \quad \frac{dy}{dx} = \frac{x^2 + 4x - 5}{x + 2} \quad \text{or} \quad \frac{dy}{dx} = \frac{x^2 + 4x + 4 - 4 - 5}{x + 2}$$

$$\text{or} \quad \frac{dy}{dx} = \frac{(x + 2)^2}{x + 2} - \frac{9}{x + 2} \quad \text{or} \quad \frac{dy}{dx} = x + 2 - \frac{9}{x + 2}$$

$$\text{or} \quad dy = \left(x + 2 - \frac{9}{x + 2} \right) dx \quad \dots(1)$$

On integrating both sides of (1), we have

$$\int dy = \int \left(x + 2 - \frac{9}{x + 2} \right) dx \quad \text{or} \quad y = \frac{x^2}{2} + 2x - 9 \log |x + 2| + c,$$

where c is an arbitrary constant, is the required general solution.

Example 28.10 Solve

$$\frac{dy}{dx} = 2x^3 - x$$

given that $y = 1$ when $x = 0$

Solution : The given differential equation is $\frac{dy}{dx} = 2x^3 - x$

$$\text{or} \quad dy = (2x^3 - x) dx \quad \dots(1)$$

On integrating both sides of (1), we get

$$\int dy = \int (2x^3 - x) dx \quad \text{or} \quad y = 2 \cdot \frac{x^4}{4} - \frac{x^2}{2} + C$$

$$\text{or} \quad y = \frac{x^4}{2} - \frac{x^2}{2} + C \quad \dots(2)$$

where C is an arbitrary constant.

Since $y = 1$ when $x = 0$, therefore, if we substitute these values in (2) we will get

$$1 = 0 - 0 + C \quad \Rightarrow \quad C = 1$$

Differential Equations

Now, on putting the value of C in (2), we get

$$y = \frac{1}{2}(x^4 - x^2) + 1 \text{ or } y = \frac{1}{2}x^2(x^2 - 1) + 1$$

which is the required particular solution.

(ii) **Differential equations of the type** $\frac{dy}{dx} = f(x) \cdot g(y)$

Consider the differential equation of the type

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

or
$$\frac{dy}{g(y)} = f(x) dx \quad \dots(1)$$

In equation (1), x's and y's have been separated from one another. Therefore, this equation is also known differential equation with variables separable.

To solve such differential equations, we integrate both sides and add an arbitrary constant on one side.

To illustrate this method, let us take few examples.

Example 28.11 Solve

$$(1 + x^2) dy = (1 + y^2) dx$$

Solution : The given differential equation

$$(1 + x^2) dy = (1 + y^2) dx$$

can be written as

$$\frac{dy}{1 + y^2} = \frac{dx}{1 + x^2} \text{ (Here variables have been separated)}$$

On integrating both sides of (1), we get

$$\int \frac{dy}{1 + y^2} = \int \frac{dx}{1 + x^2}$$

or

$$\tan^{-1} y = \tan^{-1} x + C$$

where C is an arbitrary constant.

This is the required solution.

Example 28.12 Solve

$$(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2 = 0$$

Solution : The given differential equation





$$(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2 = 0$$

can be written as $x^2(1 - y) \frac{dy}{dx} + y^2(1 + x) = 0$

or $\frac{(1 - y)}{y^2} dy = \frac{-(1 + x)}{x^2} dx$

(Variables separated)(1)

If we integrate both sides of (1), we get

$$\int \left(\frac{1}{y^2} - \frac{1}{y} \right) dy = \int \left(-\frac{1}{x^2} - \frac{1}{x} \right) dx$$

where C is an arbitrary constant.

or $-\frac{1}{y} - \log|y| = \frac{1}{x} - \log|x| + C$

or $\log \left| \frac{x}{y} \right| = \frac{1}{x} + \frac{1}{y} + C$

Which is the required general solution.

Example 28.13 Find the particular solution of

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 1}$$

when $y(0) = 3$ (i.e. when $x = 0$, $y = 3$).

Solution : The given differential equation is

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 1} \quad \text{or} \quad (3y^2 + 1) dy = 2x dx \quad (\text{Variables separated}) \quad \dots(1)$$

If we integrate both sides of (1), we get

$$\int (3y^2 + 1) dy = \int 2x dx,$$

where C is an arbitrary constant.

$$y^3 + y = x^2 + C \quad \dots(2)$$

It is given that, $y(0) = 3$.

\therefore on substituting $y = 3$ and $x = 0$ in (2), we get

$$27 + 3 = C$$

$$\therefore C = 30$$

Thus, the required particular solution is

$$y^3 + y = x^2 + 30$$

28.6.2 Homogeneous Differential Equations

Consider the following differential equations :

$$(i) \quad y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx} \quad (ii) \quad (x^3 + y^3) dx - 3xy^2 dy = 0$$

$$(iii) \quad \frac{dy}{dx} = \frac{x^3 + xy^2}{y^2 x}$$

In equation (i) above, we see that each term except $\frac{dy}{dx}$ is of degree 2

[as degree of y^2 is 2, degree of x^2 is 2 and degree of xy is $1 + 1 = 2$]

In equation (ii) each term except $\frac{dy}{dx}$ is of degree 3.

In equation (iii) each term except $\frac{dy}{dx}$ is of degree 3, as it can be rewritten as

$$y^2 x \frac{dy}{dx} = x^3 + x y^2$$

Such equations are called **homogeneous equations**.

Remarks

Homogeneous equations do not have constant terms.

For example, differential equation

$$(x^2 + 3yx) dx - (x^3 + x) dy = 0$$

is not a homogeneous equation as the degree of the function except $\frac{dy}{dx}$ in each term is not the

same. [degree of x^2 is 2, that of $3yx$ is 2, of x^3 is 3, and of x is 1]

28.6.3 Solution of Homogeneous Differential Equation :

To solve such equations, we proceed in the following manner :

- (i) write one variable = v. (the other variable).
(i.e. either $y = vx$ or $x = vy$)
- (ii) reduce the equation to separable form
- (iii) solve the equation as we had done earlier.

Example 28.14 Solve

$$(x^2 + 3xy + y^2) dx - x^2 dy = 0$$

Solution : The given differential equation is

$$(x^2 + 3xy + y^2) dx - x^2 dy = 0$$





$$\text{or } \frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} \quad \dots(1)$$

It is a homogeneous equation of degree two. (Why?)

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

\therefore From (1), we have

$$v + x \frac{dv}{dx} = \frac{x^2 + 3x.vx + (vx)^2}{x^2} \quad \text{or} \quad v + x \frac{dv}{dx} = x^2 \left[\frac{1 + 3v + v^2}{x^2} \right]$$

$$\text{or } v + x \frac{dv}{dx} = 1 + 3v + v^2 \quad \text{or} \quad x \frac{dv}{dx} = 1 + 3v + v^2 - v$$

$$\text{or } x \frac{dv}{dx} = v^2 + 2v + 1 \quad \text{or} \quad \frac{dv}{v^2 + 2v + 1} = \frac{dx}{x}$$

$$\text{or } \frac{dv}{(v+1)^2} = \frac{dx}{x} \quad \dots(2)$$

Further on integrating both sides of (2), we get

$$\frac{-1}{v+1} + C = \log|x|, \quad \text{where } C \text{ is an arbitrary constant.}$$

On substituting the value of v , we get

$$\frac{x}{y+x} + \log|x| = C \quad \text{which is the required solution.}$$

28.6.4 Differential Equation

Consider the equation

$$\frac{dy}{dx} + Py = Q \quad \dots(1)$$

where P and Q are functions of x . This is linear equation of order one.

To solve equation (1), we first multiply both sides of equation (1) by $e^{\int P dx}$ (called integrating factor) and get

$$e^{\int P dx} \frac{dy}{dx} + Py e^{\int P dx} = Q e^{\int P dx}$$

$$\text{or } \frac{d}{dx} \left(y e^{\int P dx} \right) = Q e^{\int P dx} \quad \dots(2)$$

$$\left[\because \frac{d}{dx} \left(y e^{\int P dx} \right) = e^{\int P dx} \frac{dy}{dx} + Py e^{\int P dx} \right]$$

Differential Equations

On integrating, we get

$$ye^{\int P dx} = \int Qe^{\int P dx} dx + C \quad \dots(3)$$

where C is an arbitrary constant,

or
$$y = e^{-\int P dx} \left[\int Qe^{\int P dx} dx + C \right]$$



Note : $e^{\int P dx}$ is called the integrating factor of the equation and is written as I.F in short.

Remarks

- (i) We observe that the left hand side of the linear differential equation (1) has become

$$\frac{d}{dx} \left(ye^{\int P dx} \right) \text{ after the equation has been multiplied by the factor } e^{\int P dx}.$$

- (ii) The solution of the linear differential equation

$$\frac{dy}{dx} + Py = Q$$

P and Q being functions of x only is given by

$$ye^{\int P dx} = \int Q \left(e^{\int P dx} \right) dx + C$$

- (iii) The coefficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing the equation by it throughout.

- (iv) Some differential equations become linear differential equations if y is treated as the independent variable and x is treated as the dependent variable.

For example, $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only, is also a linear differential equation of the first order.

In this case I.F. = $e^{\int P dy}$

and the solution is given by

$$x (\text{I.F.}) = \int Q. (\text{I.F.}) dy + C$$

Example 28.15 Solve

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x}$$

Solution : Here $P = \frac{1}{x}$, $Q = e^{-x}$ (Note that both P and Q are functions of x)

$$\text{I.F. (Integrating Factor)} \quad e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x \quad (x > 0)$$

On multiplying both sides of the equation by I.F., we get



$$x \cdot \frac{dy}{dx} + y = x \cdot e^{-x} \quad \text{or} \quad \frac{d}{dx}(y \cdot x) = x e^{-x}$$

Integrating both sides, we have

$$yx = \int x e^{-x} dx + C$$

where C is an arbitrary constant

$$\text{or} \quad xy = -x e^{-x} + \int e^{-x} dx + C$$

$$\text{or} \quad xy = -x e^{-x} - e^{-x} + C$$

$$\text{or} \quad xy = -e^{-x}(x+1) + C$$

$$\text{or} \quad y = -\left(\frac{x+1}{x}\right)e^{-x} + \frac{C}{x}$$

Note: In the solution $x > 0$.

Example 28.16 Solve :

$$\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$$

Solution : The given differential equation is

$$\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$$

$$\text{or} \quad \frac{dy}{dx} + y \cot x = 2 \sin x \cos x \quad \dots(1)$$

$$\text{Here} \quad P = \cot x, Q = 2 \sin x \cos x$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$

On multiplying both sides of equation (1) by I.F., we get ($\sin x > 0$)

$$\frac{d}{dx}(y \sin x) = 2 \sin^2 x \cos x$$

Further on integrating both sides, we have

$$y \sin x = \int 2 \sin^2 x \cos x dx + C$$

where C is an arbitrary constant ($\sin x > 0$)

$$\text{or} \quad y \sin x = \frac{2}{3} \sin^3 x + C, \quad \text{which is the required solution.}$$

Example 28.17 Solve $(1+y^2) \frac{dx}{dy} = \tan^{-1} y - x$

Solution : The given differential equation is

$$(1 + y^2) \frac{dx}{dy} = \tan^{-1} y - x$$

or
$$\frac{dx}{dy} = \frac{\tan^{-1} y}{1 + y^2} - \frac{x}{1 + y^2}$$

or
$$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2} \quad \dots(1)$$

which is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are the functions of y only.

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Multiplying both sides of equation (1) by I.F., we get

$$\frac{d}{dy} \left(x e^{\tan^{-1} y} \right) = \frac{\tan^{-1} y}{1 + y^2} \left(e^{\tan^{-1} y} \right)$$

On integrating both sides, we get

or
$$\left(e^{\tan^{-1} y} \right)_x = \int e^t \cdot dt + C,$$

where C is an arbitrary constant and $t = \tan^{-1} y$ and $dt = \frac{1}{1 + y^2} dy$

or
$$\left(e^{\tan^{-1} y} \right)_x = te^t - \int e^t + C$$

or
$$\left(e^{\tan^{-1} y} \right)_x = te^t - e^t + C$$

or
$$\left(e^{\tan^{-1} y} \right)_x = \tan^{-1} y e^{\tan^{-1} y} - e^{\tan^{-1} y} + C \quad (\text{on putting } t = \tan^{-1} y)$$

or
$$x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$



CHECK YOUR PROGRESS 28.2

1. (i) Is $y = \sin x$ a solution of $\frac{d^2 y}{dx^2} + y = 0$?

(ii) Is $y = x^3$ a solution of $x \frac{dy}{dx} - 4y = 0$?

2. Given below are some solutions of the differential equation $\frac{dy}{dx} = 3x$.

State which are particular solutions and which are general solutions.



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Notes

- (i) $2y = 3x^2$ (ii) $y = \frac{3}{2}x^2 + 2$
- (iii) $2y = 3x^2 + C$ (iv) $y = \frac{3}{2}x^2 + 3$
3. State whether the following differential equations are homogeneous or not ?
- (i) $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ (ii) $(3xy + y^2)dx + (x^2 + xy)dy = 0$
- (iii) $(x+2)\frac{dy}{dx} = x^2 + 4x - 9$ (iv) $(x^3 - yx^2)dy + (y^3 + x^3)dx = 0$
4. (a) Show that $y = a \sin 2x$ is a solution of $\frac{d^2y}{dx^2} + 4y = 0$
- (b) Verify that $y = x^3 + ax^2 + c$ is a solution of $\frac{d^3y}{dx^3} = 6$
5. The general solution of the differential equation $\frac{dy}{dx} = \sec^2 x$ is $y = \tan x + C$.
Find the particular solution when
- (a) $x = \frac{\pi}{4}, y = 1$ (b) $x = \frac{2\pi}{3}, y = 0$
6. Solve the following differential equations :
- (a) $\frac{dy}{dx} = x^5 \tan^{-1}(x^3)$ (b) $\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$
- (c) $(1+x^2)\frac{dy}{dx} = x$ (d) $\frac{dy}{dx} = x^2 + \sin 3x$
7. Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$, given that $y = 3$, when $x = 0$
8. Solve the following differential equations :
- (a) $(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0$ (b) $\frac{dy}{dx} = xy + x + y + 1$
- (c) $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$ (d) $\frac{dy}{dx} = e^{x-y} + e^{-y}x^2$
9. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$ when $x = 0$
10. Solve the following differential equations :
- (a) $(x^2 + y^2)dx - 2xydy = 0$ (b) $x\frac{dy}{dx} + \frac{y^2}{x} = y$

$$(c) \quad \frac{dy}{dx} = \frac{\sqrt{x^2 - y^2} + y}{x}$$

$$(d) \quad \frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$$



11. Solve: $\frac{dy}{dx} + y \sec x = \tan x$

12. Solve the following differential equations :

$$(a) \quad (1 + x^2) \frac{dy}{dx} + y = \tan^{-1} x$$

$$(b) \quad \cos^2 x \frac{dy}{dx} + y = \tan x$$

$$(c) \quad x \log x \frac{dy}{dx} + y = 2 \log x, x > 1$$

13. Solve the following differential equations:

$$(a) \quad (x + y + 1) \frac{dy}{dx} = 1$$

[Hint: $\frac{dx}{dy} = x + y + 1$ or $\frac{dx}{dy} - x = y + 1$ which is of the form $\frac{dx}{dy} + Px = Q$]

$$(b) \quad (x + 2y^2) \frac{dy}{dx} = y, y > 0 \quad [\text{Hint: } y \frac{dx}{dy} = x + 2y^2 \text{ or } \frac{dx}{dy} - \frac{x}{y} = 2y]$$

28.7 DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Till now we were dealing with the differential equations of first order. In this section, simple differential equations of second order and third order will be discussed.

28.7.1 Differential Equations of the Type $\frac{d^2y}{dx^2} = f(x)$

Consider the differential equation

$$\frac{d^2y}{dx^2} = f(x)$$

It may be noted that it is a differential equation of second order. So its general solution will contain two arbitrary constants.

Now we have, $\frac{d^2y}{dx^2} = f(x)$ (1)

or $\frac{d}{dx} \left(\frac{dy}{dx} \right) = f(x)$

Integrating both sides of (1), we get

$$\frac{dy}{dx} = \int f(x) dx + C_1, \quad \text{where } C_1 \text{ is an arbitrary constant}$$

Let $\int f(x) dx = \phi(x)$



Notes

Then
$$\frac{dy}{dx} = \phi(x) + C_1 \quad \dots(2)$$

Again on integrating both sides of (2), we get

$$y = \int \phi(x) dx + C_1 x + C_2,$$

where C_2 is another arbitrary constant. Therefore in order to find the particular solution we need two conditions. [See Example 28.19]

Example 28.18 Solve $\frac{d^2y}{dx^2} = xe^x$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} = xe^x \quad \dots(1)$$

Now integrating both sides of (1), we have

$$\frac{dy}{dx} = \int xe^x dx + C_1,$$

where C_1 is an arbitrary constant

or
$$\frac{dy}{dx} = xe^x - \int e^x dx + C_1$$

or
$$\frac{dy}{dx} = xe^x - e^x + C_1 \quad \dots(2)$$

Again on integrating both sides of (2), we get

$$y = \int (xe^x - e^x + C_1) dx + C_2,$$

where C_2 is another arbitrary constant.

or
$$y = xe^x - \int e^x dx - e^x + C_1 x + C_2$$

or
$$y = xe^x - 2e^x + C_1 x + C_2$$

which is the required general solution.

Example 28.19 Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} = x^2 + \sin 3x$$

for which $y(0) = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} = x^2 + \sin 3x \quad \dots(1)$$