

LOGARITHMS AND THEIR PROPERTIES

Definition of a logarithm: If $x > 0$ and b is a constant ($b \neq 1$), then $y = \log_b x$ if and only if $b^y = x$. In the equation $y = \log_b x$, y is referred to as the **logarithm**, b is the **base**, and x is the **argument**.

The notation $\log_b x$ is read "the logarithm (or log) base b of x ." The definition of a logarithm indicates that *a logarithm is an exponent*.

$y = \log_b x$ is the logarithmic form of $b^y = x$

$b^y = x$ is the exponential form of $y = \log_b x$

Examples of changes between logarithmic and exponential forms:

Write each equation in its exponential form.

a. $2 = \log_7 x$

b. $3 = \log_{10}(x + 8)$

c. $\log_5 125 = x$

Solution:

Use the definition $y = \log_b x$ if and only if $b^y = x$.

a.
$$\begin{array}{c} \text{Logarithms are exponents} \\ \overbrace{2 = \log_7 x \text{ if and only if } 7^2 = x} \\ \text{Base} \end{array}$$

b. $3 = \log_{10}(x + 8)$ if and only if $10^3 = (x + 8)$.

c. $\log_5 125 = x$ if and only if $5^x = 125$.

Write the following in its logarithmic form: $x = 25^{1/2}$

Solution:

Use $x = b^y$ if and only if $y = \log_b x$.

$$\begin{array}{c} \text{Exponent} \\ \overbrace{x = 25^{1/2} \text{ if and only if } \frac{1}{2} = \log_{25} x} \\ \text{Base} \end{array}$$

Equality of Exponents Theorem: If b is positive real number ($b \neq 1$) such that $b^x = b^y$, then $x = y$.

Example of Evaluating a Logarithmic Equation:

Evaluate: $\log_2 32 = x$

Solution:

$$\log_2 32 = x \quad \text{if and only if} \quad 2^x = 32$$

$$\text{Since } 32 = 2^5, \text{ we have} \quad 2^x = 2^5$$

Thus, by Equality of Exponents, $x = 5$

PROPERTIES OF LOGARITHMS:

If b , a , and c are positive real numbers, $b \neq 1$, and n is a real number, then:

1. Product: $\log_b(a \cdot c) = \log_b a + \log_b c$
2. Quotient: $\log_b \frac{a}{c} = \log_b a - \log_b c$
3. Power: $\log_b a^n = n \cdot \log_b a$
4. $\log_b 1 = 0$
5. $\log_b b = 1$
6. Inverse 1: $\log_b b^n = n$
7. Inverse 2: $b^{\log_b n} = n, n > 0$
8. One-to-One: $\log_b a = \log_b c$ if and only if $a = c$
9. **Change of Base:** $\log_b a = \frac{\log_c a}{\log_c b} = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$

Examples – Rewriting Logarithmic Expressions Using Logarithmic Properties:

Use the properties of logarithms to rewrite each expression as a single logarithm:

a. $2 \log_b x + \frac{1}{2} \log_b (x + 4)$

b. $4 \log_b (x + 2) - 3 \log_b (x - 5)$

Solution:

a. $2 \log_b x + \frac{1}{2} \log_b (x + 4)$

$= \log_b x^2 + \log_b (x + 4)^{1/2}$ **Power Property**

$= \log_b [x^2(x + 4)^{1/2}]$ **Product Property**

b. $4 \log_b (x + 2) - 3 \log_b (x - 5)$

$= \log_b (x + 2)^4 - \log_b (x - 5)^3$ **Power Property**

$= \log_b \frac{(x+2)^4}{(x-5)^3}$ **Quotient Property**

Use the properties of logarithms to express the following logarithms in terms of logarithms of x , y , and z .

a. $\log_b (xy^2)$

b. $\log_b \frac{x^2 \sqrt{y}}{z^5}$

Solution:

a. $\log_b (xy^2) = \log_b x + \log_b y^2$ **Product Property**

$= \log_b x + 2 \log_b y$ **Power Property**

b. $\log_b \frac{x^2 \sqrt{y}}{z^5}$

$= \log_b (x^2 \sqrt{y}) - \log_b z^5$ **Quotient Property**

$= \log_b (x^2 \sqrt{y}) - \log_b z^5$ **Quotient Property**

$= \log_b x^2 + \log_b \sqrt{y} - \log_b z^5$ **Product Property**

$= 2 \log_b x + \frac{1}{2} \log_b y - 5 \log_b z$ **Power Property**

Other Logarithmic Definitions:

• Definition of Common Logarithm:

Logarithms with a base of 10 are called **common logarithms**. It is customary to write $\log_{10} x$ as $\log x$.

• Definition of Natural Logarithm:

Logarithms with the base of e are called **natural logarithms**. It is customary to write $\log_e x$ as $\ln x$.

PRACTICE PROBLEMS

Evaluate:

1. $0.6^{\sqrt{3}}$ 2. $e^{3.2}$ 3. $(1.005)^{400}$ 4. $\log_4 64$ 5. $\ln 1$ 6. $\ln \sqrt{7}$

Rewrite into logarithms:

7. $2^4 = 16$ 8. $\sqrt{64} = 8$ 9. $e^4 = 54.60$

Evaluate without a calculator:

10. $\log_5 25$ 11. $\log_3 \frac{1}{81}$ 12. $\ln e^{-2}$

Use the change of base formula to evaluate the logarithms: (Round to 3 decimal places.)

13. $\log_7 3$ 14. $\log_2 \frac{1}{2}$ 15. $\log_{15} 42$

Use the properties of logarithms to rewrite each expression into lowest terms (i.e. expand the logarithms to a sum or a difference):

16. $\log 10x$ 19. $\log_4 4x^2$ 22. $\ln \frac{\sqrt{3x}}{7}$
17. $\ln \frac{xy}{z}$ 20. $\log_3 \sqrt{x-2}$
18. $\log_b \frac{x^4}{z^2}$ 21. $\ln \frac{x^5 z^2}{y^3}$

Write each expression as a single logarithmic quantity:

23. $\log 7 - \log x$ 26. $\log_2 5 + \log_2 x - \log_2 3$ 29. $\frac{1}{2} \log_5 7 - 2 \log_5 x$
24. $3 \ln x + 2 \ln y - 4 \ln z$ 27. $1 + 3 \log_4 x$
25. $\frac{3}{2} \ln x^6 - \frac{3}{4} \ln x^8$ 28. $2 \ln 8 + 5 \ln x$

Using properties of logarithms find the following values if:

- $\log_b 3 = 0.562$ $\log_b 2 = 0.356$ $\log_b 7 = 0.872$
30. $\log_b 18$ 31. $\log_b \sqrt{28}$ 32. $\log_b \frac{1}{21}$ 33. $\log_b 3b^2$ 34. $\log_b 1$

Write the exponential equation in logarithmic form:

35. $4^3 = 64$ 36. $25^{3/2} = 125$

Write the logarithmic equation in exponential form:

37. $\ln e = 1$ 38. $\log_3 \frac{1}{9} = -2$

Evaluate the following logarithms without a calculator:

39. $\log 1000$ 42. $\log_4 \frac{1}{16}$ 45. $\ln 1$
40. $\log_9 3$ 43. $\ln e^7$ 46. $\ln e^{-3}$
41. $\log_3 \frac{1}{9}$ 44. $\log_a \frac{1}{a}$

Evaluate the following logarithms for the given values of x :

47. $f(x) = \log_3 x$

a. $x = 1$

b. $x = 27$

c. $x = 0.5$

48. $g(x) = \log x$

a. $x = 0.01$

b. $x = 0.1$

c. $x = 30$

49. $f(x) = \ln x$

a. $x = e$

b. $x = \frac{1}{3}$

c. $x = 10$

50. $h(x) = \ln x$

a. $x = e^2$

b. $x = \frac{5}{4}$

c. $x = 1200$

51. $g(x) = \ln e^{3x}$

a. $x = -2$

b. $x = 0$

c. $x = 7.5$

52. $f(x) = \log_2 \sqrt{x}$

a. $x = 4$

b. $x = 64$

c. $x = 5.2$

Use the change of base formula to evaluate the following logarithms: (Round to 3 decimal places.)

53. $\log_4 9$

54. $\log_{1/2} 5$

55. $\log_{12} 200$

56. $\log_3 0.28$

Approximate the following logarithms given that $\log_5 2 \approx 0.43068$ and $\log_5 3 \approx 0.68261$:

57. $\log_5 18$

60. $\log_5 \frac{2}{3}$

58. $\log_5 \sqrt{6}$

61. $\log_5 (12)^{2/3}$

59. $\log_5 \frac{1}{2}$

62. $\log_5 (5^2 \cdot 6)$

Use the properties of logarithms to expand the expression:

63. $\log_4 6x^4$

66. $\ln \sqrt[3]{\frac{x}{5}}$

69. $\ln[\sqrt{2x}(x+3)^5]$

64. $\log 2x^{-3}$

67. $\ln \frac{x+2}{x-2}$

70. $\log_3 \frac{a^2 \sqrt{b}}{cd^5}$

65. $\log_5 \sqrt{x+2}$

68. $\ln x(x-3)^2$

Use the properties of logarithms to condense the expression:

71. $-\frac{2}{3} \ln 3y$

75. $-2(\ln 2x - \ln 3)$

79. $3 \ln x + 4 \ln y + \ln z$

72. $5 \log_2 y$

76. $4(1 + \ln x + \ln x)$

80. $\ln(x+4) - 3 \ln x - \ln y$

73. $\log_8 16x + \log_8 2x^2$

77. $4[\log_2 k - \log_2 (k-t)]$

74. $\log_4 6x - \log_4 10$

78. $\frac{1}{3}(\log_8 a + 2 \log_8 b)$

True or False? Use the properties of logarithms to determine whether the equation is true or false. If false, state why or give an example to show that it is false.

81. $\log_2 4x = 2 \log_2 x$

83. $\log 10^{2x} = 2x$

85. $\log_4 \frac{16}{x} = 2 - \log_4 x$

82. $\frac{\ln 5x}{\ln 10x} = \ln \frac{1}{2}$

84. $e^{\ln t} = t$

86. $6 \ln x + 6 \ln y = \ln(xy)^6$

Practice Problems Answers

Note: Remember that all variables that represent an argument of a logarithm must be greater than 0.

- | | | |
|--|-----------------------------------|--|
| 1. 0.413 | 33. 2.562 | 66. $\frac{1}{3}(\ln x - \ln 5)$ |
| 2. 24.533 | 34. 0 | 67. $\ln(x + 2) - \ln(x - 2)$ |
| 3. 7.352 | 35. $\log_4 64 = 3$ | 68. $\ln x + 2 \ln(x - 3)$ |
| 4. 3 | 36. $\log_{25} 125 = \frac{3}{2}$ | 69. $\frac{1}{2}(\ln 2 + \ln x) + 5 \ln(x + 3)$ |
| 5. 0 | 37. $e^1 = e$ | 70. $2 \log_3 a + \frac{1}{2} \log_3 b - \log_3 c -$
$5 \log_3 d$ |
| 6. 0.973 | 38. $3^{-2} = \frac{1}{9}$ | 71. $\ln\left(\frac{1}{3y}\right)^{2/3}$ |
| 7. $\log_2 16 = 4$ | 39. 3 | 72. $\log_2 y^5$ |
| 8. $\log_{64} 8 = \frac{1}{2}$ | 40. $\frac{1}{2}$ | 73. $\log_8 32x^3$ |
| 9. $\ln 54.60 = 4$ | 41. -2 | 74. $\log_4 \frac{3x}{5}$ |
| 10. 2 | 42. -2 | 75. $\ln \frac{9}{4x^2}$ |
| 11. -4 | 43. 7 | 76. $4 + \ln x^8$ |
| 12. -2 | 44. -1 | 77. $\log_2 \left(\frac{k}{k-t}\right)^4$ |
| 13. 0.565 | 45. 0 | 78. $\log_8 \sqrt[3]{ab^2}$ |
| 14. -1 | 46. -3 | 79. $\ln(x^3 y^4 z)$ |
| 15. 1.380 | 47. a. 0 b. 3 c. -0.631 | 80. $\ln \frac{x+4}{x^3 y}$ |
| 16. $1 + \log x$ | 48. a. -2 b. -1 c. 1.477 | 81. False. $\log_2 4x = 2 + \log_2 x$ |
| 17. $\ln x + \ln y - \ln z$ | 49. a. 1 b. -1.099 c. 2.303 | 82. False. $\ln \frac{5x}{10x} = \ln \frac{1}{2}$ |
| 18. $4 \log_b x - 2 \log_b z$ | 50. a. 2 b. 0.223 c. 7.090 | 83. True |
| 19. $1 + 2 \log_4 x$ | 51. a. -6 b. 0 c. 22.5 | 84. True |
| 20. $\frac{1}{2} \log_3 (x - 2)$ | 52. a. 1 b. 3 c. 1.189 | 85. True |
| 21. $(5 \ln x + 2 \ln z) - 3 \ln y$ | 53. 1.585 | 86. True |
| 22. $\frac{1}{2}(\ln 3 + \ln x) - \ln 7$ | 54. -2.322 | |
| 23. $\log \frac{7}{x}$ | 55. 2.132 | |
| 24. $\ln \frac{x^3 y^2}{z^4}$ | 56. -1.159 | |
| 25. $\ln x^3$ | 57. 1.7959 | |
| 26. $\log_2 \frac{5x}{3}$ | 58. 0.556645 | |
| 27. $\log_4 4x^3$ | 59. -0.43068 | |
| 28. $\ln 64x^5$ | 60. -0.25193 | |
| 29. $\log_5 \frac{\sqrt{7}}{x^2}$ | 61. 1.02931 | |
| 30. 1.48 | 62. 3.11329 | |
| 31. 0.792 | 63. $\log_4 6 + 4 \log_4 x$ | |
| 32. -1.434 | 64. $\log 2 - 3 \log x$ | |
| | 65. $\frac{1}{2} \log_5 (x + 2)$ | |