BASIC THEOREMS & RESULTS OF TRIANGLES :

(a) Two polygons are similar if (i) their corresponding angles are equal, (ii) the length of their corresponding sides are proportional. (Both condition are independent & necessary)

In case of a triangle, any one of the condition is sufficient, other satisfies automatically.

(b) **Thales Theorem (Basic Proportionality Theorem) :** In a triangle, a line drawn parallel to one side, to intersect the other sides in distinct points, divides the two sides in the same ratio.

Converse : If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side.

(c) **Similarity Theorem :**

(i) **AAA similarity** : If in two triangles, corresponding angles are equal i.e. two triangles are equiangular, then the triangles are similar.

(ii) **SSS similarity** : If the corresponding sides of two triangles are proportional, then they are similar.

(iii) **SAS similarity :** If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

(iv) If two triangles are similar then

(1) They are equiangular.

(2) The ratio of the corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes are same (converse also true)

(3) The ratio of the areas is equal to the ratio of the squares of corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes (converse also true)

(d) **Congruency theorem :**

Congruent triangles : Two triangles are conguent, iff one of them can be made to superpose on the other, so as to cover it exactly.

Sufficient-conditions (criteria) for congruence of triangles :

(i) **Side-Angle-Side (SAS) :** Two triangles are congruent, if two sides and the included angle of one triangle are equal to the corresponding sides and the included angle of other triangle.

(ii) **Angle-Side-Angle (ASA) :** Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

(iii) **Angle-Angle-Side(AAS) :** If any two angles and a non-included side of one triangle are equal to the corresponding to angles & the non-included side of the other triangle then the two triangles are congurent.

(iv) **Side-Side-Side (SSS) :** Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

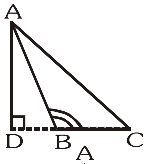
(v) **Right angle-Hypotenuse-Side(RHS) :** Two right-triangles are congruent, if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of other triangle.

(d) **Pythagoras theorem :**

(i) In a right triangle the square of hypotenuse is equal to the sum of squares of the other two sides.

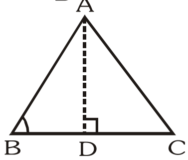
Converse : In a triangle if square of one side is equal to sum of the squares of the other two side. then the angle opposite to the side is a right angle.

(ii) In obtuse Δ :



$AC^2 = AB^2 + BC^2 + 2BC \cdot BD$ & $AC^2 > AB^2 + BC^2$

(iii) In Acute Δ :



$AC^2 = AB^2 + BC^2 - 2BC \cdot BD$ & $AC^2 < AB^2 + BC^2$

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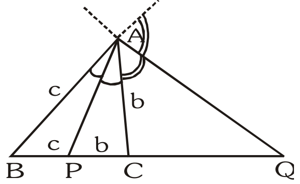
(e) In any triangle ABC, $AB^2 + AC^2 = 2(AD^2 + DC^2)$, where D is the mid point of BC

(f) The internal/external bisector of an angle of a triangle divides the opposite side internally/externally in the ratio of sides containing the angle (converse is also true). $\frac{BP}{PC} = \frac{BQ}{CQ} = \frac{AB}{AC}$



(g) The line joining the mid points of two sides of a Δ is parallel & half of the third side. (It's converse also true)

(h) (i) The diagonals of a trapezium divide each other proportionally. (converse is also true)

(ii) Any line parallel to the parallel sides of a trapezium divides the non parallel sides proportionally.

(iii) If three or more parallel lines are intersected by two transversals, then intercepts made by them on the transversals are proportional.

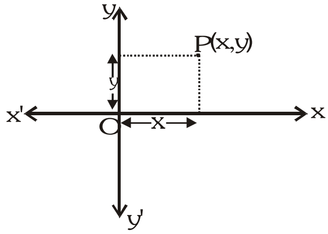
(i) In any triangle three times the sum of squares of the sides of a triangle is equal to four times the sum of the squares of its medians.

(j) The altitudes, medians and angle bisectors of a triangle are concurrent among themselves.

1. INTRODUCTION OF COORDINATE GEOMETRY :

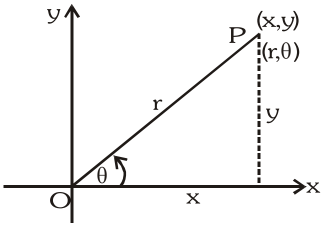
Coordinate geometry is the combination of algebra and geometry. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes. The resulting combination of analysis and geometry is referred as ***analytical geometry***.

2. CARTESIAN CO-ORDINATES SYSTEM :



In two dimensional coordinate system, two lines are used; the lines are at right angles, forming a rectangular coordinate system. The horizontal axis is the x-axis and the vertical axis is y-axis. The point of intersection O is the origin of the coordinate system. Distances along the x-axis to the right of the origin are taken as positive, distances to the left as negative. Distances along the y-axis above the origin are positive; distances below are negative. The position of a point anywhere in the plane can be specified by two numbers, the coordinates of the point, written as (x, y). The x-coordinate (or abscissa) is the distance of the point from the y-axis in a direction parallel to the x-axis (i.e. horizontally). The y-coordinate (or ordinate) is the distance from the x-axis in a direction parallel to the y-axis (vertically). The origin O is the point (0, 0).

3. POLAR CO-ORDINATES SYSTEM :



A coordinate system in which the position of a point is determined by the length of a line segment from a fixed origin together with the angle that the line segment makes with a fixed line. The origin is called the pole and the line segment is the radius vector (r).

The angle θ between the polar axis and the radius vector is called the vectorial angle. By convention, positive values of θ are measured in an anticlockwise sense, negative values in clockwise sense. The coordinates of the point are then specified as (r, θ).

If (x,y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

4. DISTANCE FORMULA AND ITS APPLICATIONS :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note :

(i) Three given points A,B and C are collinear, when sum of any two distances out of AB,BC, CA is equal to the remaining third otherwise the points will be the vertices of a triangle.

(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if AB = BC = CD = DA & AC = BD ; $AC \perp BD$

(b) Rhombus if AB = BC = CD = DA and AC $\neq$ BD ; $AC \perp BD$

(c) Parallelogram if AB = DC, BC = AD; AC $\neq$ BD ; $AC \not\perp BD$

(d) Rectangle if AB = CD, BC = DA, AC = BD ; $AC \not\perp BD$

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Illustration 1 : The number of points on x-axis which are at a distance $c(c < 3)$ from the point (2, 3) is

(A) 2 (B) 1 (C) infinite (D) no point

Solution : Let a point on x-axis is $(x_1, 0)$ then its distance from the point (2, 3)

$$= \sqrt{(x_1-2)^2+9} = c \quad \text{or} \quad (x_1-2)^2 = c^2-9$$

$$\therefore\ x_1-2 = \pm\sqrt{c^2-9} \quad \text{since } c<3 \Rightarrow c^2-9<0$$

$\therefore$ x_1 will be imaginary. **Ans. (D)**

Illustration 2 : The distance between the point $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$ is -

(A) $4a\sin\dfrac{\alpha-\beta}{2}$ (B) $2a\sin\dfrac{\alpha+\beta}{2}$ (C) $2a\sin\dfrac{\alpha-\beta}{2}$ (D) $2a\cos\dfrac{\alpha-\beta}{2}$

Solution :

$$d^2 = (a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2 = a^2(\cos\alpha-\cos\beta)^2 + a^2(\sin\alpha-\sin\beta)^2$$

$$= a^2\left\{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}\right\}^2 + a^2\left\{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right\}^2$$

$$= 4a^2\sin^2\frac{\alpha-\beta}{2}\left\{\sin^2\frac{\alpha+\beta}{2}+\cos^2\frac{\alpha+\beta}{2}\right\} = 4a^2\sin^2\frac{\alpha-\beta}{2} \Rightarrow d = 2a\sin\frac{\alpha-\beta}{2}$$

Ans. (C)

Do yourself - 1 :

(i) Find the distance between the points P(–3, 2) and Q(2, –1).

(ii) If the distance between the points P(–3, 5) and Q(–x, – 2) is $\sqrt{58}$, then find the value(s) of x.

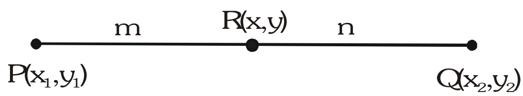
(iii) A line segment is of the length 15 units and one end is at the point (3, 2), if the abscissa of the other end is 15, then find possible ordinates.

5. SECTION FORMULA :

The co-ordinates of a point dividing a line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in the ratio m:n is given by :

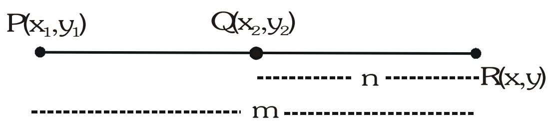
(a) **For internal division :** P - R - Q $\Rightarrow$ R divides line segment PQ, internally.

$$(x,\ y) \equiv \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$



(b) **For external division :** R - P - Q or P - Q - R $\Rightarrow$ R divides line segment PQ, externally.

$$(x,\ y) \equiv \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$



$\dfrac{(PR)}{(QR)} < 1 \Rightarrow$ R lies on the left of P & $\dfrac{(PR)}{(QR)} > 1 \Rightarrow$ R lies on the right of Q

(c) **Harmonic conjugate :** If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically ; $\dfrac{2}{AB} = \dfrac{1}{AP} + \dfrac{1}{AQ}$ i.e. AP, AB & AQ are in H.P.

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Illustration 3 : Determine the ratio in which $y - x + 2 = 0$ divides the line joining (3, –1) and (8, 9).

Solution : Suppose the line $y - x + 2 = 0$ divides the line segment joining A(3, –1) and B(8, 9) in the ratio $\lambda : 1$ at a point P, then the co-ordinates of the point P are $\left(\frac{8\lambda+3}{\lambda+1}, \frac{9\lambda-1}{\lambda+1}\right)$

But P lies on $y - x + 2 = 0$ therefore $\left(\frac{9\lambda-1}{\lambda+1}\right) - \left(\frac{8\lambda+3}{\lambda+1}\right) + 2 = 0$

$\Rightarrow \quad 9\lambda - 1 - 8\lambda - 3 + 2\lambda + 2 = 0$

$\Rightarrow \quad 3\lambda - 2 = 0 \text{ or } \lambda = \frac{2}{3}$

So, the required ratio is $\frac{2}{3} : 1$, i.e., 2 : 3 (internally) since here λ is positive.

Do yourself - 2 :

(i) Find the co-ordinates of the point dividing the join of A(1, – 2) and B(4, 7) :

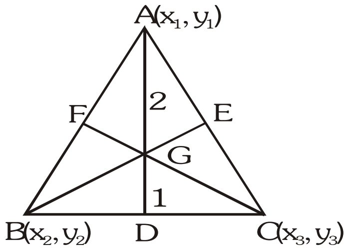
(a) Internally in the ratio 1 : 2 (b) Externally in the ratio of 2 : 1

(ii) In what ratio is the line joining A(8, 9) and B(– 7, 4) is divided by

(a) the point (2, 7) (b) the x-axis (c) the y-axis.

6. CO-ORDINATES OF SOME PARTICULAR POINTS :

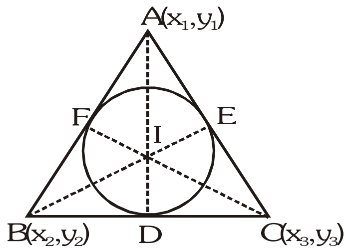
Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle ABC, then



(a) Centroid :

The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices). Centroid divides each median in the ratio of 2 : 1.

Co-ordinates of centroid $G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$



(b) Incenter :

The incenter is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of the circle touching all the sides of a triangle.

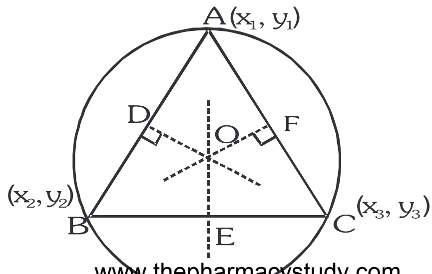
Co-ordinates of incenter $I\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

where a, b, c are the sides of triangle ABC.

Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g. $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b+c) : a, (c+a) : b, (a+b) : c$.



(c) Circumcenter :

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then $OA^2 = OB^2 = OC^2$. Also it is a centre of a circle touching all the vertices of a triangle.

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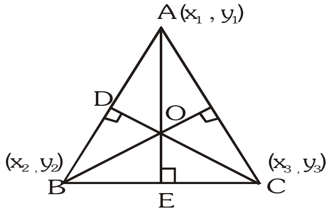
Note :

(i) If the triangle is right angled, then its circumcenter is the mid point of hypotenuse.

(ii) Co-ordinates of circumcenter $\left(\dfrac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \dfrac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$

(d) Orthocenter :

It is the point of intersection of perpendiculars drawn from vertices on the opposite sides of a triangle and it can be obtained by solving the equation of any two altitudes.



Note :

(i) If a triangle is right angled, then orthocenter is the point where right angle is formed.

(ii) Co-ordinates of circumcenter $\left(\dfrac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \dfrac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$

Remarks :

(i) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincide.

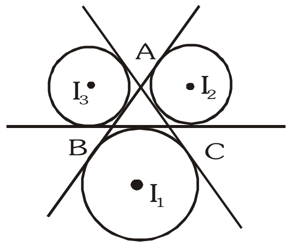
(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1

(iii) In an isosceles triangle centroid, orthocentre, incentre & circumcentre lie on the same line.

(e) Ex-centers :

The centre of a circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are

$I_1\left(\dfrac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \dfrac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$



Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and

$I_2\left(\dfrac{ax_1 - bx_2 + cx_3}{a-b+c}, \dfrac{ay_1 - by_2 + cy_3}{a-b+c}\right)$, $I_3\left(\dfrac{ax_1 + bx_2 - cx_3}{a+b-c}, \dfrac{ay_1 + by_2 - cy_3}{a+b-c}\right)$

Illustration 4 : If $\left(\dfrac{5}{3}, 3\right)$ is the centroid of a triangle and its two vertices are (0, 1) and (2, 3), then find its third vertex, circumcentre, circumradius & orthocentre.

Solution : Let the third vertex of triangle be (x, y), then

$\dfrac{5}{3} = \dfrac{x+0+2}{3} \Rightarrow x = 3$ and $3 = \dfrac{y+1+3}{3} \Rightarrow y = 5$. So third vertex is (3, 5).

Now three vertices are A(0, 1), B(2, 3) and C(3, 5)

Let circumcentre be P(h, k),

then AP = BP = CP = R (circumradius) $\Rightarrow$ $AP^2 = BP^2 = CP^2 = R^2$

$h^2 + (k-1)^2 = (h-2)^2 + (k-3)^2 = (h-3)^2 + (k-5)^2 = R^2$ .... (i)

from the first two equations, we have

$h + k = 3$ .... (ii)

from the first and third equation, we obtain

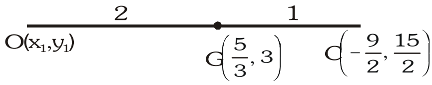
$6h + 6k = 33$ .... (iii)

On solving, (ii) & (iii), we get

$h = -\dfrac{9}{2}$, $k = \dfrac{15}{2}$

Substituting these values in (i), we have

$$R = \frac{5}{2}\sqrt{10}$$



Let $O(x_1, y_1)$ be the orthocentre, then $\dfrac{x_1 + 2\left(-\dfrac{9}{2}\right)}{3} = \dfrac{5}{3} \Rightarrow x_1 = 14$, $\dfrac{y_1 + 2\left(\dfrac{15}{2}\right)}{3} = 3$

$\Rightarrow$ $y_1 = -6$. Hence orthocentre of the triangle is (14, −6).

Illustration 5 : The vertices of a triangle are A(0, −6), B(−6, 0) and C(1,1) respectively, then coordinates of the ex-centre opposite to vertex A is :

(A) $\left(\dfrac{-3}{2}, \dfrac{-3}{2}\right)$ (B) $\left(-4, \dfrac{3}{2}\right)$ (C) $\left(\dfrac{-3}{2}, \dfrac{3}{2}\right)$ (D) (−4, 6)

Solution :

$$a = BC = \sqrt{(-6-1)^2 + (0-1)^2} = \sqrt{50} = 5\sqrt{2}$$

$$b = CA = \sqrt{(1-0)^2 + (1+6)^2} = \sqrt{50} = 5\sqrt{2}$$

$$c = AB = \sqrt{(0+6)^2 + (-6-0)^2} = \sqrt{72} = 6\sqrt{2}$$

coordinates of ex-centre opposite to vertex A will be :

$$x = \frac{-ax_1 + bx_2 + cx_3}{-a+b+c} = \frac{-5\sqrt{2}.0 + 5\sqrt{2}(-6) + 6\sqrt{2}(1)}{-5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}} = \frac{-24\sqrt{2}}{6\sqrt{2}} = -4$$

$$y = \frac{-ay_1 + by_2 + cy_3}{-a+b+c} = \frac{-5\sqrt{2}(-6) + 5\sqrt{2}.0 + 6\sqrt{2}(1)}{-5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}} = \frac{36\sqrt{2}}{6\sqrt{2}} = 6$$

Hence coordinates of ex-centre is (−4, 6) **Ans. (D)**

Do yourself - 3 :

(i) The coordinates of the vertices of a triangle are (0, 1), (2, 3) and (3, 5) :

(a) Find centroid of the triangle.

(b) Find circumcentre & the circumradius.

(c) Find Orthocentre of the triangle.

7. AREA OF TRIANGLE :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left| \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right| = \frac{1}{2}\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$$

To remember the above formula, take the help of the following method :

$$= \frac{1}{2}\begin{bmatrix} x_1 & & x_2 & & x_3 & & x_1 \\ & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_1 \end{bmatrix} = \left|\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) **Area of Equilateral triangle :** If altitude of any equilateral triangle is P, then its area = $\dfrac{P^2}{\sqrt{3}}$. If 'a' be the side of equilateral triangle, then its area = $\left(\dfrac{a^2\sqrt{3}}{4}\right)$.

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(iii) Area of quadrilateral with given vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$

Area of quad. ABCD $= \frac{1}{2}\left|\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{bmatrix}\right|$

Note : Area of a polygon can be obtained by dividing the polygon into disjoined triangles and then adding their areas.

Illustration 6 : If the vertices of a triangle are (1, 2), (4, –6) and (3, 5) then its area is

(A) $\frac{25}{2}$ sq. units (B) 12 sq. units (C) 5 sq. units (D) 25 sq. units

Solution : $\Delta = \frac{1}{2}\left[1(-6-5)+4(5-2)+3(2+6)\right] = \frac{1}{2}\left[-11+12+24\right] = \frac{25}{2}$ square units **Ans. (A)**

Illustration 7 : The point A divides the join of the points (–5, 1) and (3, 5) in the ratio k:1 and coordinates of points B and C are (1, 5) and (7, –2) respectively. If the area of ΔABC be 2 units, then k equals -

(A) 7, 9 (B) 6, 7 (C) 7, $\frac{31}{9}$ (D) 9, $\frac{31}{9}$

Solution : $A \equiv \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$

Area of $\Delta ABC = 2$ units $\Rightarrow \frac{1}{2}\left[\frac{3k-5}{k+1}(5+2)+1\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right] = \pm 2$

$\Rightarrow 14k - 66 = \pm 4(k+1)$ $\Rightarrow k = 7$ or $\frac{31}{9}$ **Ans. (C)**

Illustration 8 : Prove that the co-ordinates of the vertices of an equilateral triangle can not all be rational.

Solution : Let $A(x_1\ y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible let $x_1, y_1, x_2, y_2, x_3, y_3$ be all rational.

Now area of $\Delta ABC = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right| =$ Rational .... (i)

Since ΔABC is equilateral

$\therefore$ Area of $\Delta ABC = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(AB)^2 = \frac{\sqrt{3}}{4}\{(x_1-x_2)^2+(y_1-y_2)^2\} =$ Irrational .... (ii)

From (i) and (ii),

Rational = Irrational

which is contradiction.

Hence $x_1, y_1, x_2, y_2, x_3, y_3$ cannot all be rational.

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8. CONDITIONS FOR COLLINEARITY OF THREE GIVEN POINTS :

Three given points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear if any one of the following conditions are satisfied.

(a) Area of triangle ABC is zero i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(b) Slope of AB = slope of BC = slope of AC. i.e. $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$

(c) Find the equation of line passing through 2 given points, if the third point satisfies the given equation of the line, then three points are collinear.

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Do yourself - 4 :

(i) Find the area of the triangle whose vertices are A(1,1), B(7, – 3) and C(12, 2)

(ii) Find the area of the quadrilateral whose vertices are A(1,1) B(7, – 3), C(12,2) and D(7, 21)

(iii) Prove that the points A(a, b + c), B(b, c + a) and C(c, a + b) are collinear (By determinant method)

(iv) Prove that the points (–1, –1), (2, 3) and (8, 11) are collinear.

(v) Find the value of x so that the points (x, –1), (2, 1) and (4, 5) are collinear.

9. LOCUS :

The locus of a moving point is the path traced out by that point under one or more geometrical conditions.

(a) Equation of Locus :

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

(b) Procedure for finding the equation of the locus of a point :

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given condition as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

(iii) Eliminate the parameters, so that the eliminant contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the eliminant. The resulting equation would be the equation of the locus of P.

Illustration 9 : The ends of the rod of length ℓ moves on two mutually perpendicular lines, find the locus of the point on the rod which divides it in the ratio $m_1 : m_2$

(A) $m_1^2x^2 + m_2^2y^2 = \dfrac{\ell^2}{(m_1+m_2)^2}$ (B) $(m_2x)^2 + (m_1y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$

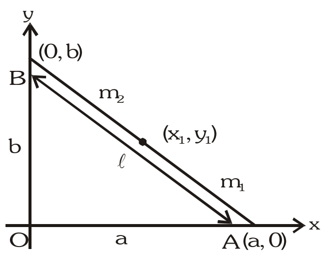
(C) $(m_1x)^2 + (m_2y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$ (D) none of these

Solution : Let (x_1, y_1) be the point that divide the rod $AB = \ell$, in the ratio $m_1 : m_2$, and $OA = a$, $OB = b$ say

$\therefore \quad a^2 + b^2 = \ell^2$ ..... (i)

Now $x_1 = \left(\dfrac{m_2a}{m_1+m_2}\right) \Rightarrow a = \left(\dfrac{m_1+m_2}{m_2}\right)x_1$

$y_1 = \left(\dfrac{m_1b}{m_1+m_2}\right) \Rightarrow b = \left(\dfrac{m_1+m_2}{m_1}\right)y_1$



putting these values in (i) $\dfrac{(m_1+m_2)^2}{m_2^2}x_1^2 + \dfrac{(m_1+m_2)^2}{m_1^2}y_1^2 = \ell^2$

$\therefore$ Locus of (x_1, y_1) is $m_1^2x^2 + m_2^2y^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$ **Ans. (C)**

Illustration 10 : A(a, 0) and B(–a, 0) are two fixed points of ΔABC. If its vertex C moves in such a way that cotA + cotB = λ, where λ is a constant, then the locus of the point C is -

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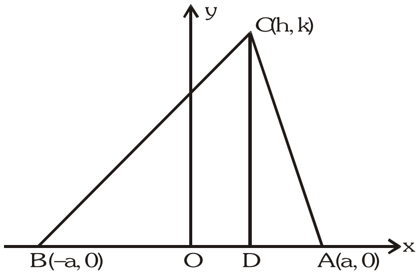
(A) $y\lambda = 2a$ (B) $y = \lambda a$ (C) $ya = 2\lambda$ (D) none of these

Solution : Given that coordinates of two fixed points A and B are (a, 0) and (–a, 0) respectively. Let variable point C is (h, k). From the adjoining figure



$$\cot A = \frac{DA}{CD} = \frac{a-h}{k}$$

$$\cot B = \frac{BD}{CD} = \frac{a+h}{k}$$

But cotA + cotB = λ, so we have

$$\frac{a-h}{k} + \frac{a+h}{k} = \lambda \Rightarrow \frac{2a}{k} = \lambda$$

Hence locus of C is $y\lambda = 2a$ **Ans. (A)**

Do yourself - 5 :

(i) Find the locus of a variable point which is at a distance of 2 units from the y-axis.

(ii) Find the locus of a point which is equidistant from both the axes.

(iii) Find the locus of a point whose co-ordinates are given by $x = at^2$, $y = 2at$, where 't' is a parameter.

10. STRAIGHT LINE :

Introduction : A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, remember that every one degree equation in variable x and y always represents a straight line i.e. $\mathbf{ax + by + c = 0}$ **;** $\mathbf{a \ \& \ b \neq 0}$ simultaneously.

(a) Equation of a line parallel to x-axis at a distance 'a' is $\mathbf{y = a}$ or $\mathbf{y = -a}$

(b) Equation of x-axis is $\mathbf{y = 0}$

(c) Equation of a line parallel to y-axis at a distance 'b' is $\mathbf{x = b}$ or $\mathbf{x = -b}$

(d) Equation of y-axis is $\mathbf{x = 0}$

Illustration 11 : Prove that every first degree equation in x, y represents a straight line.

Solution : Let ax + by + c = 0 be a first degree equation in x, y

where a, b, c are constants.

Let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be any two points on the curve represented by ax + by + c = 0. Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$

Let R be any point on the line segment joining P & Q

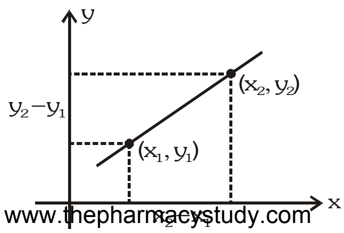
Suppose R divides PQ in the ratio λ : 1. Then, the coordinates of R are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

We have $a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c = \lambda\ 0 + 0 = 0$

$\therefore$ $R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ lies on the curve represented by ax + by + c = 0. Thus every point on the line segment joining P & Q lies on ax + by + c = 0.

Hence ax + by + c = 0 represents a straight line.

11. SLOPE OF LINE :



If a given line makes an angle $\theta(0 \le \theta < 180, \theta \neq 90)$ with the positive direction of x-axis, then slope of this line will be $\tan\theta$ and is usually denoted by the letter **m** i.e. $\mathbf{m = \tan\theta}$. If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line

$$AB = \frac{y_2 - y_1}{x_2 - x_1}$$

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Remark :

(i) If $\theta = 90$, **m does not exist** and line is parallel to **y-axis.**

(ii) If $\theta = 0$, **m = 0** and the line is parallel to **x-axis.**

(iii) Let m_1 and m_2 be slopes of two given lines (none of them is parallel to y-axis)

(a) If lines are parallel, $\mathbf{m_1 = m_2}$ and vice-versa.

(b) If lines are perpendicular, $\mathbf{m_1 m_2 = -1}$ and vice-versa

12. STANDARD FORMS OF EQUATIONS OF A STRAIGHT LINE :

(a) **Slope Intercept form** : Let m be the slope of a line and c its intercept on y-axis. Then the equation of this straight line is written as : $y = mx + c$

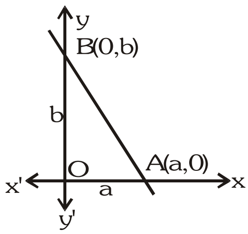
If the line passes through origin, its equation is written as $y = mx$

(b) **Point Slope form** : If m be the slope of a line and it passes through a point (x_1,y_1), then its equation is written as : $y - y_1 = m(x - x_1)$

(c) **Two point form** : Equation of a line passing through two points (x_1,y_1) and (x_2,y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{or} \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(d) **Intercept form** : If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\frac{x}{a} + \frac{y}{b} = 1$



(i) Length of intercept of line between the coordinate axes $= \sqrt{a^2 + b^2}$

(ii) Area of triangle AOB $= \frac{1}{2}OA.OB = \left|\frac{1}{2}ab\right|$

Illustration 12 : The equation of the lines which passes through the point (3, 4) and the sum of its intercepts on the axes is 14 is -

(A) $4x - 3y = 24$, $x - y = 7$
(B) $4x + 3y = 24$, $x + y = 7$
(C) $4x + 3y + 24 = 0$, $x + y + 7=0$
(D) $4x - 3y + 24 = 0$, $x - y + 7 = 0$

Solution : Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ .....(i)

This passes through (3 , 4) , therefore $\frac{3}{a} + \frac{4}{b} = 1$ .....(ii)

It is given that $a + b = 14 \Rightarrow b = 14 - a$. Putting $b = 14 - a$ in (ii) , we get

$\frac{3}{a} + \frac{4}{14 - a} = 1 \Rightarrow a^2 - 13a + 42 = 0 \quad \Rightarrow (a - 7)(a - 6) = 0 \Rightarrow a = 7 , 6$

For $a = 7$, $b = 14 - 7 = 7$ and for $a = 6$, $b = 14 - 6 = 8$

Putting the values of a and b in (i) , we get the equations of the lines

$\frac{x}{7} + \frac{y}{7} = 1$ and $\frac{x}{6} + \frac{y}{8} = 1$ or $x + y = 7$ and $4x + 3y = 24$ **Ans. (B)**

Illustration 13 : Two points A and B move on the positive direction of x-axis and y-axis respectively, such that $OA + OB = K$. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x + y)(x^2 + y^2) = Kxy$.

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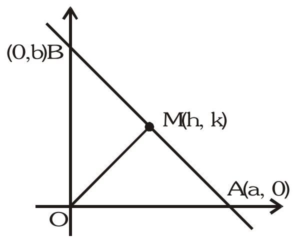
Solution : Let the equation of AB be $\frac{x}{a}+\frac{y}{b}=1$ .... (i)



given, a + b = K .... (ii)

now, $m_{AB}\ m_{OM} = -1 \Rightarrow ah = bk$ .... (iii)

from (ii) and (iii),

$a = \frac{kK}{h+k}$ and $b = \frac{hK}{h+k}$

$\therefore$ from (i) $\frac{x(h+k)}{k.K}+\frac{y(h+k)}{h.K}=1$

as it passes through (h, k)

$\frac{h(h+k)}{k.K}+\frac{k(h+k)}{h.K}=1 \Rightarrow (h+k)(h^2+k^2) = Khk$

$\therefore$ locus of (h, k) is $(x+y)(x^2+y^2) = Kxy$.

(e) **Normal form :** If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as : $\mathbf{x\cos\alpha + y\sin\alpha = p}$ (p is always positive) where $\mathbf{0 \le \alpha < 2\pi}$.

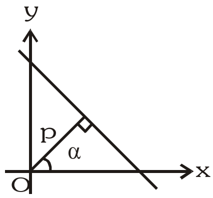
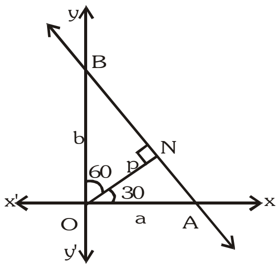


Illustration 14 : Find the equation of the straight line on which the perpendicular from origin makes an angle 30 with positive x-axis and which forms a triangle of area $\left(\frac{50}{\sqrt{3}}\right)$ sq. units with the co-ordinates axes.

Solution : $\angle NOA = 30$

Let ON = p > 0, OA = a, OB = b



In ΔONA, $\cos30 = \frac{ON}{OA} = \frac{p}{a} \Rightarrow \frac{\sqrt{3}}{2} = \frac{p}{a}$

or $a = \frac{2p}{\sqrt{3}}$

and in ΔONB, $\cos60 = \frac{ON}{OB} = \frac{p}{b} \Rightarrow \frac{1}{2} = \frac{p}{b}$

or $b = 2p$

$\because$ Area of $\Delta OAB = \frac{1}{2}\ ab = \frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)(2p) = \frac{2p^2}{\sqrt{3}}$

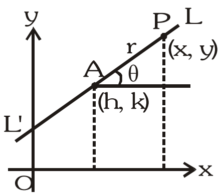
$\therefore \frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \Rightarrow p^2 = 25$

or $p = 5$

$\therefore$ Using $x\cos\alpha + y\sin\alpha = p$, the equation of the line AB is $x\cos30 + y\sin30 = 5$

or $x\sqrt{3} + y = 10$

(f) **Parametric form** : To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'.



Let AP = r, then $\mathbf{x - h = r\cos\theta}$, $\mathbf{y - k = r\sin\theta}$ & $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = \mathbf{r}$ is the equation of the straight line LAL'.

Any point P on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

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Illustration 15 : Equation of a line which passes through point A(2, 3) and makes an angle of 45 with x axis. If this line meet the line x + y + 1 = 0 at point P then distance AP is -

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{5}$

Solution : Here $x_1 = 2$, $y_1 = 3$ and $\theta = 45$ hence $\dfrac{x-2}{\cos 45^\circ} = \dfrac{y-3}{\sin 45^\circ} = r$

from first two parts $\Rightarrow$ $x - 2 = y - 3 \Rightarrow x - y + 1 = 0$

Co-ordinate of point P on this line is $\left(2+\dfrac{r}{\sqrt{2}}, 3+\dfrac{r}{\sqrt{2}}\right)$.

If this point is on line x + y + 1 = 0 then

$\left(2+\dfrac{r}{\sqrt{2}}\right) + \left(3+\dfrac{r}{\sqrt{2}}\right) + 1 = 0 \Rightarrow r = -3\sqrt{2}$; $|r| = 3\sqrt{2}$ **Ans. (B)**

Illustration 16 : A variable line is drawn through O, to cut two fixed straight lines L_1 and L_2 in A_1 and A_2, respectively.

A point A is taken on the variable line such that $\dfrac{m+n}{OA} = \dfrac{m}{OA_1} + \dfrac{n}{OA_2}$.

Show that the locus of A is a straight line passing through the point of intersection of L_1 and L_2 where O is being the origin.

Solution : Let the variable line passing through the origin is $\dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta} = r_i$ ...... (i)

Let the equation of the line L_1 is $p_1x + q_1y = 1$ ...... (ii)

Equation of the line L_2 is $p_2x + q_2y = 1$ ...... (iii)

the variable line intersects the line (ii) at A_1 and (iii) at A_2.

Let $OA_1 = r_1$.

Then $A_1 = (r_1\cos\theta, r_1\sin\theta) \Rightarrow A_1$ lies on L_1

$\Rightarrow \quad r_1 = OA_1 = \dfrac{1}{p_1\cos\theta + q_1\sin\theta}$

Similarly, $r_2 = OA_2 = \dfrac{1}{p_2\cos\theta + q_2\sin\theta}$

Let OA = r

Let co-ordinate of A are (h, k) $\Rightarrow$ $(h, k) \equiv (r\cos\theta, r\sin\theta)$

Now $\dfrac{m+n}{r} = \dfrac{m}{OA_1} + \dfrac{n}{OA_2} \Rightarrow \dfrac{m+n}{r} = \dfrac{m}{r_1} + \dfrac{n}{r_2}$

$\Rightarrow \quad m + n = m(p_1r\cos\theta + q_1r\sin\theta) + n(p_2r\cos\theta + q_2r\sin\theta)$

$\Rightarrow \quad (p_1h + q_1k - 1) + \dfrac{n}{m}(p_2h + q_2k - 1) = 0$

Therefore, locus of A is $(p_1x+q_1y - 1) + \dfrac{n}{m}(p_2x + q_2y - 1) = 0$

$\Rightarrow \quad L_1 + \lambda L_2 = 0$ where $\lambda = \dfrac{n}{m}$.

This is the equation of the line passing through the intersection of L_1 and L_2.

Illustration 17 : A straight line through P(–2, –3) cuts the pair of straight lines $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ in Q and R. Find the equation of the line if PQ. PR = 20.

Solution : Let line be $\dfrac{x+2}{\cos\theta} = \dfrac{y+3}{\sin\theta} = r$

$\Rightarrow \quad x = r\cos\theta - 2, y = r\sin\theta - 3$ .... (i)

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Now, $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ .... (ii)

Taking intersection of (i) with (ii) and considering terms of r^2 and constant (as we need PQ . PR = $r_1 . r_2$ = product of the roots)

$r^2(\cos^2\theta + 3\sin^2\theta + 4\sin\theta\cos\theta)$ + (some terms)$r + 80 = 0$

$\therefore \quad r_1.r_2 = PQ.\ PR = \dfrac{80}{\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta}$

$\therefore \quad \cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta = 4$ $\quad (\because\ PQ\ .\ PR = 20)$

$\therefore \quad \sin^2\theta - 4\sin\theta\cos\theta + 3\cos^2\theta = 0$

$\Rightarrow \quad (\sin\theta - \cos\theta)(\sin\theta - 3\cos\theta) = 0$

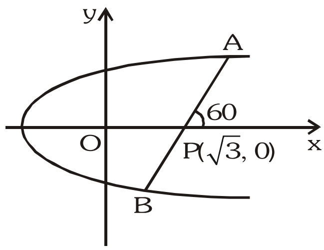
$\therefore \quad \tan\theta = 1,\ \tan\theta = 3$

hence equation of the line is $y + 3 = 1(x + 2) \Rightarrow x - y = 1$

and $y + 3 = 3(x + 2) \Rightarrow 3x - y + 3 = 0$.

Illustration 18 : If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then find the value of PA.PB {where $P \equiv (\sqrt{3}, 0)$}

Solution : Slope of line $y - \sqrt{3}x + 3 = 0$ is $\sqrt{3}$



If line makes an angle θ with x-axis, then $\tan\theta = \sqrt{3}$

$\therefore \quad \theta = 60°$

$\dfrac{x - \sqrt{3}}{\cos 60°} = \dfrac{y - 0}{\sin 60°} = r \Rightarrow \left(\sqrt{3} + \dfrac{r}{2}, \dfrac{r\sqrt{3}}{2}\right)$

be a point on the parabola $y^2 = x + 2$

then $\dfrac{3}{4}r^2 = \sqrt{3} + \dfrac{r}{2} + 2 \Rightarrow 3r^2 - 2r - 4(2 + \sqrt{3}) = 0$

$\therefore \quad PA.PB = r_1r_2 = \left|\dfrac{-4(2+\sqrt{3})}{3}\right| = \dfrac{4(2+\sqrt{3})}{3}$

Do yourself - 6 :

(i) Reduce the line $2x - 3y + 5 = 0$,

(a) In slope- intercept form and hence find slope & Y-intercept

(b) In intercept form and hence find intercepts on the axes.

(c) In normal form and hence find perpendicular distance from the origin and angle made by the perpendicular with the positive x-axis.

(ii) Find distance of point A (2, 3) measured parallel to the line $x - y = 5$ from the line $2x + y + 6 = 0$.

(g) General form : We know that a first degree equation in x and y, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \dfrac{-a}{b} = -\dfrac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $= -\dfrac{c}{a}$ and intercept by this line on y-axis $= -\dfrac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2 + b^2}$.

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13. ANGLE BETWEEN TWO LINES :

(a) If θ be the angle between two lines : $y = m_1x + c$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

Note :

(i) There are two angles formed between two lines but usually the acute angle is taken as the angle between the lines. So we shall find θ from the above formula only by taking positive value of $\tan\theta$.

(ii) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the Δ ABC found by these formulas are given by,

$$\tan A = \frac{m_1 - m_2}{1 + m_1 m_2} \; ; \; \tan B = \frac{m_2 - m_3}{1 + m_2 m_3} \quad \& \quad \tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$$

(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these line are -

(i) Parallel $\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow$ $a_1a_2+b_1b_2=0$

(iii) Coincident $\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Illustration 19 : If $x + 4y - 5 = 0$ and $4x + ky + 7 = 0$ are two perpendicular lines then k is -

(A) 3 (B) 4 (C) –1 (D) –4

Solution : $m_1 = -\frac{1}{4}$ $m_2 = -\frac{4}{k}$

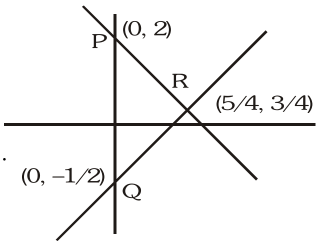
Two lines are perpendicular if $m_1 m_2 = -1$

$\Rightarrow \left(-\frac{1}{4}\right) \times \left(-\frac{4}{k}\right) = -1 \Rightarrow k = -1$ Ans. (C)

Illustration 20 : A line L passes through the points (1, 1) and (0, 2) and another line M which is perpendicular to L passes through the point (0, –1/2).The area of the triangle formed by these lines with y-axis is -

(A) 25/8 (B) 25/16 (C) 25/4 (D) 25/32

Solution : Equation of the line L is $y - 1 = \frac{-1}{1}(x - 1) \Rightarrow y = -x + 2$



Equation of the line M is $y = x - 1/2$.

If these lines meet y-axis at P (0, –1/2) and Q (0, 2) then PQ = 5/2.

Also x-coordinate of their point of intersection R = 5/4

$\therefore$ area of the $\Delta PQR = \frac{1}{2}\left(\frac{5}{2} \times \frac{5}{4}\right) = 25/16.$ Ans. (B)

Illustration 21 : If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y-axis, then the value of k is -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : A straight line is parallel to y-axis, if its y - coefficient is zero, i.e. $4 - k = 0$ i.e. $k = 4$ Ans. (D)

14. EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE :

(a) Equation of line parallel to line $ax + by + c = 0$

$ax + by + \lambda = 0$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$bx - ay + k = 0$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

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15. STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE :

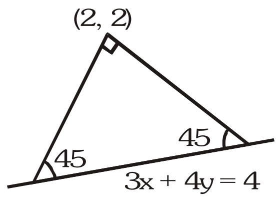
Equations of lines passing through a point (x_1,y_1) and making an angle α, with the line $y=mx+c$ is written as :

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Illustration 22 : Find the equation to the sides of an isosceles right-angled triangle, the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2).

Solution : The problem can be restated as :

Find the equations of the straight lines passing through the given point (2, 2) and making equal angles of 45 with the given straight line $3x + 4y - 4 = 0$. Slope of the line $3x + 4y - 4 = 0$ is $m_1 = -3/4$.



$\Rightarrow \quad \tan 45 = \pm\dfrac{m - m_1}{1 + m_1 m}$, i.e., $1 = \pm\dfrac{m + 3/4}{1 - \frac{3}{4}m}$

$m_A = \dfrac{1}{7}$, and $m_B = -7$

Hence the required equations of the two lines are

$y - 2 = m_A(x - 2)$ and $y - 2 = m_B(x - 2)$

$\Rightarrow \quad 7y - x - 12 = 0$ and $7x + y = 16$ **Ans.**

Do yourself - 7 :

(i) Find the angle between the lines $3x + y - 7 = 0$ and $x + 2y - 9 = 0$.

(ii) Find the line passing through the point (2, 3) and perpendicular to the straight line $4x - 3y = 10$.

(iii) Find the equation of the line which has positive y-intercept 4 units and is parallel to the line $2x - 3y - 7 = 0$. Also find the point where it cuts the x-axis.

(iv) Classify the following pairs of lines as coincident, parallel or intersecting :

(a) $x + 2y - 3 = 0$ & $-3x - 6y + 9 = 0$

(b) $x + 2y + 1 = 0$ & $2x + 4y + 3 = 0$

(c) $3x - 2y + 5 = 0$ & $2x + y - 5 = 0$

16. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

Length of perpendicular from a point (x_1,y_1) on the line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

In particular, the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \dfrac{|c|}{\sqrt{a^2 + b^2}}$

Illustration 23 : If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Solution : Let n given points be (x_i, y_i) where i = 1, 2.... n and the variable straight line is $ax + by + c = 0$.

Given that $\displaystyle\sum_{i=1}^{n}\left(\frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}}\right) = 0 \quad \Rightarrow \quad a\Sigma x_i + b\Sigma y_i + cn = 0 \ \Rightarrow\ a\frac{\Sigma x_i}{n} + b\frac{\Sigma y_i}{n} + c = 0$.

Hence the variable straight line always passes through the fixed point $\left(\dfrac{\Sigma x_i}{n}, \dfrac{\Sigma y_i}{n}\right)$. **Ans.**

Illustration 24 : Prove that no line can be drawn through the point (4, –5) so that its distance from (–2, 3) will be equal to 12.

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Solution : Suppose, if possible.

Equation of line through (4, –5) with slope of m is

$$y + 5 = m(x - 4)$$

$$\Rightarrow \quad mx - y - 4m - 5 = 0$$

Then $$\frac{|m(-2) - 3 - 4m - 5|}{\sqrt{m^2+1}} = 12$$

$$\Rightarrow \quad |-6m - 8| = 12\sqrt{(m^2+1)}$$

On squaring, $$(6m + 8)^2 = 144(m^2 + 1)$$

$$\Rightarrow \quad 4(3m + 4)^2 = 144(m^2 + 1)$$

$$\Rightarrow \quad (3m + 4)^2 = 36(m^2 + 1)$$

$$\Rightarrow \quad 27m^2 - 24m + 20 = 0 \quad \text{......... (i)}$$

Since the discriminant of (i) is $(-24)^2 - 4.27.20 = -1584$ which is negative, there is no real value of m. Hence no such line is possible.

17. DISTANCE BETWEEN TWO PARALLEL LINES :

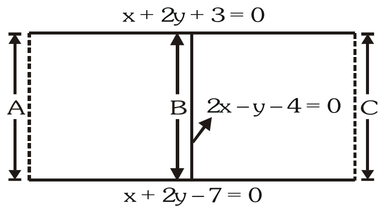
(a) The distance between two parallel lines $ax + by + c_1 = 0$ and $ax+by+c_2=0$ is $= \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$

(**Note** : The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram = $\frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides . Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

Illustration 25 : Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

Solution : Distance between the two parallel lines is $\frac{|7+3|}{\sqrt{5}} = 2\sqrt{5}$.



The equations of sides A and C are of the form $2x - y + k = 0$.

Since distance between sides A and B = distance between sides B and C $\frac{|k-(-4)|}{\sqrt{5}} = 2\sqrt{5} \Rightarrow \frac{k+4}{\sqrt{5}} = \pm 2\sqrt{5} \Rightarrow k = 6, -14.$

Hence the fourth sides of the two squares are (i) $2x - y + 6 = 0$ (ii) $2x - y - 14 = 0$. **Ans.**

Do yourself - 8 :

(i) Find the distances between the following pair of parallel lines :

(a) $3x + 4y = 13$, $3x + 4y = 3$

(b) $3x - 4y + 9 = 0$, $6x - 8y - 15 = 0$

(ii) Find the points on the x-axis such that their perpendicular distance from the line $\frac{x}{a} + \frac{y}{b} = 1$ is 'a', a, b > 0.

(iii) Show that the area of the parallelogram formed by the lines

$2x - 3y + a = 0$, $3x - 2y - a = 0$, $2x - 3y + 3a = 0$ and $3x - 2y - 2a = 0$ is $\frac{2a^2}{5}$ square units.

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18. POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE :

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

Illustration 26 : Let P $(\sin\theta, \cos\theta)$ $(0 \le \theta \le 2\pi)$ be a point and let OAB be a triangle with vertices (0, 0), $\left(\sqrt{\frac{3}{2}}, 0\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$. Find θ if P lies inside the ΔOAB.

Solution : Equations of lines along OA, OB and AB are $y = 0$, $x = 0$ and $x + y = \sqrt{\frac{3}{2}}$ respectively. Now P and B will lie on the same side of $y = 0$ if $\cos\theta > 0$. Similarly P and A will lie on the same side of $x = 0$ if $\sin\theta > 0$ and P and O will lie on the same side of $x + y = \sqrt{\frac{3}{2}}$ if $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$.

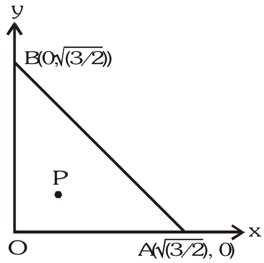
Hence P will lie inside the ΔABC, if $\sin\theta > 0$, $\cos\theta > 0$ and $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$.

Now $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$

$\Rightarrow \quad \sin(\theta + \frac{\pi}{4}) < \frac{\sqrt{3}}{2}$

i.e. $0 < \theta + \frac{\pi}{4} < \pi/3$

or $\frac{2\pi}{3} < \theta + \frac{\pi}{4} < \pi$



Since $\sin\theta > 0$ and $\cos\theta > 0$, so $0 < \theta < \frac{\pi}{12}$ or $\frac{5\pi}{12} < \theta < \frac{3\pi}{4}$.

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19. CONCURRENCY OF LINES :

(a) Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix} = 0$

(b) To test the concurrency of three lines, first find out the point of intersection of any two of the three lines. If this point lies on the remaining line (i.e. coordinates of the point satisfy the equation of the line) then the three lines are concurrent otherwise not concurrent.

Illustration 27 : If the lines $ax + by + p = 0$, $x\cos\alpha + y\sin\alpha - p = 0$ $(p \ne 0)$ and $x\sin\alpha - y\cos\alpha = 0$ are concurrent and the first two lines include an angle $\frac{\pi}{4}$, then $a^2 + b^2$ is equal to -

(A) 1 (B) 2 (C) 4 (D) p^2

Solution : Since the given lines are concurrent,

$$\begin{vmatrix} a & b & p \\ \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad a\cos\alpha + b\sin\alpha + 1 = 0$ .... (i)

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As $ax + by + p = 0$ and $x\cos\alpha + y\sin\alpha - p = 0$ include an angle $\frac{\pi}{4}$.

$$\pm\tan\frac{\pi}{4} = \frac{-\frac{a}{b}+\frac{\cos\alpha}{\sin\alpha}}{1+\frac{a}{b}\frac{\cos\alpha}{\sin\alpha}}$$

$\Rightarrow$ $-a\sin\alpha + b\cos\alpha = \pm(b\sin\alpha + a\cos\alpha)$

$\Rightarrow$ $-a\sin\alpha + b\cos\alpha = \pm 1$ [from (i)] .... (ii)

Squaring and adding (i) & (ii), we get

$a^2 + b^2 = 2$. **Ans. (B)**

Do yourself - 9 :

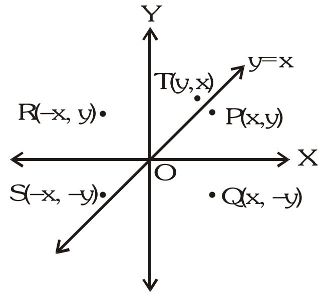
(i) Examine the positions of the points (3, 4) and (2, – 6) w.r.t. $3x - 4y = 8$

(ii) If (2, 9), (– 2, 1) and (1, –3) are the vertices of a triangle, then prove that the origin lies inside the triangle.

(iii) Find the equation of the line joining the point (2, – 9) and the point of intersection of lines $2x + 5y - 8 = 0$ and $3x - 4y - 35 = 0$.

(iv) Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33 = 0$ and $2x - 3y + \lambda = 0$ are concurrent.

20. REFLECTION OF A POINT :



Let P(x, y) be any point, then its image with respect to

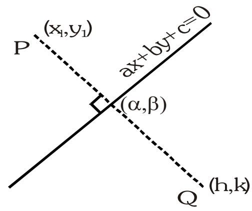
(a) x-axis is Q(x, −y)

(b) y-axis is R(−x, y)

(c) origin is S(−x,−y)

(d) line y = x is T(y, x)

(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1, y_1)$ about the line $ax + by + c = 0$ is given by following formula.



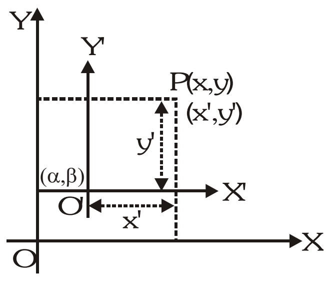
$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line $ax + by + c = 0$ is given by following formula.

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

21. TRANSFORMATION OF AXES

(a) Shifting of origin without rotation of axes :



Let P (x, y) with respect to axes OX and OY.

Let O' (α, β) is new origin with respect to axes OX and OY and let P (x', y') with respect to axes O'X' and O'Y' , where OX and O'X' are parallel and OY and O'Y' are parallel.

Then $x = x' + \alpha$, $y = y' + \beta$

or $x' = x - \alpha$, $y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

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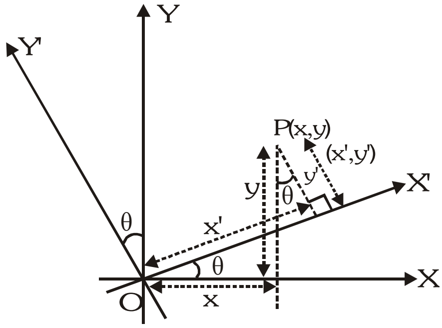
(b) Rotation of axes without shifting the origin :

Let O be the origin. Let P (x, y) with respect to axes OX and OY and let P (x', y') with respect to axes OX' and OY' where $\angle X'OX = \angle YOY' = \theta$, where θ is measured in anticlockwise direction.



then $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

Illustration 28 : Through what angle should the axes be rotated so that the equation $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$?

Solution : Let angle be θ then replacing (x, y) by ($x\cos\theta - y\sin\theta$, $x\sin\theta + y\cos\theta$)

then $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ becomes

$$9(x\cos\theta - y\sin\theta)^2 - 2\sqrt{3}\left(x\cos\theta - y\sin\theta\right)\left(x\sin\theta + y\cos\theta\right) + 7(x\sin\theta + y\cos\theta)^2 = 10$$

$$\Rightarrow \quad x^2(9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta) + 2xy(-9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta)$$

$$+ y^2(9\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta) = 10$$

On comparing with $3x^2 + 5y^2 = 5$ (coefficient of xy = 0)

We get $\quad -9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta = 0$

or $\quad \sin2\theta = -\sqrt{3}\cos2\theta \quad$ or $\quad \tan2\theta = -\sqrt{3} = \tan(180 - 60)$

or $\quad 2\theta = 120 \quad \therefore \quad \theta = 60$

Do yourself - 10 :

(i) The point (4, 1) undergoes the following transformations, then the match the correct alternatives :

	Column-I		**Column-II**
(A)	Reflection about x-axis is	(p)	(4, –1)
(B)	Reflection about y-axis is	(q)	(–4, –1)
(C)	Reflection about origin is	(r)	$\left(-\frac{12}{25}, -\frac{59}{25}\right)$
(D)	Reflection about the line y = x is	(s)	(–4, 1)
(E)	Reflection about the line 4x + 3y – 5 = 0 is	(t)	(1, 4)

(ii) On what point must the origin be shifted, if the coordinates of a point (4, 5) become (– 3, 9).

(iii) If the axes be turned through an angle $\tan^{-1}2$ (in anticlockwise direction), what does the equation $4xy - 3x^2 = a^2$ become ?

22. EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES :

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \qquad \text{..........(i)}$$

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(a) **Equation of bisector of angle containing origin :**

If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (i)

(b) **Equation of bisector of acute/obtuse angles :**

To find the equation of the bisector of the acute or obtuse angle :

(i) let ϕ be the angle between one of the two bisectors and one of two given lines. Then if $\tan\phi < 1$ i.e. $\phi < 45$ i.e. $2\phi < 90$, the angle bisector will be bisector of acute angle.

(ii) See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by -1 to make the constant terms positive.

Determine the sign of $a_1a_2 + b_1b_2$

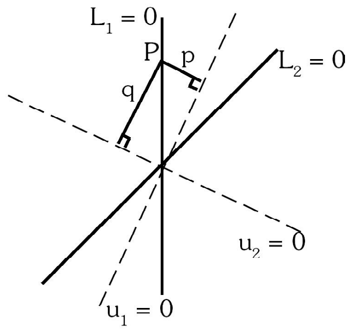
If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use − sign in eq. (1)
−	use − sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

(iii) Another way of identifying an acute and obtuse angle bisector is as follows :

Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown . If,



$|p| < |q| \Rightarrow u_1$ is the acute angle bisector .

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector .

$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular .

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

Illustration 29 : For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them.

(ii) bisector of the acute angle between them.

(iii) bisector of the angle which contains origin.

(iv) bisector of the angle which contains (1, 2).

Solution : Equations of bisectors of the angles between the given lines are

$$\frac{4x + 3y - 6}{\sqrt{4^2 + 3^2}} = \pm\frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}} \Rightarrow 9x - 7y - 41 = 0 \text{ and } 7x + 9y - 3 = 0$$

If θ is the acute angle between the line $4x + 3y - 6 = 0$ and the bisector

$9x - 7y - 41 = 0$, then $\tan\theta = \left|\dfrac{-\dfrac{4}{3} - \dfrac{9}{7}}{1 + \left(\dfrac{-4}{3}\right)\dfrac{9}{7}}\right| = \dfrac{11}{3} > 1$

Hence

(i) bisector of the obtuse angle is $9x - 7y - 41 = 0$

(ii) bisector of the acute angle is $7x + 9y - 3 = 0$

(iii) bisector of the angle which contains origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

(iv) $L_1(1, 2) = 4 \quad 1 + 3 \quad 2 - 6 = 4 > 0$
$L_2(1, 2) = 5 \quad 1 + 12 \quad 2 + 9 = 38 > 0$

+ve sign will give the required bisector, $\frac{4x+3y-6}{5} = +\frac{5x+12y+9}{13}$

$\Rightarrow \quad 9x - 7y - 41 = 0.$

Alternative :

Making c_1 and c_2 positive in the given equation, we get $-4x - 3y + 6 = 0$ and $5x + 12y + 9 = 0$

Since $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$, so the origin will lie in the acute angle.

Hence bisector of the acute angle is given by

$$\frac{-4x-3y+6}{\sqrt{4^2+3^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \quad \Rightarrow \quad 7x+9y-3=0$$

Similarly bisector of obtuse angle is $9x - 7y - 41 = 0$.

Illustration 30 : A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line mirror $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Solution : Let Q be the point of intersection of the incident ray and the line mirror, then

$x_1 - 2y_1 - 3 = 0 \quad \& \quad 3x_1 - 2y_1 - 5 = 0$

on solving these equations, we get

$x_1 = 1 \quad \& \quad y_1 = -1$

Since P(–1, –2) be a point lies on the incident ray, so we can find the image of the point P on the reflected ray about the line mirror (by property of reflection).

Let P'(h, k) be the image of point P about line mirror, then

$$\frac{h+1}{3} = \frac{k+2}{-2} = \frac{-2(-3+4-5)}{13} \quad \Rightarrow \quad h = \frac{11}{13} \text{ and } k = \frac{-42}{13}.$$

So $\quad P'\left(\frac{11}{13}, \frac{-42}{13}\right)$

Then equation of reflected ray will be

$$(y+1) = \frac{\left(\frac{-42}{13}+1\right)(x-1)}{\left(\frac{11}{13}-1\right)}$$

$\Rightarrow \quad 2y - 29x + 31 = 0$ is the required equation of reflected ray.

23. FAMILY OF LINES :

If equation of two lines be $P \equiv a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

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Illustration 31 : Prove that each member of the family of straight lines $(3\sin\theta + 4\cos\theta)x + (2\sin\theta - 7\cos\theta)y + (\sin\theta + 2\cos\theta) = 0$ (θ is a parameter) passes through a fixed point.

Solution : The given family of straight lines can be rewritten as
$(3x + 2y + 1)\sin\theta + (4x - 7y + 2)\cos\theta = 0$
or, $(4x - 7y + 2) + \tan\theta(3x + 2y + 1) = 0$ which is of the form $L_1 + \lambda L_2 = 0$
Hence each member of it will pass through a fixed point which is the intersection of
$4x - 7y + 2 = 0$ and $3x + 2y + 1 = 0$ i.e. $\left(\frac{-11}{29}, \frac{2}{29}\right)$.

Do yourself - 11 :

(i) Find the equations of bisectors of the angle between the lines $4x + 3y = 7$ and $24x + 7y - 31 = 0$. Also find which of them is (a) the bisector of the angle containing origin (b) the bisector of the acute angle.

(ii) Find the equations of the line which pass through the point of intersection of the lines $4x - 3y = 1$ and $2x - 5y + 3 = 0$ and is equally inclined to the axis.

(iii) Find the equation of the line through the point of intersection of the lines $3x - 4y + 1 = 0$ & $5x + y - 1 = 0$ and perpendicular to the line $2x - 3y = 10$.

24. PAIR OF STRAIGHT LINES :

(a) Homogeneous equation of second degree :

Let us consider the homogeneous equation of 2nd degree as

$ax^2 + 2hxy + by^2 = 0$ ....(i)

which represents pair of straight lines passing through the origin.

Now, we divide by x^2, we get

$$a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2 = 0$$

$$\frac{y}{x} = m \quad \text{(say)}$$

then $a + 2hm + bm^2 = 0$ ....(ii)

if m_1 & m_2 are the roots of equation (ii), then $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$

and also, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right|$

$$= \left|\frac{\sqrt{\frac{4h^2}{b} - \frac{4a}{b}}}{1 + \frac{a}{b}}\right| = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

These line will be :

(i) **Real and different**, if $h^2 - ab > 0$

(ii) **Real and coincident**, if $h^2 - ab = 0$

(iii) **Imaginary**, if $h^2 - ab < 0$

(iv) The condition that these lines are :

(1) At **right angles** to each other is $a + b = 0$. i.e. coefficient of **x** + coefficient of **y** = 0.

(2) **Coincident** is **h = ab**.

(3) **Equally inclined** to the axes of x is **h = 0**. i.e. coefficient of **xy** = 0.

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Homogeneous equation of 2^{nd} degree $\mathbf{ax^2 + 2hxy + by^2 = 0}$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x \;\equiv y = m_1x \;\;\&\;\; y = m_2x \quad \text{and} \quad m_1 + m_2 = -\frac{2h}{b} \;;\; m_1m_2 = \frac{a}{b}$$

These straight lines passes through the origin.

Note : A homogeneous equation of degree n represents n straight lines passing through **origin**.

(b) The combined equation of angle bisectors :

The combined equation of angle bisectors between the lines represented by homogeneous equation of 2^{nd} degree is given by $\mathbf{\frac{x^2 - y^2}{a - b} = \frac{xy}{h}}$, $\mathbf{a \neq b, h \neq 0}$.

Note :

(i) If $\mathbf{a = b}$, the bisectors are $\mathbf{x^2 - y^2 = 0}$ i.e. $\mathbf{x - y = 0, x + y = 0}$

(ii) If $\mathbf{h = 0}$, the bisectors are $\mathbf{xy = 0}$ i.e. $\mathbf{x = 0, y = 0}$.

(iii) The two bisectors are always at **right angles**, since we have coefficient of $\mathbf{x^2}$ + coefficient of $\mathbf{y^2} = 0$

(c) General Equation and Homogeneous Equation of Second Degree :

(i) The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) If θ be the angle between the lines, then $\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$

Obviously these lines are

(1) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(2) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(iii) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(iv) The product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is $\left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a - b)^2 + 4h^2}}\right|$

(v) The product of the perpendiculars drawn from the origin to the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left|\frac{c}{\sqrt{(a - b)^2 + 4h^2}}\right|$

Illustration 32 : If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then λ is equal to -

(A) 4 (B) 3 (C) 2 (D) 1

Solution : Here $a = \lambda$, $b = 12$, $c = -3$, $f = -8$, $g = 5/2$, $h = -5$

Using condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, we have

$\lambda(12)(-3) + 2(-8)(5/2)(-5) - \lambda(64) - 12(25/4) + 3(25) = 0$

$\Rightarrow \quad -36\lambda + 200 - 64\lambda - 75 + 75 = 0 \qquad \Rightarrow \quad 100\lambda = 200$

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$\therefore \quad \lambda = 2$

Ans. (C)

Illustration 33 : Show that the two straight lines $x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$ represented by the equation are such that the difference of their slopes is 2.

Solution : The given equation is $x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$ .... (i)

and general equation of second degree $ax^2 + 2hxy + by^2 = 0$ .... (ii)

Comparing (i) and (ii), we get $a = \tan^2\theta + \cos^2\theta$

$h = -\tan\theta$

and $b = \sin^2\theta$

Let separate lines of (ii) are $y = m_1x$ and $y = m_2x$

where $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

therefore, $m_1+m_2=-\dfrac{2h}{b}=\dfrac{2\tan\theta}{\sin^2\theta}$

and $m_1.m_2=\dfrac{\tan^2\theta+\cos^2\theta}{\sin^2\theta}$

$\therefore \quad m_1-m_2=\sqrt{(m_1+m_2)^2-4m_1m_2}$

$$\Rightarrow \quad \tan\theta_1-\tan\theta_2=\sqrt{\frac{4\tan^2\theta}{\sin^4\theta}-\frac{4(\tan^2\theta+\cos^2\theta)}{\sin^2\theta}}=\frac{2}{\sin^2\theta}\sqrt{\tan^2\theta-\sin^2\theta(\tan^2\theta+\cos^2\theta)}$$

$$=\frac{2\sin\theta}{\sin^2\theta}\sqrt{(\sec^2\theta-\tan^2\theta-\cos^2\theta)}=\frac{2\sin\theta}{\sin^2\theta}\sqrt{(1-\cos^2\theta)}=\frac{2}{\sin\theta}\sin\theta=2$$ **Ans.**

Illustration 34 : If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $pq = -1$.

Solution : According to the question, the equation of the bisectors of the angle between the lines

$x^2 - 2pxy - y^2 = 0$ .... (i)

is $x^2 - 2qxy - y^2 = 0$ .... (ii)

$\therefore$ The equation of bisectors of the angle between the lines (i) is $\dfrac{x^2-y^2}{1-(-1)}=\dfrac{xy}{-p}$

$\Rightarrow \quad -px^2 - 2xy + py^2 = 0$

Since (ii) and (iii) are identical, comparing (ii) and (iii), we get $\dfrac{1}{-p}=\dfrac{-2q}{-2}=\dfrac{-1}{p} \Rightarrow pq = -1$

Do yourself - 12 :

(i) Prove that the equation $x^2 - 5xy + 4y^2 = 0$ represents two lines passing through the origin. Also find their equations.

(ii) If the equation $3x^2 + kxy - 10y^2 + 7x + 19y = 6$ represents a pair of lines, find the value of k.

(iii) If the equation $6x^2 - 11xy - 10y^2 - 19y + c = 0$ represents a pair of lines, find their equations. Also find the angle between the two lines.

(iv) Find the point of intersection and the angle between the lines given by the equation :
$2x^2 - 3xy - 2y^2 + 10x + 5y + 12 = 0.$

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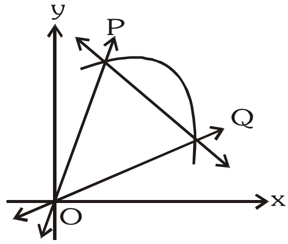
25. EQUATIONS OF LINES JOINING THE POINTS OF INTERSECTION OF A LINE AND A CURVE TO THE ORIGIN :

(a) Let the equation of curve be :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)



and straight line be

$lx + my + n = 0$ ...(ii)

Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation (i) homogenous with the help of equation of the line. Thus required equation is given by

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{\ell x+my}{-n}\right)+c\left(\frac{\ell x+my}{-n}\right)^2=0$$

$\Rightarrow$ $(an^2 + 2gln + cl^2)x^2 + 2(hn^2 + gmn + fln + clm)xy + (bn^2 + 2fmn + cm^2)y^2 = 0$ ......(iii)

All points which satisfy (i) and (ii) simultaneously, will satisfy (iii)

(b) Any second degree curve through the four points of intersection of $f(x, y) = 0$ & $xy = 0$ is given by $f(x, y) + \lambda\, xy = 0$ where $f(x, y) = 0$ is also a second degree curve.

Illustration 35 : The chord $\sqrt{6}y = \sqrt{8}px + \sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at origin then find the value of p.

Solution : $\sqrt{3}y - 2px = 1$ is the given chord. Homogenizing the equation of the curve, we get,

$py^2 - 4x(\sqrt{3}\,y - 2px)+(\sqrt{3}\,y - 2px)^2 = 0$

$\Rightarrow$ $(4p^2 + 8p)x^2 + (p + 3)y^2 - 4\sqrt{3}\,xy - 4\sqrt{3}\,pxy = 0$

Now, angle at origin is 90

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore$ $4p^2 + 8p + p + 3 = 0 \Rightarrow 4p^2 + 9p + 3 = 0$

$\therefore$ $p = \dfrac{-9 \pm \sqrt{81-48}}{8} = \dfrac{-9 \pm \sqrt{33}}{8}$.

Do yourself - 13 :

(i) Find the angle subtended at the origin by the intercept made on the curve $x^2 - y^2 - xy + 3x - 6y + 18 = 0$ by the line $2x - y = 3$.

(ii) Find the equation of the lines joining the origin to the points of intersection of the curve $2x^2 + 3xy - 4x + 1 = 0$ and the line $3x + y = 1$.

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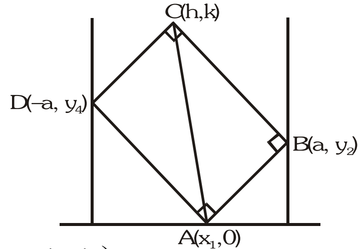
Miscellaneous Illustration :

Illustration 36 : ABCD is a variable rectangle having its sides parallel to fixed directions (say m). The vertices B and D lie on $x = a$ and $x = -a$ and A lies on the line $y = 0$. Find the locus of C.

Solution : Let A be $(x_1, 0)$, $B(a, y_2)$ and D be $(-a, y_4)$. We are given AB and AD have fixed directions and hence their slopes are constants. i.e. m & m_1 (say)



$$\therefore \quad \frac{y_2}{a - x_1} = m \text{ and } \frac{y_4}{-a - x_1} = m_1 .$$

Further, $mm_1 = -1$. Since ABCD is a rectangle.

$$\frac{y_2}{a - x_1} = m \text{ and } \frac{y_4}{-a - x_1} = -\frac{1}{m}$$

The mid point of BD is $\left(0, \frac{y_2 + y_4}{2}\right)$ and mid point of AC is $\left(\frac{x_1 + h}{2}, \frac{k}{2}\right)$, where C is taken to be (h, k). This gives $h = -x_1$ and $k = y_2 + y_4$. So C is $(-x_1, y_2 + y_4)$.

Also, $\frac{y_2}{a - x_1} = m$ and $\frac{y_4}{a + x_1} = \frac{1}{m}$ gives $m(k - y_2) = a + x_1 = m(k - m(a - x_1)) = a + x_1$

$\Rightarrow \quad mk - m^2(a - x_1) = a + x_1 \quad \Rightarrow \quad m^2(a + h) - mk + a - h = 0$

$\Rightarrow \quad (m^2 - 1)h - mk = -(m^2 + 1)a \quad \Rightarrow \quad (1 - m^2)h + mk = (m^2 + 1)a$

$\Rightarrow \quad (1 - m^2)x + my = (m^2 + 1)a$

The locus of C is $(1 - m^2)x + my = (m^2 + 1)a$.

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