Rolle's Theorem

Q→ (a) State Rolle's Theorem (2) marks
(b). verify Rolle's Theorem for this function (2)] 4 marks.

Statement → Let $f(x)$ be a function which is;

(i) continuous in $[a, b]$
(ii) Derivable in (a, b)
(iii) $f(a) = f(b)$

then $\exists$ atleast ~~number~~ one point $c \in (a\ b)$ such that
$\in$ belongs

$$f'(c) = 0.$$

Q→ Verify Rolle's Theorem for the function;
$f(x) = x^2 - 4x + 3$ in the interval $1 \leq x \leq 3$.

Sol; Given $f(x) = x^2 - 4x + 3$

(i) Being a polynomial $f(x)$ is continuous in $[1, 3]$
(ii). $f'(x) = 2x - 4$ which is clearly derivable in $(1, 3)$
(iii). $f(1) = 1^2 - 4(1) + 3 = 0$
$f(3) = 3^2 - 4(3) + 3 = 0$
So; $f(1) = f(3)$

$\sin\frac{\pi}{2} = 1$ $\quad \sin\pi = 0$ $\quad \sin 0 = 0$

$\cos\frac{\pi}{2} = 0$ $\quad \cos\pi = -1$ $\quad \cos 0 = 1$

So, all the three conditions of Roll's Theorem are satisfied.

$\Rightarrow \exists$ atleast one point $c \in (1,3)$ such that $f'(c) = 0$

$$2c - 4 = 0$$
$$2c = 4$$
$$c = 2 \in (1,3)$$

Hence Rolle's Theorem is verified.

Continuous	Incontinuous
Polynomial	Tan x
Sin x	
Cos x	
e^x	
Log	
Cubic	
Modulus	
Absolute functions	
Constant	

Q→ verify Rolle's Theorem for the function; $f(x) = e^x \sin x$ in $[0, \pi]$.

Sol→ Given $f(x) = e^x \sin x$.

i) Clearly e^x & $\sin x$ both are continuous functions and product of two continuous functions is again continuous. So $f(x) = e^x \sin x$ is continuous $[0\ \pi]$.

(ii). $f'(x) = \cancel{e^x \sin x}$ $e^x \cos x + \sin x\, e^x$ which is clearly derivable in $(0\ \pi)$.

(iii). $f(0) = e^0 \sin 0 = 1 \times 0 = 0$

$f(\pi) = e^\pi \sin \pi = e^\pi \times 0 = 0$

So; $f(0) = f(\pi)$

So; all the three condition of Rolle's Theorem are satisfied.

$\Rightarrow \exists$ atleast one point $c \in (0\ \pi)$

Such that $f'(c) = 0$

$$e^c \cos c + \sin c\, e^c = 0$$
$$e^c(\cos c + \sin c) = 0$$
$$\cos c = -\sin c$$
$$\tan c = -1$$
$$\tan c = -\tan\frac{\pi}{4} \Rightarrow \tan c = \tan\left(-\frac{\pi}{4}\right)$$

So; Rolle's Theorem is not satisfied. $\boxed{c = -\frac{\pi}{4}} \notin (0\ \pi)$

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such that $f'(c) = 0$

$e^c(\cos c + \sin c) = 0$

$\cos c = -\sin c$

$\tan c = -1$

$\tan c = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right)$

$\tan c = \tan\left(\frac{3\pi}{4}\right)$

$c = \frac{3\pi}{4} \in (0\pi)$

Roll's Theorem is verified.

Q→ Verify Rolle's Theorem for;

$f(x) = \sin 2x$ in $[0, \pi/2]$

Sol→ Given $f(x) = \sin 2x$

(i) $\sin 2x$ is continuous in $[0, \pi/2]$

(ii) $f'(x) = 2\cos 2x$ which is clearly derivable in $(0, \pi/2)$

No denominator

(iii) $f(0) = 0$

$f(\pi/2) = 0$

So $f(0) = f(\pi/2)$

So, all the three conditions of Rolle's Theorem are satisfied

$\exists$ atleast one point $c \in (0, \pi/2)$ such that $f'(c) = 0$

$f'(c) = 2\cos 2c = 0$

$\Rightarrow$ ~~2cos2c = 0~~

$\cos 2c = 0$

$\cos 2c = \cos\frac{\pi}{2}$

$2c = \frac{\pi}{2}$

Hence Rolle's theorem is verified $\boxed{c = \frac{\pi}{4}} \in (0, \pi/2)$

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II. Lagrange's Mean Value Theorem

Statement :- Let $f(x)$ be a function which is;

(i) Continuous in $[a, b]$

(ii) Derivable in (a, b)

then $\exists$ atleast one value $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Q→ Verify L.M.V theorem for the function;
$f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$
$\Rightarrow x^3 - 3x^2 + 2x$

(i) Being a polynomial; $f(x)$ is continuous in $[0, \frac{1}{2}]$

(ii) $f'(x) \Rightarrow 3x^2 - 6x + 2$ which is derivable in $(0, \frac{1}{2})$

So; ~~the~~ $\exists$ atleast one value $c \in (0, \frac{1}{2})$ such that $f'(c) = \dfrac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$

$$3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{\frac{1}{2}}$$

$$3c^2 - 6c + 2 \Rightarrow \frac{3}{\cancel{8}_4} \times \frac{\cancel{2}}{1}$$

$$3c^2 - 6c \Rightarrow \frac{3}{4} - 2$$

$$3c^2 - 6c \Rightarrow \frac{3-8}{4}$$

$f(\frac{1}{2}) \Rightarrow (\frac{1}{8}) - \frac{3}{4} + 1$

$\Rightarrow \dfrac{1 - 6 + 8}{8}$

$\Rightarrow \dfrac{3}{8}$

$f(0) = 0 - 0 = 0$

$$3c^2 - 6c \Rightarrow \frac{-5}{4}$$

$$4[3c^2 - 6c] + 5 \Rightarrow 0$$

$$12c^2 - 24c + 5 \Rightarrow 0$$

$$c = \frac{24 \pm \sqrt{576 - 4(12)(5)}}{24}$$

$$c \Rightarrow \frac{24 \pm \sqrt{336}}{24}$$

$$c \Rightarrow \frac{24 \pm 4\sqrt{21}}{24} \Rightarrow \frac{6 \pm \sqrt{21}}{6}$$

$$c \Rightarrow 1 \pm \frac{\sqrt{21}}{6} \Rightarrow 1 \pm \frac{4.5}{6}$$

$$c \Rightarrow 1 + \frac{4.5}{6} \Rightarrow \cancel{0.7}\ 1.76$$

$$c \Rightarrow 1 - \frac{4.5}{6} \Rightarrow 0.25$$

$$c \in \left(0, \frac{1}{2}\right)$$

✓ Hence ; L.M.V is verified.

⇒ Cauchy's Mean Value Theorem :

Statement :-

Let $f(x)$ and $g(x)$ be the two functions ; which are

(i) Continuous in $[a, b]$.
(ii) Derivable in (a, b)
(iii) $g'(x) \neq 0$ for any $x \in (a,b)$

then $\exists$ atleast one point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Q→ verify cauchy's Mean Theorem ;
$f(x) = \log x$ $g(x) = \frac{1}{x}$ in $[1, e]$ $e = 2.318$.

Sol. (i). clearly $f(x)$, $g(x)$ are continuous in $[1, e]$

(ii). $f'(x) = \frac{1}{x}$ $g'(x) = \frac{-1}{x^2}$ is clearly derivable in $(1, e)$

(iii). $g'(x) = \frac{-1}{x^2} \neq 0$ for any $x \in (1, e)$

So ; $\exists$ atleast one value $c \in (1, e)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(e) - f(1)}{g(e) - g(1)}$$

$$\frac{\frac{1}{c}}{\frac{-1}{c^2}} = \frac{\log e - \log 1}{\frac{1}{e} - \frac{1}{1}}$$

$$-\frac{1}{c} \times \frac{c^2}{1} = \frac{1-0}{\frac{1}{e}-1}$$

$$-c = \frac{e(1-0)}{1-e}$$

$$-c = \frac{e}{1-e}$$

$$c = \frac{-e}{1-e} \Rightarrow \boxed{\frac{e}{e-1}}$$

Consider ; $c-1$

$$\frac{e}{e-1} - 1$$

$$\frac{e-e+1}{e-1} = \frac{1}{e-1} > 0$$

$$c-1>0$$

$$\boxed{c>1}$$

$$c = \frac{e}{e-1} < e$$

$$1 < c < e$$

$$c \in (1, e)$$

$c \in (1, e)$

$c > 1$

& $c < e$

So, theorem is verified.

⇒ Taylor's Theorem [with lagrange's form of remainder].

(Imp) Q → Write the statement.
Q → Expand the equation
Q → Solve the equation

Statement: Let $f(x)$ be a function defined on $[a, b]$ such that

i) $f', f'', f''', \ldots f^{(n-1)}$ are continuous in $[a, b]$
→ (Value of n depends upon given function) → not in defination.

(ii) $f^{n}(x)$ should exist in (a, b)
~~there~~ then $\exists$ a real value
$c \in (a, b)$ such that :-

(T) form

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \cdots \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^{n}(c)$$

where $c = a + \theta h$

$f \to x^3 - 2$
$f'(x) \to 3x^2$
$f''(x) \to 6x$
$f'''(x) \to 6$
$f''''(x) \to 0$

$a < \theta < 1$

$h = b - a$

(II) form $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$

$+ \cdots + \frac{h^{n-1}}{n-1!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h)$

where $a < \theta < 1$

$h = b - a$

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Q3 Apply Taylor's Theorem to $f(x) = \sin x$ in $[\pi/2, x]$ (6m)

$a \quad b$

Sol: $f(x) = f(\pi/2) + (x - \pi/2) f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right)$

$+ \frac{\left(x - \frac{\pi}{2}\right)^3}{3!} f'''\left(\frac{\pi}{2}\right) \cdots\cdots + \frac{\left(x - \frac{\pi}{2}\right)^{n-1}}{(n-1)!} f^{n-1}\left(\frac{\pi}{2}\right)$

$+ \frac{\left(x - \frac{\pi}{2}\right)^n}{n!} f^n\left(\frac{\pi}{2} + \theta\left(x - \frac{\pi}{2}\right)\right)$

$f(x) = \sin x$

$f'(x) = \cos x = \sin\left(\frac{\pi}{2} + x\right)$

$f''(x) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + x\right) = \sin\left(\frac{2\pi}{2} + x\right) = \sin(\pi + x)$

$f'''(x) = \sin\left(\frac{3\pi}{2} + x\right)$

$f^{n-1}(x) = \sin\left[(n-1)\frac{\pi}{2} + x\right]$

$f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)$

$f(\pi/2) = \sin\frac{\pi}{2} = 1$

$f'(\pi/2) = 0$

$f''(\pi/2) = -\sin\frac{\pi}{2} = -1$

$f'''(\pi/2) = 0$

$f^{n-1}(\pi/2) = \sin\left[(n-1)\frac{\pi}{2} + \frac{\pi}{2}\right] \Rightarrow \sin\frac{n\pi}{2}$

$$f^n\left(\frac{\pi}{2}\right) = \sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right) = \sin\left((n+1)\frac{\pi}{2}\right)$$

$$\Rightarrow \sin\left[(n+1)\frac{\pi}{2} + \theta\left(x - \frac{\pi}{2}\right)\right]$$

$$\sin x = 1 + 0 + \frac{\left(x-\frac{\pi}{2}\right)^2}{2}(-1) + 0 + \cdots \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{n-1!} \sin\frac{n\pi}{2}$$

$$+ \frac{\left(x-\frac{\pi}{2}\right)^n}{n!} \sin\left[(n+1)\frac{\pi}{2} + \theta\left(x-\frac{\pi}{2}\right)\right]$$

Q→ Use Taylor's theorem to find the value of $f(2.001)$ where $f(x) = x^3 - 2x + 5$.

(2m)

$\Rightarrow f(2+.001) = f(a+h)$

$f'(x) = 3x^2 - 2$

$f''(x) = 6x$

$f'''(x) = 6$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$$

$$f(2.001) = f(2) + (0.001) f'(2) + \frac{(.001)^2}{2!} f''(2) + \frac{(.001)^3}{3!} f'''(2)$$

$\cancel{\Rightarrow 9 + 0.01 + 0.0}$

$$\Rightarrow 9 + (0.001)10 + \frac{(.001)^2}{\cancel{2}}[\cancel{12}\,6] + \frac{(.001)^3}{\cancel{6}} \times (\cancel{6})$$

$f(2) = 9$

$f'(2) = 10$

$f''(2) = 12$

$f'''(2) = 6$

$\Rightarrow 9 + .01 + .000006 + .000000001$

$\Rightarrow \boxed{9.010006001}$

→ Maclaurin's Theorem with lagrange's form of remainders.

Statement :- From Taylor's Theorem;

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$$\cdots + \frac{h^{n-1}}{n-1!} f^{n-1}(a) + \frac{h^n}{n!} f^{n}(a+\theta h)$$

where $a < \theta < 1$

Put $a = 0$; $h = x$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\cdots + \frac{x^{n-1}}{n-1!} f^{n-1}(0) + \frac{x^n}{n!} f^{n}(\theta x)$$

$0 < \theta < 1$

→ Intermediate forms

$\frac{0}{0}$, $\frac{\infty}{\infty}$, ∞^0, 0^∞, $0 \cdot \infty$, 1^∞

⇒ Intermediate / Indeterminate Forms

L' Hopital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{0}{0} \qquad \lim_{x \to a} \frac{f''(x)}{g''(x)} \neq \frac{0}{0}$$

→ Some Important results

① $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

② $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

③ $\sin x = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

④ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

⑤ $\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \frac{\sin^{-1} x}{x} = 1$

⑥ $\lim_{x \to 0} \frac{\tan x}{x} = 1$

⑦ $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$

⑧ $\log(1-x) \Rightarrow -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots$

⑨ $\lim_{x \to 0} (1+x)^{1/x} = e$

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II. Lagrange's Mean Value Theorem

Statement :- Let $f(x)$ be a function which is;

(i) Continuous in $[a, b]$

(ii) Derivable in (a, b)

then $\exists$ atleast one value $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Q→ Verify L.M.V theorem for the function;

$f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$

$\Rightarrow x^3 - 3x^2 + 2x$

(i) Being a polynomial; $f(x)$ is continuous in $[0, \frac{1}{2}]$

(ii) $f'(x) \Rightarrow 3x^2 - 6x + 2$ which is derivable in $(0, \frac{1}{2})$

So; ~~the~~ $\exists$ atleast one value $c \in (0, \frac{1}{2})$ such that $f'(c) = \dfrac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$

$$3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{\frac{1}{2}}$$

$$3c^2 - 6c + 2 \Rightarrow \frac{3}{4\cancel{8}} \times \frac{\cancel{2}}{1}$$

$$3c^2 - 6c \Rightarrow \frac{3}{4} - 2$$

$$3c^2 - 6c \Rightarrow \frac{3-8}{4}$$

$f(\frac{1}{2}) \Rightarrow (\frac{1}{8}) - \frac{3}{4} + 1$

$\Rightarrow \dfrac{1 - 6 + 8}{8}$

$\Rightarrow \dfrac{3}{8}$

$f(0) = 0 - 0 = 0$

$$3c^2 - 6c \Rightarrow \frac{-5}{4}$$

$$4[3c^2 - 6c] + 5 \Rightarrow 0$$

$$12c^2 - 24c + 5 \Rightarrow 0$$

$$c = \frac{24 \pm \sqrt{576 - 4(12)(5)}}{24}$$

$$c \Rightarrow \frac{24 \pm \sqrt{336}}{24}$$

$$c \Rightarrow \frac{24 \pm 4\sqrt{21}}{24} \Rightarrow \frac{6 \pm \sqrt{21}}{6}$$

$$c \Rightarrow 1 \pm \frac{\sqrt{21}}{6} \Rightarrow 1 \pm \frac{4.5}{6}$$

$$c \Rightarrow 1 + \frac{4.5}{6} \Rightarrow \cancel{0.7}\ 1.76$$

$$c \Rightarrow 1 - \frac{4.5}{6} \Rightarrow 0.25$$

$$c \in \left(0, \frac{1}{2}\right)$$

✓ Hence ; L.M.V is verifical.

Q→ Using L.M.V Theorem, prove that.

$$|\sin x - \sin y| \leq |x - y|$$

Sol→ Let $f(t) = \sin t$ (y, x)

(i) $f(t)$ is clearly cont in $[y, x]$

(ii) $f(t)$ is clearly derivable in (y, x)

⇒ ∃ some $c \in (y, x)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y} \quad \{\text{Using L.M.V Theorem}\}$$

$$\cos c = \frac{\sin x - \sin y}{x - y}$$

Taking Modulus on b.s.

$$|\cos c| = \frac{|\sin x - \sin y|}{|x - y|}$$

$\cos c \leq 1 \Rightarrow$

$$\frac{|\sin x - \sin y|}{|x - y|} \leq 1$$

$$|\sin x - \sin y| \leq |x - y|$$

Hence Proved.

classmate
Date
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⇒ Cauchy's Mean Value Theorem.

Statement :-

Let $f(x)$ and $g(x)$ be the two functions; which are

(i) continuous in $[a, b]$.
(ii) Derivable in (a, b)
(iii) $g'(x) \neq 0$ for any $x \in (a,b)$

then $\exists$ atleast one point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Q→ verify cauchy's Mean Theo for;
$f(x) = \log x$ $\quad g(x) = \frac{1}{x}$ in $[1, e]$ $\quad e = 2.318$.

Sol. (i). clearly $f(x)$, $g(x)$ are continuous in $[1, e]$

(ii). $f'(x) = \frac{1}{x}$ $\quad g'(x) = \frac{-1}{x^2}$ is clearly derivable in $(1, e)$

(iii). $g'(x) = \frac{-1}{x^2} \neq 0$ for any $x \in (1, e)$

So; $\exists$ atleast one value $c \in (1, e)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(e) - f(1)}{g(e) - g(1)}$$

$$\frac{\frac{1}{c}}{\frac{-1}{c^2}} = \frac{\log e - \log 1}{\frac{1}{e} - \frac{1}{1}}$$

$$-\frac{1}{c} \times \frac{c^2}{1} = \frac{1-0}{\frac{1}{e} - 1}$$

$$-c = \frac{e(1-0)}{1-e}$$

$$-c = \frac{e}{1-e}$$

$$c = \frac{-e}{1-e} \Rightarrow \boxed{\frac{e}{e-1}}$$

Consider ; $c - 1$

$$\frac{e}{e-1} - 1$$

$$\frac{e - e + 1}{e-1} = \frac{1}{e-1} > 0$$

$$c - 1 > 0$$

$$\boxed{c > 1}$$

$$c = \frac{e}{e-1} < e$$

$$1 < c < e$$

$$c \in (1, e)$$

So; Theorem is verified.

$c \in (1, e)$

$c > 1$

& $c < e$

Q→ verify Cauchy's Mean value Theorem for

(2m) $f(x) = x^2 - 1$

$g(x) = x^3$ in $[-1, 2]$

Sol; → (a) Being Polynomial clearly both $f(x)$ and $g(x)$ are continuous $[-1, 2]$.

(b). $f'(x) \Rightarrow 2x$

$g'(x) \Rightarrow 3x^2$ which is clearly derivable $(-1, 2)$

(c). $g'(x) \Rightarrow 3x^2 \neq 0$ for any $x \in (-1, 2)$

So, Cauchy's Mean Value Theorem is not verified.

Q→ $f(x) = \begin{cases} 2 & a \le x < b \\ 4 & x = b \end{cases}$

$g(x) = x$ for $x \in [a, b]$ verify Cauchy's Mean Value Theorem.

Sol→ (a) ~~$f(x)$ is constant, so clearly $f(x)$ is continuous in~~

$g(x)$ is constant

(a) $f(x) \underset{x \to b^-}{lt} = 2 \quad \underset{x \to b}{lt} \to 4$

L.H.L ≠ ~~Value~~ R.H.L

$f(x)$ is not cont in $[a, b]$

So, Cauchy's Theo. is not verified.

Q→ verify Cauchy's Mean Value Theorem for,

(i) $f(x) = \cos x$ in $[-\pi/4, \pi/4]$

$g(x) = \sin x$

(ii) $f(x) = \sqrt{x}$ in $[a, b]$; $a > 0$

$g(x) = \dfrac{1}{\sqrt{x}}$

(iii) $f(x) = x^2$ in $[1, 4]$
$g(x) = \sqrt{x}$

(iv) $f(x) = x^3$ in $[-1, 1]$
$g(x) = x^4$

(v) $f(x) = x^2$ in $[-1, 1]$
$g(x) = x^3$

⇒ Taylor's Theorem [with lagrange's form of Remainder].

Imp Q→ write the statement
Q→ Expand the equation
Q→ Solve the equation

Statement: Let $f(x)$ be a function defined on $[a, b]$ such that
i) $f', f'', f''', \ldots f^{(n-1)}$ are continuous in $[a, b]$
→ (Value of n depends upon given function) → not in defination.

(ii) $f^n(x)$ should exist in (a, b)
~~there~~ then ∃ a real value $c \in (a, b)$ such that :-

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \cdots \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c)$$

form I

where $c = a + \theta h$

$f \to x^3 - 2$
$f'(x) \to 3x^2$
$f''(x) \to 6x$
$f'''(x) \to 6$
$f''''(x) \to 0$

$a < \theta < 1$
$h = b - a$

(II) form $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$
$+ \cdots + \frac{h^{n-1}}{n-1!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h)$

where $a < \theta < 1$
$h = b - a$

Q→ If $f(x) = x^3 - 2x + 5$ find value of $f(2.001)$ using Taylor's Theorem.

H.W → (i) $f(x) = \cos x$ in $[-\pi/4, \pi/4]$
$g(x) = \sin x$

(a) Being Trigno functions clearly both $f(x)$ and $g(x)$ are continuous $[-\pi/4, \pi/4]$

(b) $f'(x) = -\sin x$, $g'(x) = \cos x$] no denomination both are derivable in $(-\pi/4, \pi/4)$

(c) $g'(x) = \cos x \neq 0$ for any $\in (-\pi/4, \pi/4)$

So; $\exists$ atleast one value $c \in (-\pi/4, \pi/4)$ such that;

$$\frac{f'(c)}{g'(c)} = \frac{f(\frac{\pi}{4}) - f(\frac{-\pi}{4})}{g(\frac{\pi}{4}) - g(\frac{-\pi}{4})}$$

$$\frac{-\sin c}{\cos c} \Rightarrow \frac{\cos(\pi/4) - \cos(-\pi/4)}{\sin(\pi/4) - \sin(-\pi/4)}$$

$\boxed{\sin(-x) = -\sin x}$
$\boxed{\cos(-x) = \cos x}$

$$-\tan c \Rightarrow \frac{\cos(\pi/4) - \cos(\pi/4)}{\sin(\pi/4) + \sin(\pi/4)}$$

$-\tan c \Rightarrow 0$

$-\tan c = \tan 0$

$\boxed{c = 0}$ # $\in (-\pi/4, \pi/4)$

(ii). $f(x) = \sqrt{x}$ in $[a, b]$; $a > 0$

$g(x) = \frac{1}{\sqrt{x}}$

(a). $f(x)$ and $g(x)$ are clearly continuous $[a, b]$

(b) $f'(x) = \frac{1}{2\sqrt{x}}$ $\quad g'(x) = \frac{-1}{2}\frac{1}{\sqrt{x^3}}$ which is clearly derivable in (a, b)

(c) $g'(x) = \frac{-1}{2}\frac{1}{\sqrt{x^3}} \neq 0$ for any $x \in (a, b)$

So; $\exists$ atleast one value $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{1/2\sqrt{c}}{-1/2\sqrt{c^3}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}$$

$$\frac{1}{\cancel{2}\sqrt{c}} \times \frac{\cancel{2}\sqrt{c^3}}{-1} = \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{ab}}}$$

$$-c = \frac{\cancel{\sqrt{b} - \sqrt{a}}}{\frac{-\cancel{(\sqrt{b} - \sqrt{a})}}{\sqrt{ab}}}$$

$$\cancel{-}c = \cancel{-}\sqrt{ab}$$

$$\boxed{c = \sqrt{ab}} \in (a, b)$$

(iii) $f(x) = x^2$ in $[1, 4]$
$g(x) = \sqrt{x}$

(a). Both are continuous $[1, 4]$
(b). $f'(x) \to 2x$ $g'(x) = \frac{1}{2\sqrt{x}}$ which is clearly derivable in $(1, 4)$

(c). $g'(x) \to \frac{1}{2\sqrt{x}} \neq 0$ for any $x \in (1, 4)$

So, $\exists$ atleast one value $c \in (1, 4)$
such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{2c}{\frac{1}{2\sqrt{x}}} = \frac{(4)^2 - (1)^2}{\sqrt{4} - \sqrt{1}}$$

$$(2c)\, 2(\sqrt{c}) = \frac{16 - 1}{2 - 1}$$

$$4c\sqrt{c} = \frac{15}{1}$$

$$\sqrt{c^3} = \frac{15}{4}$$

$$c = \frac{2.46}{1.58} = 1.55 \in (1, 4)$$

(iv). $f(x) = x^3$ in $[-1, 1]$
$g(x) = x^4$

(a). Continuous in $[-1, 1]$
(b) Derivable in $(-1, 1)$ $f'(x) \Rightarrow 3x^2$, $g'(x) = 4x^3$.
(c). $g'(x) = 4x^3 = 0$ for any $x \in (-1, 1)$
So; Cauchy's mean value Theorem is not verified.

(v) $f(x) = x^2$ in $[-1, 1]$
$g(x) = x^3$
(a). Continuous in $[-1, 1]$
(b). Derivable in $(-1, 1)$ $f'(x) = 2x$ and $g'(x) = 3x^2$
(c). $g'(x) = 3x^2 = 0$ for any $x \in (-1, 1)$
So; Cauchy's mean value Theorem is not verified.

$\pi/2 < \theta < \pi$ II: Sin, Cosec +ve	I: All +ve
$\pi < \theta < 3\pi/2$ III: Tan, Cot +ve	IV: Cos, Sec +ve, $3\pi/2 < \theta < 2\pi$

$\sin\left(\frac{\pi}{2} + x\right) = \cos x$

$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

Sin $\longleftrightarrow$ Cos
Sec $\longleftrightarrow$ Cosec
Tan $\longleftrightarrow$ Cot

$\frac{\pi \pm x}{2}$ $\frac{3\pi}{2} \pm x$

$\pi \pm x$ $2\pi \pm x$ No change

$\sin(\pi + x) = -\sin x$

$\cos(\pi + x) = -\cos x$

$\tan(\pi + x) = \tan x$

6/12

Q→ Apply Taylor's Theorem to $f(x) = \sin x$ in $[\pi/2, x]$ (6m)

Sol→ $f(x) = f(\pi/2) + (x - \pi/2) f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right)$

$+ \frac{\left(x - \frac{\pi}{2}\right)^3}{3!} f'''\left(\frac{\pi}{2}\right) - - - - + \frac{\left(x - \frac{\pi}{2}\right)^{n-1}}{(n-1)!} f^{n-1}\left(\frac{\pi}{2}\right)$

$+ \frac{\left(x - \frac{\pi}{2}\right)^n}{n!} f^n\left(\frac{\pi}{2} + \theta\left(x - \frac{\pi}{2}\right)\right)$

$f(x) = \sin x$

$f'(x) = \cos x = \sin\left(\frac{\pi}{2} + x\right)$

$f''(x) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + x\right) = \sin\left(\frac{2\pi}{2} + x\right) = \sin(\pi + x)$

$f'''(x) = \sin\left(\frac{3\pi}{2} + x\right)$

$f^{n-1}(x) = \sin\left[(n-1)\frac{\pi}{2} + x\right]$

$f^n(x) = \sin\left(\frac{n\pi}{3} + x\right)$

Sin
Cos

$f(\pi/2) = \sin\frac{\pi}{2} = 1$

$f'(\pi/2) = 0$

$f''(\pi/2) = -\sin\frac{\pi}{2} = -1$

$f'''(\pi/2) = 0$

$f^{n-1}(\pi/2) = \sin\left[(n-1)\frac{\pi}{2} + \frac{\pi}{2}\right] \Rightarrow \sin\frac{n\pi}{2}$

$$f^n\left(\frac{\pi}{2}\right) = \sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right) = \sin\left((n+1)\frac{\pi}{2}\right)$$

$$\Rightarrow \sin\left[(n+1)\frac{\pi}{2} + \theta\left(x - \frac{\pi}{2}\right)\right]$$

$$\sin x = 1 + 0 + \frac{\left(x-\frac{\pi}{2}\right)^2}{2}(-1) + 0 + \cdots \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{n-1!} \sin\frac{n\pi}{2}$$

$$+ \frac{\left(x-\frac{\pi}{2}\right)^n}{n!} \sin\left[(n+1)\frac{\pi}{2} + \theta\left(x-\frac{\pi}{2}\right)\right]$$

Q→ Use Taylor's Theorem to find the value of $f(2.001)$ where $f(x) = x^3 - 2x + 5$.

(2m)

$$\Rightarrow f(2+.001) = f(a+h) \quad \begin{cases} f'(x) = 3x^2 - 2 \\ f''(x) = 6x \\ f'''(x) = 6 \end{cases}$$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$$

$$f(2.001) = f(2) + (0.001) f'(2) + \frac{(.001)^2}{2!} f''(2) + \frac{(.001)^3}{3!} f'''(2)$$

$$\cancel{\Rightarrow 9 + 0.01 + 0.0}$$

$$\Rightarrow 9 + (0.001)10 + \frac{(.001)^2}{\cancel{2}}[\cancel{12}\,6] + \frac{[.001]^3}{\cancel{6}} \times (\cancel{6})$$

$$\Rightarrow 9 + .01 + .000006 + .000000001$$

$$\Rightarrow \boxed{9.010006001}$$

$f(2) = 9$
$f'(2) = 10$
$f''(2) = 12$
$f'''(2) = 6$

Q) Calculate the approx value of $\sqrt{26}$ using Taylor's Theorem.

$\rightarrow$ Let $a+h = \sqrt{26}$

$\sqrt{25}+\sqrt{1}=\sqrt{26}$ $a=25$; $h=1$

Sol; $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$

~~$f(\sqrt{26}) = f(\sqrt{25}) + 1$~~

Let $f(x+h) = \sqrt{x+h}$

$f(x) = \sqrt{x}$

~~$f(a) = \sqrt{a}$~~ $f'(x) = \frac{1}{2\sqrt{x}}$

$f'(x) = \frac{1}{2\sqrt{x}}$ $f''(x) = \frac{-1}{4(x)^{3/2}}$

$f'''(x) = \frac{3}{8(x)^{5/2}}$

$\sqrt{26} = \sqrt{25} + 1 \frac{1}{2\sqrt{a}} - \frac{1}{2} \times \frac{1}{4(a)^{3/2}} + \frac{1}{6} \times \frac{3}{8(a)^{5/2}}$

$\sqrt{26} = 5 + \frac{1}{10} - \frac{1}{8(25)^{3/2}} + \frac{1}{16(25)^{5/2}}$

$\sqrt{26} = 5 + 0.1 - \frac{1}{8([5^2]^{3/2})} + \frac{1}{16(5^2)^{5/2}}$

$\Rightarrow 5 + 0.1 - \frac{1}{8(125)} + \frac{1}{16 \times 3125}$

$\Rightarrow 5.1 - \frac{1}{1000} + \frac{1}{50,000}$

$\Rightarrow 5.1 - \frac{50+1}{50,000}$

$\Rightarrow 5.1 - \frac{49}{5 \times 10000}$

$\Rightarrow \frac{25.5 - 0.0049}{5} \Rightarrow \frac{25.4951}{5} \Rightarrow \boxed{5.09902}$

$\frac{125}{125}$

Q→ Expand/Express $f(x) = 4x^3 - 5x^2 + 3x - 9$ in the powers of $(x-3)$. Using Tayler's Theorem.

Let $x = 3 + x - 3$

$f(x) = f(\underset{a}{3} + \underset{h}{(x-3)})$

$$f(x) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$$

$$f(x) \Rightarrow f(3) + (x-3)(12a^2 - 10a + 3) + \frac{(x-3)^2}{2}[24a - 10] + \frac{(x-3)^3}{6}(24)$$

$$f(x) \Rightarrow 63 + (x-3)(81) + \frac{(x-3)^2}{2}[62] + (x-3)^3(4)$$

$$\cancel{\Rightarrow 63 + 81x - 243 + 31x - 93 + 4x^3 - 12}$$

$$\cancel{\Rightarrow 116x - 285}$$

$f(x) = 4x^3 - 5x^2 + 3x - 9$

$f'(x) \Rightarrow 12x^2 - 10x + 3$

$f''(x) \Rightarrow 24x - 10$

$f'''(x) \Rightarrow 24$

$f(3) \Rightarrow 108 - 45 + \cancel{9} - \cancel{9}$

$\Rightarrow \boxed{63}$

$f'(3) = 108 - 30 + 3 = \boxed{81}$

$f''(3) \Rightarrow 72 - 10 = \boxed{62}$

Maclaurin's Theorem with lagrange's form of remainders.

Statement :- From Taylor's Theorem;

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$$\cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^{n}(a+\theta h)$$

where $a < \theta < 1$

Put $a = 0$; $h = x$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\cdots + \frac{x^{n-1}}{n-1!} f^{n-1}(0) + \frac{x^n}{n!} f^{n}(\theta x)$$

$0 < \theta < 1$

Intermediate forms

$\frac{0}{0}$, $\frac{\infty}{\infty}$, ∞^0, 0^∞, $0 \cdot \infty$, 1^∞

⇒ ~~Intermediate~~ / Indeterminate Forms

L' Hopital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{0}{0} \qquad \lim_{x \to a} \frac{f''(x)}{g''(x)} = \frac{0}{0}$$

→ Some Important results

① $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

② $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

③ $\sin x = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

④ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

⑤ $\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \frac{\sin^{-1} x}{x} = 1$

⑥ $\lim_{x \to 0} \frac{\tan x}{x} = 1$

⑦ $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$

⑧ $\log(1-x) \to -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots$

⑨ $\lim_{x \to 0} (1+x)^{1/x} = e$

$\log 2 \Rightarrow 0.3014$

Q→ Evaluate $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$

Sol→ $\frac{e^0 - e^0}{\sin 0} = \frac{0}{0}$

Using L' hopital's Rule;

$$\frac{f'(x)}{g'(x)} \Rightarrow \frac{e^x - e^{-x}(-1)}{\cos x} \Rightarrow \frac{e^x + e^{-x}}{\cos x}$$

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{1+1}{1} \Rightarrow \boxed{2}$$

Q→ Evaluate $\lim_{x\to 0} \frac{1-\cos^2 x}{\sin x^2}$

$$\frac{1-(\cos 0)^2}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}$$

using L' Hopital's law

$$\frac{f'(x)}{g'(x)} = \frac{-2\cos x(-\sin x)}{\cos x^2 (2x)}$$

$$\cancel{\Rightarrow \frac{-2\cos 0(-\sin 0)}{\cos 0(0)} \Rightarrow \frac{-2(1)(-0)}{(1)(0)} \Rightarrow \frac{0}{0}}$$

$$\cancel{\frac{f''(x)}{g''(x)} \Rightarrow} \frac{\cancel{2}\cos x(\sin x)}{\cos x^2 (\cancel{2}x)}$$

$$\Rightarrow \frac{\cos x}{\cos x^2}\left(\frac{\sin x}{x}\right) \Rightarrow \frac{\cos x}{\cos x^2}(1) \Rightarrow \frac{1\times 1}{1} \Rightarrow \boxed{1}$$

Q→ $\lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$

$$\frac{1-1-2\log 2}{\sin 0} \Rightarrow \frac{1-1-2(0.30)}{0} \Rightarrow \frac{0}{0}$$

Using L'Hopital's Rule;

$$\frac{e^x - e^{-x} - 2\log(1+x)}{x\left(\frac{\sin x}{x}\right) \times x} \Rightarrow \frac{e^x - e^{-x} - 2\log(1+x)}{x^2}$$

$$\Rightarrow \frac{e^{x} - e^{-x}(-1) - 2\left(\frac{1}{1+x}\right)}{\cancel{x(\cos x) + \sin x - x} \quad 2x}$$

$$\Rightarrow \frac{1 - 1(-1) - 2\left(\frac{1}{1}\right)}{\cancel{x(1)} \quad 2(0)} \Rightarrow \frac{1+1-2}{0} \Rightarrow \frac{0}{0}$$

Again Applying L' Hopital's Rule

$$\frac{e^{x} - e^{-x} - 2\left(\frac{-1}{(1+x)^2}\right)}{2} \Rightarrow \frac{1-1+2}{2}$$

$$\Rightarrow \frac{2}{2} \Rightarrow \boxed{1}$$

$$\frac{1}{1+x} \Rightarrow (1+x)^{-1}$$

$$\Rightarrow -1(1+x)^{-2}$$

$$\Rightarrow \frac{-1}{(1+x)^2}$$

8 $\Rightarrow$ $\lim_{x \to 0} \frac{\sin^{-1}\left(\frac{1}{x}\right)}{\tan\left(\frac{1}{x}\right)} \Rightarrow \frac{0}{0}$

Let $\frac{1}{x} = h$ $\qquad \frac{1}{\infty} = 0$

$x \Rightarrow \frac{1}{h}$

$x \to 0 \Rightarrow \frac{1}{h} \to \infty$

$h \to 0$

$$\lim_{h \to 0} \frac{\sin^{-1} h}{\tan h} \times \frac{h}{h}$$

$$\lim_{h \to 0} \frac{\sin^{-1} h}{h} \times \frac{h}{\tan h} \Rightarrow 1 \times \frac{1}{\frac{\tan h}{h}} \Rightarrow 1 \times \frac{1}{1} \Rightarrow \boxed{1}$$

1 Dec

Q→ Evaluate ; $\lim_{x \to 1} \frac{x^x - x}{x - 1 - \log x} = \frac{0}{0}$

$y = x^x$

Taking log b.s

$\log y = \log x^x$

$\log y \Rightarrow x \log x$

$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x$

$\frac{1}{y}\frac{dy}{dx} \Rightarrow 1 + \log x$

$\frac{dy}{dx} = y(1 + \log x)$

$\frac{dy}{dx} = x^x(1 + \log x)$

Using L' Hopital's Rule

$\lim_{x \to 1} \frac{x^x(1+\log x) - 1}{1 - 0 - \frac{1}{x}}$

$\log 1 = 0$

$\Rightarrow \frac{x^x(1+\log x) - 1}{\frac{x-1}{x}}$

$\Rightarrow \frac{x[1(1) - 1]}{x - 1} \Rightarrow \boxed{\frac{0}{0}}$

Using L' Hopital's Rule.

$\lim_{x \to 1} \frac{x^x\left(\frac{1}{x}\right) + (1 + \log x)\, x^x(1 + \log x) - 0}{0 + \frac{1}{x}}$

$\log 0 = \infty$

$$\Rightarrow \frac{1+1}{1} \Rightarrow \frac{2}{1} \Rightarrow \boxed{2}$$

Q→ $\lim_{x\to 0} \frac{\log x^2}{\cot x^2} = \frac{\infty}{\infty}$

Using L' Hopital's Rule

$$\lim_{x\to 0} \frac{\frac{1}{x^2}(2x)}{-\operatorname{cosec}^2 x^2 (2x)}$$

$$\lim_{x\to 0} \frac{\frac{1}{x^2}}{\frac{-1}{\sin^2 x^2}} \Rightarrow \frac{1}{x^2} \times \frac{-\sin^2 x^2}{1} \Rightarrow \boxed{\frac{0}{0}}$$

Using L' Hopital's Rule.

$$\lim_{x\to 0} \frac{2\sin x^2 \cos x^2 \cdot 2x}{2x} \Rightarrow \boxed{0}$$

Q→ $\lim_{x\to\infty} \frac{x^5}{e^x} = \frac{\infty}{\infty}$

Using L' Hopital's Rule

$$\lim_{x\to\infty} \frac{5x^4}{e^x} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{20x^3}{e^x} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{60x^2}{e^x} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{120x}{e^x} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{120}{e^x} = \frac{120}{\infty} = \boxed{0}$$

Q→ $\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$

$\cancel{\rightarrow\left(\frac{1}{-1}-\frac{1}{0}\right)\Rightarrow(-1-\infty)=-(1+\infty)\Rightarrow-\infty}$

~~Using L' Hopital's Law~~

$\lim_{x\to 0}\left(\frac{x-e^x+1}{x(e^x-1)}\right)=\frac{0-1+1}{0}=\boxed{\frac{0}{0}}$

Using L' Hopital's Rule

$\lim_{x\to 0}\frac{1-e^x}{xe^x+(e^x-1)}=\boxed{\frac{0}{0}}$

Using L' Hopital's Rule

$\lim_{x\to 0}\frac{-e^x}{xe^x+e^x+e^x}\Rightarrow\frac{-\cancel{e^x}}{\cancel{e^x}(x+1+1)}\Rightarrow\boxed{\frac{-1}{2}}$

$\Rightarrow \sec^2 x - \tan^2 x = 1$

$\sec^2 x = 1 + \tan^2 x$

$\tan^2 x \Rightarrow 2\tan x \sec^2 x$

$\tan x = 0$

8 → (4m)

$$\lim_{x \to 0} \left(\cot^2 x - \frac{1}{x^2} \right)$$

$$\lim_{x \to 0} \left(\frac{1}{\tan^2 x} - \frac{1}{x^2} \right)$$

$$\lim_{x \to 0} \frac{x^2 - \tan^2 x}{x^2 \left(\frac{\tan^2 x}{x^2}\right) \times x^2} \Rightarrow \boxed{\frac{0}{0}}$$

$$\lim_{x \to 0} \frac{x^2 - \tan^2 x}{x^2 \times x^2} \Rightarrow \frac{x^2 - \tan^2 x}{x^4}$$

Using L'Hopital's Rule.

$$\lim_{x \to 0} \frac{2x - 2\tan x \sec^2 x}{4x^3}$$

$$\Rightarrow \frac{0 - 2(0)}{0} \Rightarrow \boxed{\frac{0}{0}}$$

$$\Rightarrow \frac{2x - 2\tan x (1 + \tan^2 x)}{4x^3}$$

~~using L'Hopital's Rule~~

$$\Rightarrow \frac{2x - 2\tan x - 2\tan^3 x}{4x^3}$$

$$\Rightarrow \frac{2[x - \tan x - \tan^3 x]}{4x^3} \Rightarrow \frac{x - \tan x - \tan^3 x}{2x^3}$$

Using L'Hopital's Rule;

$$\frac{1 - \sec^2 x - 3\tan^2 x \sec^2 x}{6x^2} = \boxed{\frac{0}{0}}$$

$$\frac{1 - (1 + \tan^2 x) - 3\tan^2 x (1 + \tan^2 x)}{6x^2}$$

$$\Rightarrow \frac{\cancel{1} - \cancel{1} - \tan^2 x - 3\tan^2 x - 3\tan^4 x}{6x^2}$$

$$\Rightarrow \frac{-4\tan^2 x - 3\tan^4 x}{6x^2}$$

$$\lim_{x \to 0} \frac{-\left[\tan^2 x\left(+4 + 3\tan^2 x\right)\right]}{6x^2}$$

$$\lim_{x \to 0} \Rightarrow \frac{-\left[+4 + 3\tan^2 x\right]}{6}$$

$$\Rightarrow \frac{-4+0}{6} \Rightarrow \boxed{\frac{-2}{3}}$$

4. Show that the angle of intersection of the curves $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$ is $\frac{\pi}{4}$.

   Hint: $\theta = \frac{\pi}{4}$ is point of intersection, $\tan\phi_1 = \infty \therefore \phi_1 = \frac{\pi}{2}$, $\tan\phi_2 = 1$, $\therefore \phi_2 = \frac{\pi}{4}$, $\phi_1 - \phi_2 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

5. Prove that the tangent to the logarithmic spiral $r = e^{a\theta}$ intersects the radius vector at a constant angle.

   Hint: $\frac{dr}{d\theta} = ae^{a\theta}$, $\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = a \therefore \phi = \cot^{-1} a = \text{const.}$

6. Prove that for the curve $r = ae^{m\theta^2}$, the ratio of polar subnormal to polar subtangent is proportional to θ^2.

7. Find ϕ for the following curves

   (a) $\frac{L}{r} = 1 + e\cos\theta$; (b) $r = a(1 + \cos\theta)$; (c) $\frac{2a}{r} = 1 + \cos\theta$

Ans. (a) $\cot^{-1}\left(\frac{e\sin\theta}{1+e\cos\theta}\right)$; (b) $\frac{\pi}{2} + \frac{\theta}{2}$; (c) $\frac{\pi}{2} - \frac{\theta}{2}$.

8. Show that the following pair of curves cut orthogonally:

   (a) $\frac{2a}{r} = 1 + \cos\theta$, $\frac{2b}{r} = 1 - \cos\theta$; (b) $r = a\cos\theta$, $r = a\sin\theta$; (c) $r^2\sin 2\theta = a^2$, $r^2\cos 2\theta = b^2$; (d) $r = a(1 + \cos\theta)$, $r = b(1 - \cos\theta)$.

2.4 ROLLE'S THEOREM

If $f(x)$ is (i) *continuous in $[a, b]$* (ii) *derivable in (a, b) and* (iii) *$f(a) = f(b)$, then there exists at least one value $c \in (a, b)$ such that $f'(c) = 0$.*

Proof:

I. If $f(x) = 0$ for all x, then $f'(x) = 0$ for all x.

II. Since f is continuous in $[a, b]$, it is bounded and attains its maximum M and minimum m say at two numbers c and d lying in between a and b such that

$$f(c) = M \text{ and } f(d) = m.$$

Case a: If $M = m$, then f is a constant function so that f' is zero for all x in (a, b).

Case b: If $M \neq m$, then $M = f(c) \geq f(c + h)$ for values of h both positive and negative. Then

$$\frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{for } h > 0 \qquad (13)$$

and

$$\frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{for } h < 0 \qquad (14)$$

Since f is differentiable in (a, b) from (1) and (2) as $h \to 0$, we have

$$f'(c) \leq 0 \quad \text{and} \quad f'(c) \geq 0$$

Hence $f'(c) = 0$ for some value c in (a, b). Similarly if $m = f(d) \leq f(d + h)$ for values of h both positive and negative it follows that

$$\frac{f(d+h) - f(d)}{h} \geq 0 \quad \text{for } h > 0$$

and

$$\frac{f(d+h) - f(d)}{h} \leq 0 \quad \text{for } h < 0$$

As $h \to 0$, $f'(d) \geq$ and $f'(d) \leq 0$. Hence $f'(d) = 0$.

Note 1: Geometrically, Rolle's theorem states that the tangents at c_1, c_2, c_3 are parallel to x-axis (see Fig. 2.4).

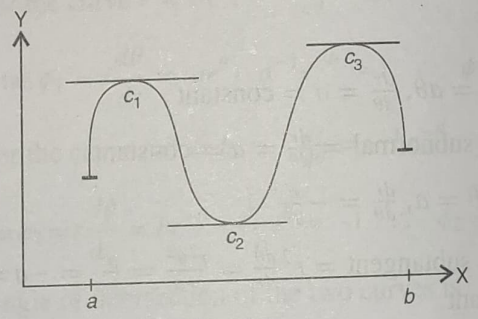

Fig. 2.4

Note 2: The function $y = f(x) = 1 - x^{\frac{2}{3}}$ in $[-1, 1]$ is not differentiable at origin O, it does not satisfy the condition of Rolle's theorem, so $f'(c) \neq 0$ for any $c \in (-1, 1)$ (refer Fig. 2.5).

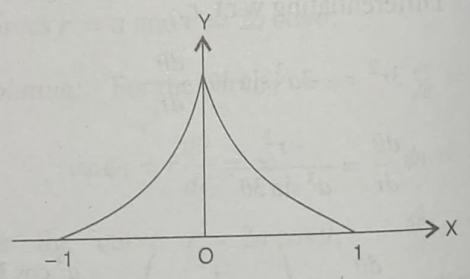

Fig. 2.5

Note 3: Alternate form: Let $b = a + h$ then $c = a + \theta h$ lies between a and b provided $0 < \theta < 1$. Thus the result of Rolle's theorem may be stated as

$$f'(c) = f(a + \theta h) = 0 \text{ with } 0 < \theta < 1.$$

Note 4: For $f(a) = f(b) = 0$, it follows from Rolle's theorem that there exists at least on $c \in (a, b)$ where $f'(c) = 0$. Thus the real root c of the equation $f'(x) = 0$ lies between a and b which are two adjacent real roots of the equation $f(x) = 0$, i.e., the real roots of equation $f'(x) = 0$ separate the real roots of the equation $f(x) = 0$.

WORKED OUT EXAMPLES

Verify Rolle's theorem for the following functions:

Example 1: $f(x) = x(x-2)e^{\frac{3x}{4}}$ in $(0, 2)$

Solution: $f(0) = 0$, $f(2) = 0$, f is continuous and differentiable, so by Rolle's theorem, $0 = f'(c)$. Here

$$f'(x) = [(x-2) + x + \frac{3}{4}(x)(x-2)]e^{\frac{3}{4}x}$$

$$= 0 \quad \text{or} \quad 3x^2 + 2x - 8 = 0$$

$\therefore c = -2$ or $\frac{8}{6}$ but $c = -2$ does not lie in $(0, 2)$ thus $c = \frac{8}{6} \in (0, 2)$.

Example 2: $f(x) = x^{2m-1}(a-x)^{2n}$ in $(0, a)$

Solution: $f(0) = f(a) = 0$. f is continuous and differentiable in $[0, a]$. By Rolle's theorem, $f'(c) = 0$ for some c in $(0, a)$.

$$f'(x) = (2m-1)x^{2m-2}(a-x)^{2n} + x^{2m-1} \cdot 2n(a-x)^{2n-1} \cdot (-1)$$

$$= x^{2m-1} \cdot (a-x)^{2n}\left[(2m-1) \cdot \frac{1}{x} - \frac{2n}{a-x}\right]$$

$$\therefore f'(c) = 0 \quad \text{if } (2m-1)\frac{1}{c} - \frac{2n}{a-c} = 0$$

or $$c = \frac{a(2m-1)}{(2m+2n-1)}.$$

Example 3: Find a root (solution) of the equation $x \ln x - 2 + x = 0$ lying in $(1, 2)$.

Solution: Choose $f(x) = (x-2)\ln x$ which is continuous, differentiable in $(1, 2)$ and further $f(1) = f(2) = 0$. Thus f satisfies all the 3 conditions of Rolle's theorem. So there exists a 'c' $\in (1, 2)$ such that $f'(c) = 0$. Here

$$f'(x) = \ln x + (x-2) \cdot \frac{1}{x}$$

$$f'(x) = \frac{x \ln x + x - 2}{x}$$

Thus $f'(c) = \frac{c \ln c + c - 2}{c} = 0$
or c is the root (solution) of the equation

$$x \ln x - 2 + x = 0.$$

Example 4: Show that the equation $f''(x) = 0$ has at least one real root between a and b if f, f', f'' are continuous in $a \le x \le b$ and the curve $y = f(x)$ crosses the x-axis at least at 3 distinct points between a and b inclusive.

Solution: Let c, d, e be the three points where $y = f(x)$ crosses the x-axis. Then $f(c) = f(d) = f(e) = 0$. Assume $a < c < d < e < b$. The function f satisfies Rolle's theorem in the two intervals (c, d) and (d, e), since f and f' are continuous and $f(c) = f(d) = f(e) = 0$. Therefore there exists at least one point in each interval (c, d) and (d, e) such that derivative is zero.
Let $c_1 \in (c, d)$ such that $f'(c_1) = 0$ and $c_2 \in (d, e)$ such that $f'(c_2) = 0$. Now the function f' satisfies Rolle's theorem because f', f'' are continuous and $f'(c_1) = f'(c_2) = 0$. Therefore by Rolle's theorem there exists a number c_3 in between c_1 and c_2 such that $f''(c_3) = 0$. Thus at least one root c_3 of the equation $f''(x) = 0$ lies in the interval (a, b).

Example 5: Without solving, show that the equation $x^4 + 2x^3 - 2 = 0$ has one and only one real root between 0 and 1.

Solution: Let $f(x) = x^4 + 2x^3 - 2$ which is continuous and differentiable in $(0, 1)$. $f'(x) = 4x^3 + 6x^2 = x^2(4x+6) = 0$ only when $x = 0$ or $x = -\frac{6}{4}$ both of which do not lie in between 0 and 1. Thus $f'(x) \neq 0$ for any $x \in (0, 1)$. By Rolle's theorem, there do not exist a and b such that $f(a) = 0$ and $f(b) = 0$ i.e., the equation $f(x) = 0$ can not have two real roots. Further $f(0) = -2$ and $f(1) = 1$ are of opposite signs and $f'(x) > 0$ for every x in $(0, 1)$. Therefore $f(x)$ crosses the x-axis exactly once. In other words, $f(x) = 0$ has one and only one root in between 0 and 1.

$(n-1)$ zeros in (a, b).

Hint: Suppose $x_1, x_2, \ldots x_n$ are the n zeros of f in (a, b) so $f(x_i) = 0$, for $i = 1, 2, \ldots n$. Then by Rolle's theorem applied to the $(n-1)$ intervals $(x_1, x_2), (x_2, x_3), \ldots (x_{n-1}, x_n) \exists c_i \ni f'(c_i) = 0$, for $i = 1, 2, \ldots n-1$.

Without solving exactly, show that the following equations have one and only one real root in the given interval.

23. $x^4 + 3x + 1 = 0$ in $(-2, -1)$.

24. $2x^3 - 3x^2 - 12x - 6 = 0$ in $(-1, 0)$.

25. If f and F are continuous in $[a, b]$ and derivable in (a, b) with $F'(x) \neq 0$ for every x in (a, b) then prove that there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{F'(c)} = \frac{f(c) - f(a)}{F(b) - F(c)}.$$

Hint: Choose $\phi(x) = f(x)[F(b) - F(a)] - [f(x) - f(a)][F(x) - F(a)]$.
$\phi(b) = f(a)[F(b) - F(a)] = \phi(a)$,
ϕ satisfies Rolle's theorem. But
$\phi'(x) = f'(x)[F(b) - F(a)] - f'(x)[F(x) - F(a)] - F'(x)[f(x) - f(a)]$, $\phi'(c) = 0$.

2.5 LAGRANGE'S MEAN VALUE THEOREM

Let f be (i) continuous in $[a, b]$ and (ii) derivable in (a, b). Then there exists at least one value $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof: Choose $\phi(x) = f(x) + x \cdot A, x \in [a, b]$. Since f and x are continuous in $[a, b]$ and derivable in (a, b), therefore ϕ is continuous in $[a, b]$ and derivable in (a, b). Now choose the unknown constant A such that $f(b) + b \cdot A = \phi(b) = \phi(a) = f(a) + a \cdot A$ so that

$$A = \frac{f(b) - f(a)}{a - b}$$

Thus ϕ satisfies all the three conditions of the Rolle's theorem. Therefore by Rolle's theorem there exists a number $c \in (a, b)$ such that

$$0 = \phi'(c) = f'(c) + A$$

Thus $f'(c) = -A = \frac{f(b) - f(a)}{b - a}$.

Alternate Form of Lagrange's Mean Value Theorem

If $b = a + h$ then

$$\frac{f(a+h) - f(a)}{a + h - a} = f'(a + \theta h)$$

$c = a + \theta h$ lies between a and $b = (a + h)$ when $0 < \theta < 1$.

Thus $f(a+h) = f(a) + hf'(a + \theta h), 0 < \theta < 1$.

Note 1: Lagrange's Mean Value (LMV) theorem is generalization of Rolle's theorem. In the special case when $f(b) = f(a)$ then LMV theorem reduces to Rolle's theorem.

Note 2: Geometrically LMV theorem states that the tangent to the curve at D is parallel to the chord AB (see Fig. 2.6).

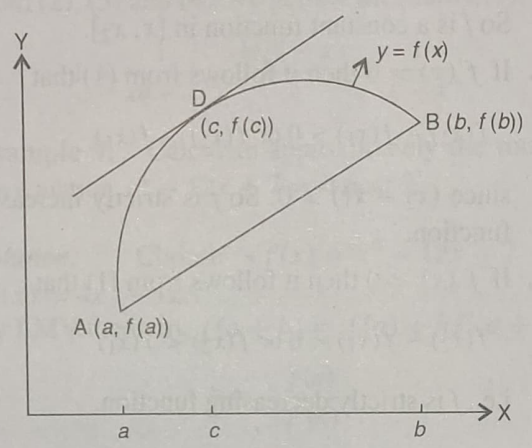

Fig. 2.6

Note 3: The average rate of change of f over the interval (a, b) given by $\frac{f(b) - f(a)}{b - a}$ is equal to $f'(c)$ which is the actual rate of change of f at some point of the interval (a, b).

Note 4: Using LMV theorem, approximate solution of equation $f(x) = 0$ can be obtained by Newton's method as follows:

Suppose $a + h$ is exact (actual) root so that

$$0 = f(a + h) = f(a) + h\, f'(a + \theta h), 0 < \theta < 1$$

Therefore $h \simeq -\frac{f(a)}{f'(a)}$.

Thus starting at a guess value 'a', h (correction) can be calculated approximately. By iteration a better root can be obtained.

Note 5: If M^* and m^* are the maximum and minimum of $f'(x)$ in $a < x < b$ then

$$m^*(b-a) < f(b) - f(a) < (b-a)M^*.$$

Note 6: Application of LMV theorem for sign of derivative.

Let $f(x)$ satisfy conditions of LMV theorem in $[a, b]$ and x_1, x_2 be any two points of $[a, b]$ such that $x_1 < x_2$. Applying LMV theorem in $[x_1, x_2]$, there exists a number $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c) \qquad (1)$$

I. If $f'(x) = 0$ for every $x \in (x_1, x_2)$ then

$$f(x_2) - f(x_1) = 0$$

or $f(x_2) = f(x_1) =$ constant

So f is a constant function in $[x, x_2]$.

II. If $f'(x) > 0$ then it follows from (1) that

$$f(x_2) - f(x_1) > 0 \text{ or } f(x_2) > f(x_1)$$

since $(x_2 - x_1) > 0$. So f is strictly increasing function.

III. If $f'(x) < 0$ then it follows from (1) that

$$f(x_2) - f(x_1) < 0 \text{ or } f(x_2) < f(x_1)$$

i.e., f is strictly decreasing function.

WORKED OUT EXAMPLES

Verify the Lagrange's Mean Value theorem:

Example 1: $f(x) = x^2$ in $(1, 5)$

Solution: f is continuous and differentiable in $(2, 3)$ so by LMV theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c in (a, b).

Here $f(5) = 25$, $f(1) = 1$, $f'(x) = 2x$. Therefore

$$\frac{25 - 1}{5 - 1} = 2c \quad \therefore c = 3 \in (1, 5).$$

Example 2: $f(x) = x^{\frac{2}{3}}, (-1, 1)$

Solution: LMV theorem is not applicable because f is not differentiable at $x = 0 \in (-1, 1)$.

Example 3: $f(x) = \cot \pi x, (-\frac{1}{2}, \frac{1}{2})$

Solution: LMV theorem is not valid because f is discontinuous at $0 \in (-\frac{1}{2}, \frac{1}{2})$.

Example 4: Deduce Lagrange's Mean Value theorem from Rolle's theorem.

Solution: Choose $g(x) = f(x) - f(a) - A(x - a)$ where A is a constant and f is continuous in $[a, b]$ and derivable in (a, b). Then $g(x)$ is also continuous in $[a, b]$ and derivable in (a, b). Further $g(a) = 0$ and determine A such that

$$g(b) = f(b) - f(a) - A(b - a) = 0$$

$$\text{Solving} \quad A = \frac{f(b) - f(a)}{b - a}$$

Thus g satisfies all the conditions of Rolle's theorem. Therefore there exists a number $c \in (a, b)$ such that

$$g'(c) = 0$$

But $g'(x) = f'(x) - A$

so $0 = g'(c) = f'(c) - A$

$$\therefore \quad f'(c) = A = \frac{f(b) - f(a)}{(b - a)}$$

which is the result of Lagrange's Mean Value theorem.

Example 5: Show that $\frac{h}{1+h^2} < \tan^{-1} h < h$ when $h \neq 0$ and $h > 0$.

Solution: Take $f(x) = \tan^{-1} x$ in $0 \le x \le h$. By LMV theorem

$$\frac{\tan^{-1} h - \tan^{-1} 0}{h - 0} = \frac{1}{1 + c^2}.$$

where $0 < c < h$.

or $\qquad \tan^{-1} h = \frac{h}{1+c^2}.$

$$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{\pi}{3} - \frac{1}{8}.$$

17. Determine the root of the equation $x^3 + 5x - 10 = 0$ which lies in (1, 2) correct to two decimal places.

Ans. 1.43

18. Using Mean Value theorem calculate approximately $\sqrt[6]{65}$.

Hint: Take $f(x) = x^{\frac{1}{6}}$ in (64, 65).

Use $f(a+h) = f(a) + hf'(a)$.

Ans. 2.0052

19. A circular hole 4 inches in diameter and 1 foot deep in a metal block is drilled out to increase the diameter to 4.12 inches. Estimate the amount of metal removed.

Hint: $V = f(x) = \pi(12)x^2$inches.
By LMV theorem

$$f(2.06) - f(2) = \text{by MVT} = 0.06\, f'(x_1)$$
$$= 0.06(24\pi x_1)$$

where $2 < x_1 < 2.06$.

Ans. 2.88π in^3

20. Find an approximate value of the root of the equation $x^3 - 2x - 5 = 0$ in (2, 3).

Hint:

$$h = -\frac{f(2)}{f'(2)} = +0.1, \text{root} = 2 + 0.1$$

$$h = -\frac{f(2.1)}{f'(2.1)} = -0.00543$$

$$\text{root} = 2.1 - 0.00543 = 2.0946.$$

Ans. 2.0946

2.6 CAUCHY'S MEAN VALUE THEOREM

Let $f(x)$ and $g(x)$ be two functions which are both derivable in $[a, b]$ and $g'(x) \neq 0$ for any value of x in $[a, b]$. Then there exists at least one value c in between a and b such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Define $\phi(x) = f(x) + Ag(x)$ where A is an unknown constant. ϕ is derivable in $[a, b]$ because f and g are derivable in $[a, b]$ by hypothesis. Choose A such that $\phi(b) = \phi(a)$ i.e.,

$$f(b) + Ag(b) = \phi(b) = \phi(a) = f(a) + Ag(a)$$

or
$$A = \frac{f(b) - f(a)}{g(a) - g(b)}$$

with $g(a) - g(b) \neq 0$. If $g(a) - g(b) = 0$ then $g(a) = g(b)$ and g satisfies conditions of Rolle's theorem and then $g'(c) = 0$ for some c. This contradicts the hypothesis that $g'(x) \neq 0$ for any x. Thus $g(a) - g(b) \neq 0$. Now the new function ϕ satisfies the conditions of Rolle's theorem. Therefore there exists at least one $c \in (a, b)$ such that

$$0 = \phi'(c) = f'(c) + Ag'(c)$$

Thus

$$\frac{f'(c)}{g'(c)} = -A = \frac{f(b) - f(a)}{g(b) - g(a)}$$

with $g(b) - g(a) \neq 0$ as $g'(c) \neq 0$ for any c.

Alternate Form of Cauchy's Mean Value Theorem

Replacing b by $a + h$,

$$\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h) - f(a)}{g(a+h) - g(a)}, 0 < \theta < 1$$

Note: Cauchy's mean value theorem is a generalization of Lagrange's mean value theorem. When $g(x) = x$, CMV theorem reduces to LMV theorem.

WORKED OUT EXAMPLES

Example 1: Verify Cauchy's mean value (CMV) theorem for the functions

i. $f(x) = x^4, g(x) = x^2$ in the interval $[a, b]$
ii. $f(x) = \ln x, g(x) = \frac{1}{x}$ in $[1, e]$.

Solution: CMV theorem states that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

i. $f(x) = x^4, g(x) = x^2, [a, b]$

$f' = 4x^3, g' = 2x$

By CMV theorem,

$$\frac{b^4 - a^4}{b^2 - a^2} = \frac{f'(c)}{g'(c)} = \frac{4c^3}{2c}$$

$$\therefore c^2 = \frac{1}{2}(a^2 + b^2) \quad \therefore c = \frac{1}{\sqrt{2}}\sqrt{a^2 + b^2} \in (a, b)$$

ii. $f' = \ln x$, $g(x) = \frac{1}{x}$, $[1, e]$

$$f' = \frac{1}{x}, \quad g' = -\frac{1}{x^2}$$

By CMV theorem,

$$\frac{\ln e - \ln 1}{\frac{1}{e} - 1} = \frac{1}{c} \cdot (-c^2) = -c$$

$$\therefore c = \frac{e}{e - 1} \in (1, e)$$

Example 2: If $0 \le a < b < \pi/2$ show that

$$0 < \cos a - \cos b < b - a.$$

Solution: Apply CMV theorem to functions $f(x) = \cos x$ and $g(x) = x$ in $[a, b]$. Then by CMV theorem

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos a - \cos b = (b - a)\sin c$$

Since cos is decreasing in $(0, \pi/2)$, so $\cos a > \cos b$ for $a < b$. Thus $\cos a - \cos b > 0$. Also in $(0, \pi/2)$ maximum value of $\sin x$ is 1. So since $c \in (a, b)$, and $\sin c < 1$

so $$(b - a)\sin c < (b - a)$$

Thus

$$0 < \cos a - \cos b < (b - a).$$

Example 3: Show that there exists a number $c \in (a, b)$ such that

$$2c\,[f(a) - f(b)] = f'(c) \cdot [a^2 - b^2]$$

when f is continuous in $[a, b]$ and derivable in (a, b).

Solution: By applying CMV theorem to the two functions $f(x)$ and $g(x) = x^2$ in $[a, b]$,

$$\frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(c)}{2c}$$

for some c in (a, b). Hence the result.

2.7 GENERALIZED MEAN VALUE THEOREM

Taylor's Theorem

Let $f^{(n-1)}$ the $(n-1)$th derivative of f is continuous in $[a, a+h]$, $f^{(n)}$ the nth derivative exists in $(a, a+h)$ and p be a given positive integer. Then there exists at least one number θ lying between 0 and 1 such that

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n \quad (1)$$

where $$R_n = \frac{h^n(1-\theta)^{n-p}}{(n-1)!\,p}f^{(n)}(a+\theta h) \quad (2)$$

and $0 < \theta < 1$

Proof: $f, f', f'', \ldots, f^{(n-2)}$ are continuous in $[a, a+h]$ by the hypothesis that $f^{(n-1)}$ is continuous in $[a, a+h]$.

Define

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \frac{(a+h-x)^3}{3!}f'''(x) + \cdots + \frac{(a+h-x)^{n-2}}{(n-2)!}f^{(n-2)}(x) + \frac{(a+h-x)^{n-1}}{(n-1)!} \times f^{(n-1)}(x) + (a+h-x)^p A \quad (3)$$

where A is unknown constant. Put $x = a+h$ in (3), we get

$$\phi(a+h) = f(a+h) + 0 + 0 \cdots + 0 \quad (4)$$

Put $x = a$ in (3), we get

$$\phi(a) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-2}}{(n-2)!}f^{(n-2)}(a) + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + h^p A \quad (5)$$

Choose the constant A such that $\phi(a) = \phi(a+h)$

Equating (3) and (4),

$$f(a+h) = \phi(a+h) = \phi(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-2}}{(n-2)!}f^{(n-2)}(a) + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + h^p A \quad (6)$$

The new function ϕ is continuous in $[a, a+h]$, derivable in $(a, a+h)$ and $\phi(a) = \phi(a+h)$ thus satisfying all the conditions of Rolle's theorem. Therefore it follows from Rolle's theorem that there exists at least one c between a and $a+h$ where

$$\phi'(c) = \phi'(a+\theta h) = 0.$$

Differentiating (3) w.r.t. x, we get

$$\phi' = f' - f' + (a+h-x)f'' + \frac{2(a+h-x)}{2!}(-1)f'' + \frac{(a+h-x)^2}{2!}f''' + \frac{3(a+h-x)^2}{3!}(-1)f''' + \cdots + \frac{(n-2)(a+h-x)^{n-3}}{(n-2)!}f^{(n-2)} + \frac{(a+h-x)^{n-2}}{(n-2)!}\cdot f^{(n-1)} + \frac{(n-1)\cdot(a+h-x)^{n-2}}{(n-1)!}(-1)f^{(n-1)} + \frac{(a+h-x)^{n-1}}{(n-1)!}f^n + p(a+h-x)^{p-1}A\cdot(-1).$$

or

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x) - pA\cdot(a+h-x)^{p-1}$$

Thus

$$0 = \phi'(a+\theta h) = \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h) - pA(1-\theta)^{p-1}h^{p-1}$$

Solving, $$A = \frac{h^{n-p}(1-\theta)^{n-p}}{(n-1)!\,p}f^{(n)}(a+\theta h) \quad (7)$$

with $h \neq 0$ and $1-\theta \neq 0$. Substituting A from (7) in (6), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^{(n)}(a+\theta h) \quad (8)$$

with $0 < \theta < 1$.

Taylor's remainder after n terms R_n due to

1. Schlomilch & Roche: $R_n = \frac{h^n(1-\theta)^{n-p}f^{(n)}(a+\theta h)}{(n-1)!p}$
2. Cauchy: $p = 1 : R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(a+\theta h)$
3. Lagrange's: $p = n : R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h)$

Putting $a + h = x$ or $h = x - a$ in (8) we get another convenient form of Taylor's theorem as

$$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n(1-\theta)^{n-p}}{(n-1)!p}f^{(n)}(a+\theta(x-a)) \quad (9)$$

with $0 < \theta < 1$.

Maclaurin's theorem is a special case of Taylor's theorem when $a = 0$. Thus putting $a = 0$ in (9)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n(1-\theta)^{n-p}}{(n-1)!p}f^{(n)}(\theta x) \quad (10)$$

Schlomilch remainder is given by $\frac{x^n(1-\theta)^{n-p}}{(n-1)!p} \cdot f^{(n)}(\theta x)$

Lagrange's Remainder for Maclaurin's Theorem: (Put p=n in Schlomilch Remainder)

$$\frac{x^n}{n!}f^{(n)}(\theta x)$$

Cauchy's Remainder for Maclaurin's Theorem: (Put p=1 in Schlomilch Remainder)

$$\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta x)$$

$$\cos\alpha = \frac{\quad}{\sqrt{a^2+b^2}}$$

2.8 TAYLOR'S SERIES AND MACLAURIN'S SERIES EXPANSIONS

From the Taylor's theorem with Lagrange's form of remainder, we get the Taylor's formula

$$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n}{n!}f^{(n)}(a+\theta(x-a)) \tag{1}$$

where $0 < \theta < 1$.

Note: Taylor's formula for $n = 1$, reduces to the Lagrange's mean value theorem (the law of means). Denote the first n terms of R.H.S. of (1) by $P_n(x)$ which is a polynomial of $(n-1)$th degree in the variable $(x-a)$ and the last term on R.H.S. of (1) by $R_n(x)$ which is the Lagrange's form of remainder after n terms i.e.,

$$P_n(x) = f(a) + (x-a)\frac{f'(a)}{1!} + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) \tag{2}$$

and

$$R_n(x) = \frac{(x-a)^n}{n!}f^n(a+\theta(x-a)) \tag{3}$$

Then $\quad f(x) = P_n(x) + R_n(x)$

or $\quad R_n(x) = f(x) - P_n(x)$

i.e., $\quad BC = AC - AB$

Thus for those values of x, for which $R_n(x)$ is small, the polynomial $P_n(x)$ yields an approximate

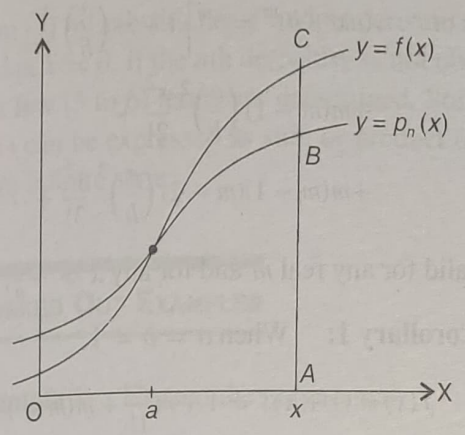

Fig. 2.7

representation of $f(x)$. When $a = 0$, the Taylor's formula (1) at origin (0, 0) reduces to

$$f(x) = f(0) + x\frac{f'(0)}{1!} + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\theta x) \quad (4)$$

(4) is known as Maclaurin's formula.

In (1) as $n \to \infty$ if $R_n(x) \to 0$, then $f(x)$ is represented by infinite series in powers of $(x - a)$ and is given by

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots \quad (5)$$

The power series (5) is known as Taylor's series. Rewriting in the summation form

$$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad (6)$$

Thus (5) or (6) is the Taylor's series expansion of $f(x)$ about the point 'a' (or in powers of $x - a$). When $a = 0$, (6) reduces to

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \quad (7)$$

The infinite series in powers of x given by (7) is called the Maclaurin's series expansion of $f(x)$ or simply Maclaurin's series of $f(x)$ (about origin).

Maclaurin Series Expansion of Some Elementary Functions

1. *Exponential function* $f(x) = e^x$.

   Differentiating n times, $f^{(n)}(x) = e^x$. So at $x = 0$, $f^{(n)}(0) = 1$. Substituting in the Maclaurin's formula (4), we obtain

   $$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}$$

   where $0 < \theta < 1$.

   For $0 < \theta < 1$, $e^{\theta x}$ is bounded for any $x (< e^x$ for $x > 0$ and < 1 for $x < 0)$. Thus the Lagrange's remainder

   $$R_n = \frac{x^n}{n!}e^{\theta x} \to 0 \text{ as } n \to \infty \text{ for any } x.$$

   Hence the Maclaurin series expansion of e^x is

   $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots; -\infty < x < \infty$$

   Therefore e^x can be evaluated to any degree of accuracy by taking sufficient number of terms.

2. *Expansion of* $f(x) = \cos x$

   Differentiating n times, $f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$, so $f^{(n)}(0) = \cos n\frac{\pi}{2}$. The remainder

   $$R_n(x) = \frac{x^n}{n!}f^{(n)}(\theta x) = \frac{x^n}{n!}\cos\left(\theta x + \frac{n\pi}{2}\right) \to 0 \text{ as } n \to \infty$$

   since $\left|\cos\left(\theta x + \frac{n\pi}{2}\right)\right| \le 1$, for all values of x.

   At $x = 0$, $f(0) = 1$, $f^{(1)}(0) = \cos\frac{\pi}{2} = 0$, $f^{(2)}(0) = \cos\pi = -1$, $f^{(3)}(0) = \cos\frac{3\pi}{2} = 0$, $f^{(4)}(0) = \cos 2\pi = 1$ and so on. Substituting these values in the Maclaurin series given by (7), we get

   $$f(x) = \cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \cdots$$
   $$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots; \quad -\infty < x < \infty.$$

3. *Expansion of* $f(x) = \sin x$.

   We know that $f^n(x) = \sin\left(x + n\frac{\pi}{2}\right)$. So $f^{(n)}(0) = \sin n\frac{\pi}{2}$ then $f(0) = 0$, $f'(0) = \sin\frac{\pi}{2} = 1$, $f^{(2)}(0) = \sin\pi = 0$, $f^{(3)}(0) = \sin\frac{3\pi}{2} = -1$, $f^{(4)}(0) = \sin 2\pi = 0$ etc.

   Substituting these values in (7), we get the Maclaurin series expansion of $f(x)$ valid for any

x as

$$f(x)=\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots, \quad -\infty<x<\infty$$

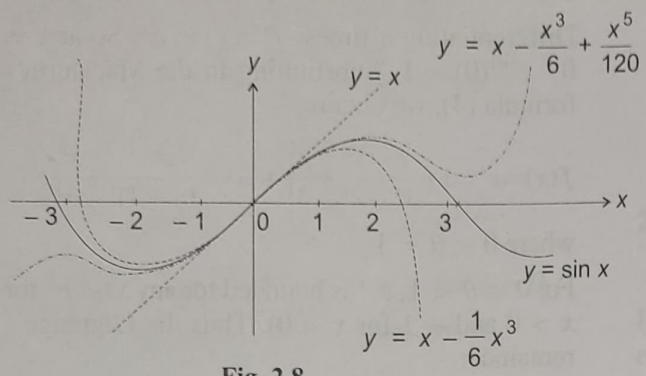

Fig. 2.8

Figure 2.8 shows the graphs of the function $f(x) = \sin x$ and the first three approximations

$$y = x,\ y = x - \frac{x^3}{3!},\ y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

4. *Expansion of* $f(x) = \sinh x$.

We know that $f^{(2n+1)}(x) = \cosh x$ and $f^{(2n)}(x) = \sinh x$ so $f^{(2n+1)}(0) = 1$, $f^{(2n)}(0) = 0$, $f(0) = 0$. Substituting in (7),

$$f(x)=\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots; \quad -\infty<x<\infty$$

5. *Expansion of* $f(x) = \cosh x$.

Since $f^{(2n)}(x) = \cosh x$ and $f^{(2n+1)}(x) = \sinh x$, we have $f^{(2n)}(0) = 1$, $f^{(2n+1)}(0) = 0$, $f(0) = 1$. From (7),

$$f(x)=\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots; \quad -\infty<x<\infty$$

6. *Expansion of* $f(x) = (ax + b)^m$ where m is any real number and $x > -\frac{b}{a}$ (i.e., $ax + b > 0$) and $a \neq 0$. The nth derivative of f is

$$f^{(n)}(x)=m(m-1)(m-2)\cdots(m-n+1)a^n(ax+b)^{m-n}$$

At $x = 0$, $f^{(n)}(0) = m(m-1)(m-2)\cdots(m-n+1)a^n \cdot b^{m-n}$ and $f(0) = b^m$.

Substituting in (7), we obtain

$$f(x) = (ax+b)^m = b^m + m \cdot a \cdot b^{m-1}x + m(m-1)a^2 \cdot b^{m-2}\frac{x^2}{2!} + m(m-1)(m-2) \cdot a^3 b^{m-3} \cdot \frac{x^3}{3!} + \cdots$$

or

$$(ax+b)^m = b^m\left[1 + m\left(\frac{a}{b}\right)\frac{x}{1!} + m(m-1)\left(\frac{a}{b}\right)^2\frac{x^2}{2!} + m(m-1)(m-2)\left(\frac{a}{b}\right)^3\frac{x^3}{3!} + \cdots\right]$$

valid for any real m and for any $x > -\frac{b}{a}, a \neq 0$.

Corollary 1: When $a = b = 1$,

$$f(x) = (1+x)^m = 1 + m\frac{x}{1!} + m(m-1)\frac{x^2}{2!} + m(m-1)(m-2)\frac{x^3}{3!} + \cdots$$

valid for any real m and for any $x > -1$.

7. *Expansion of* $f(x) = \ln(1 + x)$. We know that $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ so

$f^{(n)}(0) = (-1)^{n-1} \cdot (n-1)!$ and $f(0) = 0$. From (7),

$$f(x) = \ln(1+x) = 0 + x - \frac{x^2}{2!} + 2!\frac{x^3}{3!} - 3!\frac{x^4}{4!} + \cdots + (-1)^{n-1}(n-1)!\frac{x^n}{n!} + \cdots$$

or

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

valid for $1 + x > 0$ i.e., $x > -1$.

8. *Expansion of* $f(x) = \ln(1 - x)$.

Here $f^{(n)}(x) = \frac{(-1)^{n-1}\cdot(n-1)!(-1)^n}{(1-x)^n}$, so $f^{(n)}(0) = -(n-1)!$

From (7),

$$f(x) = \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots - \frac{x^n}{n} - \cdots$$

$$= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \cdots\right)$$

valid for $x < 1$.

Note: For any given function $f(x)$, in general, the nth derivative $f^{(n)}(x)$ can not be determined and therefore the nature of R_n not known. Therefore by assuming that $R_n \to 0$ as $n \to \infty$, formal Maclaurin's expansions of any given function is obtained

2.9 INDETERMINATE FORMS

If two functions $f(x)$ and $g(x)$ are *both* zero at $x = a$, the fraction $\frac{f(a)}{g(a)}$ is said to assume the indeterminate form $\frac{0}{0}$. Although the function $F(x) = \frac{f(x)}{g(x)}$ is undefined (indeterminate) at $x = a$, it may however approach a limit as x approaches a. The process of determining such a limit, if it exists, is known as the evaluation of indeterminate forms. The L'Hospital's rule (theorem) allows the evaluation of indeterminate forms.

L'Hospital's Rule

Let $f(x)$ and $g(x)$ be continuous in an interval (c, d) containing $x = a$. Suppose $f(a) = 0$ and $g(a) = 0$. Also the derivatives of f and g exist and $g^{i}(x) \neq 0$ in (c, d) except possibly at $x = a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{1}$$

provided the limit on R.H.S. of (1) exists.

Proof: Since $f(x)$ and $g(x)$ satisfy conditions of Cauchy's mean value theorem in (c, d), we have

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(e)}{g'(e)}$$

where $e \epsilon (c, d)$ such that $x < e < a$ or $a < e < x$. From hypothesis $f(a) = 0$, $g(a) = 0$. So

$$\frac{f(x)}{g(x)} = \frac{f'(e)}{g'(e)}$$

As $x \to a$, $e \to a$, we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note: In R.H.S. of (1) the functions $f(x)$ and $g(x)$ are differentiated w.r.t. x, separately, *not by using the quotient rule*.

Corollary 1: Suppose $f'(a) = 0$, and $g'(a) = 0$ in addition to $f(a) = 0$ and $g(a) = 0$. Then applying Cauchy's mean value theorem to $f'(x)$ and $g'(x)$, we get the L'Hospital's rule as

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(a)}{g''(a)}$$

Corollary 2: If $f(a) = 0$, $g(a) = 0$, $f'(a) = 0$, $g'(a) = 0, \ldots, f^{(n-1)}(a) = 0$, $g^{(n-1)}(a) = 0$, and $g^{(n)}(a) \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(a)}{g^{(n)}(a)}$$

Corollary 3: As $x \to \infty$ if $\frac{f(x)}{g(x)}$ assumes indeterminate form $\frac{0}{0}$ then the L'Hospital's rule is

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

Results similar to Corollary 1, and Corollary 2 are valid as $x \to \infty$.

Indeterminate Forms of the Type $\frac{\infty}{\infty}$

Suppose $f(x) = \infty$ and $g(x) = \infty$ as $x \to a$ (or as $x \to \pm\infty$). Then $\frac{f(x)}{g(x)}$ assumes the indeterminate form $\frac{\infty}{\infty}$ which can also be evaluated by L'Hospital's rule.

$$\lim_{\substack{x \to a \\ (\text{or } x \to \pm\infty)}} \frac{f(x)}{g(x)} = \lim_{\substack{x \to a \\ (\text{or } x \to \pm\infty)}} \frac{f'(x)}{g'(x)}$$

Results similar to Corollary 1 and Corollary 2 are valid in this case also.

Important Note: While evaluating the indeterminate form $\frac{\infty}{\infty}$, in many cases, it is advised to convert the indeterminate from $\frac{\infty}{\infty}$ to the form $\frac{0}{0}$ as early as possible (say by rewriting as $\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$). Otherwise the process of differentiation of the numerator $f(x)$ and denominator $g(x)$ may *never* terminate, thus complicating the problem.

Indeterminate Form $0 \cdot \infty$

As $x \to a$ (or $x \to \pm\infty$) if $f(x) \to 0$ and $g(x) \to \infty$ then the product $f(a) \cdot g(a)$ is undefined and is said to assume the indeterminate form $0 \cdot \infty$. This can be converted to the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by rewriting

$$f(x) \cdot g(x) = \frac{f(x)}{\left(\frac{1}{g(x)}\right)} \quad \text{or} \quad \frac{g(x)}{\left(\frac{1}{f(x)}\right)}$$

and apply L'Hospital's rule.

Indeterminate Form $\infty - \infty$

As $x \to a$(or $x \to \pm\infty$) if $f(x) \to \infty$ and $g(x) \to \infty$ then the difference $f(a) - g(a)$ is indeterminate of the form $\infty - \infty$. Rewriting the difference into a fraction of algebraic means, we get

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}$$

which $\frac{0}{0}$ form. Applying L'Hospital's rule, the indeterminate form $\infty - \infty$ is evaluated.

Indeterminate Forms of the Type $0^0, \infty^0, 1^\infty$

As $x \to a$ (or $x \to \neq \infty$), the expression $f(a)^{g(a)}$ is said to be (i) indeterminate form of type 0^0 if $f(x) \to 0$ and $g(x) \to 0$ (ii) indeterminate form of type ∞^0 if $f(x) \to \infty$ and $g(x) \to 0$ (iii) indeterminate form of type 1^∞ if $f(x) \to 1$ and $g(x) \to \infty$. To evaluate these forms consider.

$$y(x) = f(x)^{g(x)}$$

Taking logarithm

$$\ln y = g(x) \ln f(x)$$

which is of the form $0 \cdot \infty$ and can be evaluated as in *Indeterminate Form* $0 \cdot \infty$. Taking the limit as $x \to a$ (or $x \to \pm\infty$)

$$\lim_{x \to a} \ln y = k \text{ (say)}.$$

Then

$$k = \lim_{x \to a} (\ln y) = \ln\left(\lim_{x \to a} y\right) = \ln\left(\lim_{x \to a} f(x)^{g(x)}\right)$$

$$\therefore \lim_{x \to a} f(x)^{g(x)} = e^k.$$

Notes:

i. Maclaurin series expansion may be used to simplify the expressions.

ii. Standard limits such as $\lim_{x\to 0}\frac{\sin x}{x}=1$,

$\lim_{x\to 0}\frac{\tan x}{x}=1$, $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$ may also be used.

WORKED OUT EXAMPLES

Type $\frac{0}{0}$ form

Example 1: Evaluate

(a) $\lim_{x\to 1}\frac{1+\ln x-x}{1-2x+x^2}$ (b) $\lim_{x\to 0}\frac{e^{ax}-e^{-ax}}{\ln(1+bx)}$

(c) $\lim_{x\to\frac{\pi}{4}}\frac{(\sec^2 x-2\tan x)}{1+\cos 4x}$

Solution:

a. At $x=1$, $\frac{1+\ln x-x}{1-2x+x^2}=\frac{1+\ln 1-1}{1-2\cdot 1+1^2}=\frac{0}{0}$, indeterminate. Applying L'Hospital's rule

$$\lim_{x\to 1}\frac{1+\ln x-x}{1-2x+x^2}=\lim_{x\to 1}\frac{(1+\ln x-x)^{\text{i}}}{(1-2x+x^2)^{\text{i}}}1$$

$$=\lim_{x\to 1}\frac{0+\frac{1}{x}-1}{-2+2x}$$

which is again of the $\frac{0}{0}$ form. Applying L'Hospital's rule again, we get

$$\lim_{x\to 1}\frac{1+\ln x-x}{1-2x+x^2}=\lim_{x\to 1}\frac{(\frac{1}{x}-1)^{\text{i}}}{(-2+2x)^{\text{i}}}$$

$$=\lim_{x\to 1}\frac{-\frac{1}{x^2}}{2}=-\frac{1}{2}.$$

b. At $x=0$, $\frac{e^{ax}-e^{-ax}}{\ln(1+bx)}=\frac{1-1}{\ln 1}=\frac{0}{0}$ form.

Applying L'Hospital's rule

$$\lim_{x\to 0}\frac{e^{ax}-e^{-ax}}{\ln(1+bx)}=\lim_{x\to 0}\frac{\frac{d}{dx}(e^{ax}-e^{-ax})}{\frac{d}{dx}(\ln(1+bx))}$$

$$=\lim_{x\to 0}\frac{ae^{ax}+ae^{-ax}}{\frac{b}{1+bx}}=\frac{a+a}{\frac{b}{1}}=\frac{2a}{b}$$

c. $\lim_{x\to\frac{\pi}{4}}\frac{\sec^2 x-2\tan x}{1+\cos 4x}:\frac{0}{0}$ form. Applying the rule

$$=\lim_{x\to\frac{\pi}{4}}\frac{\frac{d}{dx}(\sec^2 x-2\tan x)}{\frac{d}{dx}(1+\cos 4x)}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{2\sec x\cdot\sec x\cdot\tan x-2\sec^2 x}{-4\sin 4x}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\sec^2 x(\tan x-1)}{-2\sin 4x}:\frac{0}{0}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{2\sec x\cdot\sec x\cdot\tan x(\tan x-1)+\sec^2 x\cdot\sec^2 x}{-8\cos 4x}$$

$$=\frac{2\cdot 2\cdot 1\cdot(1-1)+2\cdot 2}{-8\cdot(-1)}=\frac{1}{2}$$

(since $\cos 4\frac{\pi}{4}=-1$, $\sec^2\frac{\pi}{4}=2$, $\tan\frac{\pi}{4}=1$).

Example 2: Evaluate

i. $\lim_{x\to 0}\frac{\cos x-\ln(1+x)-1+x}{\sin^2 x}$

ii. $\lim_{x\to 0}\frac{x^2+2\cos x-2}{x\sin^3 x}$

iii. $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}$ is finite. Find a and the limit.

Solution:

i. $\frac{0}{0}$ form use Maclaurin series expansions for $\cos x$, $\ln(1+x)$ and $\sin x$. Then

$$\lim_{x\to 0}\frac{\cos x-\ln(1+x)-1+x}{\sin^2 x}$$

$$=\lim_{x\to 0}\frac{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-1+x}{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)^2}$$

$$=\lim_{x\to 0}\frac{-\frac{x^3}{3}+\frac{7}{24}x^4+\cdots}{x^2-\frac{x^6}{(3!)^2}+\frac{x^{10}}{(5!)^2}+\cdots-\frac{2x^4}{3!}+\frac{2x^6}{5!}+\cdots}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{3}x+\frac{7}{24}x^3+\cdots}{1-\frac{1}{3}x^2-\frac{1}{36}x^4+\cdots}=\frac{0}{1}=0.$$

ii. $\lim_{x\to 0}\frac{x^2+2\cos x-2}{x\cdot\sin^3 x}$

$$=\lim_{x\to 0}\frac{x^2+2\cos x-2}{x\cdot x^3}\frac{x^3}{\sin^3 x}$$

$$=\lim_{x\to 0}\frac{x^2+2\cos x-2}{x^4}\cdot\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^3$$

$$=\lim_{x\to 0}\frac{x^2+2\cos x-2}{x^4}\cdot 1=\frac{0}{0}$$

since $\lim_{x\to 0}\frac{\sin x}{x}=1$. Now applying L'Hospital's rule

$$=\lim_{x\to 0}\frac{2x-2\sin x}{4x^3}=\lim_{x\to 0}\frac{1-\cos x}{6x^2}=\lim_{x\to 0}\frac{\sin x}{12x}$$

$$= \frac{1}{12}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{12}\cdot 1 = \frac{1}{12}.$$

iii. At $x = 0$, $\frac{\sin 2x + a\sin x}{x^3}$ is $\frac{0}{0}$ for any a.

Applying L'Hospital's rule

$$\lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}$$

$$= \lim_{x\to 0}\frac{2\cdot\cos 2x + a\cos x}{3x^2} = \infty.$$

In order that the limit is finite, choose $a = -2$ in which case it reduces to $\frac{0}{0}$ form. Applying L'Hospital's rule

$$= \lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{-4\cos 2x + \cos x}{3} = \frac{-4+1}{3} = -1.$$

Type $\frac{\infty}{\infty}$ form

Example 3: Evaluate (i) $\lim_{x\to\infty}\frac{x^n}{e^x}$

(ii) $\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}}$ (iii) $\lim_{x\to 0}\log_{\tan x}\tan 2x$.

Solution:

i. $\frac{\infty}{\infty}$ form. Applying L'Hospital's rule repeatedly n times

$$\lim_{x\to\infty}\frac{x^n}{e^x} = \lim_{x\to 0}\frac{\frac{d^n}{dx^n}x^n}{\frac{d^n}{dx^n}e^x} = \lim_{x\to\infty}\frac{n!}{e^x} = 0.$$

ii. $\frac{\infty}{\infty}$ form. Applying L'Hospital's rule

$$\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} = \frac{1}{\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x}$$

$$= \frac{\sqrt{1+x^2}}{x} : \frac{\infty}{\infty} \text{ form.}$$

Again applying L'Hospital's rule

$$= \lim_{x\to\infty}\frac{\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x}{1}$$

$$= \lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} : \frac{\infty}{\infty} : \text{ which is the original function.}$$

Instead introduce $z = \frac{1}{x^2}$ then

$$\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} = \lim_{z\to 0}\frac{x}{x\sqrt{\frac{1}{x^2}+1}}$$

$$= \lim_{z\to 0}\frac{1}{\sqrt{z+1}} = 1.$$

iii. $\frac{\infty}{\infty}$ form.

$$\lim_{x\to 0}\log_{\tan x}\tan 2x = \lim_{x\to 0}\frac{\log\tan 2x}{\log\tan x} : \frac{\infty}{\infty}$$

Applying L'Hospital's rule

$$= \lim_{x\to 0}\frac{1}{\tan 2x}\cdot 2\cdot\sec^2 2x\cdot\frac{\tan x}{\sec^2 x} : \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0}\frac{\sin x\cdot\cos x\cdot 2}{\sin 2x\cdot\cos 2x} = \lim_{x\to 0}\frac{2\cdot\frac{1}{2}\sin 2x}{\frac{1}{2}\sin 4x}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{4x}{\sin 4x} = 1\cdot 1 = 1.$$

Type $0\cdot\infty$ form

Example 4: Evaluate

i. $\lim_{x\to 1}\log(1-x)\cdot\cot\frac{\pi x}{2}$

ii. $\lim_{x\to a}(a-x)\cdot\tan\frac{\pi x}{2a}$.

Solution:

i. $\infty\cdot 0$ forms. Rewriting in $\frac{\infty}{\infty}$ form

$$\lim_{x\to 1}\log(1-x)\cdot\cot\frac{\pi x}{2} = \lim_{x\to 1}\frac{\ln(1-x)}{\tan\frac{\pi x}{2}}.$$

Applying L'Hospital's rule

$$= \lim_{x\to 1}\frac{\frac{1}{1-x}\cdot(-1)}{\frac{\pi}{2}\cdot\sec^2\frac{\pi x}{2}} : \frac{\infty}{\infty} \text{ form.}$$

Rewriting in $\frac{0}{0}$ form

$$= \lim_{x\to 1}\frac{-2}{\pi}\frac{\cos^2\frac{\pi x}{2}}{1-x} : \frac{0}{0} \text{ form.}$$

Applying L'Hospital's rule

$$= \lim_{x\to 1} -\frac{2}{\pi}\cdot\frac{2\cdot\cos\frac{\pi x}{2}\cdot(-\sin\frac{\pi x}{2})\cdot\frac{\pi}{2}}{-1} = 0$$

ii. $0 \cdot \infty$ form. Rewriting

$$\lim_{x\to a}(a-x)\cdot\tan\frac{\pi x}{2a} = \lim_{x\to a}\frac{\tan\frac{\pi x}{2a}}{\left(\frac{1}{a-x}\right)} : \frac{\infty}{\infty} \text{ form.}$$

Applying L'Hospital's rule

$$= \lim_{x\to a}\frac{\frac{2a}{\pi}\cdot\sec^2\frac{\pi x}{2a}}{-\frac{1}{(a-x)^2}(-1)} : \frac{\infty}{\infty} \text{ form.}$$

Rewriting in $\frac{0}{0}$ form

$$= \lim_{x\to a}\frac{2a}{\pi}\cdot\frac{(a-x)^2}{\cos^2\left(\frac{\pi x}{2a}\right)} =$$

$$= \lim_{x\to a}\frac{2a}{\pi}\cdot\frac{(-2)(a-x)}{2\cdot\cos\left(\frac{\pi x}{2a}\right)\cdot\left(-\sin\frac{\pi x}{2a}\right)\cdot\frac{\pi}{2a}} : \frac{0}{0}$$

$$= \lim_{x\to a}\frac{-1}{\frac{\pi}{2a}\cdot\cos^2\frac{\pi x}{2a} - \frac{\pi}{2a}\sin^2\frac{\pi x}{2a}}$$

$$= \frac{-1}{0-\frac{\pi}{2a}} = \frac{2a}{\pi}.$$

Type $\infty - \infty$ form

Example 5: Evaluate

i. $\lim_{x\to\frac{\pi}{2}}(\sec x - \tan x)$

ii. $\lim_{x\to 1}\left[\frac{x}{x-1} - \frac{1}{\ln x}\right]$

iii. $\lim_{x\to 0}\left(\frac{1}{x^2} - \cot^2 x\right)$

Solution:

i. $\infty - \infty$ form. Rewriting

$$\lim_{x\to\frac{\pi}{2}}(\sec x - \tan x) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{\cos x}\right) : \frac{0}{0}$$

Applying L'Hospital's rule

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\sin x} = \frac{0}{1} = 0.$$

ii. $\infty - \infty$ form. Rewriting

$$\lim_{x\to 1}\left[\frac{x}{x-1} - \frac{1}{\ln x}\right] = \lim_{x\to 1}\frac{x\ln x - (x-1)}{(x-1)\ln x} : \frac{0}{0} \text{ form.}$$

$$= \lim_{x\to 1}\frac{x\cdot\frac{1}{x} + \ln x - 1}{1\cdot\ln x + (x-1)\cdot\frac{1}{x}} : \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{1+1} = \frac{1}{2}.$$

iii. $\infty - \infty$ form. Rewriting

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \cot^2 x\right) = \lim_{x\to 0}\left(\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}\right) : \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{\sin^2 x - x^2\cos^2 x}{x^4}\cdot\lim_{x\to 0}\frac{x^2}{\sin^2 x}.$$

Since $\lim_{x\to 0}\frac{\sin x}{x} = 1$,

$$= \lim_{x\to 0}\left[\frac{1}{x^2}\cdot\frac{\sin^2 x}{x^2} - \frac{\cos^2 x}{x^2}\right]\cdot 1$$

$$= \lim_{x\to 0}\left(\frac{1}{x^2}\cdot 1 - \frac{\cos^2 x}{x^2}\right) : \infty - \infty$$

$$= \lim_{x\to 0}\left(\frac{1-\cos^2 x}{x^2}\right) : \frac{0}{0} \text{ form. Applying rule}$$

$$= \lim_{x\to 0}\frac{-2\cos x\cdot(-\sin x)}{2x}$$

$$= \lim_{x\to 0}\cos x\cdot\lim_{x\to 0}\frac{\sin x}{x} = 1.1 = 1.$$

Types 0^0, ∞^0, 1^∞

Example 6: Evaluate (0^0 form)

(i) $\lim_{x\to 0}(x)^x$ (ii) $\lim_{x\to\frac{\pi}{2}}(\cos x)^{\left(\frac{\pi}{2}-x\right)}$

(iii) $\lim_{x\to 0}(x+\sin x)^{\tan x}$.

Solution:

i. 0^0 form. Put $y = x^x$. Take logarithm, then $\lim_{x\to 0}\ln y = \lim_{x\to 0} x\ln x = 0\cdot\infty$ form. Rewriting

$$= \lim_{x\to 0}\frac{\ln x}{\frac{1}{x}} : \frac{\infty}{\infty} \text{ form} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -x = 0$$

$$\therefore \lim_{x\to 0} x^x = \lim_{x\to 0} y = e^0 = 1.$$

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Section - A

Q→ 1. (a). Calculate approx. value of $\sqrt{24}$ upto 3 decimal places by Taylor's expansion.

Let $f(x) = \sqrt{x}$, where $x = 24$

Let at $h = 24$

$25 - 1 = 24$

$a = 25$, $h = -1$

From Taylor's Expansion

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$$

$f(x) = \sqrt{x}$	$f(a) = f(25) = \sqrt{25}$
$f'(x) = \frac{1}{2} x^{-1/2}$	$f'(a) = f'(25) = \frac{1}{2}(25)^{-1/2}$
$f''(x) = \frac{-1}{4} x^{-3/2}$	$f''(a) = f''(25) = \frac{-1}{4}(25)^{-3/2}$
$f'''(x) = \frac{3}{8} x^{-5/2}$	$f'''(a) = f'''(25) = \frac{3}{8}(25)^{-5/2}$

Now;

$$f(\sqrt{25-1}) = f(25) + (-1) f'(25) + \frac{(-1)^2}{2!} f''(25) + \frac{(-1)^3}{3!} f'''(25)$$

$$f(\sqrt{24}) = \sqrt{25} + (-1) \cdot \frac{1}{2} \cdot (25)^{-1/2} + \frac{1}{2}\left(\frac{-1}{4}\right)(25)^{-3/2} + \left(\frac{-1}{6}\right) \cdot \frac{3}{8}(25)^{-5/2}$$

$$f(\sqrt{24}) = 5 - \frac{1}{2}(5^2)^{-1/2} - \frac{1}{8}(5^2)^{-3/2} - \frac{1}{16}(5^2)^{-5/2}$$

$$f(\sqrt{24}) = 5 - \frac{1}{2 \times 5} - \frac{1}{8 \times 5^3} - \frac{1}{16 \times 5^5}$$

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$$f(\sqrt{24}) = 5 - \frac{1}{10} - \frac{1}{8 \times 125} - \frac{1}{16 \times 3125}$$

$$f(\sqrt{24}) = 5 - \frac{1}{10} - \frac{1}{1000} - \frac{1}{50000}$$

$$f(\sqrt{24}) = 5 - 0.1 - 0.001 - 0.00002$$

$$\boxed{f(\sqrt{24}) = 4.89898}$$ Ans

(b). $\lim_{x \to 0} \frac{x - \sin x}{\tan^3 x}$

$$\frac{0 - \sin 0}{\tan^3 0} = \frac{0-0}{0} = \frac{0}{0}$$; Indeterminate form.

By Applying L'Hopital's Rule.

$$\lim_{x \to 0} \frac{1 - \cos x}{3 \tan^2 x \sec^2 x}$$

Multiply and Divide Denominator with x^2.

$$\lim_{x \to 0} \frac{1 - \cos x}{3 \tan^2 x \sec^2 x \cdot \frac{x^2}{x^2}}$$

$$\frac{\tan^2 x}{x^2} = 1 \left[\because \lim_{x \to 0} \frac{\tan x}{x} = 1 \right]$$

$$\lim_{x \to 0} \frac{1 - \cos x}{3x^2 \sec^2 x}$$

$$\lim_{x \to 0} \frac{1}{3x^2 \sec^2 x} - \frac{\cos x}{3x^2 \sec^2 x}$$

$$\lim_{x \to 0} \frac{\cos^2 x}{3x^2} - \frac{\cos^3 x}{3x^2} = \frac{0}{0}$$

Again Applying L'Hopital's Rule.

$$\lim_{x \to 0} \frac{2(\cos x)(-\sin x)}{6x} - \frac{3\cos^2 x(-\sin x)}{6x}$$

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$$\lim_{x\to 0} -\frac{\sin x \cos x}{3x} + \frac{\sin x \cos^2 x}{2x} \quad \left[\lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

$$\lim_{x\to 0} \frac{-\cos x}{3} + \frac{\cos^2 x}{2}$$

$$\frac{-\cos 0}{3} + \frac{\cos^2 0}{2}$$

$$\frac{-1}{3} + \frac{1}{2} = \frac{-2+3}{6} = \boxed{\frac{1}{6}} \text{ Ans}$$

(c). State Taylor's Theorem with lagrange's form of remainder.

Let $f(x)$ be a function defined in $[a,b]$ such that

i) $f', f'', f''', \ldots f^{n-1}$ are continuous in $[a,b]$.

ii) $f^n(x)$ should exist in (a,b).

then $\exists$ a real value $c \in (a,b)$ such that

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \cdots$$

$$\cdots \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \boxed{\frac{(b-a)^n}{n!} f^n(c)}$$ lagrange's form of remainder

where $c = a + \theta h$; $0 < \theta < 1$

$h = b - a$

OR

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$$\cdots \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \boxed{\frac{h^n}{n!} f^n(a+\theta h)}$$ lagrange's form of remainder

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(d). Evaluate $\lim_{x \to 0} \frac{1-\cos^2 x}{\sin x^2}$

$\frac{1-\cos^2 0}{\sin 0^2} = \frac{1-1}{0} = \frac{0}{0}$; Indeterminate form.

$\lim_{x \to 0} \frac{1-\cos^2 x}{\sin x^2}$ $\left[\begin{array}{l} \sin^2 x + \cos^2 x = 1 \\ \sin^2 x = 1-\cos^2 x \end{array}\right]$

$\lim_{x \to 0} \frac{\sin^2 x}{\sin x^2}$

Applying l' Hopital's Rule

$\lim_{x \to 0} \frac{\cancel{2} \sin x \cos x}{\cos x^2 \cdot \cancel{2} x}$ $\left[\lim_{x \to 0} \frac{\sin x}{x} = 1\right]$

$\lim_{x \to 0} \frac{\cos x}{\cos x^2}$

$= \frac{\cos 0}{\cos (0)^2} = \frac{1}{1} = \boxed{1}$ Ans

(e). For what values of x and y following equation is true.

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$2x+3 = 7 \rightarrow (1)$ $\quad$ $15 = 15$

$6 = 6$ $\quad$ $2y-4 = 14 \rightarrow (2)$.

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Solving eq ①

$2x+3=7$

$2x=4$

$\boxed{x=2}$ Ans

Solving eq ②

$2y-4=14$

$2y=18$

$\boxed{y=9}$ Ans

Section → B

Q→2. Find the matrices P & Q such that PAQ is in the normal form; where $A=\begin{bmatrix}1&1&2\\1&2&3\\0&-1&-1\end{bmatrix}$.

$$A=\begin{bmatrix}1&1&2\\1&2&3\\0&-1&-1\end{bmatrix}_{3\times3}$$

$$A = I\ A\ I$$

$$A = I_{3\times3}\ A_{3\times3}\ I_{3\times3}$$

$$\begin{bmatrix}1&1&2\\1&2&3\\0&-1&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} A \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix}1&1&2\\0&1&1\\0&-1&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix} A \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 - 2C_1$

$$\begin{bmatrix}1&0&0\\0&1&-1\\0&-1&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix} A \begin{bmatrix}1&-1&-2\\0&1&0\\0&0&1\end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix}1&0&0\\0&1&-1\\0&0&-2\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\-1&1&1\end{bmatrix} A \begin{bmatrix}1&-1&-2\\0&1&0\\0&0&1\end{bmatrix}$$

$C_3 \to C_3 + C_2$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \rightarrow C_3 \times \left(\frac{-1}{2}\right)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$$

Hence; $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 1 & -1 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$

Q→3 a). For what value of x the matrix $\begin{bmatrix} 2+x & 3 & 4 \\ 1 & -1 & 2 \\ x & 1 & -5 \end{bmatrix}$ is a singular matrix.

For singular matrix, $|A| = 0$.

$$|A| = (2+x)\begin{vmatrix} -1 & 2 \\ 1 & -5 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ x & -5 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ x & 1 \end{vmatrix}$$

$$= (2+x)(5-2) - 3(-5-2x) + 4(1+x)$$

$$= 10 - 4 + 5x - 2x + 15 + 6x + 4 + 4x$$

$$= 13x + 25$$

$|A| = 0$

$13x + 25 = 0$

$13x = -25$

$$\boxed{x = \frac{-25}{13}}$$ Ans.

b). Verify Rolle's Theorem for the function $f(x) = e^x \sin x$ in $[0, \pi]$.

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Given $f(x) = e^x \sin x$

i) Clearly e^x and $\sin x$ both are continuous functions and product of two continuous functions is again continuous. So $f(x) = e^x \sin x$ is continuous $[0, \pi]$.

ii) $f'(x) = e^x \cos x + \sin x \, e^x$ which is clearly derivable in $(0, \pi)$.

iii) $f(0) = e^0 \sin 0 = 1 \times 0 = 0$

$f(\pi) = e^\pi \sin \pi = e^\pi \times 0 = 0$

So; $f(0) = f(\pi)$

So; all the three condition of Rolle's Theorem are satisfied.

Then $\exists$ atleast one point $c \in (a, b)$ such that $f'(c) = 0$

$e^c (\cos c + \sin c) = 0$

$\cos c = -\sin c$

$\tan c = -1$

$\tan c = -\tan \frac{\pi}{4}$

$\tan c = \tan \left(\pi - \frac{\pi}{4}\right)$

$\tan c = \tan \left(\frac{3\pi}{4}\right)$

$c = \frac{3\pi}{4} \in (0, \pi)$

Rolle's Theorem is verified.

Q→4.a) Evaluate $\lim_{x \to \infty} \frac{x^n}{e^x}$; $n \in N$

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Given ; $\lim_{x \to \infty} \frac{x^n}{e^x} = \frac{\infty}{\infty}$; Indeterminate form.

Applying L' Hopitals Rule,

$$\lim_{x \to \infty} \frac{n x^{n-1}}{e^x} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{n(n-1) x^{n-2}}{e^x} = \frac{\infty}{\infty}$$

$$\vdots$$

$$\lim_{x \to \infty} \frac{n!}{e^x}$$

$$\frac{n!}{e^\infty} = \boxed{0}$$

b). Evaluate $\lim_{x \to 0} \left(\cot^2 x - \frac{1}{x^2}\right)$

$$\lim_{x \to 0} \frac{1}{\tan^2 x} - \frac{1}{x^2}$$

$$\lim_{x \to 0} \frac{x^2 - \tan^2 x}{x^2 \frac{\tan^2 x}{x^2} \times x^2} = \frac{0}{0}$$; indeterminate form.

Apply L' Hopital's Rule.

$$\lim_{x \to 0} \frac{x^2 - \tan^2 x}{x^2 \times x^2} = \frac{x^2 - \tan^2 x}{x^4}$$

$$\lim_{x \to 0} \frac{2x - 2 \tan x \sec^2 x}{4x^3}$$

$$= \frac{0 - 2(0)}{0} = \frac{0}{0}$$; indeterminate form.

$$\lim_{x\to 0} \frac{2x - 2\tan x\,(1+\tan^2 x)}{4x^3}$$

$$\lim_{x\to 0} \frac{2x - 2\tan x - 2\tan^3 x}{4x^3}$$

$$\lim_{x\to 0} \frac{2\left[x - \tan x - \tan^3 x\right]}{4x^3}$$

$$\lim_{x\to 0} \frac{x - \tan x - \tan^3 x}{2x^3}$$

Using L' Hopital's Rule ;

$$\lim_{x\to 0} \frac{1 - \sec^2 x - 3\tan^2 x\,\sec^2 x}{6x^2}$$

$$\frac{1 - \sec^2 0 - 3\tan^2 0\,\sec^2 0}{6(0)^2} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{1-(1+\tan^2 x) - 3\tan^2 x\,(1+\tan^2 x)}{6x^2}$$

$$\lim_{x\to 0} \frac{1 - 1 - \tan^2 x - 3\tan^2 x - 3\tan^4 x}{6x^2}$$

$$\lim_{x\to 0} \frac{-4\tan^2 x - 3\tan^4 x}{6x^2}$$

$$\lim_{x\to 0} -\frac{\tan^2 x\,\left[4 + 3\tan^2 x\right]}{6x^2}$$

$$\lim_{x\to 0} \frac{-\left[4 + 3\tan^2 x\right]}{6}$$

$$= \frac{-4+0}{6} = \boxed{\frac{-2}{3}}$$ Ans.

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Q→5. Test whether the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ is diagonalizable or not.

Let λ is an eigen value of A; By def $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(3-\lambda)(3-\lambda) - 1\right] - 0 + 0$$

$$= (1-\lambda)\left[9 - 3\lambda - 3\lambda + \lambda^2 - 1\right]$$

$$= (1-\lambda)\left[\lambda^2 - 6\lambda + 8\right]$$

$$= (1-\lambda)(\lambda^2 - 4\lambda - 2\lambda + 8)$$

$$= (1-\lambda)(\lambda[\lambda - 4] - 2[\lambda - 4])$$

$$= (1-\lambda)(\lambda - 2)(\lambda - 4)$$

$|A - \lambda I| \Rightarrow 0$

$(1-\lambda)(\lambda-2)(\lambda-4) = 0$

$1 - \lambda = 0$ $\quad$ $\lambda - 2 = 0$ $\quad$ $\lambda - 4 = 0$

$\boxed{\lambda \to 1}$ $\quad$ $\boxed{\lambda \to 2}$ $\quad$ $\boxed{\lambda = 4}$

$[(A - \lambda I)X] = 0$

For $\lambda \Rightarrow \boxed{1}$ $\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 0$

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$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$2y - z = 0 \rightarrow$ $2y = z$, $\frac{y}{1} = \frac{z}{2}$

$-y + 2z = 0 \rightarrow (2)$

$2z = y$

$\frac{z}{1} = \frac{y}{2}$

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

For $\lambda \Rightarrow \boxed{2}$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = 0$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-x = 0$

$y - z = 0$

$-y + z = 0$

$z = y$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda \Rightarrow \boxed{4}$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right) \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = 0$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-3x = 0 \Rightarrow x = 0$

$-y - z = 0 \Rightarrow \frac{y}{-1} = \frac{z}{1}$

$-y - z = 0$

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

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Becoz we could find 4 eigen vectors however order of matrix A is 3×3 ; so given matrix is not diagonalizable as number of eigen vectors = n.