

# LAGraph Algorithms

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## Abstract

Theoretical documentation for LAGraph.

## 1 Introduction

The goal of this document is to present a notation for GraphBLAS algorithms and showcase it using important GraphBLAS algorithms.

## 2 The GraphBLAS

**Goal** The goal of GraphBLAS is to create a layer of abstraction between the graph algorithms and the graph analytics framework, separating the concerns of the algorithm developers from those of the framework developers and hardware designers. To achieve this, it builds on the theoretical framework of matrix operations on arbitrary semirings [3], which allows defining graph algorithms in the language of linear algebra [4]. To ensure portability, the GraphBLAS standard defines a C API that can be implemented on a variety of hardware including GPUs.

**Data structures** A graph with  $n$  vertices can be stored as a square adjacency matrix  $\mathbf{A} \in \text{uint}^{n \times n}$ , where rows and columns both represent vertices of the graph and element  $A(i, j)$  contains the number of edges from vertex  $i$  to vertex  $j$ . If the graph is undirected, the matrix is symmetric.

**Navigation** The fundamental step in GraphBLAS is the multiplication of an adjacency matrix with another matrix or vector over a selected semiring. For example, the operation `HasMember lor.land IsLocatedIn` computed over the “*logical or.logical and*” semiring returns a matrix representing the Places where a Forum’s members are located in. Meanwhile, when computed over the conventional arithmetic “*plus.times*” semiring, `HasMember +. × IsLocatedIn` also returns the number of such Persons. A traversal from a certain set of vertices can be expressed by using a boolean vector  $\mathbf{f}$  (often referred to as the *frontier*, *wavefront*, or *queue*) and setting `true` values for the elements corresponding to source vertices. For example, for Forums  $\mathbf{f} \in \text{bool}^{|\text{forums}|}$ , `f lor.land HasMember` returns the Persons who belong to any of the forums in  $\mathbf{f}$ . The BFS navigation step can also be captured using other semirings such as `lor.first`, where `first( $x, y$ ) =  $x$` ; `lor.second`, where `second( $x, y$ ) =  $y$` ; and `any.pair`, where `any( $x, y$ )` returns either  $x$  or  $y$ , and `pair( $x, y$ ) = 1` [2].

### 2.1 Notation

Table 1 contains the notation of GraphBLAS operations. Additionally, we use  $\mathbf{D} = \text{diag}(\mathbf{j}, \mathbf{n})$  to construct a diagonal matrix  $\mathbf{D} \leftarrow \{\mathbf{j}, \mathbf{j}, [1, 1, \dots, 1]\}$ . The elements of the matrix are  $D(\mathbf{j}, \mathbf{j}) = 1$  for  $\mathbf{j} \in \mathbf{j}$ .

#### 2.1.1 Masks

Masks  $\mathbf{C} \ll \mathbf{M} \gg$  and  $\mathbf{u} \ll \mathbf{m} \gg$  are used to selectively write to the result matrix/vector. The complements of the masks can be selected with the negation symbol, denoted with:  $\mathbf{C} \ll !\mathbf{M} \gg$  and  $\mathbf{u} \ll !\mathbf{m} \gg$ , respectively.

Masks with “replace” semantics (annihilating all elements outside the mask) are denoted with

- $\mathbf{C} \ll \mathbf{M} \gg$
- $\mathbf{C} \ll !\mathbf{M} \gg$
- $\mathbf{u} \ll \mathbf{m} \gg$
- $\mathbf{u} \ll !\mathbf{m} \gg$

The structure of the mask is denoted with:

op./method	name	notation
<b>mxm</b>	matrix-matrix multiplication	$\mathbf{C} < \mathbf{M} > + = \mathbf{A} + . \times \mathbf{B}$
<b>vxm</b>	vector-matrix multiplication	$\mathbf{w} < \mathbf{m} > + = \mathbf{u} + . \times \mathbf{A}$
<b>mxv</b>	matrix-vector multiplication	$\mathbf{w} < \mathbf{m} > + = \mathbf{A} + . \times \mathbf{u}$
<b>eWiseAdd</b>	element-wise addition set union of patterns	$\mathbf{C} < \mathbf{M} > + = \mathbf{A} + \mathbf{B}$ $\mathbf{w} < \mathbf{m} > + = \mathbf{u} + \mathbf{v}$
<b>eWiseMult</b>	element-wise multiplication set intersection of patterns	$\mathbf{C} < \mathbf{M} > + = \mathbf{A} \times \mathbf{B}$ $\mathbf{w} < \mathbf{m} > + = \mathbf{u} \times \mathbf{v}$
<b>extract</b>	extract submatrix extract column vector extract row vector extract subvector	$\mathbf{C} < \mathbf{M} > + = \mathbf{A}(\mathbf{i}, \mathbf{j})$ $\mathbf{w} < \mathbf{m} > + = \mathbf{A}(:, \mathbf{j})$ $\mathbf{w} < \mathbf{m} > + = \mathbf{A}(\mathbf{i}, :)$ $\mathbf{w} < \mathbf{m} > + = \mathbf{u}(\mathbf{i})$
<b>assign</b>	assign matrix to submatrix with mask for <b>C</b> assign scalar to submatrix with mask for <b>C</b> assign vector to subvector with mask for <b>w</b> assign scalar to subvector with mask for <b>w</b>	$\mathbf{C} < \mathbf{M} > (\mathbf{i}, \mathbf{j}) + = \mathbf{A}$ $\mathbf{C} < \mathbf{M} > (\mathbf{i}, \mathbf{j}) + = s$ $\mathbf{w} < \mathbf{m} > (\mathbf{i}) + = \mathbf{u}$ $\mathbf{w} < \mathbf{m} > (\mathbf{i}) + = s$
<b>subassign (GxB)</b>	assign matrix to submatrix with submask for <b>C</b> ( <b>i</b> , <b>j</b> ) assign scalar to submatrix with submask for <b>C</b> ( <b>i</b> , <b>j</b> ) assign vector to subvector with submask for <b>w</b> ( <b>i</b> ) assign scalar to subvector with submask for <b>w</b> ( <b>i</b> )	$\mathbf{C}(\mathbf{i}, \mathbf{j}) < \mathbf{M} > + = \mathbf{A}$ $\mathbf{C}(\mathbf{i}, \mathbf{j}) < \mathbf{M} > + = s$ $\mathbf{w}(\mathbf{i}) < \mathbf{m} > + = \mathbf{u}$ $\mathbf{w}(\mathbf{i}) < \mathbf{m} > + = s$
<b>apply</b>	apply unary operator	$\mathbf{C} < \mathbf{M} > + = \mathbf{f}((\mathbf{A}))$ $\mathbf{w} < \mathbf{m} > + = \mathbf{f}((\mathbf{u}))$
<b>select (GxB)</b>	apply select operator	$\mathbf{C} < \mathbf{M} > + = \mathbf{select}(\mathbf{A}, \mathbf{f}((\mathbf{k})))$ $\mathbf{C} < \mathbf{M} > + = \mathbf{select}(\mathbf{low} \leq \mathbf{A} \leq \mathbf{up})$ $\mathbf{w} < \mathbf{m} > + = \mathbf{select}(\mathbf{u}, \mathbf{f}((\mathbf{k})))$ $\mathbf{w} < \mathbf{m} > + = \mathbf{select}(\mathbf{low} \leq \mathbf{u} \leq \mathbf{up})$
<b>reduce</b>	reduce matrix to column vector reduce matrix to scalar reduce vector to scalar	$\mathbf{w} < \mathbf{m} > + = [\mathbf{A}]$ $s + = [\mathbf{A}]$ $s + = [\mathbf{u}]$
<b>transpose</b>	transpose	$\mathbf{C} < \mathbf{M} > + = \mathbf{A}'$
<b>kronecker</b>	Kronecker multiplication	$\mathbf{C} < \mathbf{M} > + = \mathbf{kron}(\mathbf{A}, \mathbf{B})$
<b>new</b>	new matrix new vector	$\mathbf{A} = \mathbf{TYPEPRECISION}(\mathbf{n}, \mathbf{m})$ $\mathbf{u} = \mathbf{TYPEPRECISION}(\mathbf{n})$
<b>build</b>	matrix from tuples vector from tuples	$\mathbf{C} < - \{ \mathbf{i}, \mathbf{j}, \mathbf{x} \}$ $\mathbf{w} < - \{ \mathbf{i}, \mathbf{x} \}$
<b>extractTuples</b>	extract index/value arrays	$\{ \mathbf{i}, \mathbf{j}, \mathbf{x} \} < - \mathbf{A}$ $\{ \mathbf{i}, \mathbf{x} \} < - \mathbf{u}$
<b>dup</b>	duplicate matrix duplicate vector	$\mathbf{C} < - \mathbf{A}$ $\mathbf{w} < - \mathbf{u}$
<b>extractElement</b>	extract scalar element	$s = \mathbf{A}(\mathbf{i}, \mathbf{j})$ $s = \mathbf{u}(\mathbf{i})$
<b>setElement</b>	set element	$\mathbf{C}(\mathbf{i}, \mathbf{j}) = s$ $\mathbf{w}(\mathbf{i}) = s$

Table 1: GraphBLAS operations and methods based on [1]. *Notation:* Matrices and vectors are typeset in bold, starting with uppercase (**A**) and lowercase (**u**) letters, respectively. Scalars including indices are lowercase italic (*k*, *i*, *j*) while arrays are lowercase bold italic (**x**, **i**, **j**). + and  $\times$  are the addition and multiplication operators forming a semiring and default to conventional arithmetic + and  $\times$  operators. + is the accumulation operator.

- $C < \{M\} >$
- $C < !\{M\} >$
- $u < \{m\} >$
- $u < !\{m\} >$

Combining structure and replace semantics is possible:

- $C << \{M\} >>$
- $C << !\{M\} >>$
- $u << \{m\} >>$
- $u << !\{m\} >>$

Initializing scalars, vectors, and matrices (GraphBLAS methods):

- $s = \text{fp64}()$
- $u = \text{fp32}(n)$
- $A = \text{uint16}(m, n)$
- $A = \text{int64}(k, m)$

### 3 Algorithms

LAGraph [5] implements graph algorithms using the GraphBLAS C API [6].

Here are a few algorithms that could be included in this document:

<p><b>Input:</b> <math>A, n, \text{startVertex}</math></p> <pre> 1 <b>Function</b> <i>BFS</i> 2   <math>f(\text{startVertex}) = T</math> 3   <b>for</b> <math>\text{level} = 1</math> <b>to</b> <math>n - 1</math> <b>do</b> 4     <math>s &lt; f &gt; = \text{level}</math> 5     <math>f &lt;&lt; !s &gt;&gt; = fA</math> </pre>	<p><b>Input:</b> <math>A, n, \text{startVertex}</math></p> <pre> 1 <b>Function</b> <i>ParentsBFS</i> 2   <math>f(\text{startVertex}) = 0</math> 3   <b>for</b> <math>\text{level} = 1</math> <b>to</b> <math>n - 1</math> <b>do</b> 4     <math>s &lt; f &gt; = f</math> 5     <math>f &lt;&lt; !s &gt;&gt; = fA</math> 6     <math>f \text{ any.firstj1 } A</math> </pre>	<p><b>Input:</b> <math>A, A', n, \text{startVertex}</math></p> <pre> 1 <b>Function</b> <i>DirectionOptimizingBFS</i> 2   <math>f(\text{startVertex}) = T</math> 3   <b>for</b> <math>\text{level} = 1</math> <b>to</b> <math>n - 1</math> <b>do</b> 4     <math>s &lt; f &gt; = \text{level}</math> 5     <b>if</b> <math>\text{Push}(A, f)</math> <b>then</b> 6       <math>f &lt;&lt; !s &gt;&gt; = fA</math> 7     <b>else</b> 8       <math>f &lt;&lt; !s &gt;&gt; = A'f</math> </pre>	<p><b>Input:</b> <math>A, n, \text{startVertices}</math></p> <pre> 1 <b>Function</b> <i>ConcurrentBFS</i> 2   <math>F = \text{diag}(\text{startVertices}, n)</math> 3   <b>for</b> <math>\text{level} = 1</math> <b>to</b> <math>n - 1</math> <b>do</b> 4     <math>S &lt; F &gt; = \text{level}</math> 5     <math>F &lt;&lt; !S &gt;&gt; = FA</math> </pre>
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Figure 1: Sketch algorithms of BFS variants. The  $\text{Push}((A, f))$  function makes a decision on whether it is cheaper to push or pull using heuristics based on the sparsity of the frontier  $f$  and adjacency matrix  $A$ .

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#### Algorithm 1: Multi-source breadth-first search.

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**Data:** ...  
**Result:** ...

```

1 Function MSBFS
2    $\text{Frontier} = \text{diag}(\text{sources}, n)$ 
3   for  $\text{level} = 1$  to  $n - 1$  do
4      $\text{Seen} < \text{Frontier} > = \text{level}$ 
5      $\text{Frontier} << !\text{Seen} >> = \text{Frontier any.pair } A$ 

```

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**Algorithm 2:** All-pairs shortest distance (on undirected, unweighted graphs) [7].

---

```
Data: ...
Result: ...
1 Function APD(A, n, deg)
2   Z = A
3   Z += A + . × A
4   B = {select(Z, offdiag)} // use the pattern as a Boolean matrix
5   if A == B then
6     return A
7   T = APD(B, n, deg)
8   X = T + . × A
9   Tscaled = T + . × diag((deg)
10  Xfiltered = select(X, X < Tscaled)
11  return (2 × T) MINUS Xfiltered
12 Function APD(A)
13   deg = [+A]
14   Distance = APD(A, n, deg)
15   sp = [+Distance]
```

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**Algorithm 3:** Betweenness centrality.

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```
1 Function MSBFS
   // The NumSp structure holds the number of shortest paths for each node and starting
   // vertex discovered so far.
   // Initialized to source vertices.
2   NumSp <- {s, [1, 1, ..., 1]}
   // The Frontier holds the number of shortest paths for each node and starting vertex
   // discovered so far.
   // Initialized to source vertices.
3   Frontier <NumSp>= A(s, :)
4   d = 0
   // The Sigmas matrices store frontier information for each level of the BFS phase.
   // BFS phase (forward sweep)
5   do
   // Sigmas[d](:, s) = dth level frontier from source vertex s
6   Sigmas[d] = bool(n, nsver)
7   Sigmas[d](:, :) = Frontier // Convert matrix to Boolean
8   NumSp = NumSp + Frontier // Accumulate path counts
9   Frontier << NumSp >>= A' + . × Frontier // Update frontier
10  while nvals(Frontier) > 0
11  NumSpInv = fp32(n, nsver)
12  NumSpInv = 1.0 DIV NumSp
13  BCU = fp32(n, nsver)
14  BCU(:) = 1.0 // Make BCU dense, initialize all elements to 1.0
15  W = fp32(n, nsver)
   // Tally phase (backward sweep)
16  for i = d - 1 downto 0 do
17    W << Sigmas[i] >>= NumSpInv DIV BCU
18    W << Sigmas[i - 1] >>= A + . × W // Add contributions by successors and mask with that
    // BFS level's frontier.
19    BCU += W × NumSp
   // Row reduce BCU and subtract nsver from every entry to account for 1 extra value
   // per BCU row element
20  delta = [+BCU]
21  delta MINUS = nsver
```

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**Algorithm 4:** PageRank (used in Graphalytics).

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**Data:**  $\alpha$  constant (damping factor)

**Result:** ...

```
1 Function PageRank
2   pr(:) = 1/n
3   outdegrees = [ +j A(:, j) ]
4   for k = 1 to numIterations do
5     importance = pr DIV outdegrees
6     importance = times((), importance, alpha)      // apply the times(())x, s = x · s operator
7     importance = importance + . × A
8     danglingVertexRanks < !outdegrees >= pr(:)
9     totalDanglingRank = (alpha)/(n) × [ + danglingVertexRanks ]
10    pr = (1 - alpha)/(n) + totalDanglingRank
11    pr = pr + importance
```

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**Algorithm 5:** Algebraic Bellman-Ford.

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```
1 Function SSSP
2   d(s) = 0
3   for k = 1 to n - 1 do
4     d' = d min. plus A
5     if d' == d then break
6     d <- d'
```

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**Algorithm 6:** Delta-stepping SSSP.

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**Data:**

$A, A_H, A_L \in \text{fp}(|V|, |V|)$

$s, i \in \text{uint}()$

$\Delta \in \text{fp}()$

$t, t_{\text{Req}} \in \text{fp}(|V|)$

$t_{B_i}, e \in \text{uint}(|V|)$

```
1 Function DeltaStepping
2   A_L = select(0 < A ≤ Δ)
3   A_H = select(Δ < A)
4   t(:) = ∞
5   t(s) = 0
6   while nvals(select(iΔ ≤ t)) ≠ 0 do
7     s = 0
8     tBi = select(iΔ ≤ t < (i + 1)Δ)
9     while tBi ≠ 0 do
10      tReq = A_L' + . × (t × tBi)
11      e = select(0 < e + tBi)
12      tBi = select(iΔ ≤ tReq < (i + 1)Δ) × (tReq min+ t)
13      tBi = select(iΔ ≤ tReq < (i + 1)Δ) × (tReq + min t)
14      tBi = select(iΔ ≤ tReq < (i + 1)Δ) × (tReq min+ t)
15      t = t min tReq
16      tReq = A_H' + . × (t × e)
17      t = t min tReq
18      i = i + 1
```

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**Algorithm 7:** All-pairs shortest path (Floyd–Warshall algorithm).

---

```
1 Function FloydWarshall
2   D <- A
3   for k = 1 to n do
4     D = D min [D(:, k) min. plus (D(k, :))]
```

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**Algorithm 8:** FastSV algorithm.

---

```
1 Function FastSV
2    $n = \text{nrows}(A)$ 
3    $gf = f$ 
4    $dup = gf$ 
5    $mngf = gf$ 
6    $\{i, x\} \leftarrow f$ 
7   repeat
8     // Step 1: Stochastic hooking
9      $mngf = mngf \min A$ 
10     $mngf = mngf \text{ second.min } gf$ 
11     $f(x) = f \min mngf$ 
12    // Step 2: Aggressive hooking
13     $f = f \min mngf$ 
14    // Step 3: Shortcutting
15     $f = f \min gf$ 
16    // Step 4: Calculate grandparents
17     $\{i, x\} \leftarrow f$ 
18     $gf = f(x)$ 
19    // Step 5: Check termination
20     $diff = dup \neq gf$ 
21     $sum = [+_i diff(i)]$ 
22     $dup = gf$ 
23  until  $sum == 0$ 
```

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**Algorithm 9:** Triangle count (Cohen's algorithm).

---

```
1 Function TriangleCount
2    $L = \text{tril}(A)$ 
3    $U = \text{triu}(A)$ 
4    $B = L + . \times U$ 
5    $C = B \times A$ 
6    $t = [+C]/2$ 
```

---

---

**Algorithm 10:** Triangle count (Sandia).

---

```
1 Function TriangleCount
2    $L = \text{tril}(A)$ 
3    $C < L > = L + . \times L$ 
4    $t = [+C]$ 
```

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**Algorithm 11:** Triangle count (FLAME).

---

```
1 Function TriangleCountFlame
2   for  $i = 2$  to  $n - 1$  do
3      $A_{20} = A(i + 1 : n, 0 : i - 1)$ 
4      $a_{10} = A(0 : i - 1, i)$ 
5      $a_{12} = A(i, i + 1 : n)$ 
6      $t += a_{10} + . \times A_{20} + . \times a_{12}$ 
```

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**Algorithm 12:** Local clustering coefficient.

---

```
1 Function PageRank
2    $Tri < A > = A + . \times A$  // compute triangle count matrix
3    $tri = [+Tri]$  // reduce to triangle count vector
4    $deg = [+A]$  // reduce to vertex degree vector
5    $wed = \text{perm2}((deg)$  // apply  $\text{perm2}((x) = x \cdot (x - 1)$  to get wedge count vector
6    $lcc = tri \text{DIV } wed$  // LCC vector
```

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**Algorithm 13:**  $k$ -truss algorithm.

---

```
1 Function KTruss
2    $C \leftarrow A$ 
3    $\text{nonzeros} \leftarrow \text{nvals}(C)$ 
4   for  $i = 1$  to  $n - 1$  do
5      $C \leftarrow C + .1 \text{ and } C$ 
6      $C = \text{select}(C \geq k - 2)$ 
7     if  $\text{nonzeros} == \text{nvals}(C)$  then
8       break
9      $\text{nonzeros} \leftarrow \text{nvals}(C)$ 
```

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**Algorithm 14:** Louvain algorithm (WIP).

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```
1 Function Louvain
2    $G \leftarrow G'$ 
3    $k = [+A]$ 
4    $m = (1)/(2)[+k]$ 
5    $S \leftarrow I$ 
6
7    $\text{vertices\_changed} \leftarrow \text{nvals}(k)$ 
8   while  $\text{vertices\_changed} > 0$  do
9     for  $j \in \text{range}(|V|)$  do
10       $v = G(j, :)$ 
11       $t_q = v \text{ any.pair } S$ 
12       $sr = S(j, :)$ 
13       $S(j, :) = \text{empty}$ 
14
15       $q \leftarrow k$ 
16       $q \leftarrow q \times -k(j)/m$ 
17       $q \leftarrow q + v$ 
18       $q_1 \leftarrow t_q \leftarrow q + . \times S$ 
19
20       $t = (q_1 == [\max q_1])$ 
21      while  $\text{nvals}(t) \neq 1$  do
22         $p = \text{random}() \times t$ 
23         $t = (p == [\max p])$ 
24       $S(j, :) = t$ 
25
26      if  $\text{nvals}(sr \times t) == 0$  then
27         $\text{vertices\_changed} = \text{nvals}(k)$ 
28       $\text{vertices\_changed} = \text{vertices\_changed} - 1$ 
```

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**Algorithm 15:** Community detection using label propagation (for undirected graphs).

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```
1 Function CDLP
2    $L \leftarrow \text{diag}([0, 1, \dots, n - 1])$ 
3   for  $k = 1$  to  $t$  do
4      $F = A \text{ any. second } L$  // Frequency matrix
5      $\{i, \rightarrow, x\} \leftarrow F$ 
6      $\text{merge\_sort\_pairs}(i, x)$ 
7      $\text{labels} = \text{for each row in } i, \text{ select min mode value from } x$ 
```

---

## References

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