LAGraph Algorithms

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October 7, 2020

Abstract

Theoretical documentation for LAGraph.

1 Notation

Table 1 contains the notation of GraphBLAS operations

Additionally, we use $\mathtt{D} = diag(\mathtt{J}, \mathtt{n})$ to construct a diagonal matrix $\mathtt{D} <- \{\mathtt{J}, \mathtt{J}, [1, 1, \ldots, 1]\}$. The elements of the matrix are $\mathtt{D}(\mathtt{j}, \mathtt{j}) = 1$ for $\mathtt{j} \in \mathtt{J}$.

Initializing scalars, vectors, and matrices (GraphBLAS methods):

- $\mathtt{s} = \mathtt{fp64}()$
- $\mathtt{u} = \mathtt{fp32}(\mathtt{n})$
- $\mathtt{A} = \mathtt{uint16}(\mathtt{m}, \mathtt{n})$
- $\mathtt{A} = \mathtt{int64}(\mathtt{k}, \mathtt{m})$

2 Algorithms

LAGraph [12] implements graph algorithms using the GraphBLAS C API [13].

An incomplete list of GraphBLAS algorithms:

- MSBFS [8], bidirectional BFS [8], pushpull BFS [19]
- DFS [16]
- weakly connected components [21]
- SCC (LAGraph)
- SSSP, delta-stepping [17]
- triangle count [1, 5], triangle enumeration [1], item local clustering coefficient [4]
- k-truss [5]
- betweenness centrality [12]
- closeness centrality [8]
- DNN algorithm [10, 7]
- PageRank variants (at least 2) [4], IISWC paper, ...
- Louvain [11]
- property graphs: incremental TTC case [9], SIGMOD 2014 Contest [8] Roi Lipman's talk[1]
- CFPQs based on a string of GRADES/ADBIS/other papers [2, 3, 14, 18, 15]
- Implementations: GBTL[2], SuiteSparse:GraphBLAS [6], GraphBLAST [20]

[1] http://wiki.ldbcouncil.org/pages/viewpage.action?pageId=106233859&preview=/106233859/111706128/LDBC-July-2019.pdf

[2] https://github.com/cmu-sei/gbtl

op./method	name	notation
`mxm`	matrix-matrix multiplication	$\mathbf{C}\langle\mathbf{M}\rangle+=\mathbf{A}+.\times\mathbf{B}$
`vxm`	vector-matrix multiplication	$\mathbf{w}\langle\mathbf{m}\rangle+=\mathbf{u}+.\times\mathbf{A}$
`mxv`	matrix-vector multiplication	$\mathbf{w}\langle\mathbf{m}\rangle+=\mathbf{A}+.\times\mathbf{u}$
`eWiseAdd`	element-wise addition	$\mathbf{C}\langle\mathbf{M}\rangle+=\mathbf{A}+\mathbf{B}$
	set union of patterns	$\mathbf{w}\langle\mathbf{m}\rangle+=\mathbf{u}+\mathbf{v}$
`eWiseMult`	element-wise multiplication	$\mathbf{C}\langle\mathbf{M}\rangle+=\mathbf{A}\times\mathbf{B}$
	set intersection of patterns	$\mathbf{w}\langle\mathbf{m}\rangle+=\mathbf{u}\times\mathbf{v}$
`extract`	extract submatrix	$\mathbf{C}\langle\mathbf{M}\rangle+=\mathbf{A}(\mathbf{I},\mathbf{J})$
	extract column vector	$\mathbf{w}\langle\mathbf{m}\rangle+=\mathbf{A}(:,\mathbf{j})$
	extract row vector	$\mathbf{w}\langle\mathbf{m}\rangle+=\mathbf{A}(\mathbf{i},:)$
	extract subvector	$\mathbf{w}\langle\mathbf{m}\rangle+=\mathbf{u}(\mathbf{I})$
`assign`	assign matrix to submatrix with mask for $\mathbf{C}$	$\mathbf{C}\langle\mathbf{M}\rangle(\mathbf{I},\mathbf{J})+=\mathbf{A}$
	assign scalar to submatrix with mask for $\mathbf{C}$	$\mathbf{C}\langle\mathbf{M}\rangle(\mathbf{I},\mathbf{J})+=\mathbf{s}$
	assign vector to subvector with mask for $\mathbf{w}$	$\mathbf{w}\langle\mathbf{m}\rangle(\mathbf{I})+=\mathbf{u}$
	assign scalar to subvector with mask for $\mathbf{w}$	$\mathbf{w}\langle\mathbf{m}\rangle(\mathbf{I})+=\mathbf{s}$
`subassign (GxB)`	assign matrix to submatrix with submask for $\mathbf{C}(\mathbf{I},\mathbf{J})$	$\mathbf{C}(\mathbf{I},\mathbf{J})\langle\mathbf{M}\rangle+=\mathbf{A}$
	assign scalar to submatrix with submask for $\mathbf{C}(\mathbf{I},\mathbf{J})$	$\mathbf{C}(\mathbf{I},\mathbf{J})\langle\mathbf{M}\rangle+=\mathbf{s}$
	assign vector to subvector with submask for $\mathbf{w}(\mathbf{I})$	$\mathbf{w}(\mathbf{I})\langle\mathbf{m}\rangle+=\mathbf{u}$
	assign scalar to subvector with submask for $\mathbf{w}(\mathbf{I})$	$\mathbf{w}(\mathbf{I})\langle\mathbf{m}\rangle+=\mathbf{s}$
`apply`	apply unary operator	$\mathbf{C}\langle\mathbf{M}\rangle+=\mathbf{f}(\mathbf{A})$
		$\mathbf{w}\langle\mathbf{m}\rangle+=\mathbf{f}(\mathbf{u})$
`select (GxB)`	apply select operator	$\mathbf{C}\langle\mathbf{M}\rangle+=\mathit{select}(\mathbf{A},\mathbf{f}(\mathbf{k}))$
		$\mathbf{C}\langle\mathbf{M}\rangle+=\mathit{select}(\mathbf{low}\le\mathbf{A}\le\mathbf{up})$
		$\mathbf{w}\langle\mathbf{m}\rangle+=\mathit{select}(\mathbf{u},\mathbf{f}(\mathbf{k}))$
		$\mathbf{w}\langle\mathbf{m}\rangle+=\mathit{select}(\mathbf{low}\le\mathbf{u}\le\mathbf{up})$
`reduce`	reduce matrix to column vector	$\mathbf{w}\langle\mathbf{m}\rangle+=[+\mathbf{A}]$
	reduce matrix to scalar	$\mathbf{s}+=[+\mathbf{A}]$
	reduce vector to scalar	$\mathbf{s}+=[+\mathbf{u}]$
`transpose`	transpose	$\mathbf{C}\langle\mathbf{M}\rangle+=\mathbf{A}'$
`kronecker`	Kronecker multiplication	$\mathbf{C}\langle\mathbf{M}\rangle+=\mathit{kron}(\mathbf{A},\mathbf{B})$
`new`	new matrix	$\mathbf{A}=\mathsf{TYPEPRECISION}(\mathbf{n},\mathbf{m})$
	new vector	$\mathbf{u}=\mathsf{TYPEPRECISION}(\mathbf{n})$
`build`	build matrix from index/value arrays	$\mathbf{C}\leftarrow\{\mathbf{I},\mathbf{J},\mathbf{X}\}$
	build vector from index/value arrays	$\mathbf{w}\leftarrow\{\mathbf{I},\mathbf{X}\}$
`extractTuples`	extract index/value arrays	$\{\mathbf{I},\mathbf{J},\mathbf{X}\}\leftarrow\mathbf{A}$
		$\{\mathbf{I},\mathbf{X}\}\leftarrow\mathbf{u}$
`dup`	duplicate matrix	$\mathbf{C}\leftarrow\mathbf{A}$
	duplicate vector	$\mathbf{w}\leftarrow\mathbf{u}$
`extractElement`	extract scalar element	$\mathbf{s}=\mathbf{A}(\mathbf{i},\mathbf{j})$
		$\mathbf{s}=\mathbf{u}(\mathbf{i})$
`setElement`	set element	$\mathbf{C}(\mathbf{i},\mathbf{j})=\mathbf{s}$
		$\mathbf{w}(\mathbf{i})=\mathbf{s}$

Table 1: GraphBLAS operations and methods based on [6]. *Notation:* Matrices and vectors are typeset in bold, starting with uppercase (`A`) and lowercase (`u`) letters, respectively. Scalars including indices are lowercase italic (`s`, `i`, `j`) while arrays are uppercase italic (`X`, `I`, `J`). $+$ and $\times$ are the addition and multiplication operators forming a semiring and default to conventional arithmetic $+$ and $\times$ operators. $\odot$ is the apply operator. Masks $\langle\mathbf{M}\rangle$ and $\langle\mathbf{m}\rangle$ are used to selectively write to the result matrix/vector. The complements of masks $\langle\mathbf{M}\rangle$, $\langle\mathbf{m}\rangle$ can be selected with the negation symbol, denoted with $\langle!\mathbf{M}\rangle$, $\langle!\mathbf{m}\rangle$, respectively. Masks with "replace" semantics (annihilating all elements outside the mask) are denoted with $\langle\langle\mathbf{M}\rangle\rangle$, $\langle\langle!\mathbf{M}\rangle\rangle$, $\langle\langle\mathbf{m}\rangle\rangle$, and $\langle\langle!\mathbf{m}\rangle\rangle$, The structure of the mask is denoted with $\langle\{\mathbf{M}\}\rangle$, $\langle!\{\mathbf{M}\}\rangle$, $\langle\{\mathbf{m}\}\rangle$, and $\langle!\{\mathbf{m}\}\rangle$.

Algorithm 1: Breadth-first search.

```
Function BFS
    frontier(s) = TRUE
    for level = 1 to n − 1 do
        seen <frontier>= level
        frontier << !seen >>= frontier any.pair A
```

Algorithm 2: Breadth-first search (push/pull).

```
Function BFS
    frontier(s) = TRUE
    for level = 1 to n − 1 do
        seen <frontier>= level
        push = use some heuristics to determine whether to push/pull
        if push then
            frontier << !seen >>= frontier any.pair A
        else
            frontier << !seen >>= A any.pair frontier
```

Algorithm 3: Multi-source breadth-first search.

Data: ...
Result: ...

```
Function MSBFS
    Frontier <- diag(S, n)
    for level = 1 to n − 1 do
        Seen <Frontier>= level
        Frontier << !Seen >>= Frontier any.pair A
```

Algorithm 4: Betweenness centrality.

```
Function MSBFS
    // The NumSp structure holds the number of shortest paths for each node and starting
       vertex discovered so far.
    // Initialized to source vertices.
    NumSp <- {s, [1, 1, ..., 1]}
    // The Frontier holds the number of shortest paths for each node and starting vertex
       discovered so far.
    // Initialized to source vertices.
    Frontier <NumSp>= A(s,:)
    d = 0
    // The Sigmas matrices store frontier information for each level of the BFS phase.
    // BFS phase (forward sweep)
    do
        // Sigmas[d](:, s) = d^th level frontier from source vertex s
        Sigmas[d] = bool(n, nsver)
        Sigmas[d](:,:) = Frontier                                  // Convert matrix to Boolean
        NumSp = NumSp + Frontier                                     // Accumulate path counts
        Frontier <<NumSp>>= A' +.× Frontier                                  // Update frontier
    while nvals(Frontier) > 0
    NumSpInv = fp32(n, nsver)
    NumSpInv = 1.0 DIV NumSp
    BCU = fp32(n, nsver)
    BCU(:) = 1.0                              // Make BCU dense, initialize all elements to 1.0
    W = fp32(n, nsver)
    // Tally phase (backward sweep)
    for i = d − 1 downto 0 do
        W <<Sigmas[i]>>= NumSpInv DIV BCU
        W <<Sigmas[i − 1]>>= A +.× W // Add contributions by successors and mask with that
         BFS level's frontier.
        BCU += W × NumSp
    // Row reduce BCU and subtract nsver from every entry to account for 1 extra value
       per BCU row element
    delta = [+ BCU]
    delta MINUS = nsver
```

Algorithm 5: PageRank (used in Graphalytics).

```
Data: alpha constant (damping factor)
Result: ...
Function PageRank
    pr(:) = 1/n
    outdegrees = [+_j A(:, j)]
    for k = 1 to numIterations do
        importance = pr DIV outdegrees
        importance = times(importance, alpha)          // apply the times(x, s) = x · s operator
        importance = importance +.× A
        danglingVertexRanks <!outdegrees>= pr(:)
        totalDanglingRank = [+ danglingVertexRanks(i)](alpha)/(n)
        pr(:) = (1 − alpha)/(n) + totalDanglingRank
        pr <importance>= pr + importance
```

Algorithm 6: Algebraic Bellman-Ford for SSSP.

Function *SSSP*
  $\mathtt{d}(\mathtt{s}) = 0$
  for $\mathtt{k} = 1$ **to** $\mathtt{n} - 1$ **do**
    $\mathtt{dtmp} = \mathtt{d}\ \mathtt{min.plus}\ \mathtt{A}$
    if $\mathtt{dtmp} = \mathtt{d}$ **then**
      break
    $\mathtt{d} \leftarrow \mathtt{dtmp}$

Algorithm 7: Delta-stepping SSSP.

Data:
  $\mathtt{A}, \mathtt{A_H}, \mathtt{A_L} \in \mathtt{fp}(|\mathtt{V}|, |\mathtt{V}|)$
  $\mathtt{s}, \mathtt{i} \in \mathtt{uint}()$
  $\Delta \in \mathtt{fp}()$
  $\mathtt{t}, \mathtt{t_{Req}} \in \mathtt{fp}(|\mathtt{V}|)$
  $\mathtt{t_{B_i}}, \mathtt{e} \in \mathtt{uint}(|\mathtt{V}|)$

Function *DeltaStepping*
  $\mathtt{A_L} = \mathit{select}(0 < \mathtt{A} \leq \Delta)$
  $\mathtt{A_H} = \mathit{select}(\Delta < \mathtt{A})$
  $\mathtt{t}(:) = \infty$
  $\mathtt{t}(\mathtt{s}) = 0$
  while $\mathit{nvals}(\mathit{select}(\mathtt{i}\Delta \leq \mathtt{t})) \neq 0$ **do**
    $\mathtt{s} = 0$
    $\mathtt{t_{B_i}} = \mathit{select}(\mathtt{i}\Delta \leq \mathtt{t} < (\mathtt{i}+1)\Delta)$
    while $\mathtt{t_{B_i}} \neq 0$ **do**
      $\mathtt{t_{Req}} = \mathtt{A_L}' +. \times (\mathtt{t} \times \mathtt{t_{B_i}})$
      $\mathtt{e} = \mathit{select}(0 < \mathtt{e} + \mathtt{t_{B_i}})$
      $\mathtt{t_{B_i}} = \mathit{select}(\mathtt{i}\Delta \leq \mathtt{t_{Req}} < (\mathtt{i}+1)\Delta) \times (\mathtt{t_{Req}} \underset{+}{\mathtt{min}}\, \mathtt{t})$
    $\mathtt{t_{Req}} = \mathtt{A_H}' +. \times (\mathtt{t} \times \mathtt{e})$
    $\mathtt{t} = \mathtt{t}\ \mathtt{min}\ \mathtt{t_{Req}}$
    $\mathtt{i} = \mathtt{i} + 1$

Algorithm 8: All-pairs shortest path (Floyd–Warshall algorithm).

Function *FloydWarshall*
  $\mathtt{D} \leftarrow \mathtt{A}$
  for $\mathtt{k} = 1$ **to** $\mathtt{n}$ **do**
    $\mathtt{D} = \mathtt{D}\ \mathtt{min}[\mathtt{D}(:, \mathtt{k})\ \mathtt{min.plus}\ (\mathtt{D}(\mathtt{k}, :)]$

Algorithm 9: FastSV algorithm.

```
Function FastSV
    n = nrows(A)
    gf = f
    dup = gf
    mngf = gf
    {I, X} <- f
    repeat
        // Step 1: Stochastic hooking
        mngf = mngf min A
        mngf = mngf second.min gf
        f(X) = f min mngf
        // Step 2: Aggressive hooking
        f = f min mngf
        // Step 3: Shortcutting
        f = f min gf
        // Step 4: Calculate grandparents
        {I, X} <- f
        gf = f(X)
        // Step 5: Check termination
        diff = dup ≠ gf
        sum = [+_i diff(i)]
        dup = gf
    until sum == 0
```

Algorithm 10: Triangle count (Cohen's algorithm).

```
Function TriangleCount
    L = tril(A)
    U = triu(A)
    B = L +.× U
    C = B × A
    t = [+ C]/2
```

Algorithm 11: Triangle count (Sandia).

```
Function TriangleCount
    L = tril(A)
    C <L>= L +.× L
    t = [+ C]
```

Algorithm 12: Triangle count (FLAME).

```
Function TriangleCountFlame
    for i = 2 to n − 1 do
        A_20 = A(i + 1 : n, 0 : i − 1)
        a_10 = A(0 : i − 1, i)
        a_12 = A(i, i + 1 : n)
        t += a_10 +.× A_20 +.× a_12
```

Algorithm 13: Local clustering coefficient.

```
Function PageRank
    Tri <A>= A +.× A                          // compute triangle count matrix
    tri = [+ Tri]                             // reduce to triangle count vector
    deg = [+ A]                               // reduce to vertex degree vector
    wed = perm2(deg)          // apply perm2(x) = x · (x − 1) to get wedge count vector
    lcc = tri DIV wed                                              // LCC vector
```

Algorithm 14: k-truss algorithm.

```
Function KTruss
    C <- A
    nonzeros <- nvals(C)
    for i = 1 to n − 1 do
        C <C>= C +.land C
        C = select(C ≥ k − 2)
        if nonzeros == nvals(C) then
            break
        nonzeros <- nvals(C)
```

Algorithm 15: Louvain algorithm (WIP).

```
Function Louvain
    G += G'
    k = [+ A]
    m = (1)/(2)[+ k]
    S <- I

    vertices_changed <- nvals(k)
    while vertices_changed > 0 do
        for j ∈ range(|V|) do
            v = G(j, :)
            t_q = v any.pair S
            sr = S(j, :)
            S(j, :) = empty

            q <- k
            q <k> × = − k(j)/m
            q += v
            q_1 <t_q>= q +. × S

            t = (q_1 == [max q_1])
            while nvals(t) ≠ 1 do
                p = random() × t
                t = (p == [max p])
            S(j, :) = t

            if nvals(sr × t) == 0 then
                vertices_changed = nvals(k)
            vertices_changed = vertices_changed − 1
```

Algorithm 16: Community detection using label propagation (for undirected graphs).

```
Function CDLP
    L <- diag([0, 1, . . . , n − 1])
    for k = 1 to t do
        F = A any . second L                                  // Frequency matrix
        {I, _, X} <- F
        merge_sort_pairs(I, X)
        labels = for each row in I, select min mode value from X
```

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