LAGraph Algorithms

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Abstract

Theoretical documentation for LAGraph.

1 Notation

Table 1 contains the notation of GraphBLAS operations

Additionally, we use $\mathbf{D} = diag(J, n)$ to construct a diagonal matrix $\mathbf{D} \leftarrow \{J, J, [1, 1, \ldots, 1]\}$. The elements of the matrix are $\mathbf{D}(j, j) = 1$ for $j \in J$.

Initializing scalars, vectors, and matrices (GraphBLAS methods):

- let: $s \in \mathbb{Q}_{64}$
- let: $\mathbf{u} \in \mathbb{Q}_{32}^{n}$
- let: $\mathbf{A} \in \mathbb{N}_{16}^{m \times n}$
- let: $\mathbf{A} \in \mathbb{Z}_{64}^{k \times m}$

2 Algorithms

LAGraph [12] implements graph algorithms using the GraphBLAS C API [13].

An incomplete list of GraphBLAS algorithms:

- MSBFS [8], bidirectional BFS [8], pushpull BFS [19]
- DFS [16]
- weakly connected components [21]
- SCC (LAGraph)
- SSSP, delta-stepping [17]
- triangle count [1, 5], triangle enumeration [1], item local clustering coefficient [4]
- k-truss [5]
- betweenness centrality [12]
- closeness centrality [8]
- DNN algorithm [10, 7]
- PageRank variants (at least 2) [4], IISWC paper, ...
- Louvain [11]
- property graphs: incremental TTC case [9], SIGMOD 2014 Contest [8] Roi Lipman's talk[1]
- CFPQs based on a string of GRADES/ADBIS/other papers [2, 3, 14, 18, 15]
- Implementations: GBTL[2], SuiteSparse:GraphBLAS [6], GraphBLAST [20]

[1] http://wiki.ldbcouncil.org/pages/viewpage.action?pageId=106233859&preview=/106233859/111706128/LDBC-July-2019.pdf

[2] https://github.com/cmu-sei/gbtl

op./method	name	notation
`mxm`	matrix-matrix multiplication	$\mathbf{C}\langle\mathbf{M}\rangle\odot{=}\mathbf{A}\oplus.\otimes\mathbf{B}$
`vxm`	vector-matrix multiplication	$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}\mathbf{u}\oplus.\otimes\mathbf{A}$
`mxv`	matrix-vector multiplication	$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}\mathbf{A}\oplus.\otimes\mathbf{u}$
`eWiseAdd`	element-wise addition	$\mathbf{C}\langle\mathbf{M}\rangle\odot{=}\mathbf{A}\oplus\mathbf{B}$
	set union of patterns	$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}\mathbf{u}\oplus\mathbf{v}$
`eWiseMult`	element-wise multiplication	$\mathbf{C}\langle\mathbf{M}\rangle\odot{=}\mathbf{A}\otimes\mathbf{B}$
	set intersection of patterns	$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}\mathbf{u}\otimes\mathbf{v}$
`extract`	extract submatrix	$\mathbf{C}\langle\mathbf{M}\rangle\odot{=}\mathbf{A}(I,J)$
	extract column vector	$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}\mathbf{A}(:,j)$
	extract row vector	$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}\mathbf{A}(i,:)$
	extract subvector	$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}\mathbf{u}(I)$
`assign`	assign matrix to submatrix with mask for $\mathbf{C}$	$\mathbf{C}\langle\mathbf{M}\rangle(I,J)\odot{=}\mathbf{A}$
	assign scalar to submatrix with mask for $\mathbf{C}$	$\mathbf{C}\langle\mathbf{M}\rangle(I,J)\odot{=}s$
	assign vector to subvector with mask for $\mathbf{w}$	$\mathbf{w}\langle\mathbf{m}\rangle(I)\odot{=}\mathbf{u}$
	assign scalar to subvector with mask for $\mathbf{w}$	$\mathbf{w}\langle\mathbf{m}\rangle(I)\odot{=}s$
`subassign (GxB)`	assign matrix to submatrix with submask for $\mathbf{C}(I,J)$	$\mathbf{C}(I,J)\langle\mathbf{M}\rangle\odot{=}\mathbf{A}$
	assign scalar to submatrix with submask for $\mathbf{C}(I,J)$	$\mathbf{C}(I,J)\langle\mathbf{M}\rangle\odot{=}s$
	assign vector to subvector with submask for $\mathbf{w}(I)$	$\mathbf{w}(I)\langle\mathbf{m}\rangle\odot{=}\mathbf{u}$
	assign scalar to subvector with submask for $\mathbf{w}(I)$	$\mathbf{w}(I)\langle\mathbf{m}\rangle\odot{=}s$
`apply`	apply unary operator	$\mathbf{C}\langle\mathbf{M}\rangle\odot{=}f(\mathbf{A})$
		$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}f(\mathbf{u})$
`select (GxB)`	apply select operator	$\mathbf{C}\langle\mathbf{M}\rangle\odot{=}select(\mathbf{A},f(k))$
		$\mathbf{C}\langle\mathbf{M}\rangle\odot{=}select(low\leq\mathbf{A}\leq up)$
		$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}select(\mathbf{u},f(k))$
		$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}select(low\leq\mathbf{u}\leq up)$
`reduce`	reduce matrix to column vector	$\mathbf{w}\langle\mathbf{m}\rangle\odot{=}[\oplus_j\,\mathbf{A}(:,j)]$
	reduce matrix to scalar	$s\odot{=}[\oplus_{ij}\,\mathbf{A}(i,j)]$
	reduce vector to scalar	$s\odot{=}[\oplus_i\,\mathbf{u}(i)]$
`transpose`	transpose	$\mathbf{C}\langle\mathbf{M}\rangle\odot{=}\mathbf{A}^\top$
`kronecker`	Kronecker multiplication	$\mathbf{C}\langle\mathbf{M}\rangle\odot{=}kron(\mathbf{A},\mathbf{B})$
`new`	new matrix	let: $\mathbf{A}\in\mathsf{TYPE}^{n\times m}_{\mathsf{PRECISION}}$
	new vector	let: $\mathbf{u}\in\mathsf{TYPE}^{n}_{\mathsf{PRECISION}}$
`build`	build matrix from index/value arrays	$\mathbf{C}\leftarrowtail\{I,J,X\}$
	build vector from index/value arrays	$\mathbf{w}\leftarrowtail\{I,X\}$
`extractTuples`	extract index/value arrays	$\{I,J,X\}\rightarrowtail\mathbf{A}$
		$\{I,X\}\rightarrowtail\mathbf{u}$
`dup`	duplicate matrix	$\mathbf{C}\leftarrowtail\mathbf{A}$
	duplicate vector	$\mathbf{w}\leftarrowtail\mathbf{u}$
`extractElement`	extract scalar element	$s=\mathbf{A}(i,j)$
		$s=\mathbf{u}(i)$
`setElement`	set element	$\mathbf{C}(i,j)=s$
		$\mathbf{w}(i)=s$

Table 1: GraphBLAS operations and methods based on [6]. *Notation:* Matrices and vectors are typeset in bold, starting with uppercase ($\mathbf{A}$) and lowercase ($\mathbf{u}$) letters, respectively. Scalars including indices are lowercase italic (s, i, j) while arrays are uppercase italic (X, I, J). $\oplus$ and $\otimes$ are the addition and multiplication operators forming a semiring and default to conventional arithmetic $+$ and $\times$ operators. $\odot$ is the apply operator. Masks $\langle\mathbf{M}\rangle$ and $\langle\mathbf{m}\rangle$ are used to selectively write to the result matrix/vector. The complements of masks $\langle\mathbf{M}\rangle$, $\langle\mathbf{m}\rangle$ can be selected with the negation symbol, denoted with $\langle\neg\mathbf{M}\rangle$, $\langle\neg\mathbf{m}\rangle$, respectively. Masks with "replace" semantics (annihilating all elements outside the mask) are denoted with $\langle\langle\mathbf{M}\rangle\rangle$, $\langle\langle\neg\mathbf{M}\rangle\rangle$, $\langle\langle\mathbf{m}\rangle\rangle$, and $\langle\langle\neg\mathbf{m}\rangle\rangle$, The structure of the mask is denoted with $\langle\{\mathbf{M}\}\rangle$, $\langle\neg\{\mathbf{M}\}\rangle$, $\langle\{\mathbf{m}\}\rangle$, and $\langle\neg\{\mathbf{m}\}\rangle$.

Algorithm 1: Breadth-first search.

```
1 Function BFS
2     frontier(s) = TRUE
3     for level = 1 to n − 1 do
4         seen⟨frontier⟩ = level
5         frontier⟪¬seen⟫ = frontier any.pair A
```

Algorithm 2: Breadth-first search (push/pull).

```
1 Function BFS
2     frontier(s) = TRUE
3     for level = 1 to n − 1 do
4         seen⟨frontier⟩ = level
5         push = use some heuristics to determine whether to push/pull
6         if push then
7             frontier⟪¬seen⟫ = frontier any.pair A
8         else
9             frontier⟪¬seen⟫ = A any.pair frontier
```

Algorithm 3: Multi-source breadth-first search.

```
  Data: ...
  Result: ...
1 Function MSBFS
2     Frontier ↤ diag(S, n)
3     for level = 1 to n − 1 do
4         Seen⟨Frontier⟩ = level
5         Frontier⟪¬Seen⟫ = Frontier any.pair A
```

Algorithm 4: Betweenness centrality.

Function *MSBFS*
  // The **NumSp** structure holds the number of shortest paths for each node and starting vertex discovered so far.
  // Initialized to source vertices.
  $\mathbf{NumSp} \leftarrow \{s, [1, 1, \ldots, 1]\}$
  // The **Frontier** holds the number of shortest paths for each node and starting vertex discovered so far.
  // Initialized to source vertices.
  $\mathbf{Frontier}\langle\mathbf{NumSp}\rangle = \mathbf{A}(s, :)$
  $d = 0$
  // The **Sigmas** matrices store frontier information for each level of the BFS phase.
  // BFS phase (forward sweep)
  do
    // $\mathbf{Sigmas}[d](:, s) = d^{\text{th}}$ level frontier from source vertex s
    let: $\mathbf{Sigmas}[d] \in \mathbb{B}^{n \times nsver}$
    $\mathbf{Sigmas}[d](:, :) = \mathbf{Frontier}$ // Convert matrix to Boolean
    $\mathbf{NumSp} = \mathbf{NumSp} \oplus \mathbf{Frontier}$ // Accumulate path counts
    $\mathbf{Frontier}\langle\langle\mathbf{NumSp}\rangle\rangle = \mathbf{A}^{\top} \oplus.\otimes \mathbf{Frontier}$ // Update frontier
  while $nvals(\mathbf{Frontier}) > 0$
  let: $\mathbf{NumSpInv} \in \mathbb{Q}_{32}^{n \times nsver}$
  $\mathbf{NumSpInv} = 1.0 \oslash \mathbf{NumSp}$
  let: $\mathbf{BCU} \in \mathbb{Q}_{32}^{n \times nsver}$
  $\mathbf{BCU}(:) = 1.0$ // Make **BCU** dense, initialize all elements to 1.0
  let: $\mathbf{W} \in \mathbb{Q}_{32}^{n \times nsver}$
  // Tally phase (backward sweep)
  for $i = d - 1$ **downto** 0 **do**
    $\mathbf{W}\langle\langle\mathbf{Sigmas}[i]\rangle\rangle = \mathbf{NumSpInv} \oslash \mathbf{BCU}$
    $\mathbf{W}\langle\langle\mathbf{Sigmas}[i-1]\rangle\rangle = \mathbf{A} \oplus.\otimes \mathbf{W}$ // Add contributions by successors and mask with that BFS level's frontier.
    $\mathbf{BCU} \oplus= \mathbf{W} \otimes \mathbf{NumSp}$
  // Row reduce **BCU** and subtract $nsver$ from every entry to account for 1 extra value per **BCU** row element
  $\mathbf{delta} = [\oplus_j \mathbf{BCU}(:, j)]$
  $\mathbf{delta} \ominus= nsver$

Algorithm 5: PageRank (used in Graphalytics).

Data: α constant (damping factor)
Result: ...
Function *PageRank*
  $\mathbf{pr}(:) = 1/n$
  $\mathbf{outdegrees} = [\oplus_j \mathbf{A}(:, j)]$
  for $k = 1$ **to** $numIterations$ **do**
    $\mathbf{importance} = \mathbf{pr} \oslash \mathbf{outdegrees}$
    $\mathbf{importance} = times(\mathbf{importance}, \alpha)$ // apply the $times(x, s) = x \cdot s$ operator
    $\mathbf{importance} = \mathbf{importance} \oplus.\otimes \mathbf{A}$
    $\mathbf{danglingVertexRanks}\langle\neg\mathbf{outdegrees}\rangle = \mathbf{pr}(:)$
    $totalDanglingRank = [\oplus_i \mathbf{danglingVertexRanks}(i)(\otimes)]\frac{\alpha}{n}$
    $\mathbf{pr}(:) = \frac{1-\alpha}{n} \oplus totalDanglingRank$
    $\mathbf{pr}\langle\mathbf{importance}\rangle = \mathbf{pr} \oplus \mathbf{importance}$

Algorithm 6: Algebraic Bellman-Ford for SSSP.

Function *SSSP*
- $\mathbf{d}(s) = 0$
- **for** $k = 1$ **to** $n - 1$ **do**
  - $\mathbf{dtmp} = \mathbf{d}\ \mathsf{min.plus}\ \mathbf{A}$
  - **if** $\mathbf{dtmp} = \mathbf{d}$ **then**
    - **break**
  - $\mathbf{d} \leftarrowtail \mathbf{dtmp}$

Algorithm 7: Delta-stepping SSSP.

Data:
- $\mathbf{A}, \mathbf{A_H}, \mathbf{A_L} \in \mathbb{Q}^{|V| \times |V|}$
- $s, i \in \mathbb{N}$
- $\Delta \in \mathbb{Q}$
- $\mathbf{t}, \mathbf{t_{Req}} \in \mathbb{Q}^{|V|}$
- $\mathbf{t_{B_i}}, \mathbf{e} \in \mathbb{N}^{|V|}$

Function *DeltaStepping*
- $\mathbf{A_L} = select(0 < \mathbf{A} \leq \Delta)$
- $\mathbf{A_H} = select(\Delta < \mathbf{A})$
- $\mathbf{t}(:) = \infty$
- $\mathbf{t}(s) = 0$
- **while** $nvals(select(i\Delta \leq \mathbf{t})) \neq 0$ **do**
  - $s = 0$
  - $\mathbf{t_{B_i}} = select(i\Delta \leq \mathbf{t} < (i+1)\Delta)$
  - **while** $\mathbf{t_{B_i}} \neq 0$ **do**
    - $\mathbf{t_{Req}} = \mathbf{A_L}^{\top} \oplus . \otimes (\mathbf{t} \otimes \mathbf{t_{B_i}})$
    - $\mathbf{e} = select(0 < \mathbf{e} \oplus \mathbf{t_{B_i}})$
    - $\mathbf{t_{B_i}} = select(i\Delta \leq \mathbf{t_{Req}} < (i+1)\Delta) \otimes (\mathbf{t_{Req}} \underset{\oplus}{\mathsf{min}}\, \mathbf{t})$
  - $\mathbf{t_{Req}} = \mathbf{A_H}^{\top} \oplus . \otimes (\mathbf{t} \otimes \mathbf{e})$
  - $\mathbf{t} = \mathbf{t}\ \mathsf{min}\ \mathbf{t_{Req}}$
  - $i = i + 1$

Algorithm 8: All-pairs shortest path (Floyd–Warshall algorithm).

Function *FloydWarshall*
- $\mathbf{D} \leftarrowtail \mathbf{A}$
- **for** $k = 1$ **to** n **do**
  - $\mathbf{D} = \mathbf{D}\ \mathsf{min}[\mathbf{D}(:, k)\ \mathsf{min.plus}\ (\mathbf{D}(k, :)]$

Algorithm 9: FastSV algorithm.

```
Function FastSV
    n = nrows(A)
    gf = f
    dup = gf
    mngf = gf
    {I, X} ↤ f
    repeat
        // Step 1: Stochastic hooking
        mngf = mngf min A
        mngf = mngf second.min gf
        f(X) = f min mngf
        // Step 2: Aggressive hooking
        f = f min mngf
        // Step 3: Shortcutting
        f = f min gf
        // Step 4: Calculate grandparents
        {I, X} ↤ f
        gf = f(X)
        // Step 5: Check termination
        diff = dup ≠ gf
        sum = [⊕_i diff(i)]
        dup = gf
    until sum == 0
```

Algorithm 10: Triangle count (Cohen's algorithm).

```
Function TriangleCount
    L = tril(A)
    U = triu(A)
    B = L ⊕.⊗ U
    C = B ⊗ A
    t = [⊕_ij C(i, j)]/2
```

Algorithm 11: Triangle count (Sandia).

```
Function TriangleCount
    L = tril(A)
    C⟨L⟩ = L ⊕.⊗ L
    t = [⊕_ij C(i, j)]
```

Algorithm 12: Triangle count (FLAME).

```
Function TriangleCountFlame
    for i = 2 to n − 1 do
        A_20 = A(i + 1 : n, 0 : i − 1)
        a_10 = A(0 : i − 1, i)
        a_12 = A(i, i + 1 : n)
        t ⊕= a_10 ⊕.⊗ A_20 ⊕.⊗ a_12
```

Algorithm 13: Local clustering coefficient.

```
Function PageRank
    Tri⟨A⟩ = A ⊕.⊗ A                          // compute triangle count matrix
    tri = [⊕_j Tri(:, j)]                     // reduce to triangle count vector
    deg = [⊕_j A(:, j)]                       // reduce to vertex degree vector
    wed = perm2(deg)          // apply perm2(x) = x · (x − 1) to get wedge count vector
    lcc = tri ⊘ wed                                                  // LCC vector
```

Algorithm 14: k-truss algorithm.

Function *KTruss*
- $\mathbf{C} \leftarrow \mathbf{A}$
- $nonzeros \leftarrow nvals(C)$
- **for** $i = 1$ **to** $n - 1$ **do**
  - $\mathbf{C}\langle\mathbf{C}\rangle = \mathbf{C} \oplus.\, \mathsf{land}\, \mathbf{C}$
  - $\mathbf{C} = select(\mathbf{C} \geq k - 2)$
  - **if** $nonzeros == nvals(C)$ **then**
    - **break**
  - $nonzeros \leftarrow nvals(C)$

Algorithm 15: Louvain algorithm (WIP).

Function *Louvain*
- $\mathbf{G} \oplus\!= \mathbf{G}^\top$
- $\mathbf{k} = [\oplus_j \mathbf{A}(:, j)]$
- $m = \frac{1}{2}[\oplus_i \mathbf{k}(i)]$
- $\mathbf{S} \leftarrow \mathbf{I}$
- $vertices\_changed \leftarrow nvals(\mathbf{k})$
- **while** $vertices\_changed > 0$ **do**
  - **for** $j \in range(|V|)$ **do**
    - $\mathbf{v} = \mathbf{G}(j, :)$
    - $\mathbf{t_q} = \mathbf{v}\ \mathsf{any.pair}\ \mathbf{S}$
    - $\mathbf{sr} = \mathbf{S}(j, :)$
    - $\mathbf{S}(j, :) = \text{empty}$
    - $\mathbf{q} \leftarrow \mathbf{k}$
    - $\mathbf{q}\langle\mathbf{k}\rangle \otimes\!= -\mathbf{k}(j)/m$
    - $\mathbf{q} \oplus\!= \mathbf{v}$
    - $\mathbf{q_1}\langle\mathbf{t_q}\rangle = \mathbf{q} \oplus.\otimes \mathbf{S}$
    - $\mathbf{t} = (\mathbf{q_1} == [\mathsf{max}_i\, \mathbf{q_1}(i)])$
    - **while** $nvals(t) \neq 1$ **do**
      - $\mathbf{p} = random() \otimes \mathbf{t}$
      - $\mathbf{t} = (\mathbf{p} == [\mathsf{max}_i\, \mathbf{p}(i)])$
    - $\mathbf{S}(j, :) = \mathbf{t}$
    - **if** $nvals(\mathbf{sr} \otimes \mathbf{t}) == 0$ **then**
      - $vertices\_changed = nvals(\mathbf{k})$
    - $vertices\_changed = vertices\_changed - 1$

Algorithm 16: Community detection using label propagation (for undirected graphs).

Function *CDLP*
- $\mathbf{L} \leftarrow diag([0, 1, \ldots, n - 1])$
- **for** $k = 1$ **to** t **do**
  - $\mathbf{F} = \mathbf{A}\ \mathsf{any}\ .\ \mathsf{second}\ \mathbf{L}$ `// Frequency matrix`
  - $\{I, \_, X\} \leftarrow \mathbf{F}$
  - $merge\_sort\_pairs(I, X)$
  - **labels** = for each row in I, select min mode value from X

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