

LAGraph Algorithms

LAGraph Working Group

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Abstract

Theoretical documentation for LAGraph.

1 Notation

Table 1 contains the notation of GraphBLAS operations

Additionally, we use $\mathbf{D} = \text{diag}(\mathbf{J}, \mathbf{n})$ to construct a diagonal matrix $\mathbf{D} \leftarrow \{\mathbf{J}, \mathbf{J}, [1, 1, \dots, 1]\}$. The elements of the matrix are $\mathbf{D}(\mathbf{j}, \mathbf{j}) = 1$ for $\mathbf{j} \in \mathbf{J}$.

Initializing scalars, vectors, and matrices (GraphBLAS methods):

- $\mathbf{s} = \text{fp64}()$
- $\mathbf{u} = \text{fp32}(\mathbf{n})$
- $\mathbf{A} = \text{uint16}(\mathbf{m}, \mathbf{n})$
- $\mathbf{A} = \text{int64}(\mathbf{k}, \mathbf{m})$

2 Algorithms

LAGraph [12] implements graph algorithms using the GraphBLAS C API [13].

An incomplete list of GraphBLAS algorithms:

- MSBFS [8], bidirectional BFS [8], pushpull BFS [19]
- DFS [16]
- weakly connected components [21]
- SCC (LAGraph)
- SSSP, delta-stepping [17]
- triangle count [1, 5], triangle enumeration [1], item local clustering coefficient [4]
- k -truss [5]
- betweenness centrality [12]
- closeness centrality [8]
- DNN algorithm [10, 7]
- PageRank variants (at least 2) [4], IISWC paper, ...
- Louvain [11]
- property graphs: incremental TTC case [9], SIGMOD 2014 Contest [8] Roi Lipman's talk¹
- CFPQs based on a string of GRADES/ADBIS/other papers [2, 3, 14, 18, 15]
- Implementations: GBTL², SuiteSparse:GraphBLAS [6], GraphBLAST [20]

¹<http://wiki.ldbcouncil.org/pages/viewpage.action?pageId=106233859&preview=/106233859/111706128/LDBC-July-2019.pdf>

²<https://github.com/cmu-sei/gbtl>

op./method	name	notation
mxm	matrix-matrix multiplication	$\mathbf{C} \langle \mathbf{M} \rangle + = \mathbf{A} + . \times \mathbf{B}$
vxm	vector-matrix multiplication	$\mathbf{w} \langle \mathbf{m} \rangle + = \mathbf{u} + . \times \mathbf{A}$
mxv	matrix-vector multiplication	$\mathbf{w} \langle \mathbf{m} \rangle + = \mathbf{A} + . \times \mathbf{u}$
eWiseAdd	element-wise addition set union of patterns	$\mathbf{C} \langle \mathbf{M} \rangle + = \mathbf{A} + \mathbf{B}$ $\mathbf{w} \langle \mathbf{m} \rangle + = \mathbf{u} + \mathbf{v}$
eWiseMult	element-wise multiplication set intersection of patterns	$\mathbf{C} \langle \mathbf{M} \rangle + = \mathbf{A} \times \mathbf{B}$ $\mathbf{w} \langle \mathbf{m} \rangle + = \mathbf{u} \times \mathbf{v}$
extract	extract submatrix extract column vector extract row vector extract subvector	$\mathbf{C} \langle \mathbf{M} \rangle + = \mathbf{A}(\mathbf{I}, \mathbf{J})$ $\mathbf{w} \langle \mathbf{m} \rangle + = \mathbf{A}(:, \mathbf{j})$ $\mathbf{w} \langle \mathbf{m} \rangle + = \mathbf{A}(\mathbf{i}, :)$ $\mathbf{w} \langle \mathbf{m} \rangle + = \mathbf{u}(\mathbf{I})$
assign	assign matrix to submatrix with mask for \mathbf{C} assign scalar to submatrix with mask for \mathbf{C} assign vector to subvector with mask for \mathbf{w} assign scalar to subvector with mask for \mathbf{w}	$\mathbf{C} \langle \mathbf{M} \rangle (\mathbf{I}, \mathbf{J}) + = \mathbf{A}$ $\mathbf{C} \langle \mathbf{M} \rangle (\mathbf{I}, \mathbf{J}) + = s$ $\mathbf{w} \langle \mathbf{m} \rangle (\mathbf{I}) + = \mathbf{u}$ $\mathbf{w} \langle \mathbf{m} \rangle (\mathbf{I}) + = s$
subassign (GxB)	assign matrix to submatrix with submask for $\mathbf{C}(\mathbf{I}, \mathbf{J})$ assign scalar to submatrix with submask for $\mathbf{C}(\mathbf{I}, \mathbf{J})$ assign vector to subvector with submask for $\mathbf{w}(\mathbf{I})$ assign scalar to subvector with submask for $\mathbf{w}(\mathbf{I})$	$\mathbf{C}(\mathbf{I}, \mathbf{J}) \langle \mathbf{M} \rangle + = \mathbf{A}$ $\mathbf{C}(\mathbf{I}, \mathbf{J}) \langle \mathbf{M} \rangle + = s$ $\mathbf{w}(\mathbf{I}) \langle \mathbf{m} \rangle + = \mathbf{u}$ $\mathbf{w}(\mathbf{I}) \langle \mathbf{m} \rangle + = s$
apply	apply unary operator	$\mathbf{C} \langle \mathbf{M} \rangle + = \mathbf{f}(\mathbf{A})$ $\mathbf{w} \langle \mathbf{m} \rangle + = \mathbf{f}(\mathbf{u})$
select (GxB)	apply select operator	$\mathbf{C} \langle \mathbf{M} \rangle + = \text{select}(\mathbf{A}, \mathbf{f}(\mathbf{k}))$ $\mathbf{C} \langle \mathbf{M} \rangle + = \text{select}(\text{low} \leq \mathbf{A} \leq \text{up})$ $\mathbf{w} \langle \mathbf{m} \rangle + = \text{select}(\mathbf{u}, \mathbf{f}(\mathbf{k}))$ $\mathbf{w} \langle \mathbf{m} \rangle + = \text{select}(\text{low} \leq \mathbf{u} \leq \text{up})$
reduce	reduce matrix to column vector reduce matrix to scalar reduce vector to scalar	$\mathbf{w} \langle \mathbf{m} \rangle + = [\mathbf{A}]$ $s + = [\mathbf{A}]$ $s + = [\mathbf{u}]$
transpose	transpose	$\mathbf{C} \langle \mathbf{M} \rangle + = \mathbf{A}'$
kroncker	Kronecker multiplication	$\mathbf{C} \langle \mathbf{M} \rangle + = \text{kron}(\mathbf{A}, \mathbf{B})$
new	new matrix new vector	$\mathbf{A} = \text{TYPEPRECISION}(\mathbf{n}, \mathbf{m})$ $\mathbf{u} = \text{TYPEPRECISION}(\mathbf{n})$
build	build matrix from index/value arrays build vector from index/value arrays	$\mathbf{C} \leftarrow \{\mathbf{I}, \mathbf{J}, \mathbf{X}\}$ $\mathbf{w} \leftarrow \{\mathbf{I}, \mathbf{X}\}$
extractTuples	extract index/value arrays	$\{\mathbf{I}, \mathbf{J}, \mathbf{X}\} \leftarrow \mathbf{A}$ $\{\mathbf{I}, \mathbf{X}\} \leftarrow \mathbf{u}$
dup	duplicate matrix duplicate vector	$\mathbf{C} \leftarrow \mathbf{A}$ $\mathbf{w} \leftarrow \mathbf{u}$
extractElement	extract scalar element	$s = \mathbf{A}(\mathbf{i}, \mathbf{j})$ $s = \mathbf{u}(\mathbf{i})$
setElement	set element	$\mathbf{C}(\mathbf{i}, \mathbf{j}) = s$ $\mathbf{w}(\mathbf{i}) = s$

Table 1: GraphBLAS operations and methods based on [6]. *Notation:* Matrices and vectors are typeset in bold, starting with uppercase (\mathbf{A}) and lowercase (\mathbf{u}) letters, respectively. Scalars including indices are lowercase italic (s , i , j) while arrays are uppercase italic (\mathbf{X} , \mathbf{I} , \mathbf{J}). $+$ and \times are the addition and multiplication operators forming a semiring and default to conventional arithmetic $+$ and \times operators. \odot is the apply operator. Masks $\langle \mathbf{M} \rangle$ and $\langle \mathbf{m} \rangle$ are used to selectively write to the result matrix/vector. The complements of masks $\langle \mathbf{M} \rangle$, $\langle \mathbf{m} \rangle$ can be selected with the negation symbol, denoted with $\langle !\mathbf{M} \rangle$, $\langle !\mathbf{m} \rangle$, respectively. Masks with “replace” semantics (annihilating all elements outside the mask) are denoted with $\langle \langle \mathbf{M} \rangle \rangle$, $\langle \langle !\mathbf{M} \rangle \rangle$, $\langle \langle \mathbf{m} \rangle \rangle$, and $\langle \langle !\mathbf{m} \rangle \rangle$. The structure of the mask is denoted with $\langle \{\mathbf{M}\} \rangle$, $\langle !\{\mathbf{M}\} \rangle$, $\langle \{\mathbf{m}\} \rangle$, and $\langle !\{\mathbf{m}\} \rangle$.

Algorithm 1: Breadth-first search.

```
1 Function BFS
2   frontier(s) = TRUE
3   for level = 1 to n - 1 do
4     seen < frontier >= level
5     frontier << !seen >> = frontier any.pair A
```

Algorithm 2: Breadth-first search (push/pull).

```
1 Function BFS
2   frontier(s) = TRUE
3   for level = 1 to n - 1 do
4     seen < frontier >= level
5     push = use some heuristics to determine whether to push/pull
6     if push then
7       frontier << !seen >> = frontier any.pair A
8     else
9       frontier << !seen >> = A any.pair frontier
```

Algorithm 3: Multi-source breadth-first search.

Data: ...

Result: ...

```
1 Function MSBFS
2   Frontier <- diag(S, n)
3   for level = 1 to n - 1 do
4     Seen < Frontier >= level
5     Frontier << !Seen >> = Frontier any.pair A
```

Algorithm 4: Betweenness centrality.

```
1 Function MSBFS
2   // The NumSp structure holds the number of shortest paths for each node and starting
   // vertex discovered so far.
3   // Initialized to source vertices.
4   NumSp <- {s, [1, 1, ..., 1]}
5   // The Frontier holds the number of shortest paths for each node and starting vertex
   // discovered so far.
6   // Initialized to source vertices.
7   Frontier < NumSp >= A(s, :)
8   d = 0
9   // The Sigmas matrices store frontier information for each level of the BFS phase.
10  // BFS phase (forward sweep)
11  do
12    // Sigmas[d](:, s) = dth level frontier from source vertex s
13    Sigmas[d] = bool(n, nsver)
14    Sigmas[d](:, :) = Frontier // Convert matrix to Boolean
15    NumSp = NumSp + Frontier // Accumulate path counts
16    Frontier << NumSp >>= A' + . × Frontier // Update frontier
17  while nvals(Frontier) > 0
18  NumSpInv = fp32(n, nsver)
19  NumSpInv = 1.0 DIV NumSp
20  BCU = fp32(n, nsver)
21  BCU(:) = 1.0 // Make BCU dense, initialize all elements to 1.0
22  W = fp32(n, nsver)
23  // Tally phase (backward sweep)
24  for i = d - 1 downto 0 do
25    W << Sigmas[i] >>= NumSpInv DIV BCU
26    W << Sigmas[i - 1] >>= A + . × W // Add contributions by successors and mask with that
    // BFS level's frontier.
27    BCU + = W × NumSp
28  // Row reduce BCU and subtract nsver from every entry to account for 1 extra value
    // per BCU row element
29  delta = [+ BCU]
30  delta MINUS = nsver
```

Algorithm 5: PageRank (used in Graphalytics).

Data: *alpha* constant (damping factor)

Result: ...

```
1 Function PageRank
2   pr(:) = 1/n
3   outdegrees = [+j A(:, j)]
4   for k = 1 to numIterations do
5     importance = pr DIV outdegrees
6     importance = times(importance, alpha) // apply the times(x, s) = x · s operator
7     importance = importance + . × A
8     danglingVertexRanks < !outdegrees >= pr(:)
9     totalDanglingRank = [+ danglingVertexRanks(i)](alpha)/(n)
10    pr(:) = (1 - alpha)/(n) + totalDanglingRank
11    pr < importance >= pr + importance
```

Algorithm 6: Algebraic Bellman-Ford for SSSP.

```
1 Function SSSP
2    $d(s) = 0$ 
3   for  $k = 1$  to  $n - 1$  do
4      $dtmp = d \text{ min.plus } A$ 
5     if  $dtmp = d$  then
6       break
7      $d \leftarrow dtmp$ 
```

Algorithm 7: Delta-stepping SSSP.

Data:

$A, A_H, A_L \in \text{fp}(|V|, |V|)$

$s, i \in \text{uint}()$

$\Delta \in \text{fp}()$

$t, t_{\text{Req}} \in \text{fp}(|V|)$

$t_{B_i}, e \in \text{uint}(|V|)$

```
1 Function DeltaStepping
2    $A_L = \text{select}(0 < A \leq \Delta)$ 
3    $A_H = \text{select}(\Delta < A)$ 
4    $t(:) = \infty$ 
5    $t(s) = 0$ 
6   while  $nvals(\text{select}(i\Delta \leq t)) \neq 0$  do
7      $s = 0$ 
8      $t_{B_i} = \text{select}(i\Delta \leq t < (i+1)\Delta)$ 
9     while  $t_{B_i} \neq 0$  do
10       $t_{\text{Req}} = A_L' + . \times (t \times t_{B_i})$ 
11       $e = \text{select}(0 < e + t_{B_i})$ 
12       $t_{B_i} = \text{select}(i\Delta \leq t_{\text{Req}} < (i+1)\Delta) \times (t_{\text{Req}} \min_+ t)$ 
13       $t_{\text{Req}} = A_H' + . \times (t \times e)$ 
14       $t = t \min t_{\text{Req}}$ 
15       $i = i + 1$ 
```

Algorithm 8: All-pairs shortest path (Floyd–Warshall algorithm).

```
1 Function FloydWarshall
2    $D \leftarrow A$ 
3   for  $k = 1$  to  $n$  do
4      $D = D \text{ min}[D(:, k) \text{ min.plus } (D(k, :))]$ 
```

Algorithm 9: FastSV algorithm.

```
1 Function FastSV
2    $n = \text{nrows}(A)$ 
3    $gf = f$ 
4    $dup = gf$ 
5    $mngf = gf$ 
6    $\{I, X\} \leftarrow f$ 
7   repeat
8     // Step 1: Stochastic hooking
9      $mngf = mngf \min A$ 
10     $mngf = mngf \text{ second.min } gf$ 
11     $f(X) = f \min mngf$ 
12    // Step 2: Aggressive hooking
13     $f = f \min mngf$ 
14    // Step 3: Shortcutting
15     $f = f \min gf$ 
16    // Step 4: Calculate grandparents
17     $\{I, X\} \leftarrow f$ 
18     $gf = f(X)$ 
19    // Step 5: Check termination
20     $diff = dup \neq gf$ 
21     $sum = [+_i diff(i)]$ 
22     $dup = gf$ 
23  until  $sum == 0$ 
```

Algorithm 10: Triangle count (Cohen's algorithm).

```
1 Function TriangleCount
2    $L = \text{tril}(A)$ 
3    $U = \text{triu}(A)$ 
4    $B = L + . \times U$ 
5    $C = B \times A$ 
6    $t = [+ C] / 2$ 
```

Algorithm 11: Triangle count (Sandia).

```
1 Function TriangleCount
2    $L = \text{tril}(A)$ 
3    $C \leftarrow L = L + . \times L$ 
4    $t = [+ C]$ 
```

Algorithm 12: Triangle count (FLAME).

```
1 Function TriangleCountFlame
2   for  $i = 2$  to  $n - 1$  do
3      $A_{20} = A(i + 1 : n, 0 : i - 1)$ 
4      $a_{10} = A(0 : i - 1, i)$ 
5      $a_{12} = A(i, i + 1 : n)$ 
6      $t += a_{10} + . \times A_{20} + . \times a_{12}$ 
```

Algorithm 13: Local clustering coefficient.

```
1 Function PageRank
2    $Tri \leftarrow A + . \times A$  // compute triangle count matrix
3    $tri = [+ Tri]$  // reduce to triangle count vector
4    $deg = [+ A]$  // reduce to vertex degree vector
5    $wed = \text{perm2}(deg)$  // apply  $\text{perm2}(x) = x \cdot (x - 1)$  to get wedge count vector
6    $lcc = tri \text{ DIV } wed$  // LCC vector
```

Algorithm 14: k -truss algorithm.

```
1 Function KTruss
2    $C \leftarrow A$ 
3    $\text{nonzeros} \leftarrow \text{nvals}(C)$ 
4   for  $i = 1$  to  $n - 1$  do
5      $C < C > = C + .1$  and  $C$ 
6      $C = \text{select}(C \geq k - 2)$ 
7     if  $\text{nonzeros} == \text{nvals}(C)$  then
8       break
9      $\text{nonzeros} \leftarrow \text{nvals}(C)$ 
```

Algorithm 15: Louvain algorithm (WIP).

```
1 Function Louvain
2    $G \leftarrow G'$ 
3    $k = [+A]$ 
4    $m = (1)/(2)[+k]$ 
5    $S \leftarrow I$ 
6
7    $\text{vertices\_changed} \leftarrow \text{nvals}(k)$ 
8   while  $\text{vertices\_changed} > 0$  do
9     for  $j \in \text{range}(|V|)$  do
10       $v = G(j, :)$ 
11       $t_q = v \text{ any.pair } S$ 
12       $sr = S(j, :)$ 
13       $S(j, :) = \text{empty}$ 
14
15       $q \leftarrow k$ 
16       $q < k > \times = -k(j)/m$ 
17       $q \leftarrow v$ 
18       $q_1 < t_q > = q + . \times S$ 
19
20       $t = (q_1 == [\max q_1])$ 
21      while  $\text{nvals}(t) \neq 1$  do
22         $p = \text{random}() \times t$ 
23         $t = (p == [\max p])$ 
24       $S(j, :) = t$ 
25
26      if  $\text{nvals}(sr \times t) == 0$  then
27         $\text{vertices\_changed} = \text{nvals}(k)$ 
28       $\text{vertices\_changed} = \text{vertices\_changed} - 1$ 
```

Algorithm 16: Community detection using label propagation (for undirected graphs).

```
1 Function CDLP
2    $L \leftarrow \text{diag}([0, 1, \dots, n - 1])$ 
3   for  $k = 1$  to  $t$  do
4      $F = A \text{ any. second } L$  // Frequency matrix
5      $\{I, \rightarrow, X\} \leftarrow F$ 
6      $\text{merge\_sort\_pairs}(I, X)$ 
7      $\text{labels} = \text{for each row in } I, \text{ select min mode value from } X$ 
```

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