Directed Angles

Graphiel

26 November 2024

Table of contents

§1 The basics

Definition 1.1. The *directed angle* $\measuredangle(l, m)$ is defined as the angle starting on l "going" until m *counterclockwise.*

The measurement of *directed angles* are measured **modulo 180°**, hence,

$$180° = 0$$

$$150° = -30°$$

$$90° = -90°$$

$$[...]$$

Also, notice that the *directed angle* $\measuredangle(ABC)$ can be written as

$$\measuredangle\left(\overline{AB}, \overline{BC}\right)$$

this is indeed very useful for a more intuitive understanding of "what the hell we are doing".

§2 Main properties

Fact 2.1. $\measuredangle(XYX) = 0$.

Fact 2.2. $\measuredangle(l, m) + \measuredangle(m, l) = 180°$.

Fact 2.3. $\measuredangle(ABC) = -\measuredangle(CBA)$.

Proof. By the Definition 1.1 of *directed angles*, we have that

$$\measuredangle(ABC) = \measuredangle\left(\overline{AB}, \overline{BC}\right)$$

$$\measuredangle(CBA) = \measuredangle\left(\overline{CB}, \overline{BA}\right) = \measuredangle\left(\overline{BC}, \overline{AB}\right)$$

since *directed angles* are measured *counterclockwise* from the first line to the second one, then it's clear that these two angles are measured oppositely. □

Fact 2.4. $\measuredangle PBA = \measuredangle PBC$ if and only if A, B, C are collinear.

Proof. Let A, B, C and P be all distinct non-collinear points. Notice that, if $\measuredangle PBA = \measuredangle PBC$, then we have two cases, hence,

$$\text{I.} \quad A \cong C \text{ (absurd by definition)}.$$

$$\text{II.} \quad \measuredangle PBA, \measuredangle PBC \notin \{0, 180°\} \text{ and } \measuredangle PBA + \measuredangle PBC = 180° = 0.$$

Notice that I is absurd by definition and that II is the only remaining option. Indeed, since they share the same side $\overline{PB}$, then A, B, C are in fact, collinear. □

Fact 2.5. $\measuredangle ABC + \measuredangle CBD = \measuredangle ABD$.

§3 Other useful properties

Theorem 3.1. *A quadrilateral $ABCD$ is cyclic if and only if $\measuredangle ABC = \measuredangle CDA$.*

Proof. Since a convex quadrilateral has opposite angles from which both together sum up to 180°, then $\angle ABC + \angle ADC = 180°$. If we denote that using *directed angles*, we get the following

$$\measuredangle ABC + \measuredangle ADC = 180°$$

but directed angles are taken by **modulo 180°**, therefore

$$\measuredangle ABC + \measuredangle ADC = 0$$

also, notice that $\measuredangle ADC = -\measuredangle CDA$, therefore

$$\measuredangle ABC - \measuredangle CDA = 0 \therefore \measuredangle ABC = \measuredangle CDA$$

just as we wished to prove. □

Theorem 3.2. *(Triangle sum)* $\measuredangle ABC + \measuredangle ACB + \measuredangle BAC = 0$

Proof. We can prove this easily by noticing that *directed angles* are measured by **modulo 180°**, which means that $180° = 0$ (see Definition 1.1).
However, if you are more skeptical and is wondering "are you sure that they necessarily sum up to 180°?" then the proof is left as an exercise[1]. □

Theorem 3.3. *If $AB \parallel CD$, then $\measuredangle ABC + \measuredangle BCD = 0$.*

Proof. This lefts us with two cases, hence,

I. Quadrilateral $ABCD$ is convex.

II. Quadrilateral $ABCD$ is not convex.

On case I, we have that

$$\measuredangle BCD = \measuredangle CBP \left(B \in \overline{AP} \land B \neq P\right) \therefore \measuredangle ABC + \measuredangle BCD = 180° = 0 \quad \blacksquare$$

On case II, we have that

$$\measuredangle ABC = -\measuredangle BCD \therefore \measuredangle ABC + \measuredangle BCD = 0 \quad \blacksquare$$

just as we wished to prove. □

[1] Try looking at "Figure 1.4A" on Evan Chen's Geometry book.