Shortlist 2010/G1

A Geometry problem

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Problem 0.1. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $\overline{AP} = \overline{AQ}$.

See Figure 1.

Solution. First, notice that

$$\angle APQ = \angle APB = \angle C.$$

Then notice, because $AFDC$ is a cyclic quadrilateral, that

$$\measuredangle C = \measuredangle AFD$$

since we're trying to prove that $\angle AQP = \angle APQ$ and $\angle AFP = \angle APQ$, then $\angle AQP$ would be equal to $\angle AFP$ which means that $AFQP$ would be a cyclic quadrilateral.
Let's prove it:

$$\measuredangle AFD = \measuredangle AFQ$$

$$\measuredangle AFQ = \measuredangle APQ$$

so $AFQP$ is, indeed, cyclic. This gives us

$$\angle AFP = \angle AQP = \angle APQ$$

just as we ought to prove. □