Notes of EGMO

Chapter 1 - Angle Chasing

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Table of contents

§1 Cyclic Quadrilaterals

Let $ABCD$ be a convex quadrilateral. Then the following are equivalent:

Proposition 1.1. $ABCD$ is cyclic.

Proposition 1.2. $\angle ABC + \angle CDA = 180°$.

Proposition 1.3. $\angle ABD = \angle ACD$.

Let's prove that if Proposition 1.1 is true, then Proposition 1.2 and Proposition 1.3 are also true, for that we will need the following Lemma 1.1.1 (after proving it, too):

§1.1 The Inscribed Angle Theorem

Lemma 1.1.1. *The angle $\angle ABC$ inscribed in the circumference Φ measures exactly half of the central correspondent angle $\angle AOC$.*

Proof. Let $\angle ABC = \alpha$ be a angle inscribed in a circumference Φ with center O. Notice that $\triangle\, BOA$ and $\triangle\, BOC$ are both isosceles $\left(\overline{OA} = \overline{OB} = \overline{OC}\right)$, therefore

$$\angle ABC = \angle OBA + \angle OBC$$

$$\angle AOC = 2(\angle OBA + \angle OBC)$$

$$\therefore \angle AOC = 2 \cdot \angle ABC$$

as we wished to proved. □

§1.2 Back to the origins

We can now use Lemma 1.1.1 to prove that Proposition 1.1 is equivalent to Proposition 1.2 and Proposition 1.3.

Theorem 1.2.1. *(Proposition 1.2) On a cyclic quadrilateral $ABCD$, $\angle ABC + \angle CDA = 180°$.*

Proof. Let O be the center of $(ABCD)$, notice that $2 \cdot \angle ABC = \angle AOC$ and $2 \cdot \angle CDA = \angle COA$. The following is, indeed, obvious

$$\angle AOC = \angle COA = 180° \therefore \angle AOC + \angle COA = 360°$$

$$2(\angle ABC + \angle CDA) = \angle AOC + \angle COA = 360° \therefore \angle ABC + \angle CDA = 180°$$

just as we wished to prove. □

Theorem 1.2.2. *(Proposition 1.3) On a cyclic quadrilateral $ABCD$, $\angle ABD = \angle ACD$.*

Proof. Take a look at Lemma 1.1.1,

$$\angle ABD = \angle ACD \because 2 \cdot \angle ABD = \angle AOD = 2 \cdot \angle ACD$$

just as we wished to prove. □

Remark 1.2.3. See my notes (or Evan's) on *directed angles*. They are very useful when speaking of cyclic quadrilaterals.

§2 The Orthic Triangle

This one comes in handy in relation to the previous section, for it stands as a nice exercise.

Lemma 2.1. *On $\triangle ABC$, let D, E and F denote the foot of the perpendiculars from A, B and C, respectively. Let $H = AD \cap BE \cap CF$ (H is called the* orthocenter[1] *of $\triangle ABC$), then*

Theorem 2.2. There are six cyclic quadrilaterals on this configuration.

Theorem 2.3. Point H is the incenter of $\triangle DEF$.

Let's take a look at Theorem 2.2 first:

Proof. If you drew the diagram, you may had quickly noticed that $AEHF$, $BDHF$ and $CDHE$ are all cyclic quadrilaterals, so there are only three other remaining to find. Now let's take a look at quadrilaterals $AFDC$, $AEBD$ and $BFEC$ and choose, say, $AFDC$ first, then notice that

$$\angle ADF = \angle FBH = 90° - \angle A$$

$$\angle FCA = 90° - \angle A \therefore \angle ADF = \angle FCA \therefore AFDC \text{ is cyclic.}$$

We can use the same "algorithm" to prove the "cyclicness" of $AEBD$ and $BFEC$ too. □

Now let's proceed to Theorem 2.3:

Proof. Let's take any angle formed by one of the vertexes of $\triangle DEF$ and do a simple angle chasing,

$$\angle ADF = \angle ABH$$

quadrilateral $AEBD$ is cyclic, therefore

$$\angle ABE = \angle ABH = \angle ADE$$

since $\angle ADF = \angle ADE$, then DH is indeed the bisector of $\angle FDE$, just as we wished to prove. □

§3 The Incenter/Excenter Lemma

Lemma 3.1. *Let ABC be a triangle with incenter I. Ray AI meets (ABC) again at L. Let I_A be the reflection of I over L. Then,*

Theorem 3.2. The points I, B, C, and I_A lie on a circle with diameter $\overline{II_A}$ and center L. In particular, $\overline{LI} = \overline{LB} = \overline{LC} = \overline{LI_A}$.

Theorem 3.3. Rays BI_A and CI_A bisect the exterior angles of $\triangle ABC$.

Let's prove Theorem 3.2 first:

Proof. By definition, points A, B, C, L are concyclic, therefore

$$\angle LAC = \angle LBC = \angle BAL = \angle BCL$$

[1]We will take for granted that this point exists (i.e., that lines AD, BE and CF all intersect at this common point H).

therefore $\triangle BLC$ is isosceles with base $\overline{BC}$ (i.e., $\overline{LB} = \overline{LC}$). Next, notice that triangles BIL and CIL are also isosceles ($\because \angle ICL = \angle CIL = \frac{\angle A + \angle C}{2}$ and $\angle IBL = \angle BIL = \frac{\angle A + \angle B}{2}$). Finally, we have that

$$\overline{LI} = \overline{LB} = \overline{LC} = \overline{LI_A}$$

Since $(I - L - I_A)$, then $\overline{II_A}$ is indeed the diameter of $(IBCI_A)$. □

Now we must prove Theorem 3.3, which can be easily done with simple angle chasing:

Proof. Extend rays AB and AC, take a look at $(IBCI_A)$, then it's obvious that

$$\angle CBP = \angle A + \angle C \text{ where } (P \in AB, AB \subset AP, P \neq B)$$

$$\angle CBI_A = \angle I_AIC = \frac{\angle A + \angle C}{2}$$

$$\therefore \angle I_ABP = \angle CBI_A \iff BI_A \text{ is the bisector of } \angle CBP$$

To prove that CI_A is the bisector of BCQ where $(Q \in AC, AC \subset AP, P \neq C)$ is the same thing. □

Remark 3.4. The point I_A is called the *excenter* of $\triangle ABC$. As you could see on this section (and will see again soon), this theorem is very useful and deserves it's title of *lemma*.

§4 Tangent to Circles and Phantom Points

§4.1 The Tangent-Chord Theorem

On a circumference (ABC), let P be a point that lies outside (ABC), then the following are all equivalent:

Proposition 4.1.1. *Line AP is tangent to (ABC).*

Proposition 4.1.2. $OA \perp AP$.

Proposition 4.1.3. $\angle PAB = \angle ACB$.

Let's prove that, if Proposition 4.1.1 is true then Proposition 4.1.2 and Proposition 4.1.3 are also true. Shall we begin with Proposition 4.1.2 first?

Proof. Assuming that line AP is tangent to (ABC), notice that OA is the smallest distance from O to A, therefore $OA \perp AP$. □

This one was tough. Now shall we prove Proposition 4.1.3?

Proof. Take a look at triangle AOB, it is isosceles of base $\overline{AB}$. Let $\angle AOB = 2 \cdot \alpha$, then $\angle OAB = \angle OBA = 90° - \alpha$. Since $\angle OAP = 90°$ and $\angle OAP = \angle PAB + \angle OAB = \angle PAB + 90° - \alpha$, then $\angle PAB = \alpha$ and also $\angle ACB = \frac{\angle AOB}{2} = \frac{2\alpha}{2} = \alpha$, therefore

$$\angle PAB = \angle ACB$$

just as we wished to prove. □

§4.2 P-phantom Points?!

Alright, ladies and gentleman, there's no other way to show why this is so useful than with a nice example problem.

Here's my loose definition of *phantom points*:

Definition 4.2.1. A *phantom point* P is a point not well defined, usually established when we want to prove collinearity, or perhaps when we want to prove that this *phantom point* is the same as other already established point on the diagram.

Now let's clear the way using a nice example problem:

Example 4.2.2. Let ABC be an acute triangle with circumcenter O, and let K be a point such that KA is tangent to (ABC) and $\angle KCB = 90°$. Point D lies on $\overline{BC}$ such that $KD \parallel AB$. Show that line $\overline{DO}$ passes through A.

Solution. First, let's create a *phantom point* D' such that $D' = AO \cap BC$, if we somehow prove that $D \cong D'$ then, by definition of D', we would also prove that $A - O - D$. So, forget about D and focus on D', not assuming anything other than it's definition. Notice that

$$\angle B = \angle KAC$$

$$\text{Quadrilateral } AD'CK \text{ is cyclic} \therefore \angle KAC = \angle KD'C \therefore \angle B = \angle KD'C$$

$$\therefore KD' \parallel AB$$

since there's only one line passing through K which is parallel to AB, then D' and D are unique, i.e.,

$$D \cong D'$$

just as we wished to prove. □