Vision Algorithms for Mobile Robotics

Lecture 03
Camera Calibration

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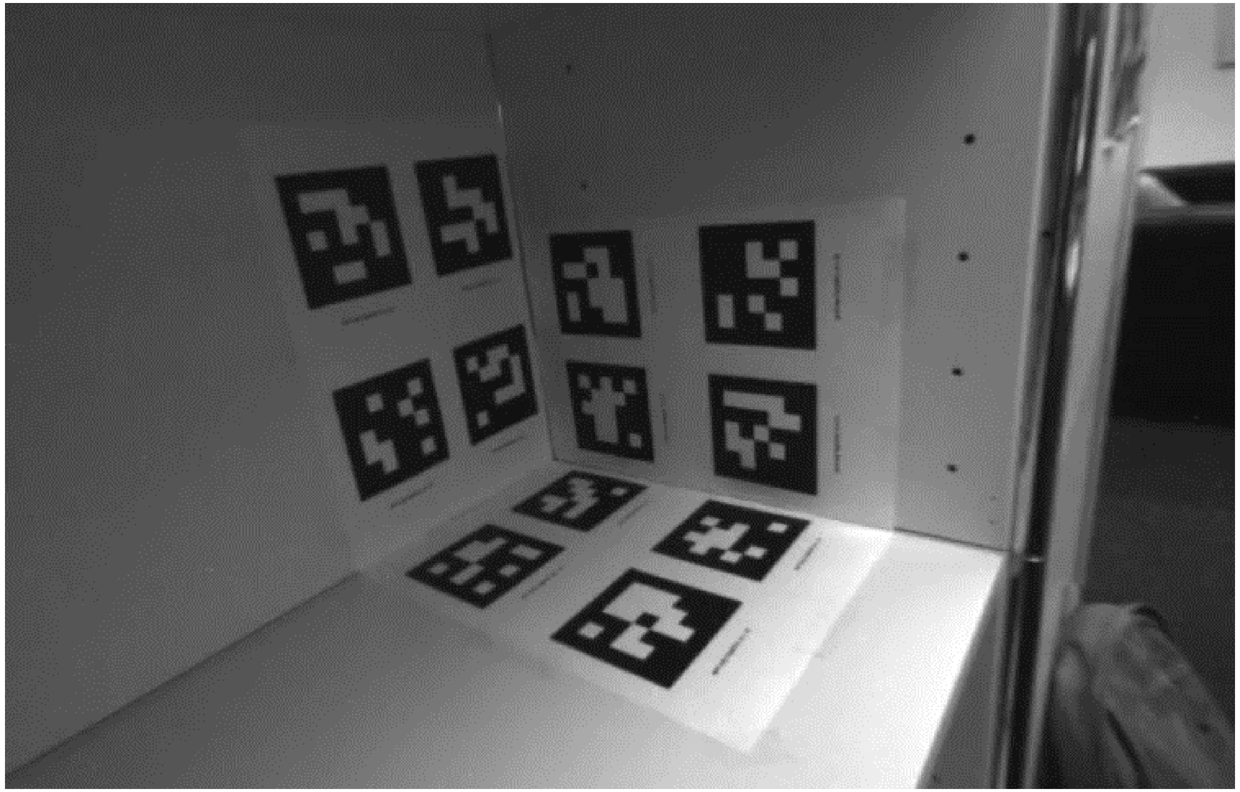
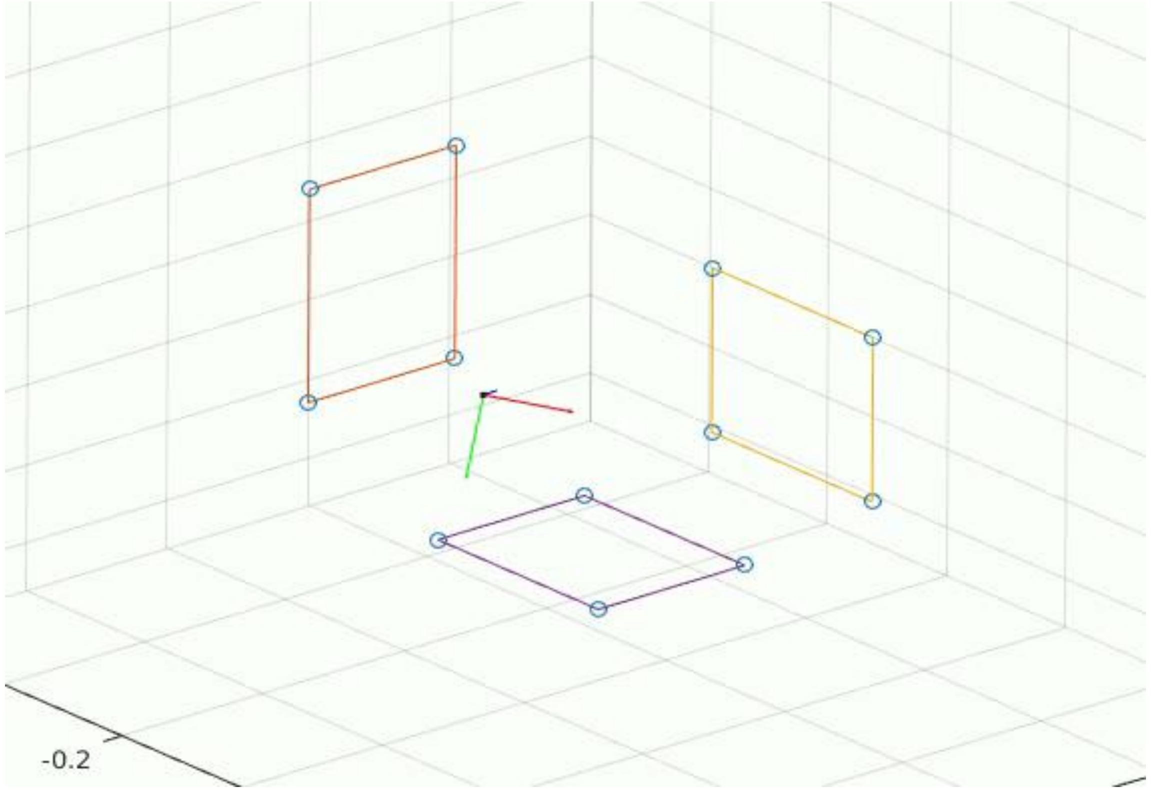
Lab Exercise 2 – This afternoon

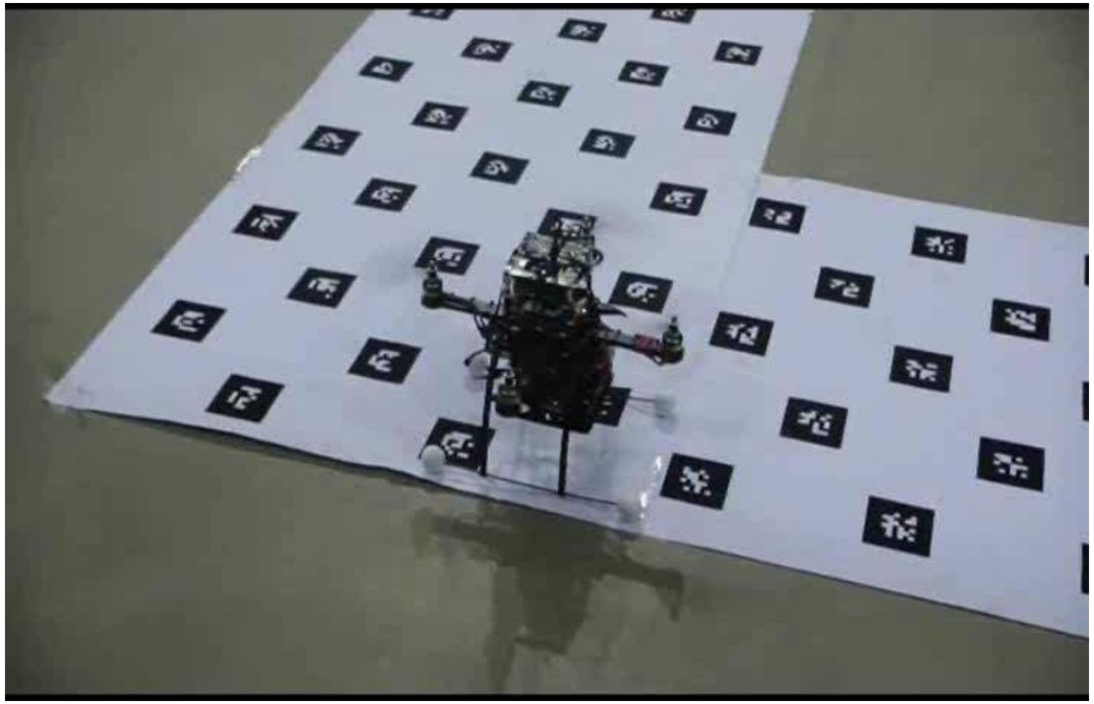
- Work description: your first camera motion estimator using DLT algorithm





Goal of today’s lecture

- Learn how to calibrate a camera
- Study the foundational algorithms for camera localization



Two applications of the camera localization algorithms covered in this lecture: drone navigation & Microsoft Hololens

Today’s Outline

- Camera calibration
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Camera Calibration

- Calibration is the process to determine the **intrinsic** (K plus lens distortion) **and extrinsic** (R, T) parameters of a camera. For now, we will **neglect the lens distortion** and see later how it can be determined.
- The solution for K, R, T can be found by applying the perspective projection equation:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K \left[R \mid T \right] \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

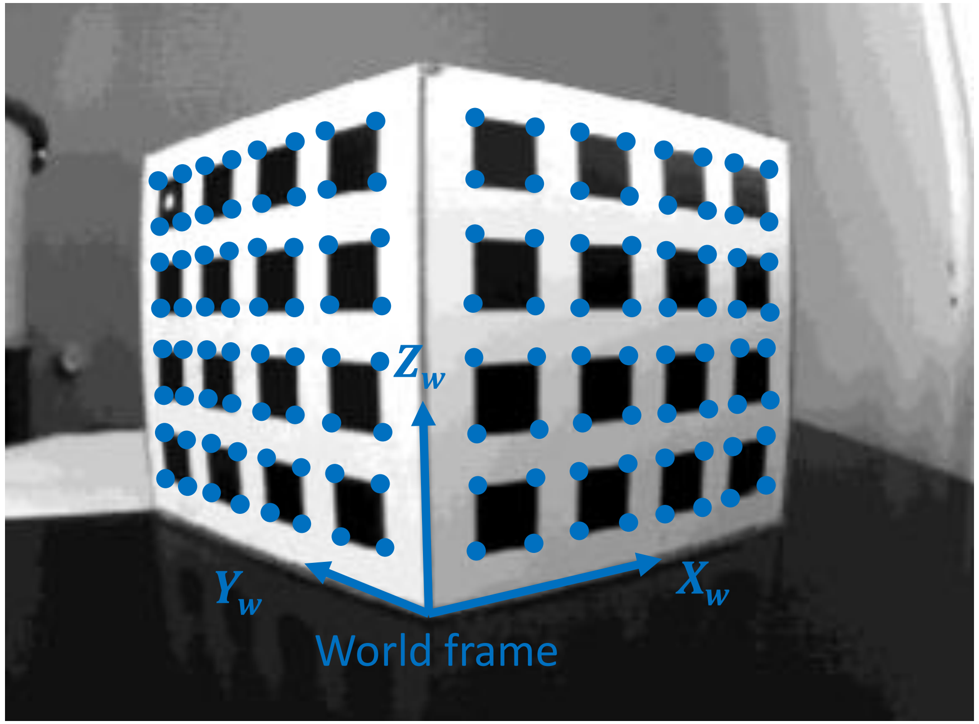
- There are two popular methods:
  - **Tsai's method**: uses 3D objects
  - **Zhang's method**: uses planar grids

Today’s Outline

- Camera calibration
  - Tsai’s method: From 3D objects
  - Zhang’s method: from planar grids
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Tsai's Method: Calibration from 3D Objects

- This method was proposed in 1987 by Tsai and consists of measuring the 3D position of $n \geq 6$ **control points** on a 3D calibration target and the **2D coordinates of their projection** in the image.



Tsai, Roger Y. (1987) "A Versatile Camera Calibration Technique for High-Accuracy 3D Machine Vision Metrology Using Off-the-Shelf TV Cameras and Lenses,'' *IEEE Journal of Robotics and Automation*, 1987. PDF.

Applying the Direct Linear Transform (DLT) algorithm

The idea of the DLT is to rewrite the perspective projection equation as a **homogeneous linear equation** and solve it by standard methods. Let's write the perspective equation for a generic 3D-2D point correspondence:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K\left[R \,\middle|\, T\right] \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_u r_{11} + u_0 r_{31} & \alpha_u r_{12} + u_0 r_{32} & \alpha_u r_{13} + u_0 r_{33} & \alpha_u t_1 + u_0 t_3 \\ \alpha_v r_{21} + v_0 r_{31} & \alpha_v r_{22} + v_0 r_{32} & \alpha_v r_{23} + v_0 r_{33} & \alpha_v t_2 + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

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$$\Rightarrow \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

What are the assumptions behind this this substitution?

Applying the Direct Linear Transform (DLT) algorithm

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = M \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

where m_i^{T} is the i-th row of M

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_1^{\mathrm{T}} \\ m_2^{\mathrm{T}} \\ m_3^{\mathrm{T}} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Applying the Direct Linear Transform (DLT) algorithm

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_1^{\mathrm{T}} \\ m_2^{\mathrm{T}} \\ m_3^{\mathrm{T}} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}}_{\boldsymbol{P}}$$

Conversion back from homogeneous coordinates to pixel coordinates leads to:

$$u = \frac{\lambda u}{\lambda} = \frac{m_1^{\mathrm{T}} \cdot P}{m_3^{\mathrm{T}} \cdot P} \qquad \Rightarrow \qquad (m_1^{\mathrm{T}} - u_i m_3^{\mathrm{T}}) \cdot P_i = 0$$

$$v = \frac{\lambda v}{\lambda} = \frac{m_2^{\mathrm{T}} \cdot P}{m_3^{\mathrm{T}} \cdot P} \qquad\qquad (m_2^{\mathrm{T}} - v_i m_3^{\mathrm{T}}) \cdot P_i = 0$$

Applying the Direct Linear Transform (DLT) algorithm

- By re-arranging the terms, we obtain

$$\begin{aligned}(m_1^{\mathrm{T}} - u_i m_3^{\mathrm{T}}) \cdot P_i = 0 \\ (m_2^{\mathrm{T}} - v_i m_3^{\mathrm{T}}) \cdot P_i = 0\end{aligned} \Rightarrow \begin{pmatrix} P_1^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_1^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_1^{\mathrm{T}} & -v_1 P_1^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- For n points, we can stack all these equations into a big matrix:

$$\begin{pmatrix} P_1^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_1^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_1^{\mathrm{T}} & -v_1 P_1^{\mathrm{T}} \\ & \vdots & \\ P_n^{\mathrm{T}} & 0^{\mathrm{T}} & -u_n P_n^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_n^{\mathrm{T}} & -v_n P_n^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Applying the Direct Linear Transform (DLT) algorithm

$$\underbrace{\begin{pmatrix} X_w^1 & Y_w^1 & Z_w^1 & 1 & 0 & 0 & 0 & 0 & -u_1X_w^1 & -u_1Y_w^1 & -u_1Z_w^1 & -u_1 \\ 0 & 0 & 0 & 0 & X_w^1 & Y_w^1 & Z_w^1 & 1 & -v_1X_w^1 & -v_1Y_w^1 & -v_1Z_w^1 & -v_1 \\ & & & & & & & \vdots & & & & \\ X_w^n & Y_w^n & Z_w^n & 1 & 0 & 0 & 0 & 0 & -u_nX_w^n & -u_nY_w^n & -u_nZ_w^n & -u_n \\ 0 & 0 & 0 & 0 & X_w^n & Y_w^n & Z_w^n & 1 & -v_nX_w^n & -v_nY_w^n & -v_nZ_w^n & -v_n \end{pmatrix}}_{\text{Q (this matrix is known)}} \underbrace{\begin{pmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{pmatrix}}_{\text{M (this matrix is unknown)}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{Q} \cdot \mathbf{M} = 0$$

Applying the Direct Linear Transform (DLT) algorithm

$$Q \cdot M = 0$$

Minimal solution

- $Q_{(2n\times 12)}$ should have rank 11 to have a unique (up to a scale) *non-zero* solution M
- Because each 3D-to-2D point correspondence provides 2 independent equations, then $5+\frac{1}{2}$ point correspondences are needed (in practice **6 point** correspondences!)

Over-determined solution

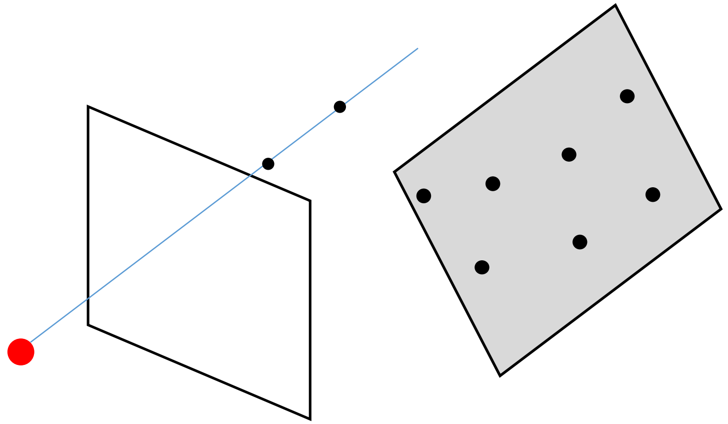
- For $n \geq 6$ points, a solution is the **Least Square solution**, which minimizes the sum of squared residuals, $||QM||^2$, subject to the constraint $||M||^2 = 1$. It can be solved through Singular Value Decomposition (SVD). The solution is the eigenvector corresponding to the smallest eigenvalue of the matrix $Q^T Q$ (because it is the unit vector x that minimizes $||Qx||^2 = x^T Q^T Q x$.
- Matlab instructions:
  - `[U,S,V] = SVD(Q);`
  - `M = V(:,12);`

Applying the Direct Linear Transform (DLT) algorithm

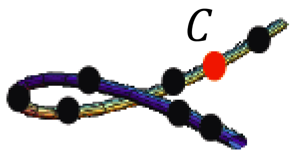
Degenerate configurations

$$\mathrm{Q} \cdot \mathrm{M} = 0$$

1. Points lying on a **plane** and/or along a single **line** passing through the **center of projection**



2. Camera and points on a **twisted cubic** (i.e., smooth curve in 3D space of degree 3)



Applying the Direct Linear Transform (DLT) algorithm

- Once we have determined M, we can recover the intrinsic and extrinsic parameters by remembering that:

$$\mathrm{M} = \mathrm{K}(\mathrm{R} \mid \mathrm{T})$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

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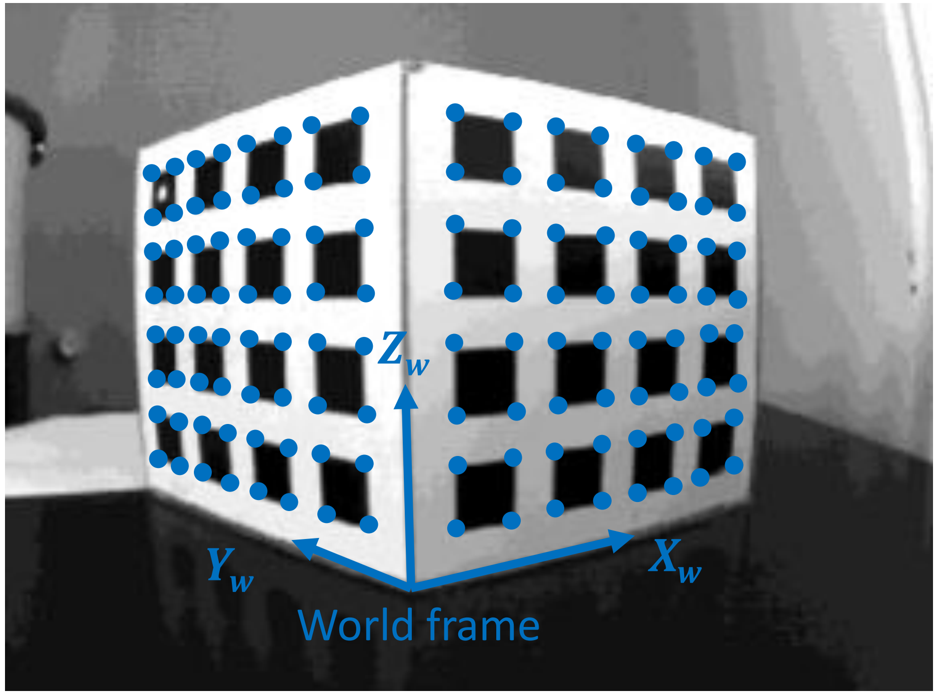
- However, notice that we are not enforcing the constraint that $\boldsymbol{R}$ **is orthogonal, i.e.,** $\boldsymbol{R} \cdot \boldsymbol{R}^{\boldsymbol{T}} = \boldsymbol{I}$
- To do this, we can use the so-called **QR factorization of** $\boldsymbol{M}$, which decomposes M into a R (orthogonal), T, and an upper triangular matrix (i.e., K)
- What if K is known (calibrated camera)?

Example of Tsai's Calibration Results

Recommendation: use many **more than 6 points** (ideally more than 20) **and non coplanar**



α_u	α_u/α_v	K_{12}	u_0	v_0	**Average Reprojection error**
1673.3	1.0063	1.39	379.96	305.78	0.365

Why is this ratio not 1?

What is this?

What is this?

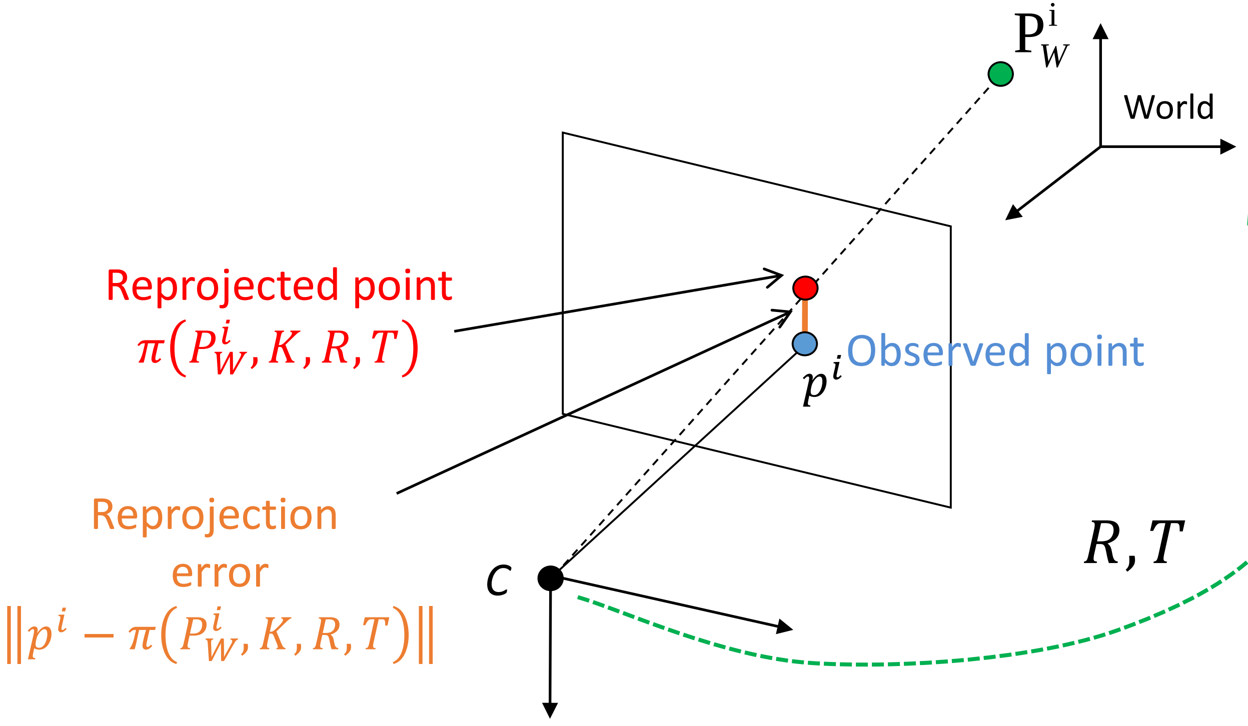
Corners can be detected with accuracy < 0.1 pixels (see Lecture 5)

How can we estimate the lens distortion parameters?

How can we enforce $\alpha_u = \alpha_v$ and $K_{12} = 0$?

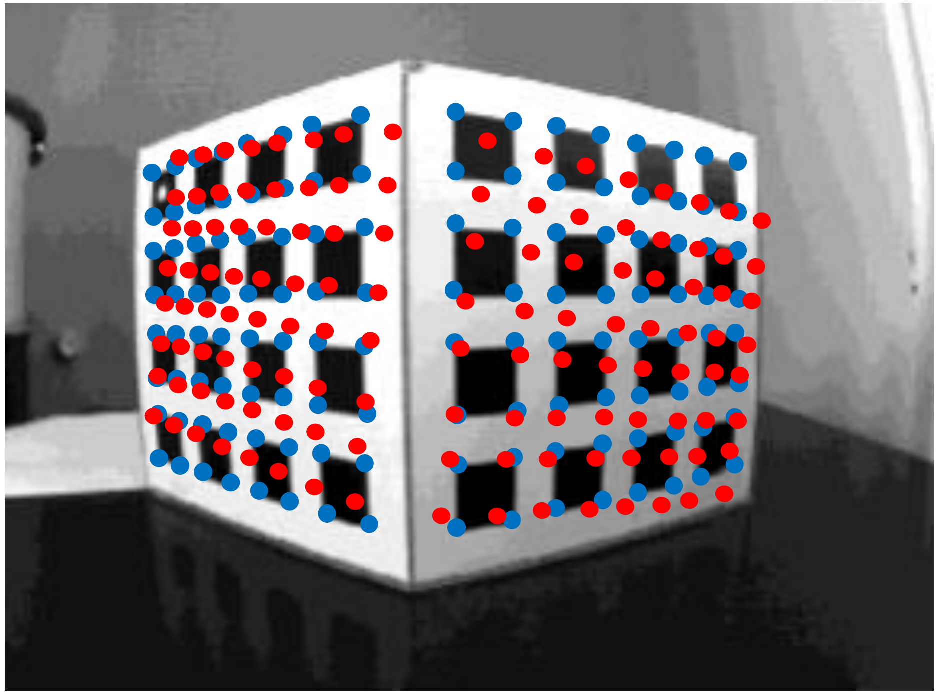
Reprojection Error

- The reprojection error is the **Euclidean distance** (in pixels) between an **observed image point** and the **corresponding** 3D point **reprojected** onto the camera frame.
- The reprojection error gives us a **quantitative measure of the accuracy** of the calibration (**ideally it should be zero**).



Reprojection Error

- The reprojection error can be used to assess the quality of the camera calibration
- What reprojection error is acceptable?
- What are the sources of the reprojection error?
- How can we further improve the calibration parameters?



Control points (observed points)

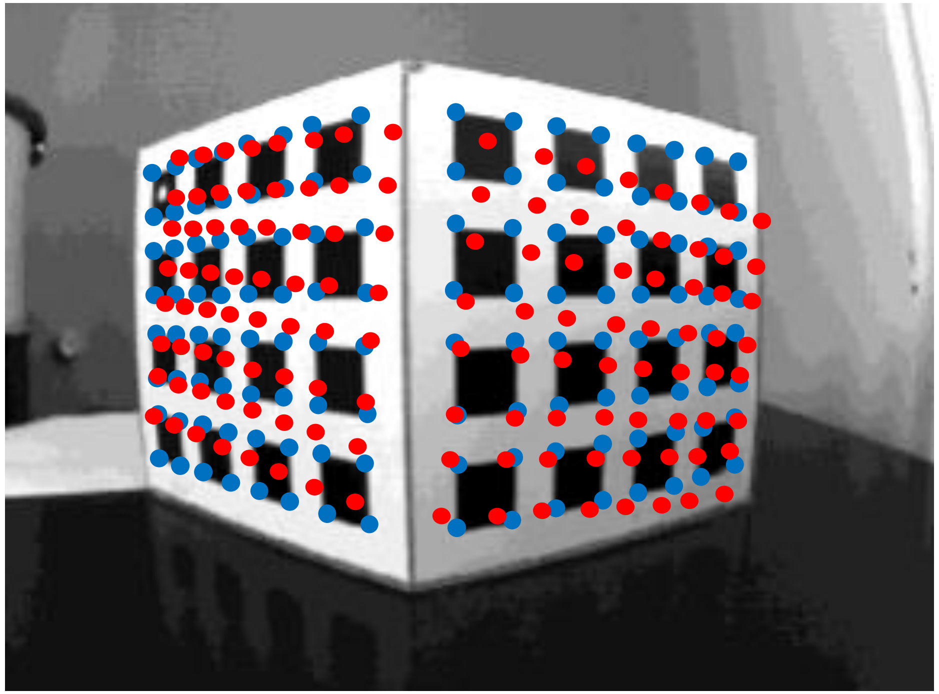
Reprojected points $\pi\left(P_W^i, K, R, T\right)$

Non-Linear Calibration Refinement

- The calibration parameters K, R, T determined by the DLT can be refined by minimizing the following cost:

$$K, R, T, lens\ distortion = argmin_{K,R,T,lens} \sum_{i=1}^{n} \left\| p^i - \pi\left(P_W^i, K, R, T\right) \right\|^2$$

- This time we also include the **lens distortion** (can be set to 0 for initialization)
- Can be minimized using **Levenberg–Marquardt** (more robust than Gauss-Newton to local minima)



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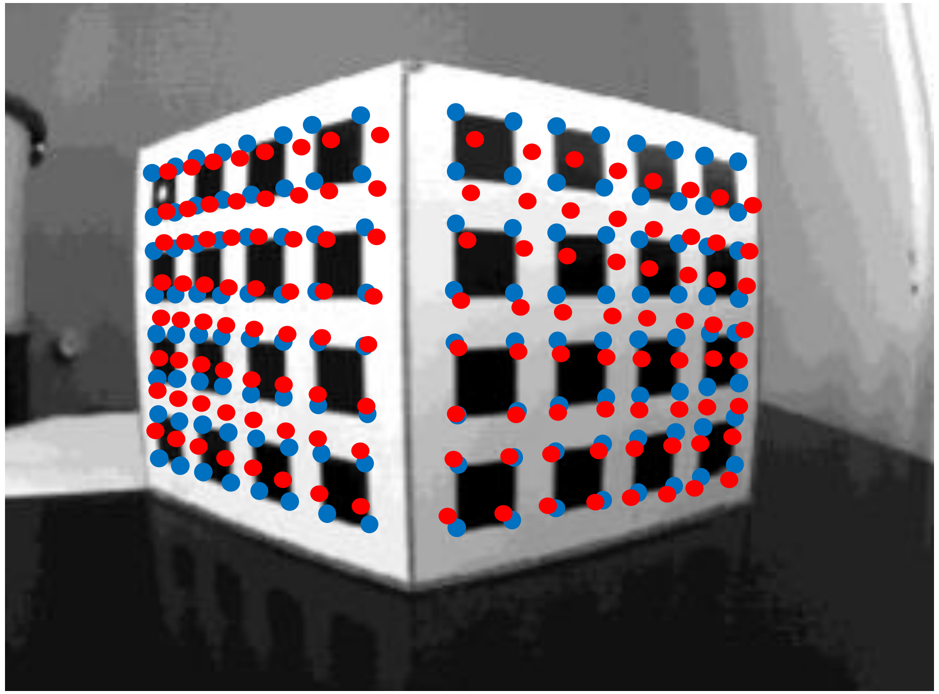

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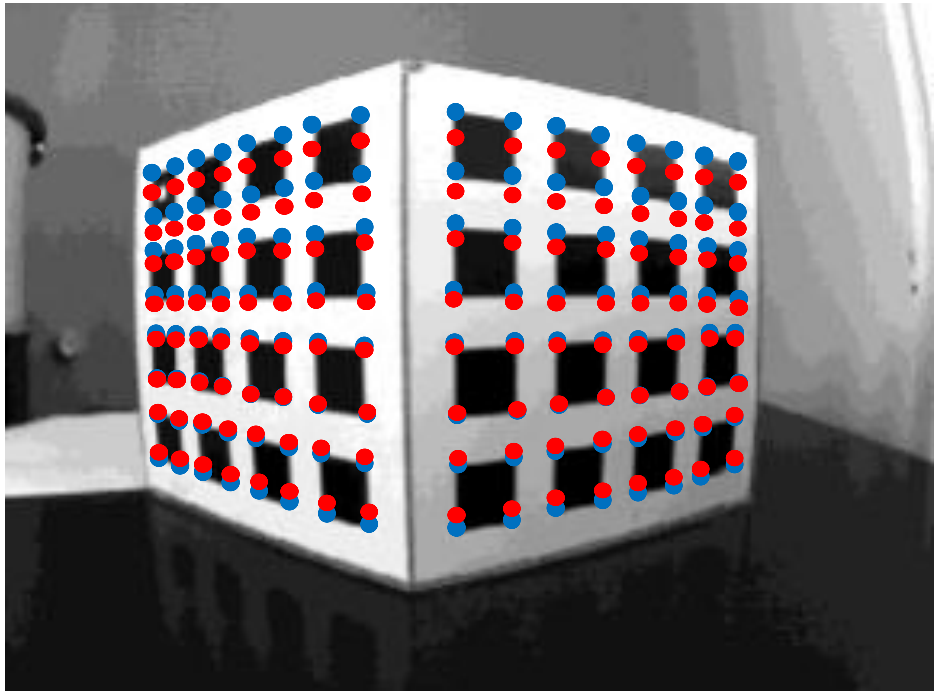

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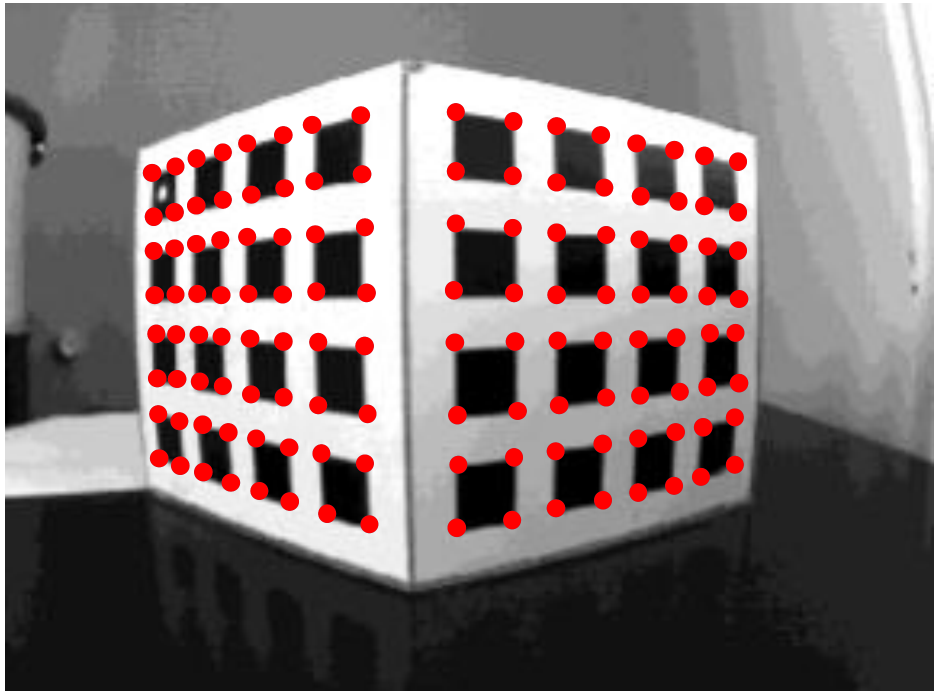

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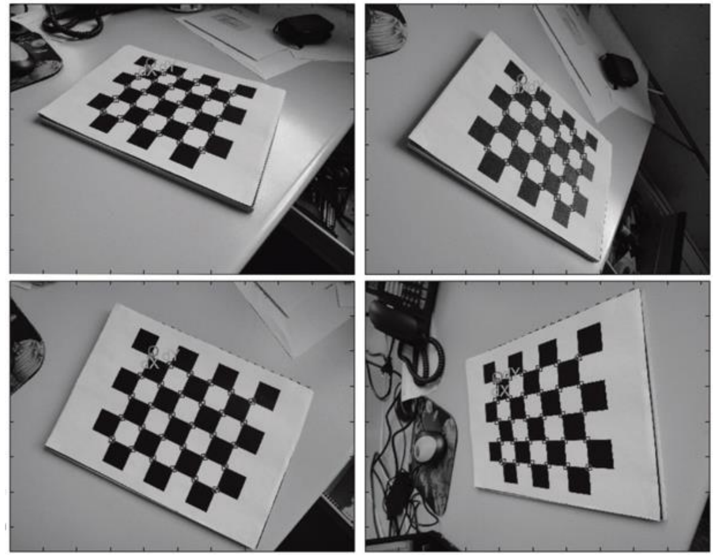
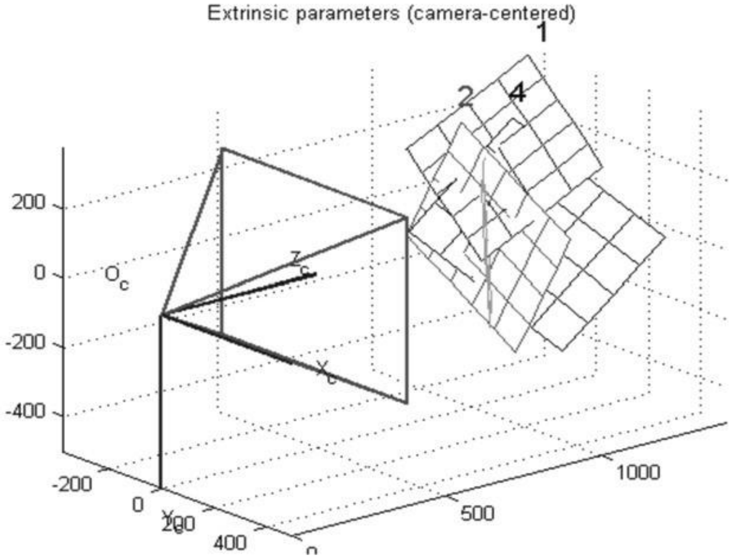
Today’s Outline

- Camera calibration
  - Tsai’s method: From 3D objects
  - Zhang’s method: from planar grids
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Zhang’s Algorithm: Calibration from Planar Grids

- **Tsai’s calibration** requires that the world’s 3D points are non-coplanar, which is **not very practical**
- **Today’s camera calibration toolboxes** (Matlab, OpenCV) use **multiple views** of a **planar grid** (e.g., a checker board)
- They are based on a method developed in 2000 by Zhang (Microsoft Research)





Zhang, A flexible new technique for camera calibration, IEEE Transactions on Pattern Analysis and Machine Intelligence, 2000. PDF.

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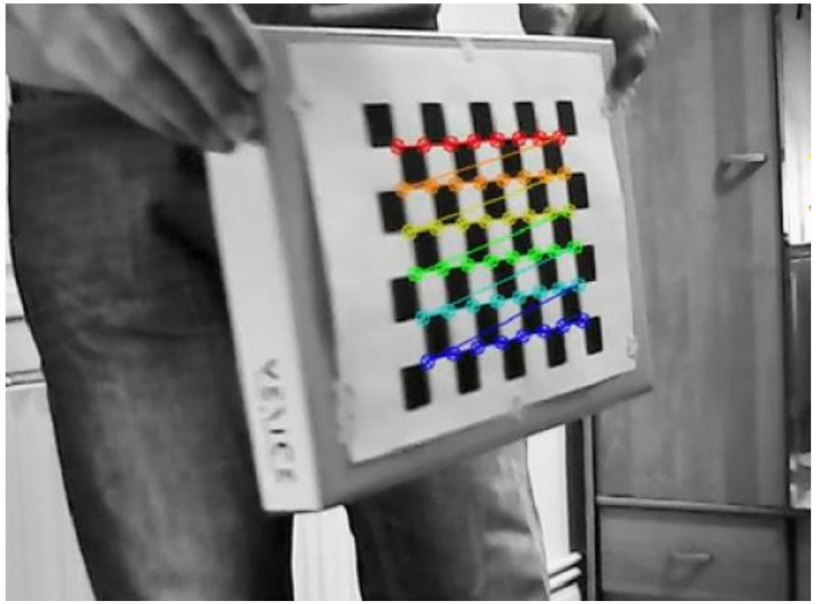

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Applying the Direct Linear Transform (DLT) algorithm

As in Tsai's method, we start by writing the perspective projection equation (again, we neglect the radial distortion). However, in **Zhang's method the points are all coplanar**, i.e., $\boldsymbol{Z_w = 0}$, and thus we can write:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K\left[R \mid T\right] \cdot \begin{bmatrix} X_w \\ Y_w \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow$$

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$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

Applying the Direct Linear Transform (DLT) algorithm

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$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = H \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

This matrix is called Homography

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_1^{\mathrm{T}} \\ h_2^{\mathrm{T}} \\ h_3^{\mathrm{T}} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

where h_i^{T} is the i-*th* row of H

Applying the Direct Linear Transform (DLT) algorithm

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$$v = \frac{\lambda v}{\lambda} = \frac{h_2^{\mathrm{T}} \cdot P}{h_3^{\mathrm{T}} \cdot P} \qquad \qquad (h_2^{\mathrm{T}} - v_i h_3^{\mathrm{T}}) \cdot P_i = 0$$

Applying the Direct Linear Transform (DLT) algorithm

- By re-arranging the terms, we obtain:

$$\begin{aligned}(h_1^{\mathrm{T}} - u_i h_3^{\mathrm{T}}) \cdot P_i = 0 \\ (h_2^{\mathrm{T}} - v_i h_3^{\mathrm{T}}) \cdot P_i = 0\end{aligned} \quad \Rightarrow \quad \begin{aligned}P_i^{\mathrm{T}} \cdot h_1 + 0 \cdot h_2^{\mathrm{T}} - u_i P_i^{\mathrm{T}} \cdot h_3^{\mathrm{T}} = 0 \\ 0 \cdot h_1^{\mathrm{T}} + P_i^{\mathrm{T}} \cdot h_2^{\mathrm{T}} - v_i P_i^{\mathrm{T}} \cdot h_3^{\mathrm{T}} = 0\end{aligned} \quad \Rightarrow \begin{pmatrix} P_i^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_i^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_i^{\mathrm{T}} & -v_1 P_i^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- For n points (from a **single view**), we can stack all these equations into a big matrix:

$$\underbrace{\begin{pmatrix} P_1^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_1^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_1^{\mathrm{T}} & -v_1 P_1^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ P_n^{\mathrm{T}} & 0^{\mathrm{T}} & -u_n P_n^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_n^{\mathrm{T}} & -v_n P_n^{\mathrm{T}} \end{pmatrix}}_{\text{Q (this matrix is } \textbf{known}\text{)}} \underbrace{\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}}_{\text{H (this matrix is } \textbf{unknown}\text{)}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad \mathrm{Q} \cdot \mathrm{H} = 0$$

Applying the Direct Linear Transform (DLT) algorithm

$$\mathrm{Q} \cdot \mathrm{H} = 0$$

Minimal solution

- $Q_{(2n\times 9)}$ should have rank 8 to have a unique (up to a scale) non-trivial solution H
- Each point correspondence provides 2 independent equations
- Thus, a minimum of **4 non-collinear points** is required

Over-determined solution

- $n \geq 4$ points
- It can be solved through Singular Value Decomposition (SVD) (same considerations as before)

How to recover K, R, T

- H can be decomposed by recalling that:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}$$

- Differently from Tsai's, the decomposition of H into K, R, T requires **at least two views**
- In practice **the more views the better**, e.g., 20-50 views spanning the entire field of view of the camera for the best calibration results
- Notice that now **each view j has a different homography** H^j (and so a different R^j and T^j). However, $\boldsymbol{K}$ **is the same for all views**:

$$\begin{bmatrix} h_{11}^j & h_{12}^j & h_{13}^j \\ h_{21}^j & h_{22}^j & h_{23}^j \\ h_{31}^j & h_{33}^j & h_{33}^j \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11}^j & r_{12}^j & t_1^j \\ r_{21}^j & r_{22}^j & t_2^j \\ r_{31}^j & r_{32}^j & t_3^j \end{bmatrix}$$

How to recover K, R, T from H and from multiple views?

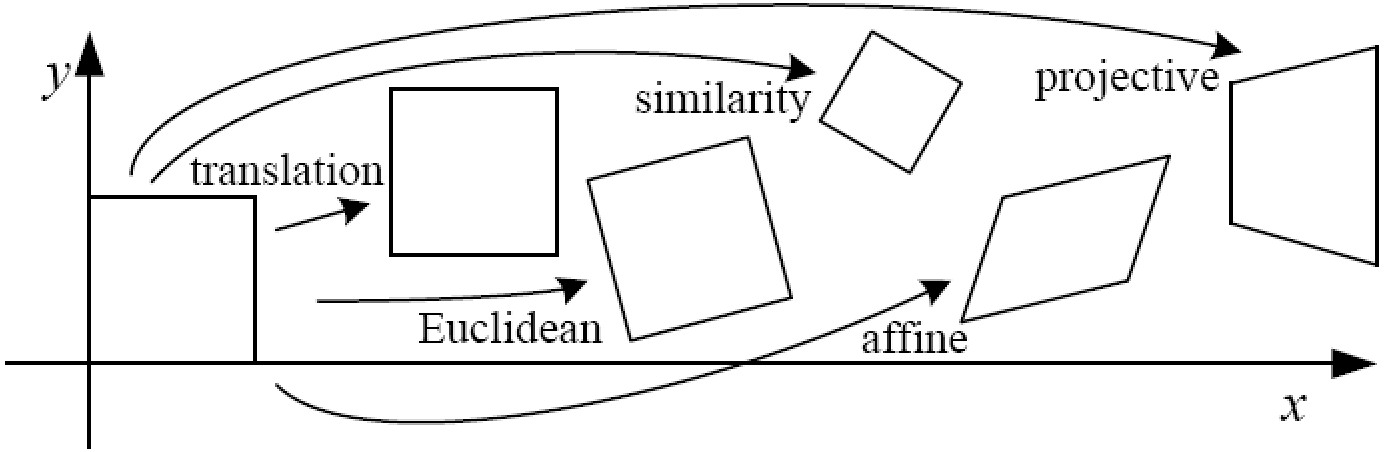
Won’t be asked at the exam ☺

1. Estimate the homography H_i for each i-th view using the DLT algorithm.

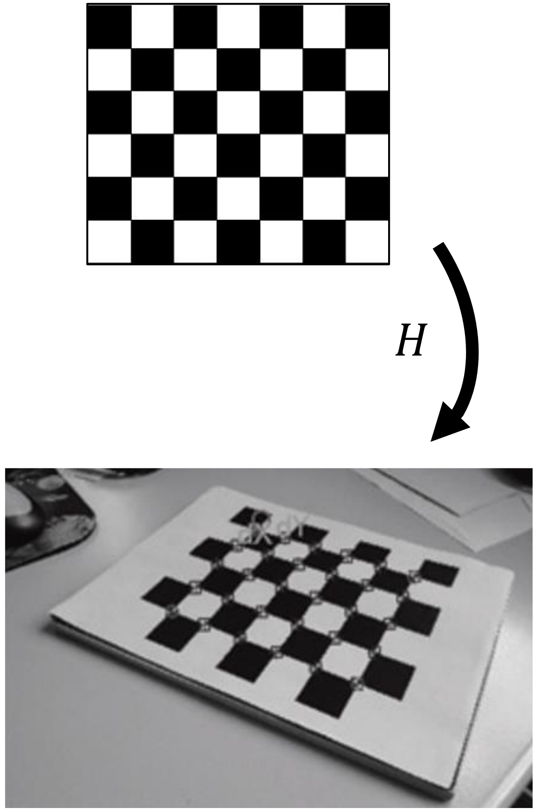
2. Determine the intrinsics K of the camera from a set of homographies:
    1. Each homography $H_i \sim K(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t})$ provides two *linear* equations in the 6 entries of the matrix $\mathrm{B} \coloneqq K^{-\top}K^{-1}$. Letting $\boldsymbol{w}_1 \coloneqq K\boldsymbol{r}_1$, $\boldsymbol{w}_2 \coloneqq K\boldsymbol{r}_2$, the rotation constraints $\boldsymbol{r}_1^\top \boldsymbol{r}_1 = \boldsymbol{r}_2^\top \boldsymbol{r}_2 = 1$ and $\boldsymbol{r}_1^\top \boldsymbol{r}_2 = 0$ become $\boldsymbol{w}_1^\top B \boldsymbol{w}_1 - \boldsymbol{w}_2^\top B \boldsymbol{w}_2 = 0$ and $\boldsymbol{w}_1^\top B \boldsymbol{w}_2 = 0$.
    2. Stack $2N$ equations from N views, to yield a linear system $A\boldsymbol{b} = \boldsymbol{0}$. Solve for $\boldsymbol{b}$ (i.e., B) using the Singular Value Decomposition (SVD).
    3. Use Cholesky decomposition to obtain K from B.

3. The extrinsic parameters for each view can be computed using K:
$\boldsymbol{r}_1 \sim \lambda K^{-1} H_i(:,1)$, $\boldsymbol{r}_2 \sim \lambda K^{-1} H_i(:,2)$, $\boldsymbol{r}_3 = \boldsymbol{r}_1 \times \boldsymbol{r}_2$ and $T_i = \lambda K^{-1} H_i(:,3)$, with $\lambda = 1/K^{-1}H_i(:,1)$.
Finally, build $R_i = (\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ and enforce rotation matrix constraints.

Types of 2D Transformations



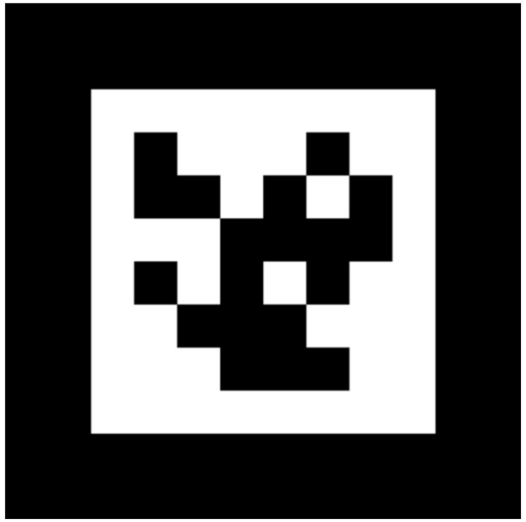
Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\left[\, \boldsymbol{I} \mid \boldsymbol{t} \,\right]_{2\times 3}$	2	orientation $+\cdots$	
rigid (Euclidean)	$\left[\, \boldsymbol{R} \mid \boldsymbol{t} \,\right]_{2\times 3}$	3	lengths $+\cdots$	
similarity	$\left[\, s\boldsymbol{R} \mid \boldsymbol{t} \,\right]_{2\times 3}$	4	angles $+\cdots$	
affine	$\left[\, \boldsymbol{A} \,\right]_{2\times 3}$	6	parallelism $+\cdots$	
projective	$\left[\, \tilde{\boldsymbol{H}} \,\right]_{3\times 3}$	8	straight lines	



This matrix is called **Homography**

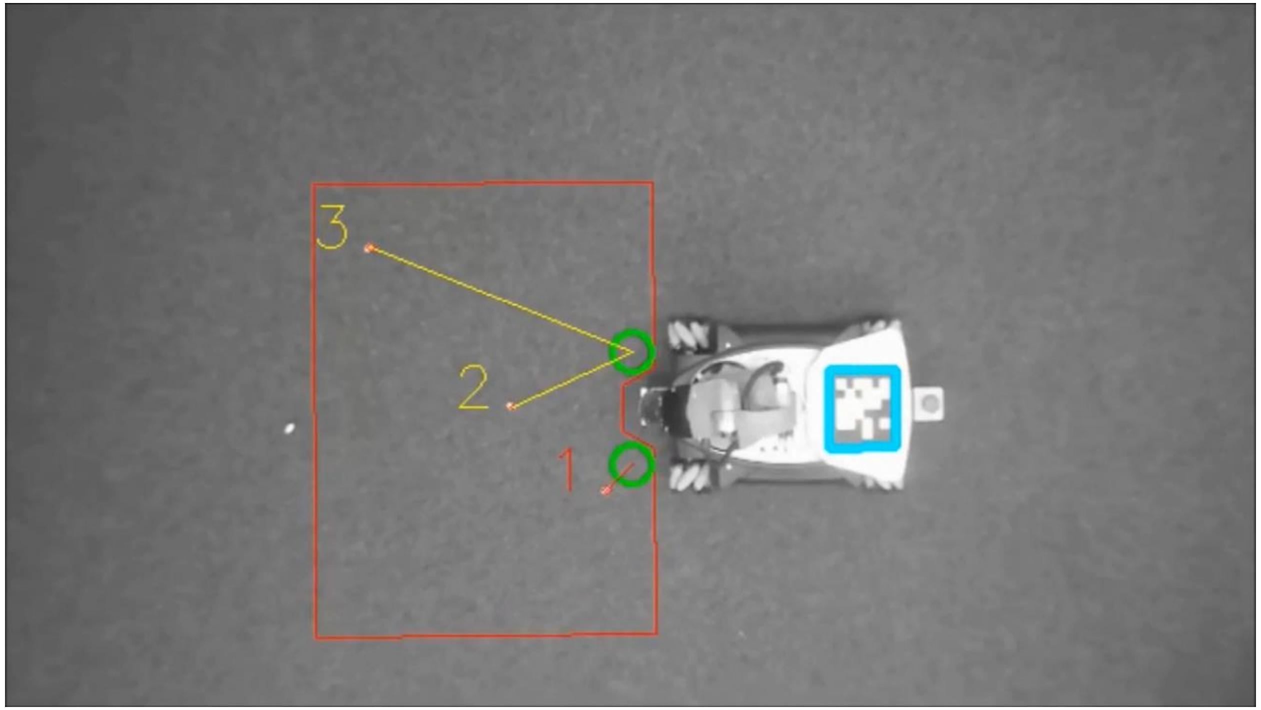
Application to Augmented Reality

- Today, there are thousands of application of Zhang's algorithm, e.g. Augmented Reality (AR)
- See AprilTag or ARuco Markers

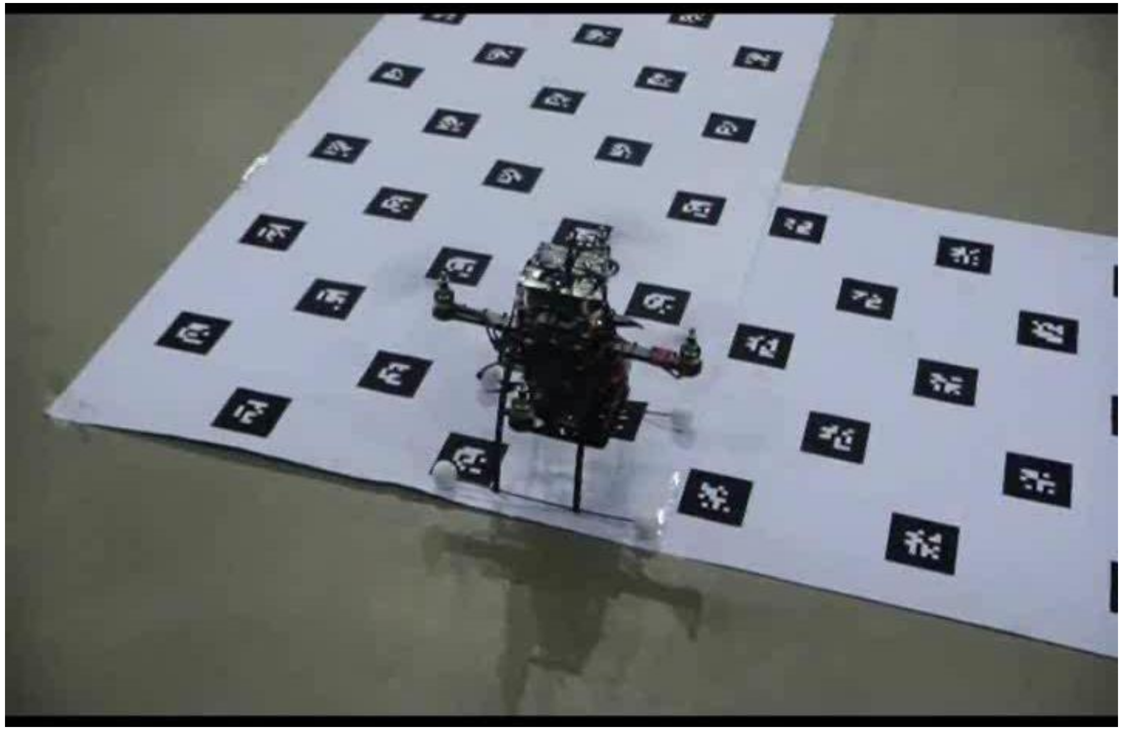


Application to Robotics

- Do we need to know the size of the tag?
  - For Augmented Reality?
  - For Control?



My lab. Video.



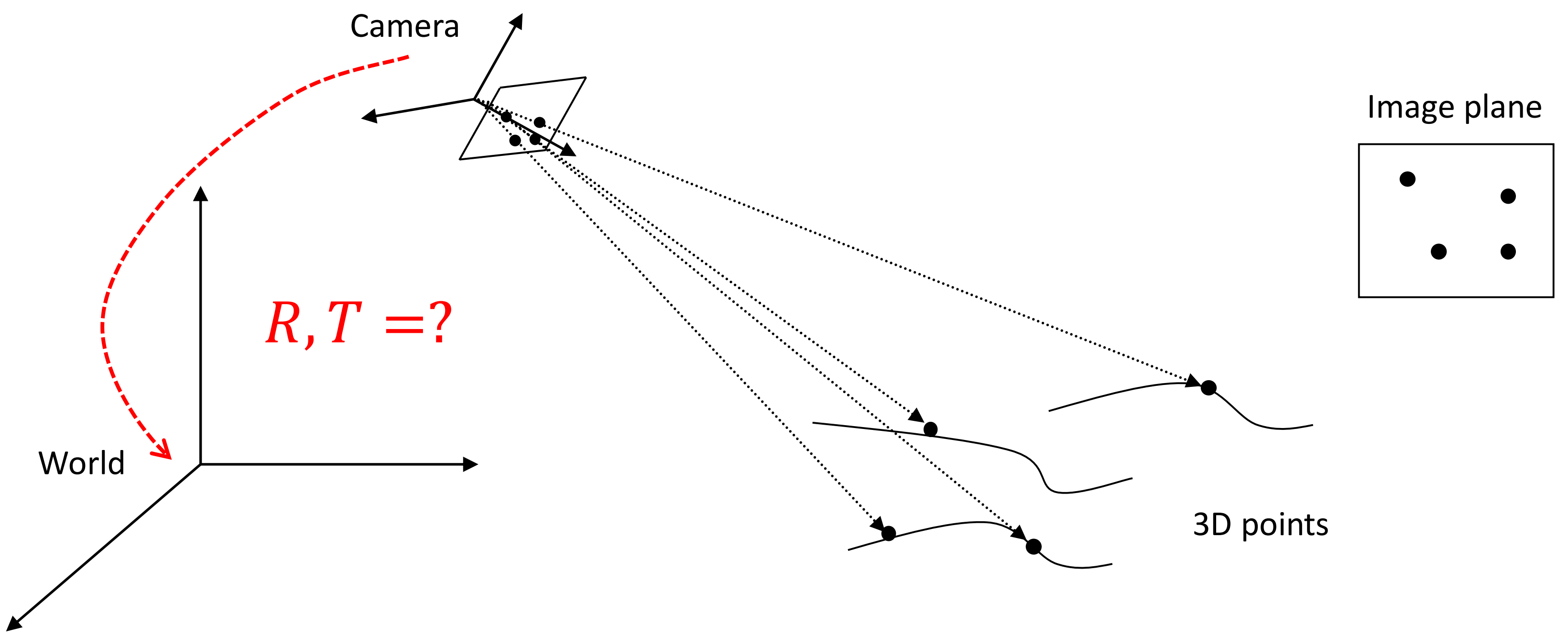
Marc Pollefeys' lab. Video.

Today’s Outline

- Camera calibration
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Camera Localization (or Perspective from n Points: PnP)

- This is the problem of determining the **6DoF pose of a camera** (position and orientation) with respect to the world frame **from a set of 3D-2D point correspondences**.
- It assumes the **camera** to be **already calibrated**
- The **DLT can be used** to solve this problem **but is suboptimal**. We want to study **algebraic solutions** to the problem.

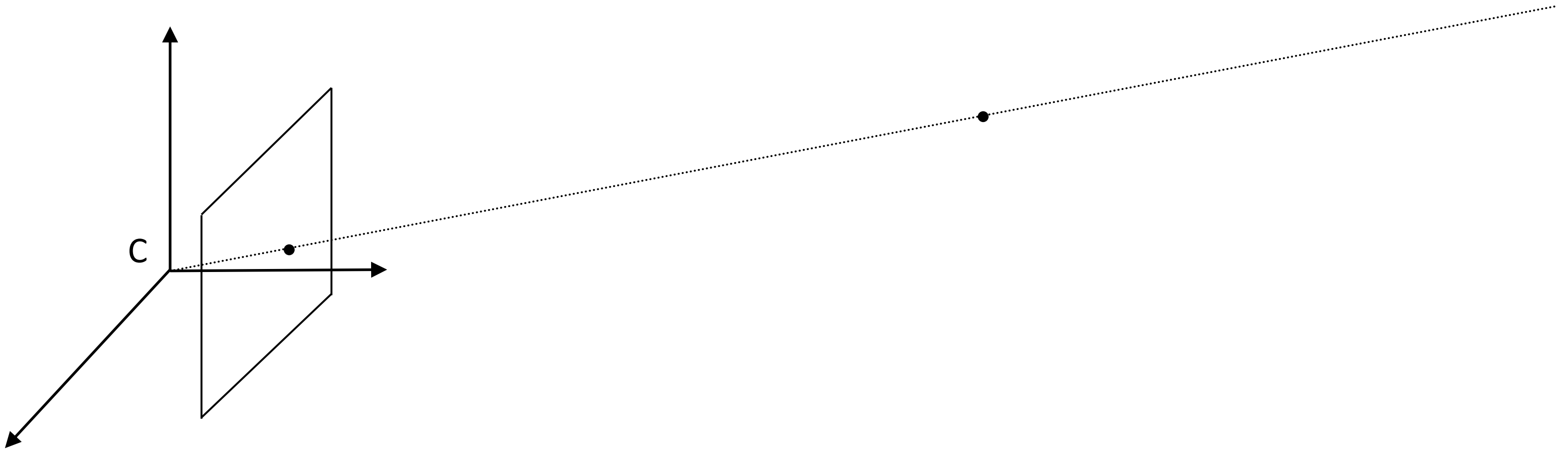


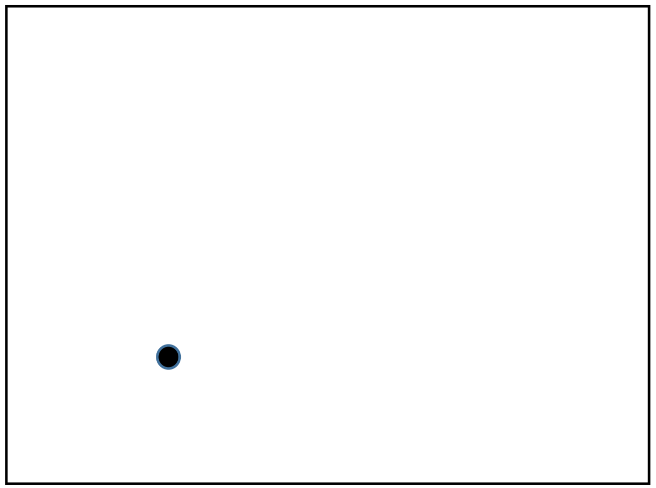
How Many Points are Enough?

- **1 Point**:
  - infinite solutions
- **2 Points**:
  - infinitely many solutions, but bounded
- **3 Points** (non collinear):
  - up to 4 solution
- **4 Points**:
  - Unique solution

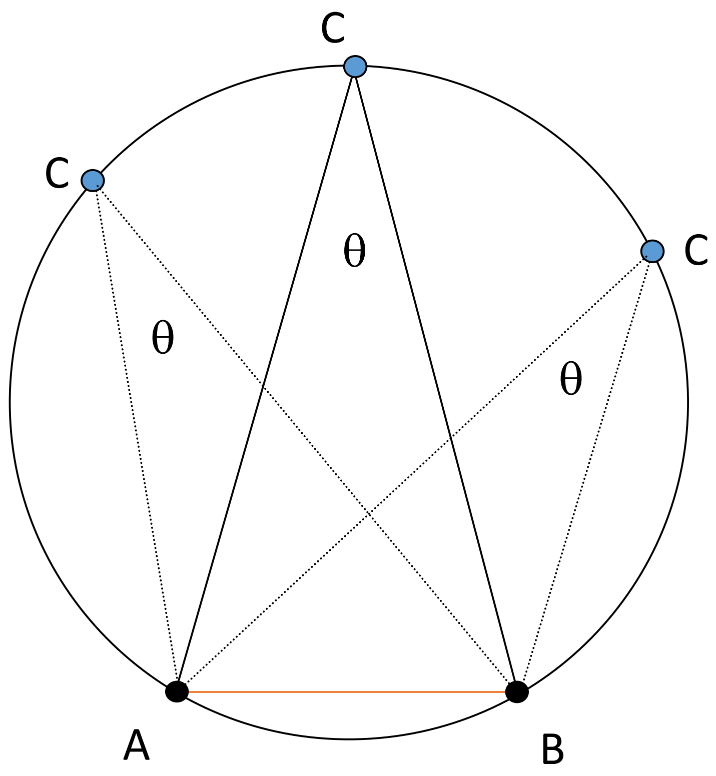
1 Point

- **1 Point**:
  - infinite solutions





2 Points



- **2 Points**:
  - infinite solutions, but bounded

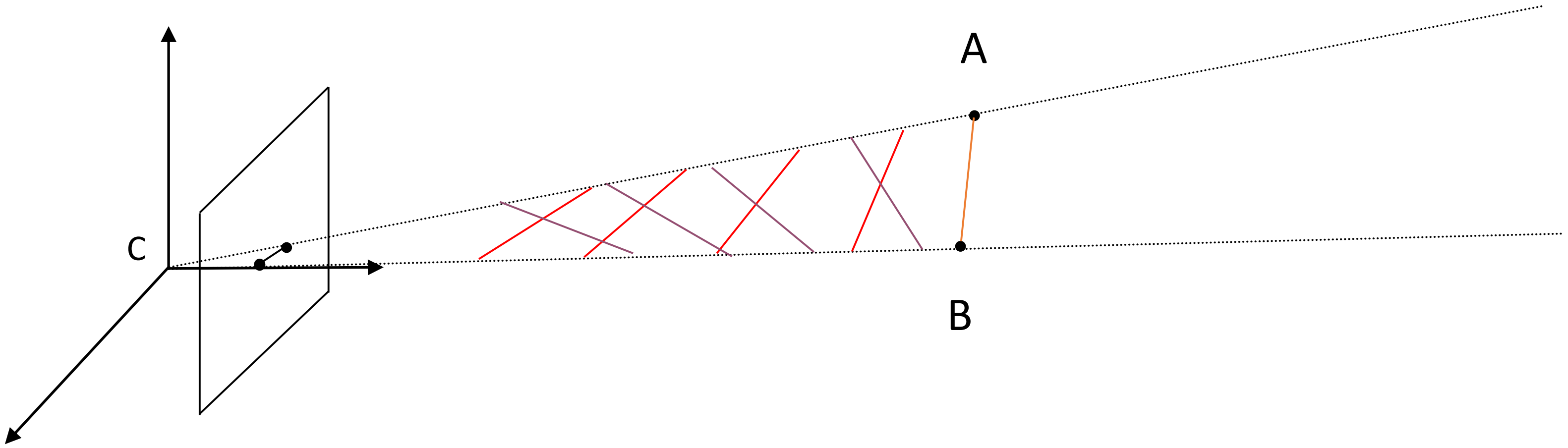
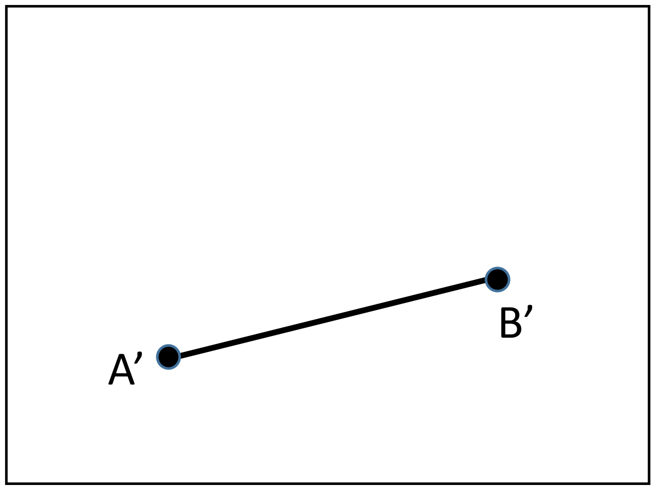


Image plane



3 Points (P3P problem)

- **3 Points** (non collinear):
  - up to 4 solution

From Carnot's Theorem:

$$s_1^2 = L_B^2 + L_A^2 - 2L_B L_A \cos\theta_{AB}$$

$$s_2^2 = L_A^2 + L_C^2 - 2L_A L_C \cos\theta_{AC}$$

$$s_3^2 = L_B^2 + L_C^2 - 2L_B L_C \cos\theta_{BC}$$

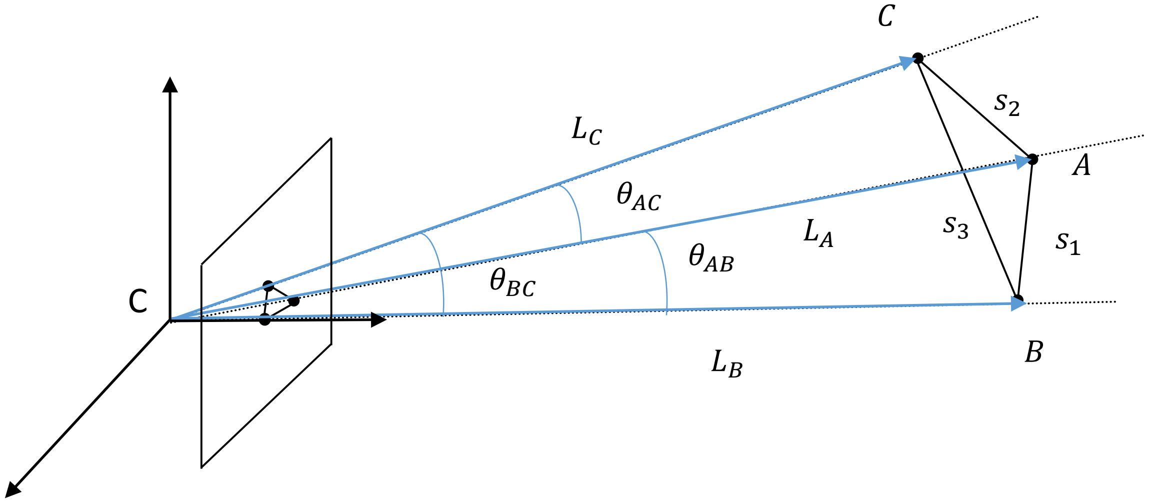
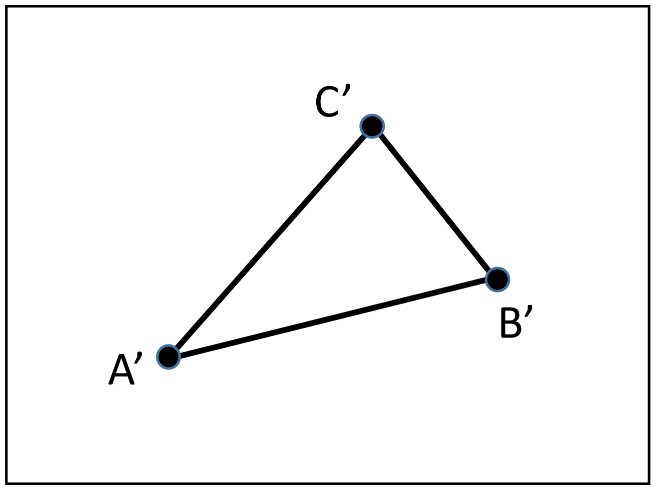


Image plane



Algebraic Approach: reduce to 4th order equation

$$s_1^2 = L_B^2 + L_A^2 - 2L_B L_A \cos\theta_{AB}$$

$$s_2^2 = L_A^2 + L_C^2 - 2L_A L_C \cos\theta_{AC}$$

$$s_3^2 = L_B^2 + L_C^2 - 2L_B L_C \cos\theta_{BC}$$

- It is known that $\boldsymbol{n}$ **independent polynomial equations**, in $\boldsymbol{n}$ **unknowns**, can have no more **solutions** than the **product of their respective degrees**. Thus, the system can have a maximum of **8** solutions. However, because every term in the system is either a constant or of second degree, for every real positive solution **there is a negative solution**.
- Thus, with 3 points, there are at most **4 valid (positive) solutions**.

M. A.Fischler and R. C.Bolles. Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography. Graphics and Image Processing, 1981. PDF.

Algebraic Approach: reduce to 4th order equation

$$s_1^2 = L_B^2 + L_A^2 - 2L_B L_A \cos\theta_{AB}$$

$$s_2^2 = L_A^2 + L_C^2 - 2L_A L_C \cos\theta_{AC}$$

$$s_3^2 = L_B^2 + L_C^2 - 2L_B L_C \cos\theta_{BC}$$

- By defining $\boldsymbol{x} = \boldsymbol{L_B}/\boldsymbol{L_A}$, it can be shown that the system can be reduced to a 4th order equation:

$$G_0 + G_1 x + G_2 x^2 + G_3 x^3 + G_4 x^4 = 0$$

How can we disambiguate the 4 solutions? How do we determine R and T?

- A 4th point can be used to disambiguate the solutions. A classification of the four solutions and the determination of R and T from the point distances was given Gao's algorithm, implemented in OpenCV (*solvePnP_P3P*)

Gao, Hou, Tang, Cheng. Complete Solution Classification for the Perspective-Three-Point Problem. IEEE Transactions on Pattern Analysis and Machine Intelligence, 2003. PDF.

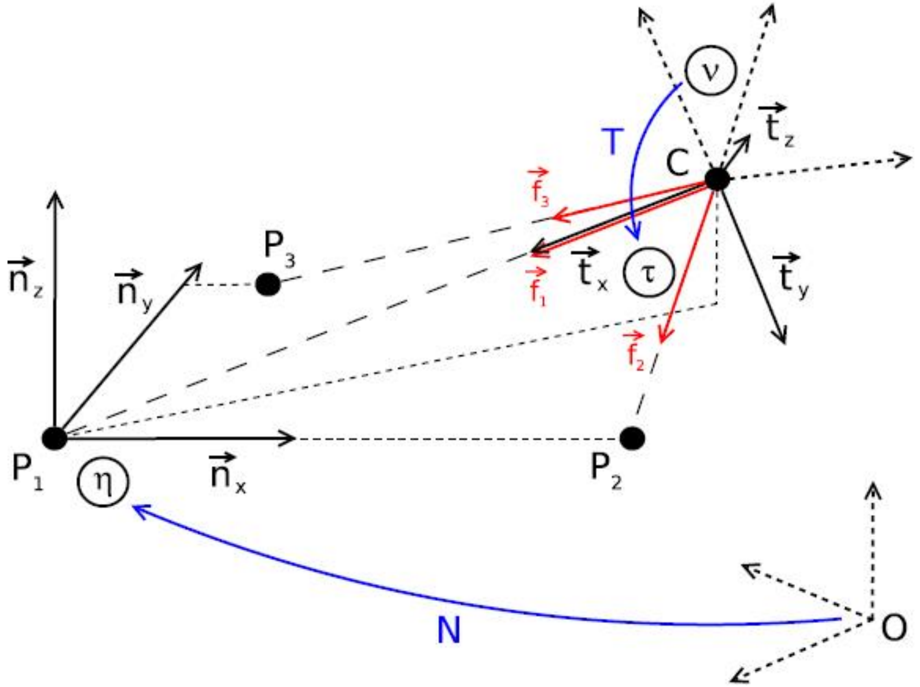
Modern Solution to P3P

Won't be asked at the exam ☺

A more **modern version of P3P** was developed by Kneip in 2011 and **directly solves for the camera's pose** (not distances from the points). This solution inspired the algorithm currently used in OpenCV (*solvePnP_AP3P*), by Ke'17, which consists of two steps:

1. Eliminate the camera's position and the features' distances to yield a system of 3 equations in *the camera's orientation alone*.
2. Successively eliminate two of the unknown 3-DOFs (angles) algebraically and arrive at a ***quartic polynomial equation***.

- Outperforms previous methods in terms of speed, accuracy, and robustness to close-to-singular cases.



Kneip, Scaramuzza, Siegwart. A Novel Parameterization of the Perspective-Three-Point Problem for a Direct Computation of Absolute Camera Position and Orientation. IEEE Conference on Computer Vision and Pattern Recognition (CVPR), 2011. PDF.

Ke, Roumeliotis. An Efficient Algebraic Solution to the Perspective-Three-Point Problem. CVPR'17. PDF.

SWISS MADE

Solution to PnP for $n \geq 4$

Won't be asked at the exam ☺

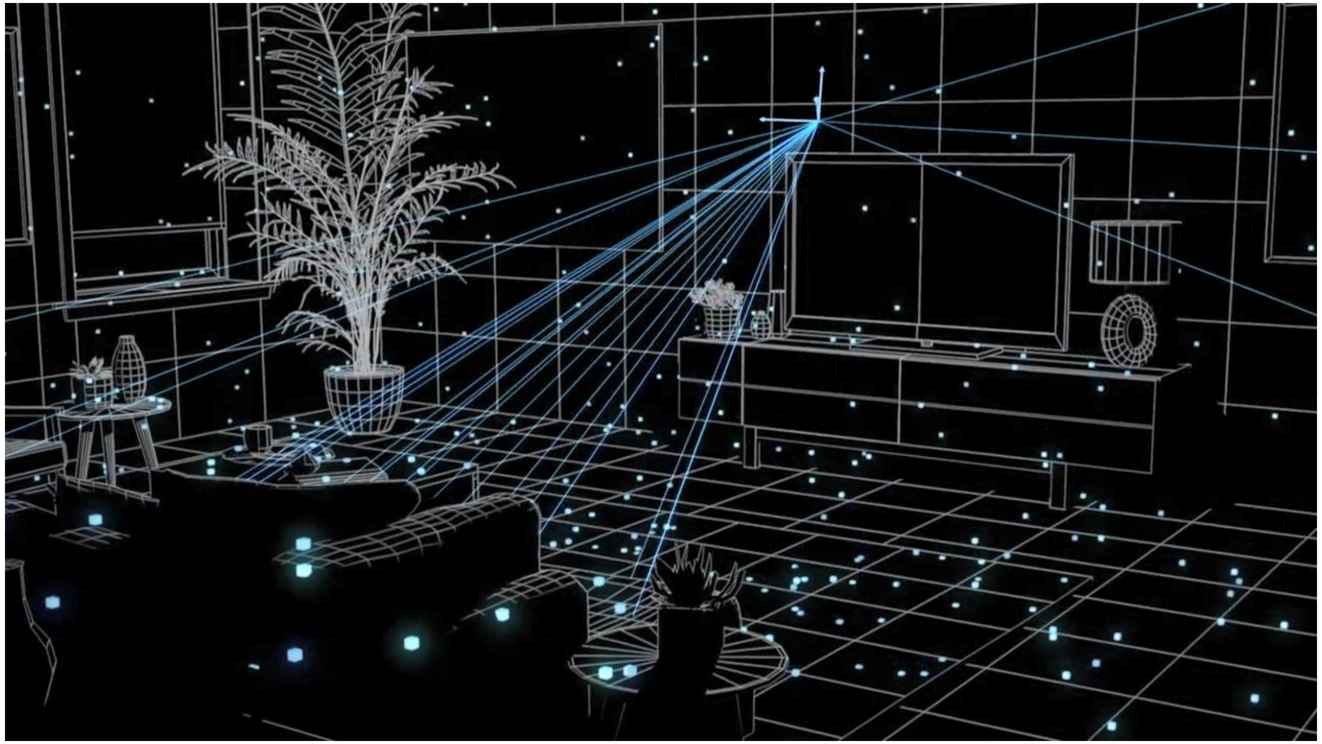
An efficient algebraic solution to the PnP problem for $n \geq 4$ was developed by Lepetit in 2009 and coined **EPnP** (Efficient PnP) and can be found in OpenCV (solvePnP_EPnP)

- EPnP expresses the *n* world's points as a weighted sum of **four virtual control points**
- The coordinates of these virtual control points become the **unknowns of the problem**, which can be solved in $O(n)$ time by solving a **constant number** of **quartic polynomial equations**
- The final pose of the camera is then solved from the control points

Lepetit, Moreno Noguer, Fua, EPnP: An Accurate *O(n)* Solution to the P*n*P Problem, International Journal of Computer Vision. PDF.

Application to Monocular Localization

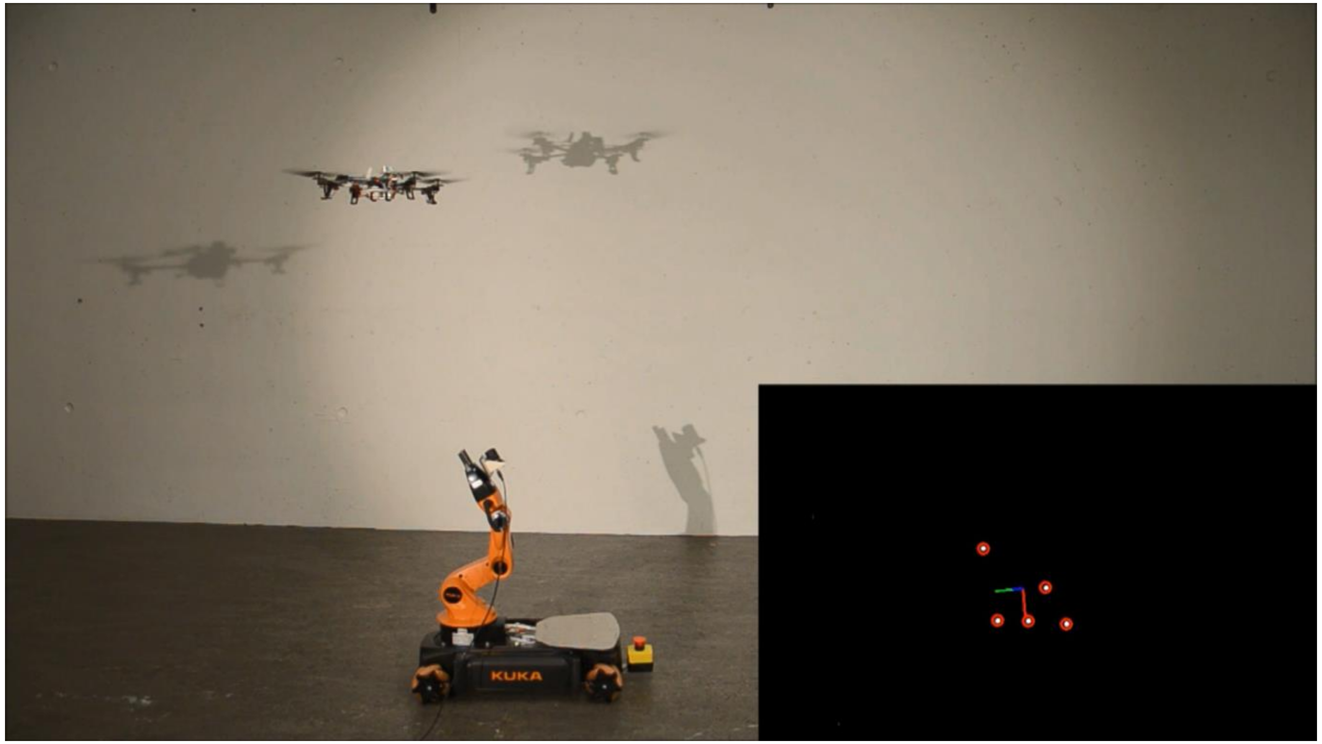
Localization: Given a 3D point cloud (map), determine the pose of the camera



Video of Oculus Insight (the VIO used in Oculus Quest): built by former Zurich-Eye team, today Oculus Zurich.
Dr. Christian Forster (Oculus Zurich & co-founder of Zurich-Eye) will give a lecture on Nov. 26

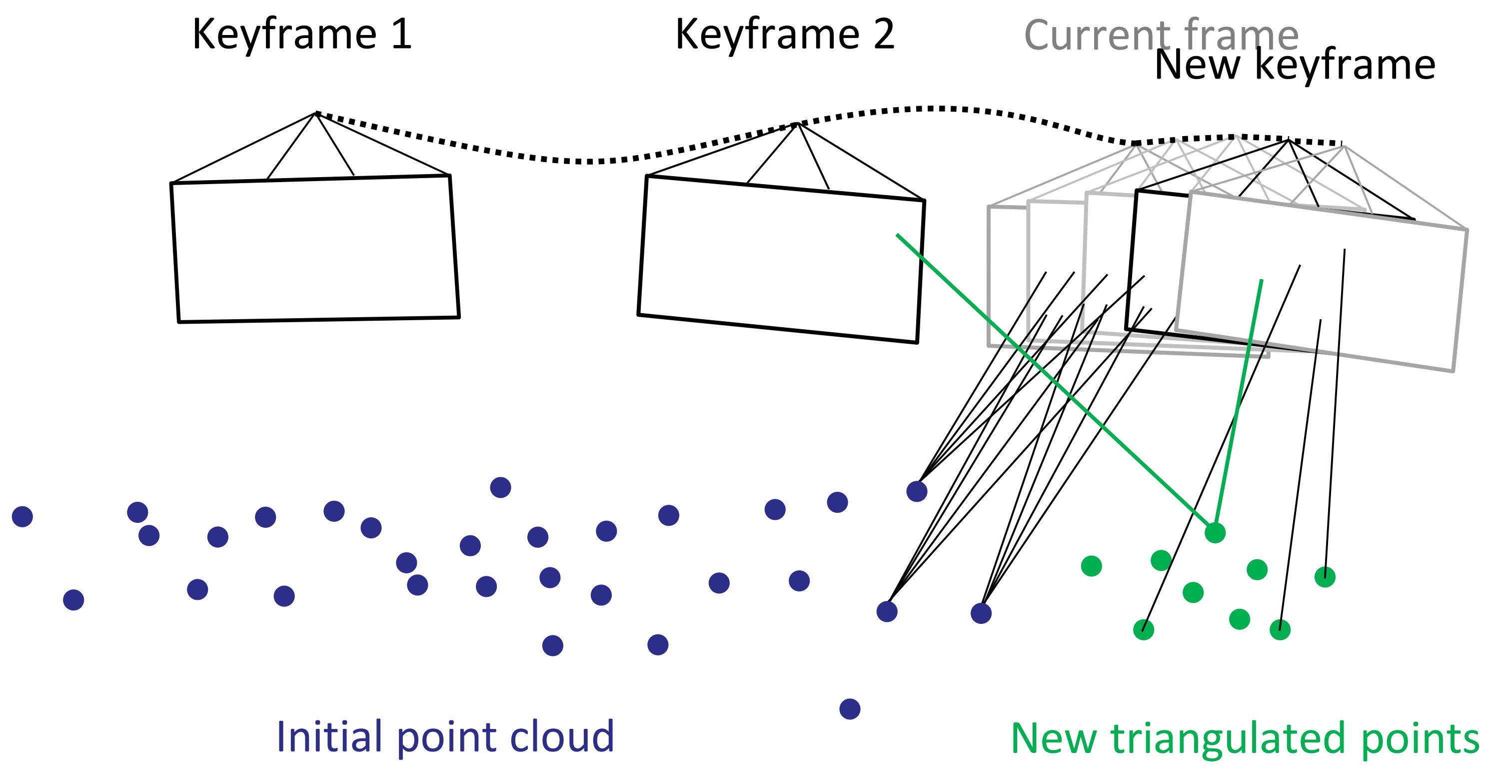
Application to Multi-Robot mutual Localization

Here, the drone carries 5 LEDs that are used by the ground robot to control the drone's position relative to it



Faessler, Mueggler, Schwabe, Scaramuzza. A Monocular Pose Estimation System based on Infrared LEDs.
IEEE International Conference on Robotics and Automation (ICRA), Hong Kong, 2014. [PDF](). [Video]().

Application to Monocular Visual Odometry



Robust Estimation in Presence of Outliers

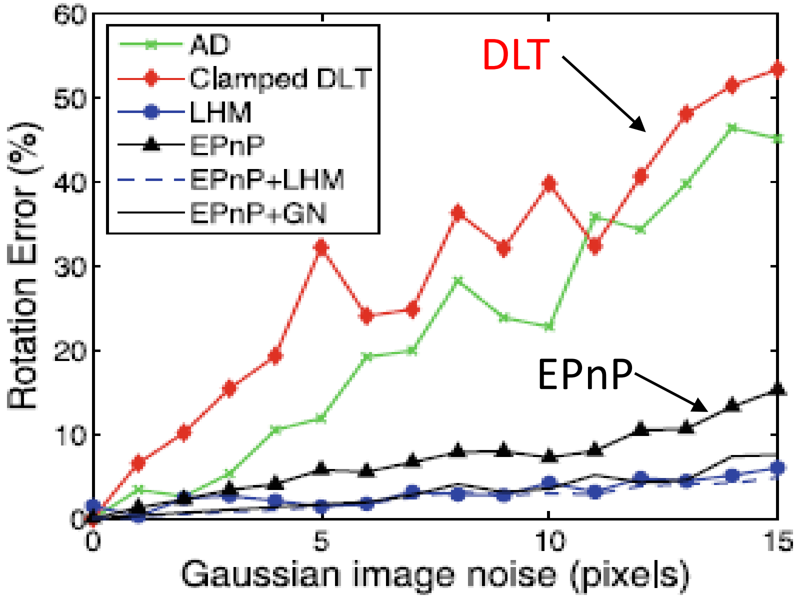
- **All P*n*P problems** (solved by DLT, EPnP, or P3P algorithms) are prone to errors if there are outliers in the set of 3D-2D point correspondences.
- The **RANSAC** algorithm (**Lecture 08**) can be used, in conjunction with the PnP algorithm, to **remove the outliers**.
- P*n*P with RANSAC can be found in OpenCV's (*solvePnPRansac*)

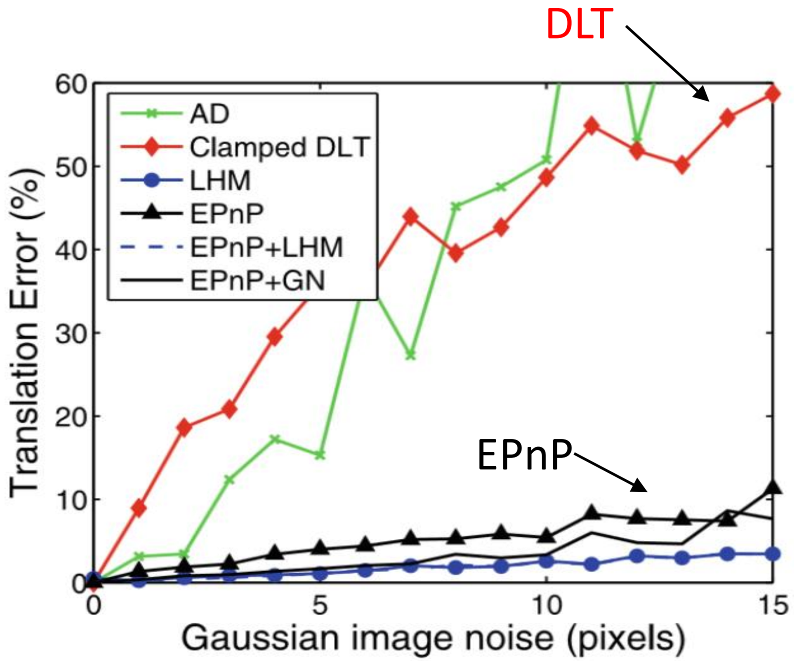
EPnP vs. DLT

If a camera is calibrated, only R and T need to be determined. In this case, **should we use DLT or EPnP**?

EPnP vs. DLT: Accuracy vs noise

EPnP is more up to $\mathbf{10} \times$ **more accurate** and **more efficient** than DLT

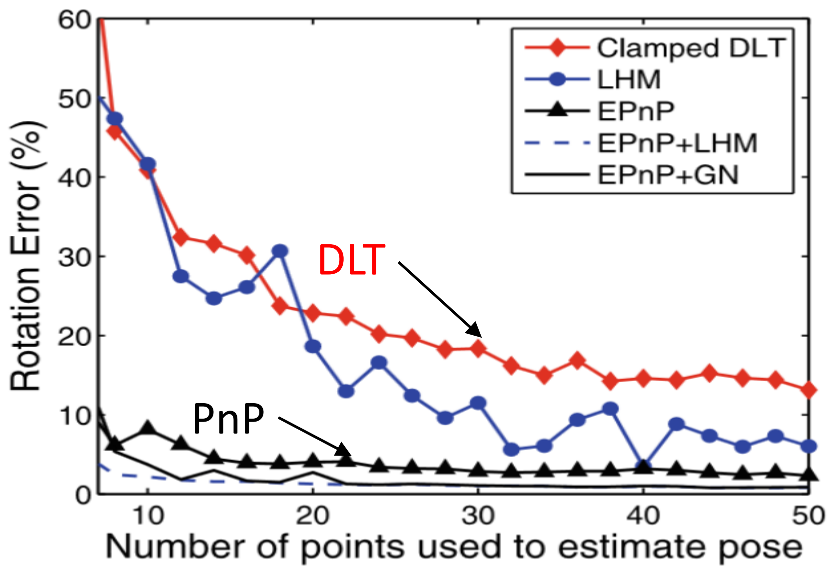


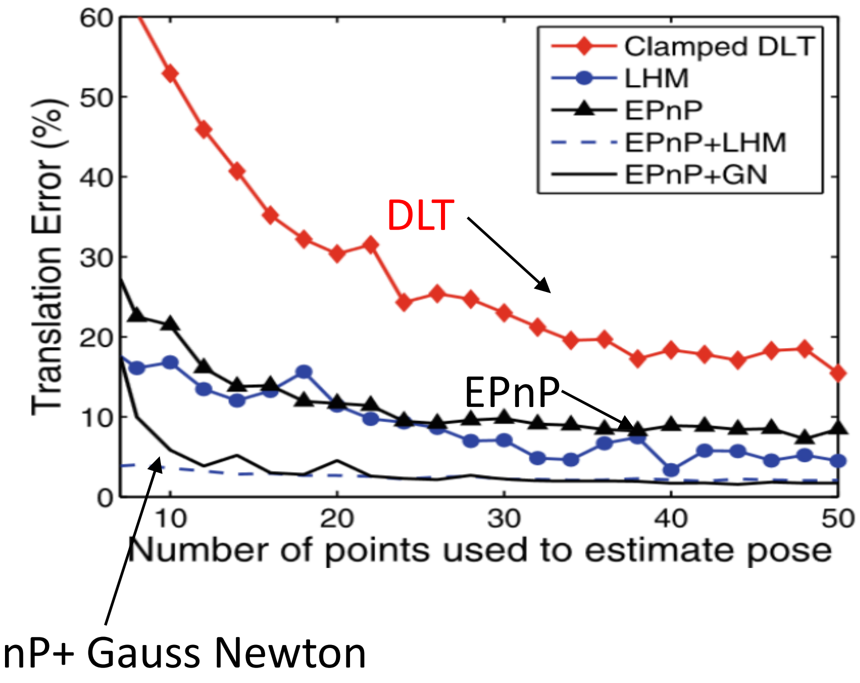


Plots from
Lepetit, Moreno Noguer, Fua, EP*n*P: An Accurate *O(n)* Solution to the P*n*P Problem, International Journal of Computer Vision. [PDF](PDF).

EPnP vs. DLT: Accuracy vs number of points

EPnP is more up to $\mathbf{10\times}$ **more accurate** and **more efficient** than DLT

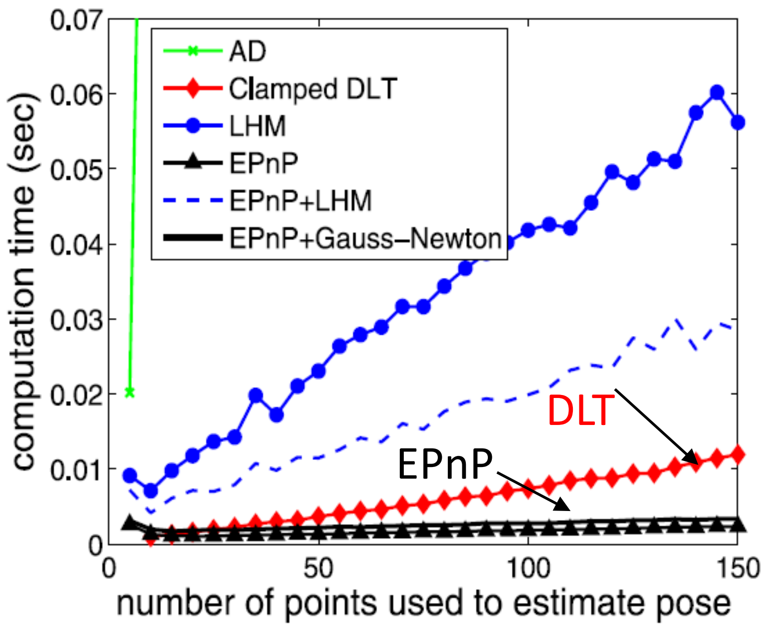




Plots from
Lepetit, Moreno Noguer, Fua, EP*n*P: An Accurate *O(n)* Solution to the P*n*P Problem, International Journal of Computer Vision. [PDF](PDF).

EPnP vs. DLT: Timing

EPnP is more up to $\mathbf{10} \times$ **more accurate** and **more efficient** than DLT



Plots from
Lepetit, Moreno Noguer, Fua, EP*n*P: An Accurate *O(n)* Solution to the P*n*P Problem, International Journal of Computer Vision. [PDF](PDF).

PnP problem: Recap

Calibrated camera (i.e., instrinc parameters are known)	**Uncalibrated camera** (i.e., intrinsic parameters unknown)
Either DLT or EPnP can be used	**Only DLT** can be used

EPnP: minimum number of points: **3 (P3P) +1** for disambiguation
DLT: Minimum number of points: **4 if coplanar, 6 if non-coplanar**

The output of both DLT and EPnP can be refined via **non-linear optimization** by minimizing the sum of squared reprojection errors

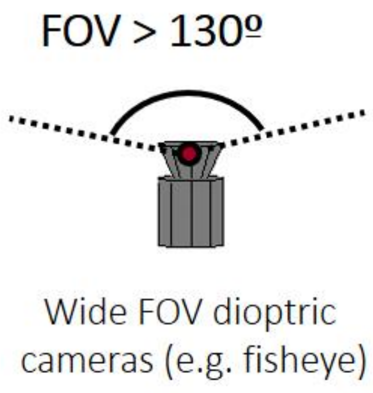
Today’s Outline

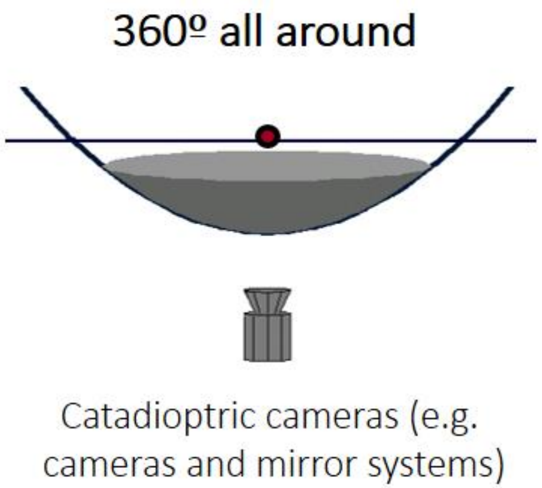
- Camera calibration
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Overview on Omnidirectional Cameras

Fisheye

Catadioptric

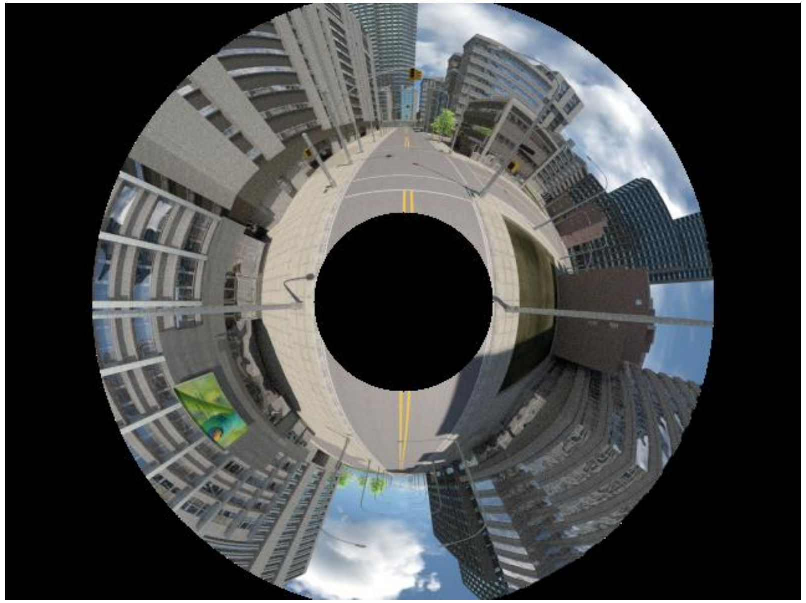




Camera View Comparison

Perspective

Fisheye

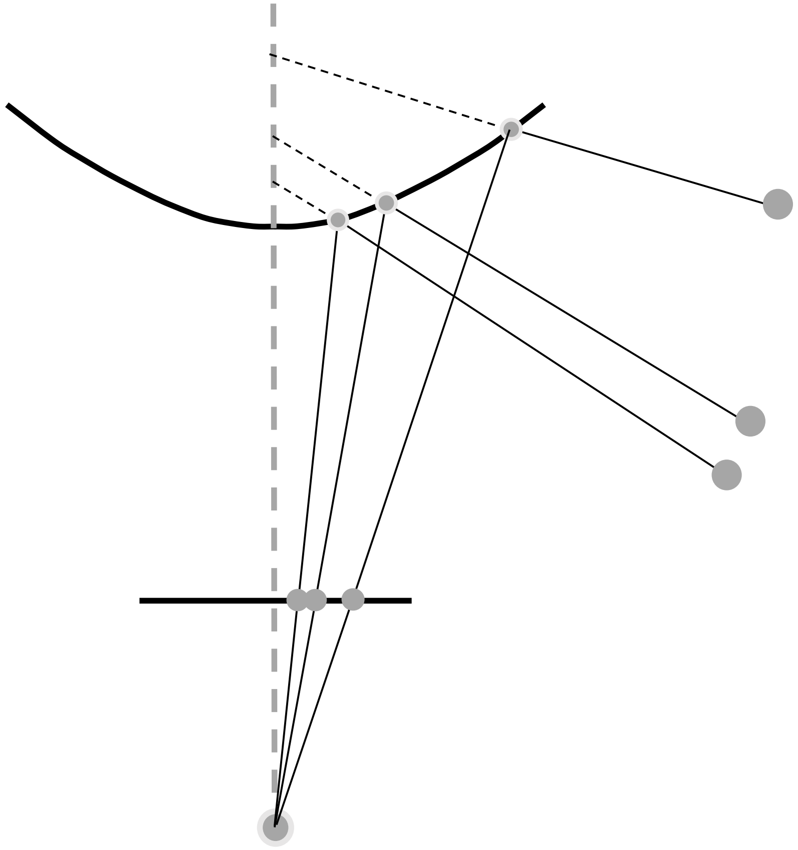


Catadioptric

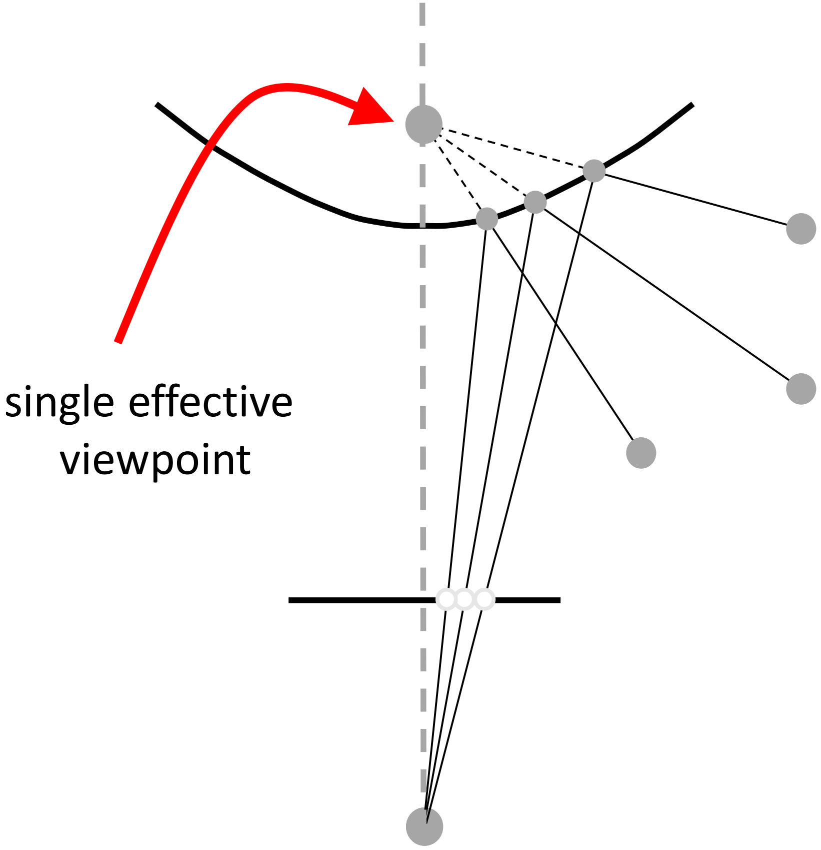
Zhang, Rebecq, Forster, Scaramuzza. Benefit of Large Field-of-View Cameras for Visual Odometry.
IEEE International Conference on Robotics and Automation (ICRA), 2016. PDF.

Central vs Non-Central Omnidirectional Cameras

Non-Central projection system
Rays do not intersect in a single point

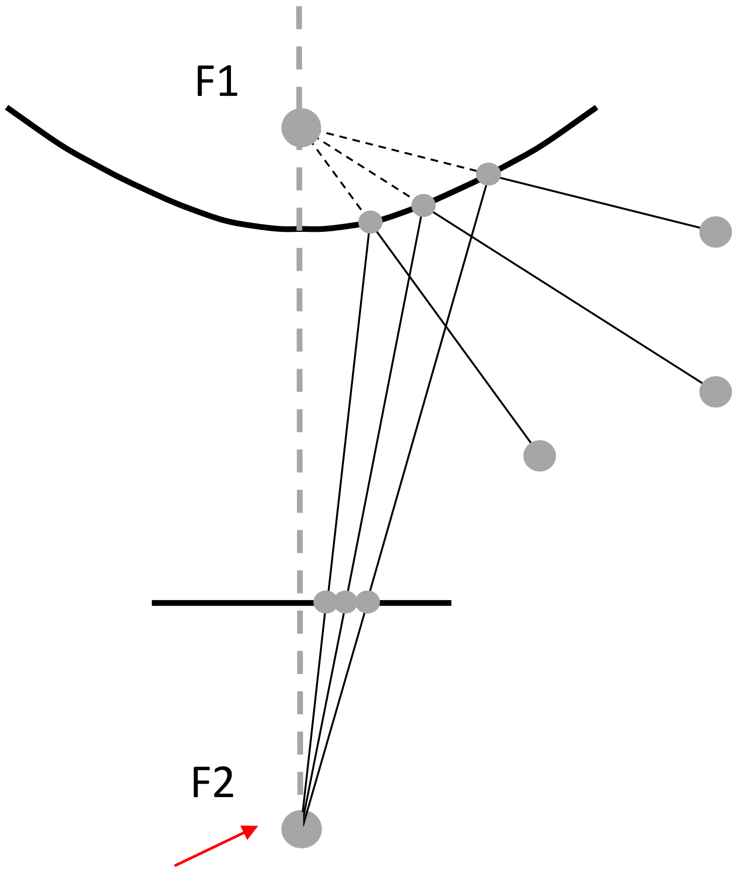


Central projection system
Rays intersect in a single point



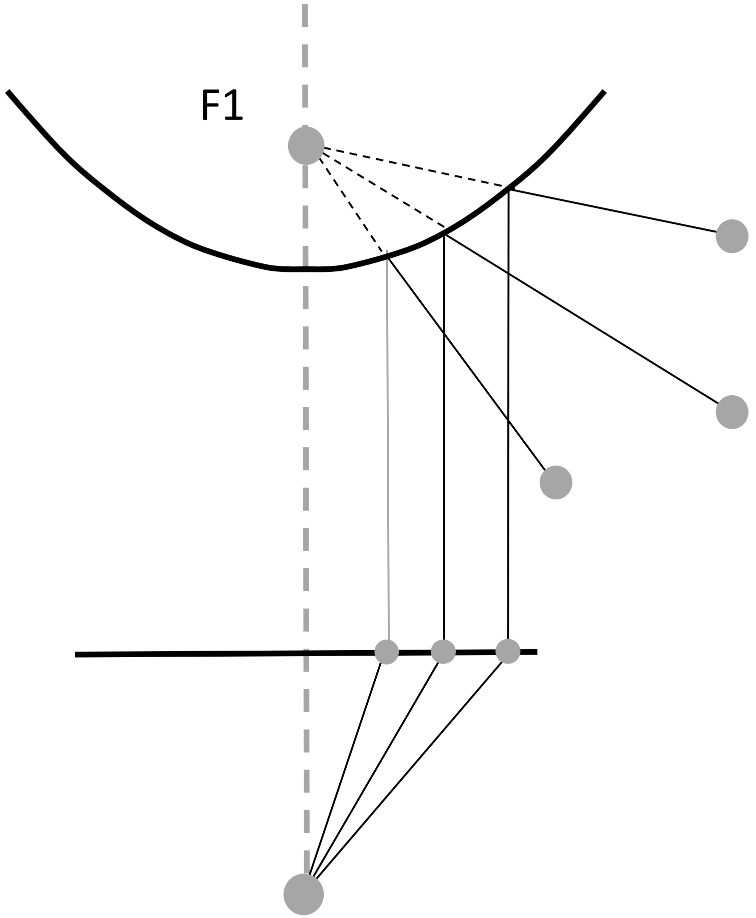
Central Omnidirectional Cameras

Hyperbola + Perspective camera



NB: one of the foci of the hyperbola must lie in the camera's center of projection

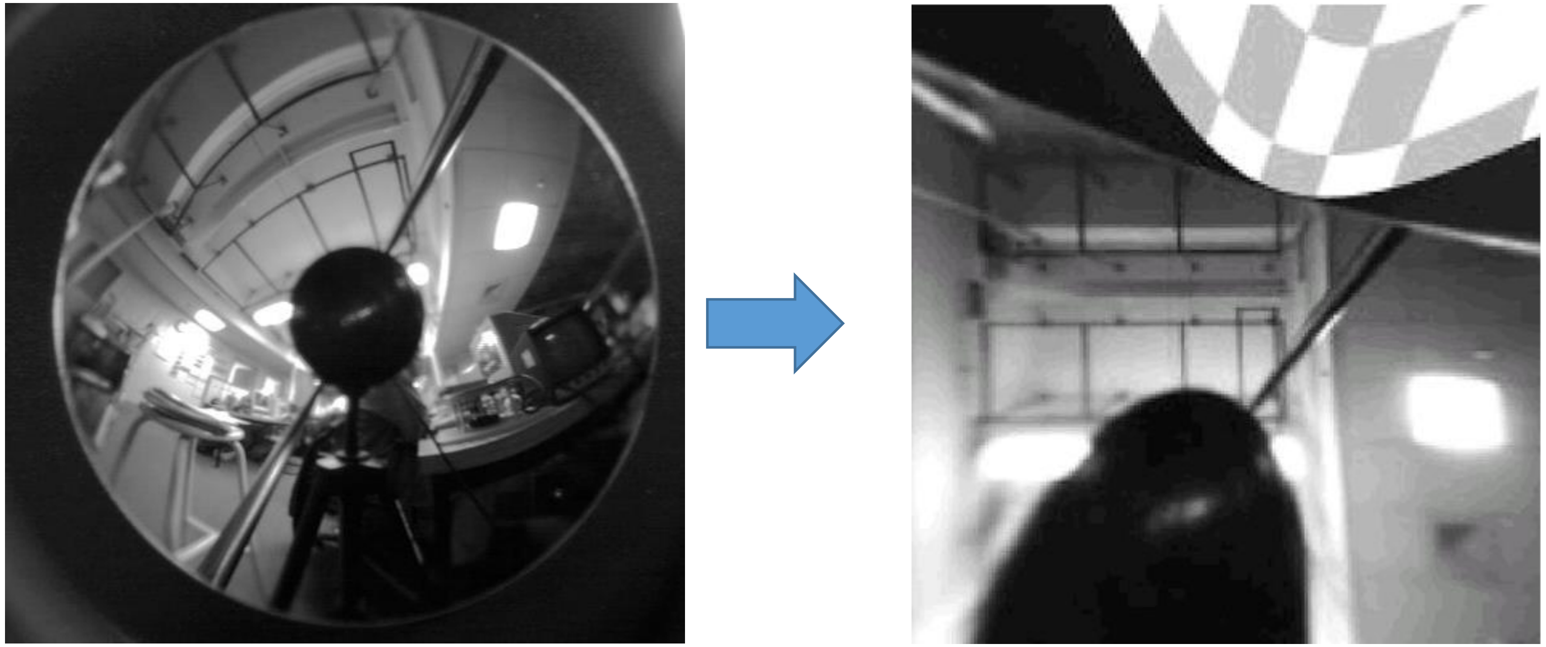
Parabola + Orthographic lens



Why do we prefer central cameras?

Because we can:

- Apply standard algorithms valid for perspective geometry.
- Unwarp parts of an image into a perspective one
- Transform image points into normalized vectors on the unit sphere



Unified Omnidirectional Camera Model
(for Fisheye and Catadioptric cameras)

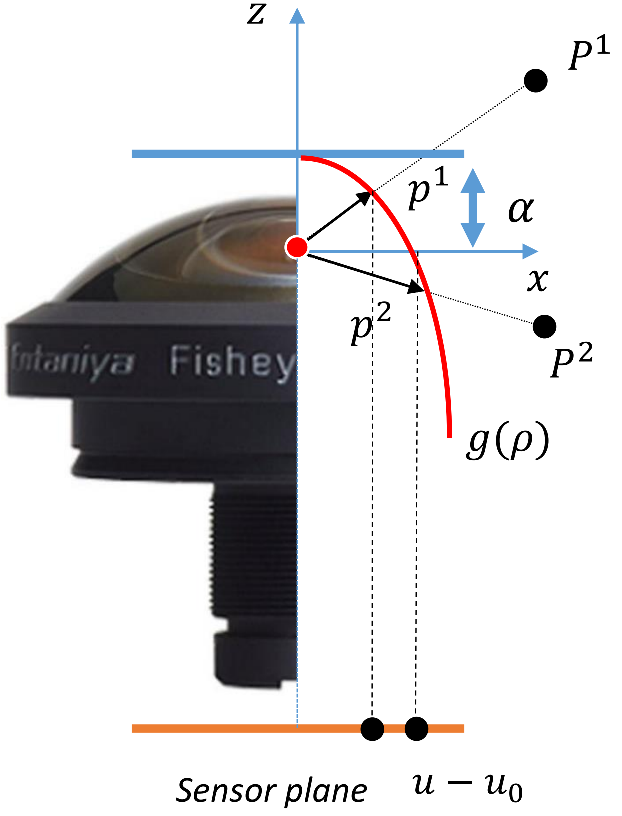
- We describe the *image projection function* by means of a polynomial, whose coefficients are the parameters to be estimated
- The coefficients, intrinsics, and extrinsics are then found via DLT

$$\lambda \cdot \mathbf{p} = \frac{\lambda}{\alpha} \cdot \begin{bmatrix} u - u_0 \\ v - v_0 \\ g(\rho) \end{bmatrix} = \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix}$$

$$g(\rho) = \alpha + a_1\rho + a_2\rho^2 + \ldots + a_N\rho^N$$

$$\rho = \sqrt{(u - u_0)^2 + (v - v_0)^2}$$

When $a_i = 0$ then we get a pinhole camera

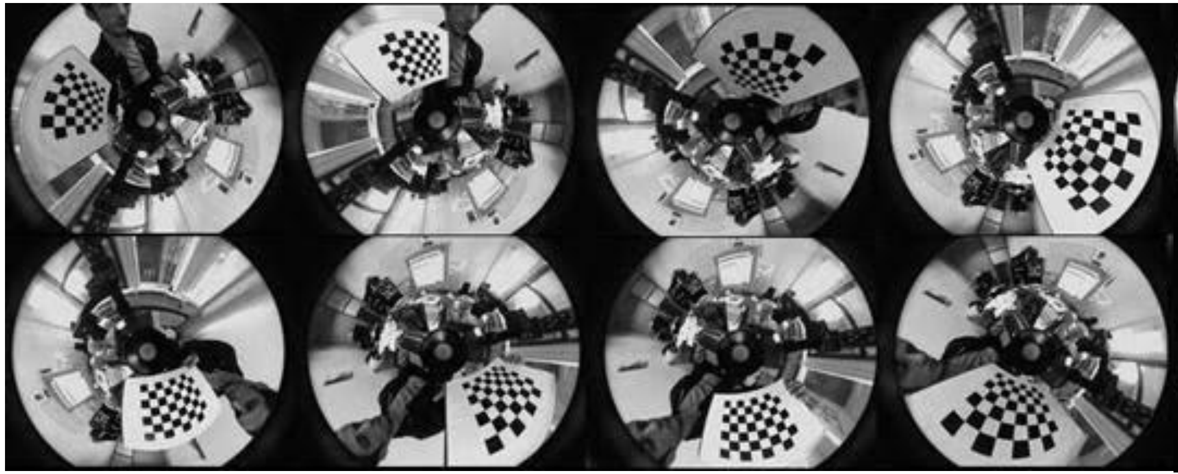


Scaramuzza, Martinelli, Siegwart. A Flexible Technique for Accurate Omnidirectional Camera Calibration and Structure from Motion. IEEE International Conference on Computer Vision Systems (ICVS), 2006. PDF.

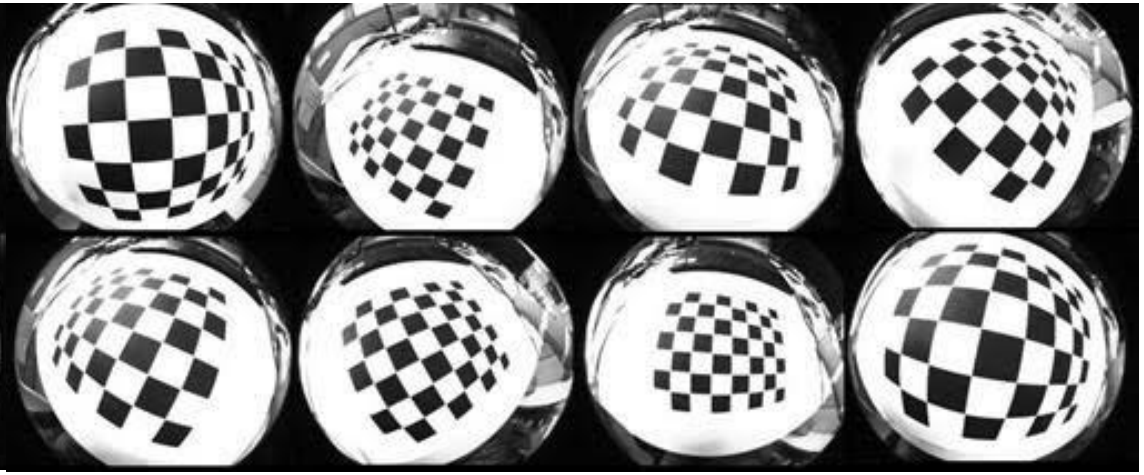
Scaramuzza, *Omnidirectional Camera*, chapter of Encyclopedia of Computer Vision, Springer, 2014. PDF

OCamCalib: Omnidirectional Camera Calibration Toolbox

- Released in 2006, OCamCalib is the standard toolbox for calibrating wide angle cameras (fisheye and catadioptric)
- Since 2015, included in the Matlab Computer Vision Toolbox



Example calibration images of a catadioptric camera

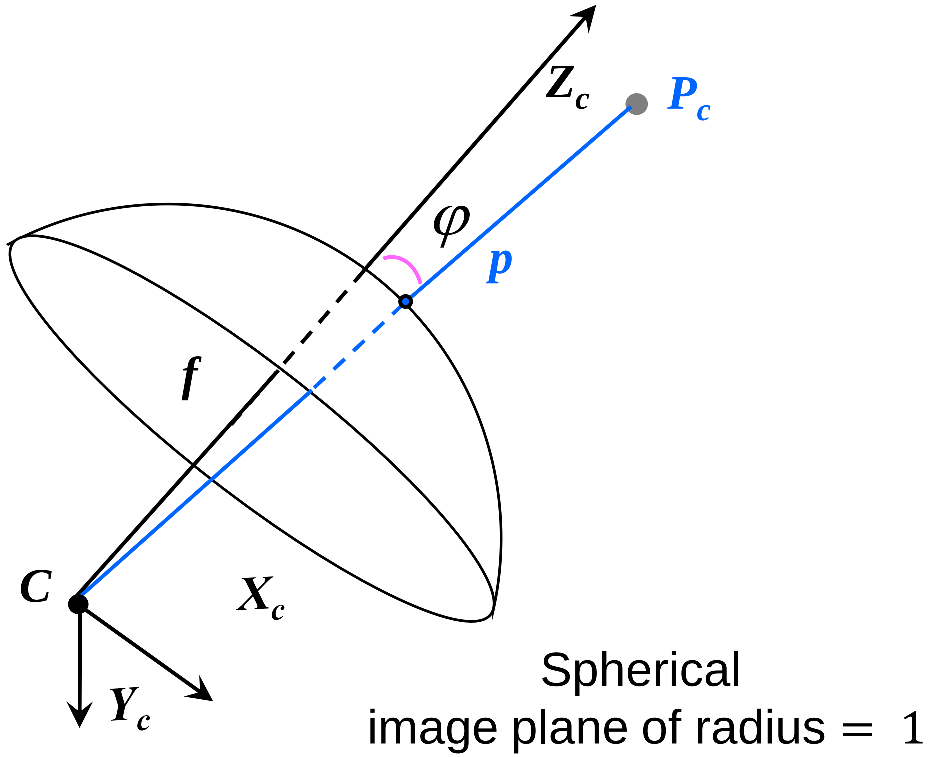


Example calibration images of a fisheye camera

Scaramuzza, Martinelli, Siegwart, *A toolbox for easily calibrating omnidirectional cameras*, IROS'06. PDF

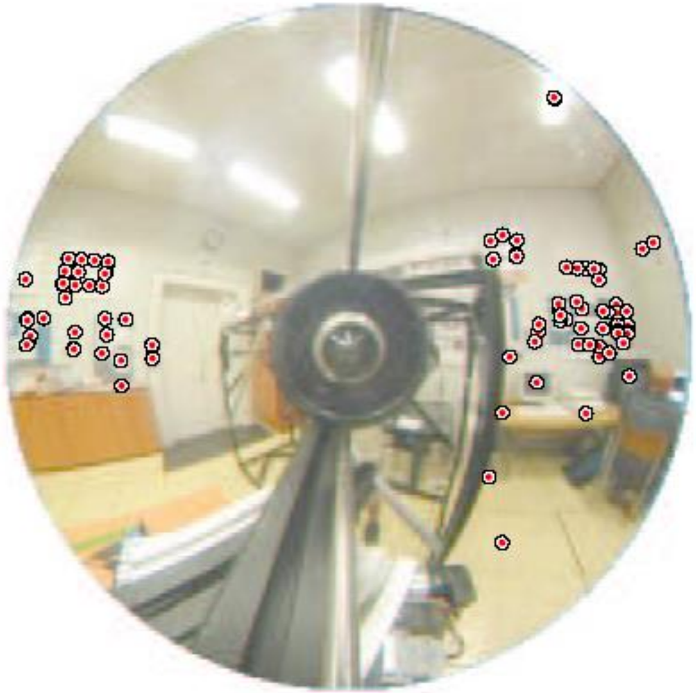
Projection of Image Points on the Unit Sphere

- Always possible after the camera has been calibrated

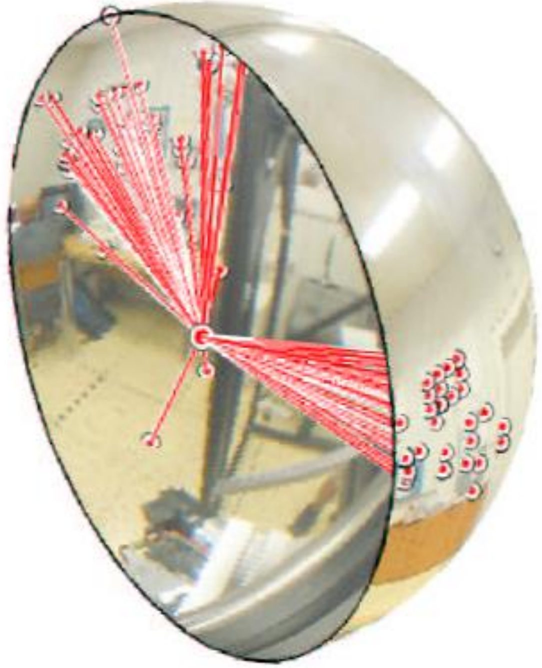


Projection of Image Points on the Unit Sphere

- Always possible after the camera has been calibrated



Points



Rays

Summary (things to remember)

- Calibration from 3D objects: DLT algorithm
- Calibration from planar grids: DLT algorithm using homography projection
- Reprojection Error and non linear optimization
- P3P algorithm
- DLT vs EPNP comparison
- Readings: Chapter 2.1 of Szeliski book, 1st Edition
- Omnidirectional cameras
  - Central vs non central projection
  - Unified (spherical) model for perspective and omnidirectional cameras
- Reading: Chapter 4 of Autonomous Mobile Robots book: link

Understanding Check

Are you able to:

- Describe the differences between Tsai's and Zhang's calibration methods
- Explain and derive the DLT in both Tsai's and Zhang's methods? What is the minimum number of point correspondences they require?
- Describe the general PnP problem and derive the behavior of its solutions?
- Explain the working principle of the P3P algorithm?
- What is the reprojection error and how is it used for refining the calibration?
- Define central and non central omnidirectional cameras?
- What kind of mirrors ensure central projection?