# Continuous Random Variables

**General Information**

- A function $f\colon \mathbb{R} \to \mathbb{R}$ is a *probability mass function* (pdf) of a continuous random variable $X$ iff $f$ is nonnegative and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
- For any probability mass function $f$, we have $\mathrm{P}(a \leq X \leq b) = \int_a^b f(x)\,dx$. Whether the inequality is strict or nonstrict does not affect the above identity.
- A *mode* of $X$ is any value $m$ such that $f(m)$ is maximum.
- A *cumulative distribution function* (cdf) $F\colon \mathbb{R} \to [0,1]$ of a random variable $X$ is defined by
$$F(x) \coloneqq P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx.$$
- When writing out the cdf as a piecewise function, we explicitly write out the range of values for each case. We reserve the use of "otherwise" for pdf's.
- Any cdf is continuous and nondecreasing.
- Let $X$ be a continuous random variable with cdf $F$. To find the pdf $g$ of any $y(X)$, we first find its cdf, then differentiate. We achieve this by reverse engineering $y(X) \leq y$ to find an inequality that relates $X$ with $y$. E.g. $e^X \leq y$ iff $X \leq \ln(y)$.
- A *median* of $X$ is any value $m$ such that $\mathrm{P}(X \leq m) = F(m) = 1/2$.
- Mean/Expectation:
$$\mu = \mathrm{E}(X) \coloneqq \int_{-\infty}^{\infty} x f(x)\,dx \qquad \text{and} \qquad \mathrm{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$
- Important property:
$$\mathrm{E}(a g(X) \pm b h(x)) = a\,\mathrm{E}(g(X)) \pm \mathrm{E}(h(X)).$$
- Variance:
$$\mathrm{Var}(X) \coloneqq \mathrm{E}(X^2) - [\mathrm{E}(X)]^2.$$
- Important property:
$$\mathrm{Var}(aX \pm b) = a^2\,\mathrm{Var}(X).$$

# Special Continuous Random Variables

**Definition 2.1**

A continuous random variable $X$ has a *normal distribution* with mean $\mu$ and standard deviation $\sigma$, denoted by $X \sim \mathrm{N}(\mu, \sigma^2)$, iff its pdf $f$ is such that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

**General Information**

- A normal distribution is symmetrical about the line $x = \mu$. That is

  $$\mathrm{P}(X \leq \mu - \delta) = \mathrm{P}(X \geq \mu + \delta)$$

  for each $\delta > 0$. Note that the mean, median, and mode coincide with $\mu$.
- Properties of the normal distribution. Let $X$ and $Y$ be independent, such that $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Y \sim \mathrm{N}(m, s^2)$. Then, for any $n \in \mathbb{N}$ and $x$, $y \in \mathbb{R}$,
  - $nX \sim \mathrm{N}(n\mu, n^2\sigma^2)$,
  - $X_1 + X_2 + \cdots + X_n \sim \mathrm{N}(n\mu, n\sigma^2)$,
  - $aX \pm bY \sim \mathrm{N}(a\mu \pm bm, a^2\sigma^2 + b^2s^2)$.
- Question phrasing may be misleading at times. Try to use some inference as to what exactly does the setter mean.

**Example 2.1**

"The mass of the padding is 30% of the mass of a randomly selected light bulb of mass $L$. Find the probability that a light bulb with padding has mass $c$."
Then for any light bulb of mass $L_1$, the mass of the padding is $0.3L_2$ (and *not* $0.3L_1$). i.e. we are to find $\mathrm{P}(L_1 + 0.3L_2)$.

- A variable $Z \sim \mathrm{N}(0, 1)$ is said to follow the *standard* normal distribution.

  *Note*: $Z$ is reserved for this purpose.
- Let $X \in \mathrm{N}(\mu, \sigma^2)$. Then, $\frac{X-\mu}{\sigma}$ follows the standard normal distribution.
- A continuous random variable $X$ has a *uniform distribution* over the interval $(a, b)$, which is denoted by $X \sim \mathrm{U}(a, b)$, iff its pdf $f$ is such that

  $$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$
- What `Tail` do we select for `invNorm`?

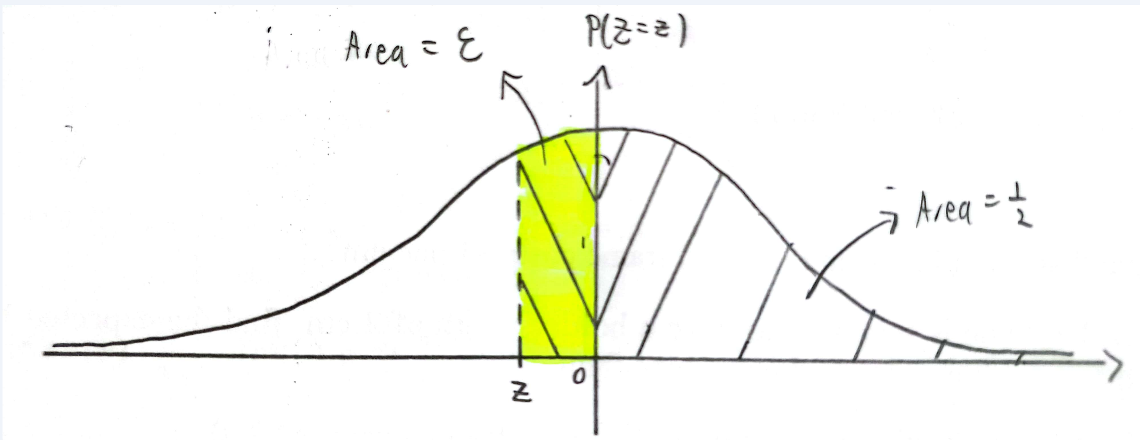
  | | |
  |---|---|
  | $\mathrm{P}(X < x) = p$ | `LEFT` |
  | $\mathrm{P}(-x < X < x) = p$ | `CENTER` |
  | $\mathrm{P}(X > x) = p$ | `RIGHT` |

- When using `invNorm` on an inequality, what should the sign be? For simplicity, we write $\mathscr{L}(p) = \texttt{invNorm}(p, 0, 1, \texttt{RIGHT})$, and $\mathscr{R}(p) = \texttt{invNorm}(p, 0, 1, \texttt{LEFT})$. Then,

| | |
|---|---|
| $\mathrm{P}(Z > z) \geq p$ | $z \leq \mathscr{L}(p)$ |
| $\mathrm{P}(Z > z) \leq p$ | $z \geq \mathscr{L}(p)$ |
| $\mathrm{P}(Z < z) \geq p$ | $z \geq \mathscr{R}(p)$ |
| $\mathrm{P}(Z < z) \leq p$ | $z \leq \mathscr{R}(p)$ |

**Example 2.2**

Suppose we want to find the least integer value of $m$ for which $\mathrm{P}(Z > 1 - m) \geq 1/2$. Then, using `invNorm (RIGHT)`, we infer that $z \leq 0$, *not* $z \geq 0$. An illustration:




- A continuous random variable $Y$ has an (negative) exponential distribution, which we denote with $Y \sim \mathrm{Exp}(\lambda)$, iff its pdf $g$ is such that

$$g(Y) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Expectation and variance:

| Distribution | Expectation | Variance |
|---|---|---|
| $X \sim \mathrm{U}(a, b)$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ |
| $Y \sim \mathrm{Exp}(\lambda)$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ |

*Note*: We need to remember the expectation and variance for the uniform distribution, as it is not provided in the MF26 formula sheet (unlike all other distributions).

- *Warning*: The G.C. tends to incorrectly process an integral if its upper and lower bounds contain $\pm$E99.

# Sampling and Estimation

**Definition 3.1**

A sample is a finite subset of the population.

**Definition 3.2**

A random sample is a sample selected such that each member of the population has an equal probability of being selected.

**Definition 3.3**

Any statistic $T$ derived from a random sample and used to estimated an unknown population parameter $\theta$ is known as an *estimator*. It is an *unbiased* estimator iff $\mathrm{E}(T) = \theta$. If $T$ is unbiased we commonly write $\hat{\theta}$ for $T$.

**General Information**

- Either write $\hat{\mu}$ or write out "Unbiased estimate of the population mean $\mu$, $\bar{x} = \ldots$" Same holds for other population parameters $\theta$.
- Estimators you should know:
-

| Parameter | Estimator | Unbiased? | Formula |
|---|---|---|---|
| Population Mean $\mu$ | Sample Mean $\overline{X}$ | ✓ | $\dfrac{X_1 + X_2 + \cdots + X_n}{n}$ |
| Population Variance $\sigma^2$ | Sample Variance $\sigma_n^2$ | × | $\dfrac{\sum (X_i - \overline{X})^2}{n}$<br>$\dfrac{\sum X_i^2}{n} - \overline{X}^2$ |
| | $S^2$ | ✓ | $\dfrac{n}{n-1}\sigma_n^2$<br>$\dfrac{\sum (X_i - \overline{X})^2}{n-1}$<br>$\dfrac{1}{n-1}\left[\sum X_i^2 - \dfrac{(\sum X_i)^2}{n}\right]$ |
| Population Proportion $p$ | Sample Proportion $P_s$ | ✓ | $\dfrac{X}{n}$ |

- Let $X$ be a random variable following *any distribution*, and suppose we have a random sample $X_1, X_2, \ldots, X_n$ of size $n \geq 50$.

  Then by CLT (Central Limit Theorem), since $n \geq 50$ is large,

  $$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right) \qquad \text{and} \qquad X_1 + X_2 + \cdots + X_n \sim \mathrm{N}(n\mu, \nu\sigma^2)$$

  *approximately.*

- Assumptions when using CLT:
  - The sample is random.
  - Each $X_i$ is independent and identically distributed.
- Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$ is known and we pick a *particular* sample. Let the unbiased estimates for the population mean and variance of this sample be $\bar{x}$ and $s^2$, respectively. Then,

| Distribution | Is An Approximation? |
|---|---|
| $\overline{X} \sim \mathrm{N}(\mu, \sigma^2)$ | No |
| $\overline{X} \sim \mathrm{N}(\bar{x}, \sigma^2)$ | Yes |
| $\overline{X} \sim \mathrm{N}(\mu, s^2)$ | Yes |
| $\overline{X} \sim \mathrm{N}(\bar{x}, s^2)$ | Yes |

So, if we obtain any of the latter three in solving a question, we must write "$X \sim \mathrm{N}(_,_)$approximately" (even though we knew $X$ *exactly* follows a normal distribution!)

- Pooled estimators you should know. First assume we have two populations, from which we select a random sample of size $n_1$ and $n_2$. We let $\overline{X}_1$ and $S_1^2$ denote the sample mean and unbiased estimator for variance, respectively. Similarly define $\overline{X}_2$ and $S_2^2$.

| Parameter | Unbiased Pooled Estimator |
|---|---|
| Mean | $\hat{\mu} = \dfrac{n_1\overline{X}_1 + n_2\overline{X}_2}{n_1+n_2}$ |
| Variance | $S_p^2 = \dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$ |

- Let $0 < \alpha < 1$

The following definition is found in Hogg-McKean-Craig. Similar definitions are also found in Wackerly-Mendenhall-Schaefer and Nitis Mukhopadhyay.

**Definition 3.4**

Let $X_1, X_2, \ldots, X_n$ be a sample on a random variable $X$, where $X$ has pdf $f(x;\theta)$, $\theta \in \Omega$. Let $0 < \alpha < 1$ be specified. Let $L = L(X_1, X_2, \ldots, X_n)$ and $U = U((X_1, X_2, \ldots, X_n))$ be two statistics. We say that the interval $(L, U)$ is a $(1-\alpha)100\%$ *confidence interval* for $\theta$ iff

$$1 - \alpha = P_\theta[\theta \in (L, U)].$$

That is, the probability that the interval contains $\theta$ is $1-\alpha$, which is called the *confidence coefficient* or *confidence level* of the interval.

- We cannot write "a $1-\alpha$ (e.g. 0.95) confidence interval". The $1-\alpha$ must always be expressed as a *percentage.*

- Let $\hat{\theta}$ be a statistic that is normally distributed with mean $\theta$ and standard error $\sigma_{\hat{\theta}}$. We see that
$$\frac{\hat{\theta}-\theta}{\sigma_{\hat{\theta}}} = Z \sim \mathrm{N}(0,1).$$
Rewriting $\mathrm{P}(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}) = 1-\alpha$ gives
$$\mathrm{P}(\hat{\theta} - z_{1-\alpha/2}\sigma_{\hat{\theta}} < \theta < \hat{\theta} + z_{1-\alpha/2}\sigma_{\hat{\theta}}) = 1-\alpha.$$
Hence, a $(1-\alpha)100\%$ confidence interval for $\theta$ is
$$(\hat{\theta} - z_{1-\alpha/2}\sigma_{\hat{\theta}}\,,\ \hat{\theta} + z_{1-\alpha/2}\sigma_{\hat{\theta}}).$$
(Wackerly-Mendenhall-Schaefer)

- Let $0 < \alpha < 1$ and $X_1, X_2, \ldots, X_n$ be a sample on a random variable $X$ with mean $\mu$, where

$n$ is large. Then, an approximate $(1-\alpha)100\%$ confidence interval for $\mu$ is

$$\left(\bar{x} - z_{1-\alpha/2}\frac{s}{\sqrt{n}}\,,\ \bar{x} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}\right).$$

When the variance $\sigma^2$ is known, we can replace $s$ with $\sigma$. If the distribution of $X$ is known to be normal, in addition to $\sigma^2$ being known exactly, then the confidence interval is exact; it is not just an approximation.

(Hogg-McKean-Craig)

- Let $X$ be a Bernoulli random variable with probability of success $p$, where $X$ is 1 or 0 if the outcome is success or failure, respectively. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the distribution of $X$, where $n$ is large. Let $\hat{p} = \overline{X}$ be the sample proportion of successes. Then, an approximate $(1-\alpha)100\%$ confidence interval for $p$ is given by

$$\left(\hat{p} - z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\,,\ \hat{p} + z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right).$$

(Letting $Y = X_1 + X_2 + \cdots + X_n \sim \mathrm{B}(n,p)$ gives $\hat{p} = Y/n$, which is the phrasing used by the school's notes.)

(Hogg-McKean-Craig)

**Note**

Standard phrasing for the interpretation of a $(1-\alpha)100\%$ confidence interval $(a,b)$.

The probability that the interval $(a,b)$ contains the true value of the [population mean/proportion in context] is $1-\alpha$.

**Note**

Standard phrasing for what is a $(1-\alpha)100\%$ confidence interval for $\theta$?

It is an interval which has probability $1-\alpha$ of containing the true value of $\theta$.

**Note**

Standard phrasing for whether mean/proportion in context has likely increased/decreased, when given suitable confidence intervals.

1. There is no conclusive result.

   Since the old and new $(1-\alpha)\%$ confidence intervals overlap, we are unable to conclude whether the [mean/proportion in context] has decreased or not. Hence, it is inconclusive from these figures as to whether the [context (e.g. an awareness campaign)] has been effective.

2. It has likely increased/decreased.

   The old $(1-\alpha)\%$ confidence interval is to the left/right of the new $(1-\alpha)\%$ confidence interval, such that they do not overlap. So, can conclude that the [mean/proportion in context] likely increased/decreased. Hence, these figures suggests that the [context (e.g. an awareness campaign)] has been effective.

**G.C. Skills**

Calculating statistics (i.e. $\bar{x}$, $s$, etc) by G.C. given data for a sample.

1. Keying in the data: `stat` $\Longrightarrow$ `1:Edit` $\Longrightarrow$ Key in the data into one of the lists $\mathtt{L}_i$.
2. Calculating the statistic: `stat` $\Longrightarrow$ `CALC` $\Longrightarrow$ `1-Var Stats` (`List:`$\mathtt{L}_i$) $\Longrightarrow$ `Calculate`.
3. Getting the statistic for further calculations: `vars` $\Longrightarrow$ `5:Statistics` $\Longrightarrow$ Select the desired statistic.

**G.C. Skills**

Calculating the symmetric confidence interval by G.C.

| | |
|---|---|
| Mean: | `stat` $\Longrightarrow$ `TESTS` $\Longrightarrow$ `7:ZInterval...` |
| Proportion: | `stat` $\Longrightarrow$ `TESTS` $\Longrightarrow$ `A:1-PropZInt...` |

# Correlation and Linear Regression

**Note**

A good scatter diagram should follow the guidelines below.

- The relative position of each point on the scatter diagram should be clearly shown.
- The range of values for the set of data should be clearly shown by marking out the extreme $x$ and $y$ values on the corresponding axis.
- The axes should be labeled clearly with the variables.

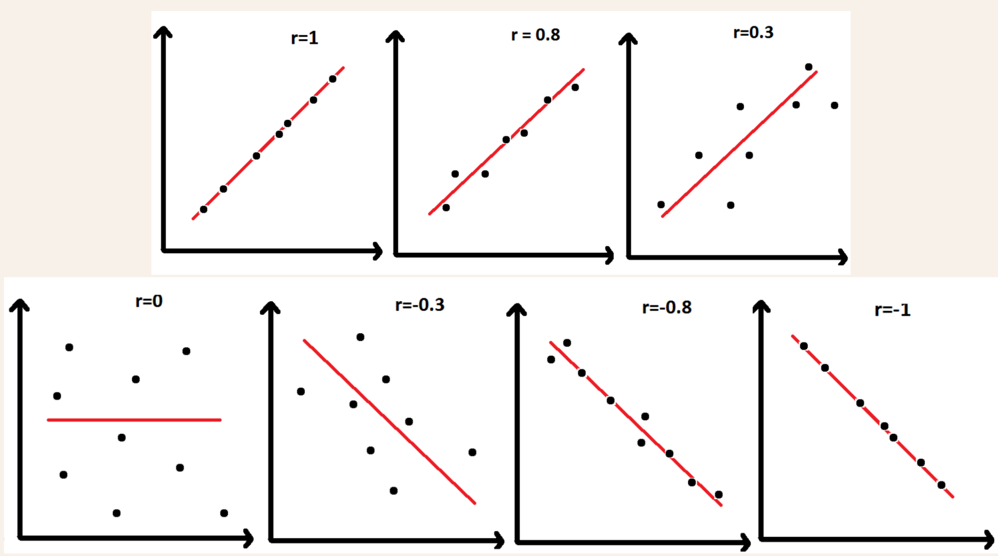
**General Information**

- The Product Moment Correlation Coefficient is a measure of the linear correlation between two variables. It is defined by

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2\sum(y-\bar{y})^2}} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sqrt{\left[\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}\right]\left[\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}\right]}},$$

  which takes on a value from 0 to 1.

- When $r = 0$, there is no linear relationship. But, a nonlinear relationship may be present. Additionally, the regression lines are perpendicular.
- The closer the value of $r$ is to 1 (or -1), the stronger the positive (or negative) linear correlation. Furthermore, the regression lines coincide.




- The regression line of $y$ on $x$ minimises the sum of squares deviation (error) in the $y$-direction. (i.e. we are assuming $x$ is the independent variable whose values are known exactly.) It is given by

$$y = \bar{y} + b(x-\bar{x}), \qquad \text{where} \qquad b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2} = \frac{\sum xy - \dfrac{\sum x\sum y}{n}}{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}.$$

- The point $(\bar{x}, \bar{y})$ always lies on both the regression lines of $y$ on $x$, and $x$ on $y$.
- Say we are given the value of one variable, and asked to approximate the the value of the other variable. Then, we should always use the line of the *dependent* variable on the *independent*.
- Estimations should not be taken for data outside the range of the sample provided, even if the value of $r$ is close to 1.