Linear Algebra

Definition 1.1

A vector space (or linear space) V over a field $\mathbb{F}$ consists of a set on which two operations (called addition and multiplication respectively here) are defined so that;

(A) (V is Closed Under Addition) For all $\mathbf{x}, \mathbf{y} \in V$, there exists a unique element $\mathbf{x} + \mathbf{y} \in V$.

(M) (V is Closed Under Scalar Multiplication) For all elements $a \in \mathbb{F}$ and elements $\mathbf{x} \in V$, there exists a unique element $a\mathbf{x} \in V$.

Such that the following properties hold:

(VS 1) (Commutativity of Addition) For all $\mathbf{x}, \mathbf{y} \in V$, we have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

(VS 2) (Associativity of Addition) For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

(VS 3) (Existance of The Zero/Null Vector) There exists an element in V denoted by $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$.

(VS 4) (Existance of Additive Inverses) For all elements $\mathbf{x} \in V$, there exists an element $\mathbf{y} \in V$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$.

(VS 5) (Multiplicative Identity) For all elements $x \in V$, we have $1\mathbf{x} = \mathbf{x}$, where 1 denotes the multiplicative identity in $\mathbb{F}$.

(VS 6) (Compatibility of Scalar Multiplication with Field Multiplication) For all elements $a, b \in \mathbb{F}$ and elements $\mathbf{x} \in V$, we have $(ab)\mathbf{x} = a(b\mathbf{x})$.

(VS 7) (Distributivity of Scalar Multiplication over Vector Addition) For all elements $a \in \mathbb{F}$ and elements $\mathbf{x}, \mathbf{y} \in V$, we have $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

(VS 8) (Distributivity of Scalar Multiplication over Field Addition) For all elements $a, b \in \mathbb{F}$, and elements $\mathbf{x} \in V$, we have $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

Theorem 1.2

Let V be a vector space and W a subset of V. Then W is a subspace of V iff the following 3 conditions hold for the operations defined in V.

(a) $\mathbf{0} \in W$

(b) $\mathbf{x} + \mathbf{y} \in W$ whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$.

(c) $c\mathbf{x} \in W$ whenever $c \in \mathbb{F}$ and $\mathbf{x} \in W$.

Definition 1.3

A subset S of a vector space V *generates* (or *spans*) V iff $\mathrm{span}(S) = V$. In this case, we also say that the vectors of S generate (or span) V.

Definition 1.4

Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a *linear combination* of vectors of S iff there exists a finite number of vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$ in $\mathbb{F}$ such that

$$v = \sum_{i=1}^{n} a_i u_i.$$

In this case we also say that v is a linear combination of $u_1, u_2, \ldots, u_n$ and call $a_1, a_2, \ldots, a_n$ the *coefficients* of the linear combination

Definition 1.5

A set subset S of a vector space V is called *linearly dependent* iff there exists a finite number of distinct vectors $u_1, u_2, \ldots, u_n$in S and scalars $a_1, a_2, \ldots, a_n$ not all zero, such that

$$a_1 u_1 + a_2 u_2 + a_n u_n = \mathbf{0}.$$

Definition 1.6

A *basis* β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Theorem 1.7: The Rank-Nullity Theorem.

For any vector spaces V and W, and a linear operator $T\colon V \to W$, it holds that

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$

General Information

- Let $\mathbf{A}$ be an $m \times n$ matrix, and $\mathbf{a}_j$ its jth column. For any $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top$,

$$\mathbf{A}\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{a}_j.$$

- Let $\mathbf{A}$ and $\mathbf{B}$ be matrices having n rows. For any matrix $\mathbf{M}$ with n columns, we have

$$\left(\mathbf{A} \mid \mathbf{B}\right) = \left(\mathbf{MA} \mid \mathbf{MB}\right).$$

Definition 1.8

A system $\mathbf{Ax} = \mathbf{b}$ is *homogeneous* iff $\mathbf{b} = 0$; otherwise it is *nonhomogeneous*.

Theorem 1.9

For any matrix, its row space, column space, and rank are identical.

Theorem 1.10

A system $\mathbf{Ax} = \mathbf{b}$ of m linear equations in n unknowns has a solution space of dimension $n - \operatorname{rank}(A)$.

Definition 1.11

A system $\mathbf{Ax} = \mathbf{b}$ of linear equations is *consistent* iff its solution set is nonempty; otherwise it is *inconsistent*.

Theorem 1.12: The Rouché-Capelli Theorem.

A system $\mathbf{Ax} = \mathbf{b}$ is consistent iff $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b})$.

Definition 1.13

A matrix is said to be in *reduced row echelon form* iff

- Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).
- The first nonzero entry in each row is the only nonzero entry in its column.
- The first nonzero entry in each row is 1 and it occurs in a column to the right of the first

nonzero entry in the preceding row.

- Gaussian elimination.
  - In the forward pass, the augmented matrix is transformed into an upper triangular matrix in which the first nonzero entry of each row is 1 and it occurs in a column to the right of the first nonzero entry of each preceding row.
  - In the backward pass, the upper triangular matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.
- Gaussian elimination always reducecs a matrix to its rref form.
- Let $\mathbf{A}$ be an invertible $n \times n$ matrix. Then, for some elementary row matrices $\mathbf{E}_1$ to $\mathbf{E}_p$,
$$\mathbf{E}_p\mathbf{E}_{p-1}\dots\mathbf{E}_1(\mathbf{A} \,|\, \mathbf{I}_n) = \mathbf{A}^{-1}(\mathbf{A} \,|\, \mathbf{I}_n) = (\mathbf{I}_n \,|\, \mathbf{A}^{-1}).$$
In other words, we can perform Gaussian elimination, so that $(\mathbf{A} \,|\, \mathbf{I}_n) \to (\mathbf{I}_n \,|\, \mathbf{A}^{-1})$.
- Let $\mathbf{A} \coloneqq \begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{pmatrix}$ be $m \times n$ matrix, and $\mathbf{A}' \coloneqq \begin{pmatrix}\mathbf{a}'_1 & \mathbf{a}'_2 & \cdots & \mathbf{a}'_n\end{pmatrix}$ its rref. Then, $\{\mathbf{a}_{k_1}, \mathbf{a}_{k_2}, \dots, \mathbf{a}_{k_m}\}$ is linearly independent iff $\{\mathbf{a}'_{k_1}, \mathbf{a}'_{k_2}, \dots, \mathbf{a}'_{k_m}\}$ is. Moreover, the row space of $\mathbf{A}$ and $\mathbf{A}'$ are clearly identical.
- Finding a basis for an intersection of subspaces. Let V and W be subspaces of $\mathbb{F}^n$ generated by the columns of the $n \times m$ matrix $\mathbf{A}$ and $n \times k$ matrix $\mathbf{B}$, respectively. Find a basis for the subspace $V \cap W$.
  1. First notice that $\mathbf{v} \in V \cap W$ iff
  $$\mathbf{v} = \mathbf{A}\mathbf{x}_1 = \mathbf{B}\mathbf{x}_2$$
  for some $\mathbf{x}_2 \in \mathbb{F}^m$ and $\mathbf{x}_2 \in \mathbb{F}^k$. That is,
  $$\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}\begin{pmatrix}\mathbf{x_1} \\ -\mathbf{x_2}\end{pmatrix} = \mathbf{0}.$$
  So, equivalently, we write
  $$\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}\mathbf{y} = \mathbf{0}.$$
  for some $\mathbf{y} \in \mathbb{F}^{m+k}$. As such, by row reducing $\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}$, we find a basis
  $$\beta \coloneqq \left\{\begin{pmatrix}\mathbf{u_1} \\ \mathbf{u_1'}\end{pmatrix}, \begin{pmatrix}\mathbf{u_2} \\ \mathbf{u_2'}\end{pmatrix}, \dots, \begin{pmatrix}\mathbf{u_r} \\ \mathbf{u_r'}\end{pmatrix}\right\},$$
  where $\mathbf{u}_i \in \mathbb{F}^m$ and $\mathbf{u}_i \in \mathbb{F}^k$. Now, a generating set for $V \cap W$ is
  $$\Gamma \coloneqq \{\mathbf{Au_1}, \mathbf{Au_2}, \dots, \mathbf{Au_r}\}.$$
  Alternatively, another generating set for $V \cap W$ is
  $$\Delta \coloneqq \{\mathbf{Bu_1'}, \mathbf{Bu_2'}, \dots, \mathbf{Bu_r'}\}.$$
  From here, it is simple to choose bases $\gamma \subseteq \Gamma$ and $\delta \subseteq \Delta$ for $V \cap W$.
  (Naturally, it holds that $\mathbf{Au_i} + \mathbf{Bu_i'} = 0$.)
  2. An alternative method. By row reduction, we can calculate
  $$\begin{aligned} r \coloneqq \dim(V \cap W) &= \dim(U) + \dim(V) - \dim(U + V), \\ &= \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}) - \operatorname{rank}\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}, \\ &= \operatorname{rank}\left(\mathbf{A}^\top\right) + \operatorname{rank}\left(\mathbf{B}^\top\right) - \operatorname{rank}\begin{pmatrix}\mathbf{A}^\top \\ \mathbf{B}^\top\end{pmatrix}. \end{aligned}$$
  Then, a basis for $V \cap W$ can be formed by choosing r linearly independent columns of $\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}$, or rows of $\begin{pmatrix}\mathbf{A}^\top \\ \mathbf{B}^\top\end{pmatrix}$.

3. Another alternative, probably the best option! Skip the row reduction of $\mathbf{A}$ and $\mathbf{B}$ in the above method. We just reduce

$$\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix} \to \begin{pmatrix}\mathbf{A}' & \mathbf{B}'\end{pmatrix}.$$

Let $\mathbf{c_i}$ and $\mathbf{c_i'}$ be the ith column of $\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}$ and $\begin{pmatrix}\mathbf{A}' & \mathbf{B}'\end{pmatrix}$, respectively. We compare the columns of A' and B' to find (with relative ease) a basis $\beta' \coloneqq \left\{\mathbf{c'_{i_1}}, \mathbf{c'_{i_2}}, \ldots, \mathbf{c'_{i_r}}\right\}$ for the intersection of the column spaces of A' and B'. Then, $\beta \coloneqq \{\mathbf{c_{i_1}}, \mathbf{c_{i_2}}, \ldots, \mathbf{c_{i_r}}\}$ is a basis for $V \cap W$ (the intersection of the column spaces of A and B).

4. A fourth method for when I learn about orthogonal complements.

Definition 1.14

Let $\mathbf{A} \in \mathrm{M}_{n\times n}(\mathbb{F})$. If $n = 1$, so that $A = (a_1 1)$, we define $\det(\mathbf{A}) \coloneqq a_1 1$. For $n \geq 2$, we define $\det(\mathbf{A})$ recursively as

$$\det(\mathbf{A}) \coloneqq \sum_{j=1}^{n} (-1)^{1+j} \mathbf{A}_{1j} \cdot \det(\widetilde{\mathbf{A}}_{1j}).$$

THe scalar $\det(\mathbf{A})$ is called the *determinant* of $\mathbf{A}$ and is also denoted by $|\mathbf{A}|$. The scalar

$$(-1)^{i+j} \det(\widetilde{\mathbf{A}}_{1j})$$

is called the cofactor of the entry of $\mathbf{A}$ in row i, column j.

- A matrix $\mathbf{A}$ is invertible iff its determinant is nonzero.

Theorem 1.15

The determinant $\det\colon \mathrm{M}_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is an alternating n-linear function. The former (alternating) means that for $\mathbf{A} \in \mathrm{M}_{n\times n}(\mathbb{F})$ and any $\mathbf{B}$ obtained from $\mathbf{A}$ by interchanging any two rows of $\mathbf{A}$,

$$\det(\mathbf{B}) = -\det(\mathbf{A}).$$

The latter (n-linearity) means that, for any scalar $k \in \mathbb{F}$ and vectors $\mathbf{u}, \mathbf{v}, \mathbf{a}_i \in \mathbb{F}^n$,

$$\det\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_{r-1}\mathbf{u} + k\mathbf{v}\\ \mathbf{a}_{r+1}\\ \vdots\\ \mathbf{a}_n\end{pmatrix} = \det\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_{r-1}\mathbf{u}\\ \mathbf{a}_{r+1}\\ \vdots\\ \mathbf{a}_n\end{pmatrix} + k\det\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_{r-1}\mathbf{v}\\ \mathbf{a}_{r+1}\\ \vdots\\ \mathbf{a}_n\end{pmatrix}.$$

(In fact, it can be shown that $\det\colon \mathrm{M}_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is the *unique* alternating n-linear function, such that $\det(\mathbf{I}) = 1$.)

Corollary 1.16

Let $\mathbf{A} \in \mathrm{M}_{n\times n}(\mathbb{F})$. Then, for any matrix $\mathbf{B}$ obtained by adding a scalar multiple of one row/column of $\mathbf{A}$ to another, $\det(\mathbf{B}) = \det(\mathbf{A})$.

Theorem 1.17

The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if $\mathbf{A} \in \mathrm{M}_{n\times n}(\mathbb{F})$, then for any integer $1 \le i \le n$,

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{i+j} \mathbf{A}_{ij} \cdot \det(\widetilde{\mathbf{A}}_{ij}).$$

Here, $\widetilde{\mathbf{A}}_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained from $\mathbf{A}$ by deleting its ith row and jth column.

Corollary 1.18

The determinant of any triangular matrix is the product of its diagonals.

Theorem 1.19

Let A be an $n \times n$ matrix. Then,

$$\det(\mathbf{A}) = \det(\mathbf{A}^\top).$$

So, the determinant of a square matrix can also be evaluated by cofactor expansion along any column.

Theorem 1.20

Let $\mathbf{A}$ be an invertible $n \times n$ matrix. Then,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \operatorname{adj}(A),$$

where $\operatorname{adj}(\mathbf{A})$ is the adjugate/classical adjoint of $\mathbf{A}$. That is, the matrix whose (i,j)th entry is the (j,i)th cofactor $(-1)^{j+i}\det(\widetilde{\mathbf{A}}_{ji})$

Theorem 1.21

For any $\mathbf{A}, \mathbf{B} \in \mathrm{M}_{n\times n}(\mathbb{F})$, we have $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$.

Definition 1.22

A linear operator T on a finite-dimensional vector space V is called *diagonalisable* iff there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix $\mathbf{A}$ is called diagonalisable iff $L_{\mathbf{A}}$ is diagonalisable.

Definition 1.23

Let T be a linear operator on a vector space V. A nonzero vector $\mathbf{v} \in V$ is called an *eigenvector* of T iff there exists a scalar λ such that $T(\mathbf{v}) = \lambda\mathbf{v}$. The scalar λ is called the *eigenvalue* corresponding to the eigenvector $\mathbf{v}$.
Let $\mathbf{A}$ be in $\mathrm{M}_{n\times n}(\mathbb{F})$. A nonzero vector $v \in \mathbb{F}^n$ is called an *eigenvector* of $\mathbf{A}$ iff v is an eigenvector of $L_{\mathbf{A}}$; that is, iff $\mathbf{A}v = \lambda v$ for some scalar λ. The scalar λ is called the eigenvalue of $\mathbf{A}$ corresponding to the eigenvector v.

Definition 1.24

Let $\mathbf{A} \in \mathrm{M}_{n\times n}(\mathbb{F})$. The polynomial $f(t) = \det(\mathbf{A} - \lambda\mathbf{I}_n)$ is called the *characteristic polynomial* of $\mathbf{A}$.

- A matrix $\mathbf{A} \in \mathrm{M}_{n\times n}(\mathbb{F})$ is diagonalizable iff there exists an ordered basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for $\mathbb{F}^n$ consisting of eigenvectors of $\mathbf{A}$, i.e. a eigenbasis. Furthermore, if $\mathbf{Q}$ is the $n \times n$ matrix whose jth column is $\mathbf{v}_j$, then $\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}\mathbf{Q}$ is a diagonal matrix such that d_{jj} is the eigenvalue of A corresponding to $\mathbf{v}_j$. The matrix $\mathbf{Q}$ is said to *diagonalise* $\mathbf{A}$.

- Hence, we obtain the following procedure to diagonalise a 3×3 matrix $\mathbf{A}$ with three distinct eigenvalues.

1. Find the eigenvalues λ_1, λ_2, and λ_3 of $\mathbf{A}$. They are just the roots of the characteristic polynomial of $\mathbf{A}$. This can be done using the GC.
2. Find an eigenvector $\mathbf{v}_j$ corresponding to each eigenvalue λ_j by finding the nullspace of $\mathbf{A} - \lambda_j \mathbf{I}$.
3. Let $\mathbf{Q} = \left(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\right)$. Then,
$$\mathbf{D} := \mathbf{Q}^{-1} A \mathbf{Q}$$
is a diagonal matrix.

Note

Let $\mathbf{A}$ be a 3×3 real matrix with the eigenvalue λ. Then, the cross product of two nonzero rows/columns of $\mathbf{A} - \lambda \mathbf{I}$ is an eigenvector of $\mathbf{A}$.

Theorem 1.25: The Cayley-Hamiliton Theorem.

Let T be a linear operator on a finite dimensional vector space V, and let $f(t)$ be the characteristic polynomial of T. Then $f(T) = T_0$, the zero transformation. That is, T "satisfies" its characteristic equation.

Corollary 1.26: The Cayley-Hamiliton Theorem for Matrices.

Let A be an $n \times n$ matrix, and let $f(t)$ be the characteristic polynomial of A. Then, $f(A) = O$, the $n \times n$ zero matrix.

G.C. Skills

Finding eigenvalues of a matrix $\mathbf{A}$ using the GC.

1. `2nd` $\Longrightarrow$ `x`$^{-1}$ (`matrix`) $\Longrightarrow$ Key in the matrix $\mathbf{A} - t\mathbf{I}$, e.g. into `[A]`.
2. Plot $\mathtt{Y}_1 = \det(\mathtt{[A]})$.
3. `2nd` $\Longrightarrow$ `trace` $\Longrightarrow$ `2:zero` $\Longrightarrow$ Find the roots.

Numerical Methods

General Information

- The parity of the degree of a real polynomial is the same as that of its number of real roots.
- Let the real polynomial p given by $p(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_0$ have coefficients $a_n > 0$ and $a_0 < 0$. Then, it has at least one positive and one negative root.
- Suppose we have some function $f\colon \mathbb{R} \to \mathbb{R}$ with a root α, whose value we want to approximate. There are three ways to obtain this approximation.
  1. Linear interpolation on an interval $[a, b]$ containing α. We let $x_0 := b$ and
  $$x_{i+1} := \frac{a|f(x_i)| + x_i|f(a)|}{|f(a)| + |f(x_i)|}.$$
     – *Additional notes.*
  2. Fixed-point Iteration. First select a function $F\colon \mathbb{R} \to \mathbb{R}$, such that $F(\alpha) = \alpha$, and choose some initial approximation x_0 to α. Then, we recursively define $x_{n+1} := F(x_n)$. The desired convergence behavior is for x_n to approach α.
     – *Additional notes.*

G.C. Skills

Linear interpolation: finding an approximation to a root in $[a, b]$ up to n decimal places.

1. $Y_1 = f(x)$,
2. $a \to A$ and $b \to B$,
3. $\dfrac{B|Y_1(A)| + A|Y_1(B)|}{|Y_1(A)| + |Y_1(B)|}$,
4. Ans $\to A$ or B (choose the one that has the opposite sign to Ans),
5. Repeat steps 4 to 5,
6. Terminate this process when the approximations are consistent up to n decimal places.