

Linear Algebra

Definition 1.1

A vector space (or linear space) V over a field \mathbb{F} consists of a set on which two operations (called addition and multiplication respectively here) are defined so that;

- (A) (V is Closed Under Addition) For all $\mathbf{x}, \mathbf{y} \in V$, there exists a unique element $\mathbf{x} + \mathbf{y} \in V$.
- (M) (V is Closed Under Scalar Multiplication) For all elements $a \in \mathbb{F}$ and elements $\mathbf{x} \in V$, there exists a unique element $a\mathbf{x} \in V$.

Such that the following properties hold:

- (VS 1) (Commutativity of Addition) For all $\mathbf{x}, \mathbf{y} \in V$, we have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
- (VS 2) (Associativity of Addition) For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
- (VS 3) (Existence of The Zero/Null Vector) There exists an element in V denoted by $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$.
- (VS 4) (Existence of Additive Inverses) For all elements $\mathbf{x} \in V$, there exists an element $\mathbf{y} \in V$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$.
- (VS 5) (Multiplicative Identity) For all elements $x \in V$, we have $1\mathbf{x} = \mathbf{x}$, where 1 denotes the multiplicative identity in \mathbb{F} .
- (VS 6) (Compatibility of Scalar Multiplication with Field Multiplication) For all elements $a, b \in \mathbb{F}$ and elements $\mathbf{x} \in V$, we have $(ab)\mathbf{x} = a(b\mathbf{x})$.
- (VS 7) (Distributivity of Scalar Multiplication over Vector Addition) For all elements $a \in \mathbb{F}$ and elements $\mathbf{x}, \mathbf{y} \in V$, we have $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.
- (VS 8) (Distributivity of Scalar Multiplication over Field Addition) For all elements $a, b \in \mathbb{F}$, and elements $\mathbf{x} \in V$, we have $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

General Information

- Let V be a vector space and W a subset of V . Then W is a subspace of V iff the following 3 conditions hold for the operations defined in V .
 - (a) $\mathbf{0} \in W$
 - (b) $\mathbf{x} + \mathbf{y} \in W$ whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$.
 - (c) $c\mathbf{x} \in W$ whenever $c \in \mathbb{F}$ and $\mathbf{x} \in W$.
- A subset S of a vector space V *generates* (or *spans*) V iff $\text{span}(S) = V$. In this case, we also say that the vectors of S generate (or span) V .
- Let V be a vector space and S a nonempty subset of V . A vector $v \in V$ is called a *linear combination* of vectors of S iff there exists a finite number of vectors u_1, u_2, \dots, u_n in S and scalars a_1, a_2, \dots, a_n in \mathbb{F} such that

$$v = \sum_{i=1}^n a_i u_i.$$

In this case we also say that v is a linear combination of u_1, u_2, \dots, u_n and call a_1, a_2, \dots, a_n the *coefficients* of the linear combination

- A set subset S of a vector space V is called *linearly dependent* iff there exists a finite number of distinct vectors u_1, u_2, \dots, u_n in S and scalars a_1, a_2, \dots, a_n not all zero, such that

$$a_1 u_1 + a_2 u_2 + \dots + a_n u_n = \mathbf{0}.$$

- A *basis* β for a vector space V is a linearly independent subset of V that generates V . If β is a basis for V , we also say that the vectors of β form a basis for V .
- The Rank-Nullity Theorem: For any vector spaces V and W , and a linear operator $T: V \rightarrow W$, it holds that

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

- For any matrix, its row space, column space, and rank are identical.
- A system $\mathbf{Ax} = \mathbf{b}$ is *homogeneous* iff $\mathbf{b} = \mathbf{0}$; otherwise it is *nonhomogeneous*.
- A system $\mathbf{Ax} = \mathbf{b}$ of m linear equations in n unknowns has a solution space of dimension $n - \text{rank}(A)$.
- A system $\mathbf{Ax} = \mathbf{b}$ of linear equations is *consistent* iff its solution set is nonempty; otherwise it is *inconsistent*.
- The Rouché-Capelli theorem: A system $\mathbf{Ax} = \mathbf{b}$ is consistent iff $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b})$.
- A matrix is said to be in *reduced row echelon form* iff

- Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).
- The first nonzero entry in each row is the only nonzero entry in its column.
- The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

- Gaussian elimination.
 - In the forward pass, the augmented matrix is transformed into an upper triangular matrix in which the first nonzero entry of each row is 1 and it occurs in a column to the right of the first nonzero entry of each preceding row.
 - In the backward pass, the upper triangular matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.
- Let \mathbf{A} be an $m \times n$ matrix, and \mathbf{a}_j its j th column. For any $\mathbf{x} = (x_1 \ x_2 \ \dots \ x_n)^\top$,

$$\mathbf{Ax} = \sum_{j=1}^n x_j \mathbf{a}_j.$$

- Let \mathbf{A} and \mathbf{B} be matrices having n rows. For any matrix \mathbf{M} with n columns, we have

$$(\mathbf{A} \mid \mathbf{B}) = (\mathbf{MA} \mid \mathbf{MB}).$$

- Finding a basis for an intersection of subspaces. Let V and W be subspaces of \mathbb{F}^n generated by the columns of the $n \times m$ matrix \mathbf{A} and $n \times k$ matrix \mathbf{B} , respectively. To find a basis for the subspace $V \cap W$, first notice that $\mathbf{v} \in V \cap W$ iff

$$\mathbf{v} = \mathbf{Ax}_1 = \mathbf{Bx}_2$$

for some $\mathbf{x}_2 \in \mathbb{F}^m$ and $\mathbf{x}_2 \in \mathbb{F}^k$. That is,

$$(\mathbf{A} \ \mathbf{B}) \begin{pmatrix} \mathbf{x}_1 \\ -\mathbf{x}_2 \end{pmatrix} = \mathbf{0}.$$

So, equivalently, we write

$$(\mathbf{A} \ \mathbf{B}) \mathbf{y} = \mathbf{0}.$$

for some $\mathbf{y} \in \mathbb{F}^{m+k}$. As such, by row reducing $(\mathbf{A} \ \mathbf{B})$, we find a basis

$$\beta := \left\{ \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}'_1 \end{pmatrix}, \begin{pmatrix} \mathbf{u}_2 \\ \mathbf{u}'_2 \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}'_r \end{pmatrix} \right\},$$

where $\mathbf{u}_i \in \mathbb{F}^m$ and $\mathbf{u}_i \in \mathbb{F}^k$. Now, a generating set for $V \cap W$ is

$$\Gamma := \{\mathbf{A}\mathbf{u}_1, \mathbf{A}\mathbf{u}_2, \dots, \mathbf{A}\mathbf{u}_r\}.$$

Alternatively, another generating set for $V \cap W$ is

$$\Delta := \{\mathbf{B}\mathbf{u}'_1, \mathbf{B}\mathbf{u}'_2, \dots, \mathbf{B}\mathbf{u}'_r\}.$$

From here, it is simple to choose bases $\gamma \subseteq \Gamma$ and $\delta \subseteq \Delta$ for $V \cap W$.

(Naturally, it holds that $\mathbf{A}\mathbf{u}_i + \mathbf{B}\mathbf{u}'_i = \mathbf{0}$.)

- An alternative method. By row reduction, we can calculate

$$\begin{aligned} r := \dim(V \cap W) &= \dim(U) + \dim(V) - \dim(U + V), \\ &= \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}) - \text{rank}(\mathbf{A} \ \mathbf{B}), \\ &= \text{rank}(\mathbf{A}^\top) + \text{rank}(\mathbf{B}^\top) - \text{rank} \begin{pmatrix} \mathbf{A}^\top \\ \mathbf{B}^\top \end{pmatrix}. \end{aligned}$$

Then, a basis for $V \cap W$ can be formed by choosing r linearly independent columns of $(\mathbf{A} \ \mathbf{B})$, or rows of $\begin{pmatrix} \mathbf{A}^\top \\ \mathbf{B}^\top \end{pmatrix}$.

- **Another alternative.** Skip the row reduction of \mathbf{A} and \mathbf{B} in the above method. We just reduce

$$(\mathbf{A} \ \mathbf{B}) \rightarrow (\mathbf{A}' \ \mathbf{B}').$$

Let \mathbf{c}_i and \mathbf{c}'_i be the i th column of $(\mathbf{A} \ \mathbf{B})$ and $(\mathbf{A}' \ \mathbf{B}')$, respectively. We compare the columns of \mathbf{A}' and \mathbf{B}' to find (with relative ease) a basis $\beta' := \{\mathbf{c}'_{i_1}, \mathbf{c}'_{i_2}, \dots, \mathbf{c}'_{i_r}\}$ for the intersection of the column spaces of \mathbf{A}' and \mathbf{B}' . Then, $\beta := \{\mathbf{c}_{i_1}, \mathbf{c}_{i_2}, \dots, \mathbf{c}_{i_r}\}$ is a basis for $V \cap W$ (the intersection of the column spaces of \mathbf{A} and \mathbf{B}).

- **A fourth method** for when I learn about orthogonal complements.
- The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if $\mathbf{A} \in M_{n \times n}(\mathbb{F})$, then for any integer $1 \leq i \leq n$,

$$\det(\mathbf{A}) = \sum_{j=1}^n (-1)^{i+j} \mathbf{A}_{ij} \cdot \det(\tilde{\mathbf{A}}_{ij}).$$

Here, $\tilde{\mathbf{A}}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained from \mathbf{A} by deleting its i th row and j th column.

- The determinant of a square matrix can also be evaluated by cofactor expansion along any column, since

$$\det(\mathbf{A}) = \det(\mathbf{A}^\top).$$

- A matrix \mathbf{A} is invertible iff its determinant is nonzero.
- Let \mathbf{A} be an invertible $n \times n$ matrix. Then, for some elementary row matrices \mathbf{E}_1 to \mathbf{E}_p ,

$$\mathbf{E}_p \mathbf{E}_{p-1} \dots \mathbf{E}_1 (\mathbf{A} \mid \mathbf{I}_n) = \mathbf{A}^{-1} (\mathbf{A} \mid \mathbf{I}_n) = (\mathbf{I}_n \mid \mathbf{A}^{-1}).$$

In other words, we can perform Gaussian elimination, so that $(\mathbf{A} \mid \mathbf{I}_n) \rightarrow (\mathbf{I}_n \mid \mathbf{A}^{-1})$.

- Alternatively,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A}),$$

where $\text{adj}(\mathbf{A})$ is the adjugate / classical adjoint of \mathbf{A} . That is, the matrix whose (i, j) th entry is the (j, i) th cofactor $(-1)^{j+i} \det(\tilde{\mathbf{A}}_{ji})$

Numerical Methods

General Information

- The parity of the degree of a real polynomial is the same as that of its number of real roots.
- Let the real polynomial p given by $p(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \dots + a_0$ have coefficients $a_n > 0$ and $a_0 < 0$. Then, it has at least one positive and one negative root.
- Suppose we have some function $f: \mathbb{R} \rightarrow \mathbb{R}$ with a root α , whose value we want to approximate. There are three ways to obtain this approximation.

1. Linear interpolation on an interval $[a, b]$ containing α . We let $x_0 := b$ and

$$x_{i+1} := \frac{a|f(x_i)| + x_i|f(a)|}{|f(a)| + |f(x_i)|}.$$

– *Additional notes.*

2. Fixed-point Iteration. First select a function $F: \mathbb{R} \rightarrow \mathbb{R}$, such that $F(\alpha) = \alpha$, and choose some initial approximation x_0 to α . Then, we recursively define $x_{n+1} := F(x_n)$. The desired convergence behavior is for x_n to approach α .

– *Additional notes.*

G.C. Skills

Linear interpolation: finding an approximation to a root in $[a, b]$ up to n decimal places.

1. $Y_1 = f(x)$,
2. $a \rightarrow A$ and $b \rightarrow B$,
3. $\frac{B|Y_1(A)| + A|Y_1(B)|}{|Y_1(A)| + |Y_1(B)|}$,
4. $\text{Ans} \rightarrow A$ or B (choose the one that has the opposite sign to Ans),
5. Repeat steps 4 to 5,
6. Terminate this process when the approximations are consistent up to n decimal places.