A-Levels Math Notes

Grass

Contents

Inequalities and Equations

1.1 Solving Inequalities

1.1.1 Rational Inequalities

General Methods

1. Quadratic formula for factorisation / finding roots (of polynomial).
2. Completing the square.
3. Discriminant/Completing the Sq to eliminate* factors which are *always* positive or negative (e.g. removing $x^2 - 3x + 4$). *Note to include coefficient of x^2 in argument.*
4. GC (include sketch).
5. *Rational Functions*[a]: Move everything to one side (+,−), then use number line.
6. Number line (more complicated functions).

[a]Fractions of Polynomials

Important Notes

- ☐ Eliminating Factors — *only*[a] works for $c = 0$ in $f(x) \geq c$ or $f(x) \leq c$.
- ☐ Discriminant — include coefficient of x^2 in argument.
- ☐ When using factor elimination to remove some $f(x)$, we only need to say that "$f(x)$ is negative"[b].
- ☐ Rational functions — exclude the values that causes division by zero to occur.
- ☐ With inequalities, be really careful about multiplication! If $x > y$ *and* $z > 0$, then $xz > yz$.
- ☐ Cross multiplication preserves/reverses order for $\frac{x}{y} < \frac{x'}{y'}$ iff y and y' are *both* positive or negative.[c]
- ☐ Squaring preserves/reverses the order of $x < y$ iff x and y are *both* positive or negative.
- ☐ Can't necessarily use differentiation to solve if qns asks for algebraic method.
- ☐ Safer to graph out the two functions separately!
- ☐ Be careful about whether to include equality! Don't forget to account for it!

[a]Counterexample: $P(x) = x(3x^2 - 9x + 10) \leq 2$ iff $x \leq 2$ is false. E.g.: $P(1.8) = 6.336 \not\leq 2$.
[b]Source: Comment on Assignment A1
[c]Otherwise, note the counterexample $\frac{1}{2} < \frac{1}{-3}$.

1.2 Modulus Inequalities

Fact

Given $x \in \mathbb{R}$, we have that

- $|x| \geq 0$,
- $|x^2| = |x|^2 = x^2$,
- $\sqrt{x^2} = |x|$.

And as long as $x \in \mathbb{R}^+$,

- $\sqrt{x}^2 = |x|$.

Useful Properties

For every $x, k \in \mathbb{R}$:

(a) $|x| < k$ iff[a] $-k < x < k$.

(b) $|x| > k$ iff $x < -k$ or $x > k$.

(of course, similarly applies for the non-strict ordering $\leq$)

[a] Notice that $k > 0$ here since $0 \leq |x| < k$.

Important Notes

- Note that when solving for $|x| = y$, $|x| < y$, etc, y must be greater than or equal to 0. In other words, there may be solutions we will need to reject.
  (For $<$, equality is of course not allowed.)

Important Notes

△ Carelessness: Look at the question carefully! If they ask for a *set* of values, then rmb to give it as a *set*!

△ Exponentiation and Logarithms: Simply use ln and avoid $\log_c$ for $c < 1$.[a]

[a] Order is *Preserved* under exponentiation/logarithms if the base is *larger than* one. Otherwise, when it is *less than* one, the order is *reversed*. https://www.desmos.com/calculator/gd8z5fa0bg

1.3 System of Linear Equations

Things

χ For more complicated real-world-context qns, try playing around with the values (e.g. use simult eqns) first. It may work out nicer than expected.

1.4 Summary

G.C. Skills

1. Plotting curves $y = f(x)$ in G.C.
2. How to use simultaneous equation solver.

Important Notes

- ☐ Eliminating Factors — *only*[a] works for $c = 0$ in $f(x) \geq c$ or $f(x) \leq c$.
- ☐ Discriminant — include coefficient of x^2 in argument.
- ☐ When using factor elimination to remove some $f(x)$, we only need to say that "$f(x)$ is negative"[b].
- ☐ Rational functions — exclude the values that causes division by zero to occur.
- ☐ With inequalities, be really careful about multiplication! If $x > y$ *and* $z > 0$, then $xz > yz$.
- ☐ Cross multiplication preserves order for $\frac{x}{y} < \frac{x'}{y'}$ iff y and y' are *both* positive or negative.[c]
- ☐ Squaring preserves/reverses order for $x < y$ iff x and y are *both* positive or negative.
- ☐ Can't necessarily use differentiation to solve if qns asks for algebraic method.
- ☐ Safer to graph out the two functions separately!
- ☐ Be careful about whether to include equality! Don't forget to account for it!
- ☐ Note that when solving for $|x| = y$, $|x| < y$, etc, y must be greater than or equal to 0. In other words, there may be solutions we will need to reject.
  (For $<$, equality is of course not allowed.)
- ☐ Carelessness: Look at the question carefully! If they ask for a *set* of values, then rmb to give it as a *set*!
- ☐ Exponentiation and Logarithms: Simply use ln and avoid $\log_c$ for $c < 1$.[d]
- ☐ For more complicated real-world-context qns, try playing around with the values (e.g. use simult eqns) first. It may work out nicer than expected.

[a] Counterexample: $P(x) = x(3x^2 - 9x + 10) \leq 2$ iff $x \leq 2$ is false. E.g.: $P(1.8) = 6.336 \nleq 2$.
[b] Source: Comment on Assignment A1
[c] Otherwise, note the counterexample $\frac{1}{2} < \frac{1}{-3}$.
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Sequences and Series

2.1 Binomial Theorem and Series

Theorem 2.1.1: The Binomial Theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r,$$

where $n \in \mathbb{Z}^+$.

Theorem 2.1.2: The Binomial Series

$$(1+x)^p = \sum_{r=0}^{\infty} \binom{p}{r} x^r,$$

where $p \in \mathbb{Q}$, $|x| < 1$, *and*

$$\binom{p}{r} := \frac{p(p-1)\cdots(p-r+1)}{r!}.$$

Corollary 2.1.3

Clearly,

$$(a+x)^p = a^p \left(1 + \frac{x}{a}\right)^p = a^p \sum_{r=0}^{\infty} \binom{p}{r} \frac{x^r}{a^r},$$

under the same conditions.

Fact

We can expand $(a+x)^p$ in descending powers of x by using $(a+x)^p = x^p \left(1 + \frac{a}{x}\right)^p$.

Note

Sometimes computing a couple terms can be useful in finding a pattern (e.g. to get the coefficient of x^k explicitly).

2.2 APGP

Basics

	AP	GP
u_n	$u_n = S_n - S_{n-1}$	
	$u_n = a + (n-1)d$	$u_n = ar^{n-1}$
S_n	$S_n = \frac{n}{2}[2a + (n-1)d]$ $= \frac{n}{2}(a + \ell)$	$S_n = \frac{a(1-r^n)}{1-r}$ $= \frac{a(r^n-1)}{r-1}$
S_∞	Divergent	Converges to $S_\infty = \frac{a}{1-r}$ when $\lvert r\rvert < 1$.
Prove AP/GP	I Show $u_n - u_{n-1}$ is a constant/independent of n. II Show $u_n = a+(n-1)d$ explicitly.	I Show $\frac{u_n}{u_{n-1}}$ is constant/independent of n. II Show $u_n = ar^{n-1}$ explicitly
Mean	$u_{n+1} = \frac{u_n + u_{n+2}}{2}$. (Arithmetic Mean)	$\frac{u_{n+1}}{u_n} = \frac{u_{n+2}}{u_{n+1}}$ $u_{n+1}^2 = u_n \cdot u_{n+2}$ (Geometric Mean)

Important Notes

Applications: Write out a few terms in a table and observe the trend.[a]

[a] You can literally say "By observing a trend, . . . "

G.C. Skills

Table function[a]

1. Enter eqn into GC.
2. 2nd graph to show table
3. 2nd tblset for setup options

[a] E.g.: By G.C., $n \geq 182$.

2.3 Summation

Fact

$$\begin{aligned}
\sum_{i=m}^{n} f(i) + g(i) &= \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i) \\
\sum_{i=m}^{n} af(i) &= a \sum_{i=m}^{n} f(i) \\
\sum_{i=m}^{n} a &= (n-m+1)a, \text{ for any constant a} \\
\sum_{i=m}^{n} f(i) &= \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i)
\end{aligned}$$

Note

- Look out for sums being AP and GPs.
- Results to be provided:
  - $\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$
  - $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$

2.4 Method of Differences

General Info

$$\sum_{i=1}^{n} u_i = \sum_{r=1}^{n} f(r) - f(r-1) = f(n) - f(0).$$

- Explain convergence of a function $h(x) = f(x) + g(x)$: As $n \to \infty$, $f(x) \to 0$ and $g(x) \to 0$. Hence, $h(x)$ converges to...

G.C. Skills

Know how to generate sequences using the two methods:

1. Table method — Present by showing two consecutive values of n so that the values of the sequence are of opposite signs. E.g.:

n	S_n
182	$561.28 < 0$
183	$-1935.91 < 0$

2. 2nd stat seq (& we can use operations on seq, e.g. sum)

Recurrence Relations

Necessities

1. Recurrence relation is *homogenous* if constant (b below) is zero.
2. First order linear recurrence relation: $u_n = au_{n-1} + b$, with $a \neq 0$.
3. Second order *homogenous* linear recurrence relation: $u_n = a_1 u_{n-1} + a_2 u_{n-2}$, $a_2 \neq 0$.
4. Solving RRs in general:
   (a) Continually expand u_n in terms of u_{n-1}, then in terms of u_{n-2}, $\ldots$, till explicit formula is obtained (e.g. u_1).
   (b) Use a_1 to generate a_2, a_3, $\ldots$, a_n.
5. Solving 1st order RRs, $u_{n+1} = au_n + b$:
   (a) Iteration — Essentially technique 4(a). Will need to use G.P. formula at the end.
   (b) Rewriting RR + Using G.P. Formulas ((c) is better)
      i. Write RR as $u_n - k = a(u_{n-1} - k)$, where $k = \frac{b}{1-a}$. Let $v_n = u_n - k$.
      ii. $\frac{v_n}{v_{n-1}} = a$, a const. and $\{v_n\}$ is a G.P. with 1st term $v_1 - k$ and common ratio a.
      iii. So, $v_n = (u_1 - k)a^{n-1}$, and accordingly, $u_n = v_n + k = (u_1 - k)a^{n-1} + k$.
   (c) $\star$ Let $u_n = Aa^n + \frac{b}{1-a}$. Then solve for the constant A with info provided.
6. Solving 2nd order (homogenous) RRs, $u_{n+2} = au_{n+1} + bu_n$:
   Assume $u_n = m^n$, then $m^2 - am - b = 0$ (is the *characteristic/auxillary equation* of the RR). Solve for the roots, say m_1 and m_2. Then, the general solution for u_n is

$$u_n = \begin{cases} Am_1^n + Bm_2^n & \text{if } m_1, m_2 \in \mathbb{R}, \text{ and } m_1 \neq m_2, \\ (C + Dn)m_1^n & \text{if } m_1, m_2 \in \mathbb{R}, \text{ and } m_1 = m_2, \\ r^n[A\cos(n\theta) + B\sin(n\theta)] & \text{if } m_1, m_2 \in \mathbb{C},\ m_1 = re^{i\theta}, \text{ and } m_2 = re^{-i\theta}. \end{cases}$$

Induction

Let $P(x)$ be the statement that "$\ldots$".

When $n = 1$, $\ldots$

$\implies P(1)$ is true.

Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$.

Then, $\ldots$

$\implies P(k+1)$ is true.

Therefore, since $P(1)$ is true and $P(k)$ true $\implies P(k+1)$ true, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Differentiation

Definition

(i) A function f is called (strictly) increasing on an interval I iff $f'(x) > 0$ for all $x \in I$.

(ii) A function f is called monotonically increasing on an interval I iff $f'(x) \geq 0$ for any $x \in I$.

Things To Know

1. How to sketch the graph of the integral or[a] derivative of a function f.
2. Relationship btw. a function f and its derivative, f':

$y = f(x)$	$y = f'(x)$
Vertical asymptote at $x = a$	Vertical asymptote at $x = a$.
Horizontal asymptote at $y = a$	Horizontal asymptote $y = 0$.

3. Recap:

$f(x)$	$f'(x)$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$, $\lvert x\rvert < a$
$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2-x^2}}$, $\lvert x\rvert < a$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a+x^2}$, $x \in \mathbb{R}$
$\log_a(f(x))$	$\frac{1}{x\ln(a)}$
a^x	$a^x \ln(a)$

4. Implicit differentiation: $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$. ★ Makes life much easier (e.g. finding $f^{(n)}(x)$).
5. Parametric Differentiation: $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$.
6. Small angle approximation:
   (a) $\sin(x) \approx x$,
   (b) $\cos(x) \approx 1 - \frac{x^2}{2}$,
   (c) $\tan(x) \approx x$.
7. Maclaurin Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x$.

[a] Of course, provided that f is integrable/differentiable.

Integration Techniques

6.1 Basic Integration (IBS, IBP, etc)

1. Factor Formulae ★ (must *rmb*):
   (a) $\sin(mx)\cos(nx) = \frac{1}{2}[\sin((m+n)x) + sin((m-n)x)]$,
   (b) $\cos(mx)\cos(nx) = \frac{1}{2}[\cos((m+n)x) + cos(m-n)x]$,
   (c) $\sin(mx)\sin(nx) = -\frac{1}{2}[\cos((m+n)x) - \cos((m-n)x)]$.
2. Common classes of integrals:
   (a) Apply partial fractions:
   $$\int \frac{f(x)}{g(x)}\,dx.$$
   (b) Split $px+q$, then complete the square:
   $$\int \frac{px+1}{\sqrt{ax^2+bx+c}}\,dx \quad \text{or} \quad \int \frac{px+1}{ax^2+bx+c}\,dx$$
3. Integration by Substitution:
   $$\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du.$$
4. Use Pythagoras' Theorem / draw a right-angled triangle to help with trig conversions, e.g.: $\tan(\theta)$ to $\frac{x+1}{\sqrt{2-(x+1)^2}}$.
5. Integration by Parts:
   $$\text{Let } u = g(x),\ \frac{dv}{dx} = h(x), \quad \frac{du}{dx} = g'(x),\ v = \int h(x)\,dx.$$
   $$\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx.$$

6.2 Areas & Volumes

1. Volume of revolution when rotated about x-axis:

   (a) The disc method
   $$\int_{x_1}^{x_2} \pi y^2 \, dx = \int_{x=x_1}^{x=x_1} \pi y^2 \frac{dx}{dt} \, dt.$$

   (b) The shell method:
   $$\int_{x_1}^{x_2} 2\pi yx \, dy.$$

2. Arc length:
   $$\int_{x_1}^{x_2} \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \, dx = \int_{y_1}^{y_2} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy = \int_{t_1}^{t_2} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \, dt = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

3. Surface area of revolution when rotated about x-axis:
   $$\int_{x_1}^{x_2} 2\pi y \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \, dx = \int_{y_1}^{y_2} 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \, dx = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

   ★ Rotating about x-axis $\Longrightarrow$ y in integrand
   Rotating about y-axis $\Longrightarrow$ x in integrand.

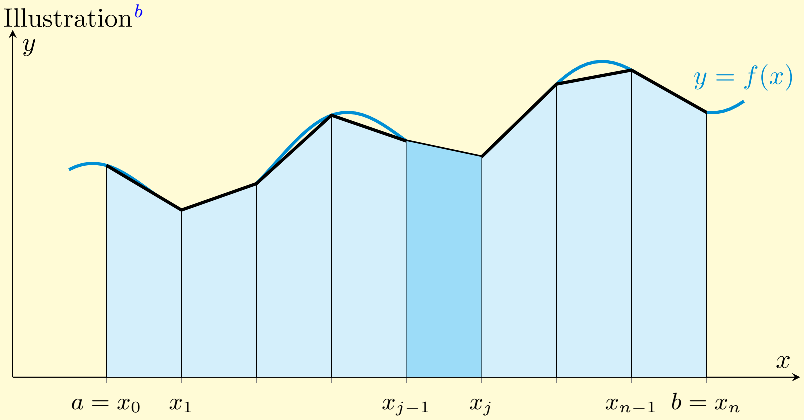
6.3 Numerical Methods

6.3.1 Trapezium Rule

1. Formula[a]:

$$\int_a^b y\,dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

2. Illustration[b]



3. Error:

   (a) Concave upwards, i.e. ($f'(x)$ is increasing / $f''(x) > 0$) $\implies$ overestimation.

   (b) Concave downwards, i.e. ($f'(x)$ is decreasing / $f''(x) < 0$) $\implies$ underestimation.

[a] For n intervals (i.e. (n+1)ordinates) of width $h := (b-a)/n$

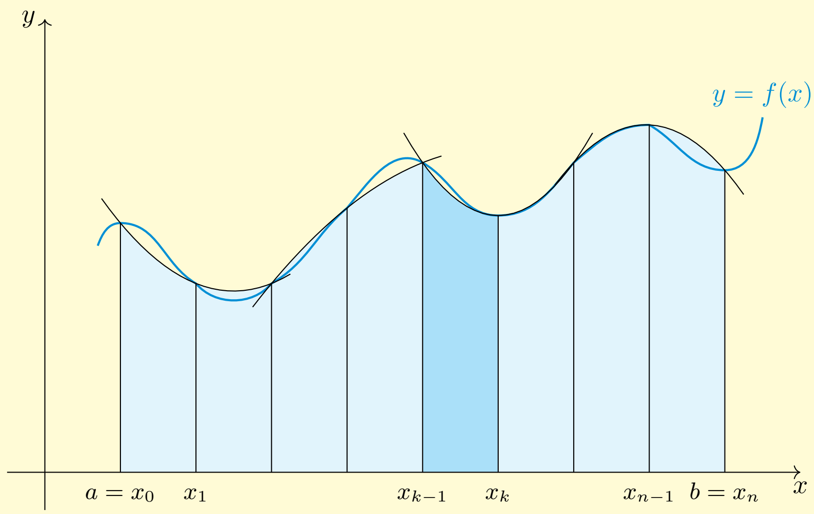
[b] Trapezium rule tikzpicture credit: https://tex.stackexchange.com/a/110618

6.3.2 Simpson's Rule

1. Formula[a][b]:

$$\int_a^b y\,dx = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

2. Illustration[c]



Accuracy of the Trapezium rule vs Simpson's Rule: "Simpson's Rule uses *quadratic curves* to interpolate the points on the curve so it usually *gives a better approximation* to the actual curve than the trapezium rule which uses *straight lines* to interpolate the ordinates."

[a]For n intervals (i.e. (n+1)ordinates) of width $h := (b-a)/n$

[b]Number of intervals n should be *even*, that of ordinates *odd*.

[c]Simpson's rule tikzpicture credit: `https://tex.stackexchange.com/a/439119`

Complex Numbers

7.1 Complex Number I

1. Find the square root of $x + iy$: Let $\sqrt{x+iy} = a + bi$. Then square both sides & solve.
2. Simplifying fractions: multiply by the denominator's conjugate
$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \cdots .$$
3. Polynomials:
   (a) Fundamental Theorem of Algebra: If $p(z) := \sum_{i=0}^{n} a_i z^i$ is a polynomial of degree $n \geq 1$ with complex coefficients, then there exists complex numbers c_i for each $1 \leq i \leq n$ such that
   $$p(z) = a_n \prod_{i=1}^{n} (z - c_i).$$
   (b) If a polynomial in real coefficients only has root $a + bi$, then $a - bi$ is another root.

Example 7.1

Find the roots of $iz^2 + 2z + 3i = 0$.

$$z^2 - 2iz + 3 = 0$$
$$z = \frac{2i \pm \sqrt{(2i)^2 - 4(1)(3)}}{2(1)} = i \pm \frac{\sqrt{-16}}{2} = i \pm 2i$$

So, $z = 3i$ or $z = -i$.

Example 7.2: N2010/2/1

One root of the equation $x^4 + 4x^3 + ax + b = 0$, where a and b are real, is $x = -2 + i$. Find the values of a and b and the other roots.

Substitute $-2 + i$ into the equation:

$$(-2+i)^4 + 4(-2+i)^3 + (-2+i)^2 + a(-2+i) + b = 0$$
$$-12 + 16i = 2a - b - ai$$
$$a = -16, \quad 2a - b = -12$$

Therefore, $a = -16$, $b = -20$.

Since all the coefficients of the polynomial are real (**explain**), $-2 - i$ is another root. Now, $x^4 + 4x^3 + ax + b = (x - (-2+i))(x - (-2-i))(cx + d)$ for some $c, d \in \mathbb{R}$.

Accordingly, substitute $x = 0$, then $x = 2$, and solve. Alternatively, notice $x^4 + 4x^3 + ax + b = (x^2 - 2(-2)x + ((-2)^2 + 1^2))(x^2 + cx + d) = (x^2 + 4x + 5)(x^2 + 4x + 5)$. Either ways, we have $c = 0$ and $d = -4$. As such, the last two roots are $x = -2 \pm i$ and $x = \pm 2$.

   (c) Simultaneous equations: Solve as usual.
   (d) Properties of modulus: $|z_1^x z_2^y| = |z_1|^x |z_2|^y$, for any $x, y \in \mathbb{R}$.

(e) Properties of arguments (same as log): $\arg(z) \in (-\pi, \pi]$ and $\arg(z_1^x z_2^y) = n \arg(z_1) + m \arg(z_2)$ for any $x, y \in \mathbb{R}$.

(f) Polar form: $z = re^{i\theta}$.

(g) Polar/Trigonometric form: $z = r[\cos(\theta) + i \sin(\theta)]$.

Note

Show that the value of w^n is either 2^n or 2^{-n} for integers n.

Then we **must** show that $w^n = \cdots = \begin{cases} 2^n(1) = 2^n & \text{if } n \text{ even,} \\ 2^n(-1) = -2^n & \text{if } n \text{ odd.} \end{cases}$

7.2 Complex Numbers II

Theorem 7.2.1: De Moivre's Theorem

Let z be a complex number, n an integer, and θ an angle. Suppose $z = re^{i\theta}$. Then,

$$z^n = e^{i\theta} = r^n[\cos(n\theta) + i \sin n\theta].$$

1. All nth roots of any complex number are the same distance r from the origin and have the same angular separation, π/n.

Graphing Techniques

8.1 Graphing 'Familiar' Functions and Asymptotic bois

Definition

1. **Lines of Symmetry**: A *line of symmetry* of a function is a line, such that the function is a reflection of itself about that line.
2. **Horizontal Asymptotes**: A (horizontal) line $g(x) = c$ is the *horizontal asymptote* of the curve $f(x)$ iff $\lim_{x\to\infty} f(x) = c$ (or with $-\infty$ instead of ∞).[a]
3. **Vertical Asymptotes**: A (vertical) line $x = c$ is a *vertical asymptote* of the curve $f(x)$ iff $\lim_{x\to c} f(x) = \infty$ or $-\infty$.
4. **Oblique Asymptotes**: A line $g(x) = mx + c$ — where $m \neq 0$ — is an *oblique asymptote* of the curve $f(x)$ iff $\lim_{x\to\infty}[f(x) - g(x)] = 0$ (or with $-\infty$ instead of ∞).

[a] Otherwise notated by $f(x) \to c$ as $x \to \infty$.

Curve Sketching (Rational Funcs)

S Stationary points

I Intersection with axes

A Asymptotes

i Know how to sketch the graphs of $y = \frac{ax+b}{cx+d}$ and $y = \frac{ax^2+bx+c}{dx+e}$.

ii Rectangular Hyperbolas $\left(\text{of the form } y = \frac{ax+b}{cx+d}\right)$:

- *Two* asymptotes, namely $x = -\frac{d}{c}$ and $y = \frac{a}{c}$.
- *Two* lines of symmetry with gradients ± 1 *and* pass through the intersection point of the aforementioned two asymptotes.

iii *If* $n = \deg P = \deg Q$, then

- $y = R(x)$ is the *horizontal* asymptote of $\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}$.
- Equivalently, $y = \frac{\text{coeff}_P(x^n)}{\text{coeff}_Q(x^n)}$ is a *horizontal* asymptote.[a]

iv If $\deg P = \deg Q + 1$, then $R(x)$ is an *oblique* asymptote of $\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}$.

v Write down asymptotes and lines of symmetry.[b] If none are present indicate with "No lines of symmetry."

[a] E.g.: $y = \frac{1}{15}$ is a horizontal asymptote of $y = \frac{1x^2+2x-3}{(5x+1)(3x+2)}$.

[b] E.g.:

Asymptotes: $x = 4$, $y = 20$.

Lines of Symmetry: $y = x + 16$, $y = -x + 24$.

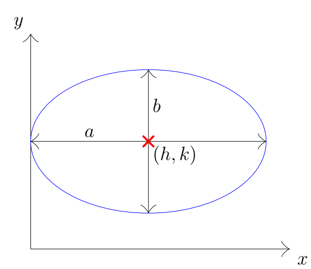
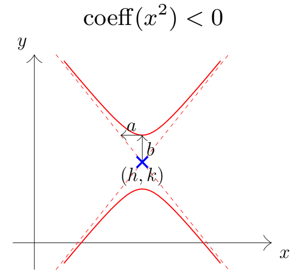
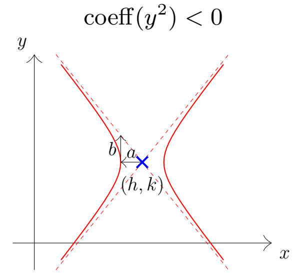
Important Notes

- — *Can* explicitly write out the asymptotes and lines of symmetry (or if they are not present) to be safe.
- — Using the discriminant intelligently can result in nice answers.
- — Know how to use the G.C. Transfrm app.[a][b]

[a] Thingy that allows you to vary the value of some parameter A for a function $f(Ax)$.
[b] E.g.: Solve for the values of k being a positive *integer*. We can use the app to visually see where the curves intersect.

8.2 Conics

"Tikz is pain, PGFPlots is suffering" — Wise Man.

	Ellipses	Hyperbolas
Standard Forms	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
General Equation	$ax^2 + by^2 + cx^2 + dx + e = 0$, where $\operatorname{sgn}(a) = \operatorname{sgn} b$.	$ax^2 + by^2 + cx^2 + dex + e = 0$, where $\operatorname{sgn}(a) \neq \operatorname{sgn} b$.
Center	(h, k)	
Vertical 'Radius' (variables here from *standard form*!)	b	
Horizontal 'Radius' (variables here from *standard form*!)	a	
Vertical Vertices (variables here from *standard form*!)	$(h, k \pm b)$	
Horizontal Vertices (variables here from *standard form*!)	$(h \pm a, k)$	
Shape		coeff$(x^2) < 0$  coeff$(y^2) < 0$ 
Asymptotes (No need to rmb!)	-	$y = k \pm \frac{b(x-h)}{a}$
Lines of Symmetry	$x = h$, $y = k$	

General Info

- To find asymptote of hyperbolas, just solve

$$\frac{(x-h)^2}{a^2} = \frac{(y-k)^2}{b^2}.$$

- Label vertices *or* radii, together with the center and asymptotes.

8.3 Parametric Equations

Important Notes

- Check the qns for any *restrictions* on the parameter! And modify that of the G.C.'s accordingly (Tmin & Tmax).
- Vary the $t - step$ or resolution (of cartesian form) as necessary (when the graph is oddly jagged)0.
- Think carefully for tricker qns[a].

[a] E.g.: Simult eqns can be useful in converting frm parametric to cartesian form.

8.4 Summary

G.C. Skills

1. Plot conics with the two ways.
2. Know G.C. functions like finding axial intercepts.

Important Notes

-☐- *Can* explicitly write out the asymptotes and lines of symmetry (or if they are not present) to be safe.

-☐- Using the discriminant intelligently can result in nice answers.

-☐- Know how to use the G.C. Transfrm app.[a b]

-☐- Check the qns for any *restrictions* on the parameter! And modify that of the G.C.'s accordingly (Tmin & Tmax).

-☐- Vary the $t - step$ or resolution (of cartesian form) as necessary (when the graph is oddly jagged)0.

-☐- Think carefully for tricker qns[c].

[a] Thingy that allows you to vary the value of some parameter A for a function $f(Ax)$.
[b] E.g.: Solve for the values of k being a positive *integer*. We can use the app to visually see where the curves intersect.
[c] E.g.: Simult eqns can be useful in converting frm parametric to cartesian form.

8.5 Scaling, Translations, and Reflections

Playing With x		
Function	x is replaced with	(Horizontal) Transformation
$f(x+a)$	$x+a$	Translate a units in the positive ($a \leq 1$) O/R negative x-direction ($a \geq 1$).
$f(-x)$	$-x$	Reflect about the y-axis
$f(ax)$	ax	Scale parallel to the x-axis by a scale factor of $\frac{1}{a}$ if[1] $a \geq 1$.
Playing With $f(x)$		
Function / Change to $f(x)$		(Vertical) Transformation
$f(x)+a$		Translate a units in the positive ($a \geq 1$) O/R negative y-direction ($a \leq 1$).
$-f(x)$		Reflect about the x-axis.
$af(x)$		Scale parallel to the y-axis by scale factor a.

G.C. Skills

Transfrm app (allows you to vary a parameter of a function[a])

[a] E.g.: The variable A in $y = Ax + b$

Important Notes

Transform x

Translation ⇨ Scaling / Reflection

↓

Transform y

Scaling / Reflection ⇨ Translation

8.6 $|f(x)|$ and $f(|x|)$

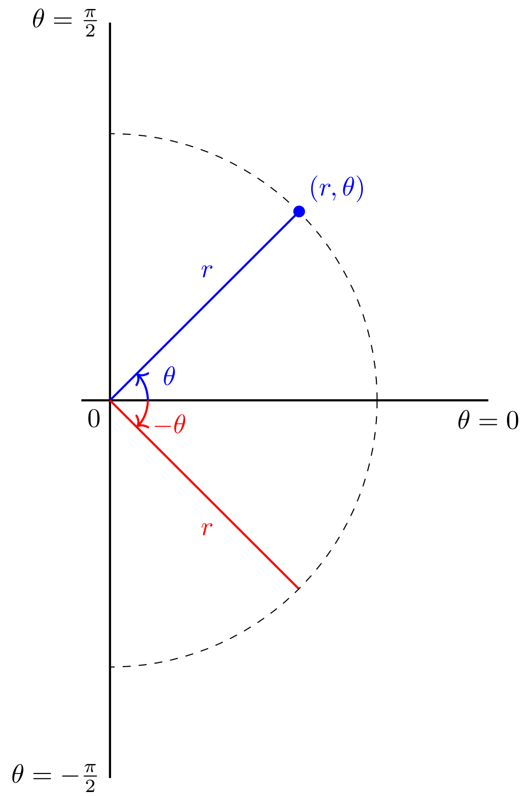
Basics

➻ $f(|x|)$ ➢ Graph of 'negative side' is a reflection of the 'positive side' (across the y-axis).

8.7 $y = \frac{1}{f(x)}$

Conditions	Results
$f(x) > 0$	$\frac{1}{f(x)} > 0$
$f(x) < 0$	$\frac{1}{f(x)} < 0$
Vertical Asymptote at $x = c$	$\frac{1}{f(x)}$ *tends* to 0 * $\frac{1}{f(x)}$ is undefined at $x = c$
$\frac{df}{dx} = -\frac{d}{dx}\left(\frac{1}{f(x)}\right)$ i.e. when $f(x)$ increases, $\frac{1}{f(x)}$ decreases.	
(a, b) is a *minimum* pt	$\left(a, \frac{1}{b}\right)$ is a *maximum* pt
(a, b) is a *maximum* pt	$\left(a, \frac{1}{b}\right)$ is a *minimum* pt

Polar Curves



Definition

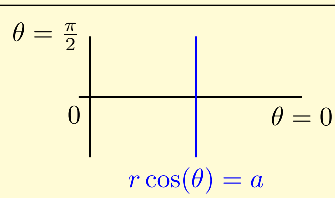
1. The *pole* is the origin, i.e. the point 0.
2. The *initial line / polar axis* is the *half line* $\theta = 0$.

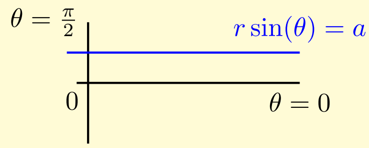
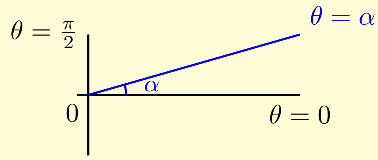
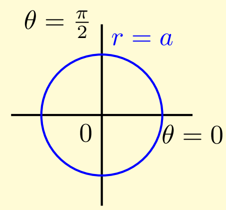
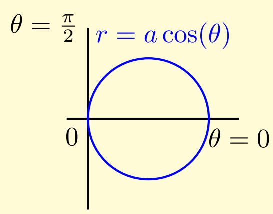
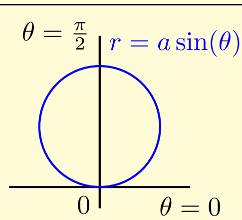
General Info

- Coordinate Conversion

$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$	$x = r\cos(\theta)$ $y = r\sin(\theta)$

- Standard Functions

Polar Equation	Cartesian Equation
	$x = a$

	$y = a$
	$y = x\tan(\alpha)$
	$x^2 + y^2 = a^2$
	$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$
	$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$

- Tangent lines at the pole are obtained by solving $r = 0$.
- Know how to find range of r and θ (given a func/eqn).
- $r = f(\theta)$ is symmetrical about the polar (horizontal) axis iff $f(\theta) = f(-\theta)$.
  - △ r is a function of $\cos(n\theta)$[a] *only* $\implies$ lines of symmetry: $n\theta = 0, \pi, 2\pi, \ldots, m\pi, \ldots$
- $r = f(\theta)$ is symmetrical about the vertical line $\theta = \frac{\pi}{2}$ iff the equation $f(\theta) = f(\pi - \theta)$.
  - △ r is a function of $\sin(n\theta)$ *only* $\implies$ lines of symmetry: $n\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, \frac{(2m+1)\pi}{2}, \ldots$
- $r = f(\theta)$ is symmetrical about the pole iff[b] (r, θ) is a point on the curve whenever $(-r, \theta)$ is.
- R-formula may be necessary
- Area of a sector: $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta$, where $\alpha < \beta$.
- Arc length$= \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, \mathrm{d}\theta$

---

[a] E.g.: $r = a\sqrt{(4 + \sin^2(\theta))^2 \cos(\theta)}$

[b] In other words, $r = f(\theta)$ is unchanged when r is replaced by $-r$.

Important Notes

1. r is *always*[a] ≥ 0 in our syllabus.
2. No need to fully expand; a final answer such as $(x^2 + y^2)^2 = 3y(x^2 + y^2) - 4y^2$ suffices.
3. Polar curve sketching essentials:
   (a) Shape of curve
   (b) Intersection(s) with ('axial') half lines
   (c) Nothing else *unless* the qns asks for it

   - ☐ Be careful of the sharpness / smoothness of points! Points supposed to be sharp should be sufficiently so, and those supposed to be smooth should be sufficiently rounded / not look sharp.
   - ☐ Check whether the polar axes are tangents at pole. Use this information to draw accurate graphs.
   - ☐ Best to add a small dotted line to show tangentiality at intercepts.
   - ☐ Careful about constants like a in $r = a\sin(\theta)$ for axial intercepts.
   - ☐ No need to state points at the pole unless they are 'axial', i.e. $\theta = 0$, or $\frac{\pi}{2}$, etc.
4. When finding maximum / minimum y values $\left(\frac{dy}{d\theta} = 0\right)$, we need to check its nature (1st/2nd deriv test). However, this is unnecessary for max / min r values.
5. For stuff like $\frac{dy}{dx}$, try to keep it in polar form if possible instead of converting to cartesian form.
6. As usual, be *careful*! E.g. Which values need to be rejected.
7. When reflecting/rotating, a diagram may be useful in finding the necessary angle/expression to replace θ with. E.g.:
   (a) In the case of reflecting about $r = \theta$ or $y = x$, $(r, \theta) \to (r, \frac{\pi}{2} - \theta)$.
   (b) Reflect about the half-line $\theta = \frac{\pi}{2} \implies (r, \theta) \to (r, \pi - \theta)$.

[a] in some situations it can be negative

G.C. Skills

1. Nice polar $\implies$ Zoom fit + Zoom square
2. Simply press alpha trace 1 to get r_1. In fact, this works for the other modes available in the GC as well.
3. We can type $\frac{d}{d\theta} r_1\big|_{\theta=\theta}$ info formulas (like the one for arc length) without having to manually differentiate it!

Conic Sections

Essentials

- Eccentricity, $e := \frac{\text{distance of } P \text{ from focus}}{\text{distance of } P \text{ from directrix}}$.
  - $e = 0$: Circle
  - $0 < e < 1$: Ellipse
  - $e = 1$: Parabola
  - $e > 1$: Hyperbola

Conic	Parabolas		Ellipses		Hyperbolas	
Equation	$x^2 = 4py$	$y^2 = 4px$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ b > a$	$\frac{x^2}{a^2} - \frac{y^2}{b^2}$	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
Foci	$(0, p)$	$(p, 0)$	$(\pm c, 0)$	$(0, \pm c)$	$(\pm c, 0)$	$(0, \pm c)$
a, b, c	N.A.		$c^2 = a^2 - b^2$	$c^2 = b^2 - a^2$	$c^2 = a^2 + b^2$	
Directrices	$y = -p$	$x = -p$	$x = \pm\frac{a}{e} = \pm\frac{a^2}{c}$	$y = \pm\frac{b}{e} = \pm\frac{b^2}{c}$	$x = \pm\frac{a}{e} = \pm\frac{a^2}{c}$	$y = \pm\frac{b}{e} = \pm\frac{b^2}{c}$
e	$e = 1$		$0 < e < 1$		$e > 1$	
	N.A.		$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \frac{c}{b} = \frac{\sqrt{b^2 - a^2}}{b} = \sqrt{1 - \frac{a^2}{b^2}}$	$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{1 + \frac{b^2}{a^2}}$	$e = \frac{c}{b} = \frac{\sqrt{a^2 + b^2}}{b} = \sqrt{1 + \frac{a^2}{b^2}}$
Reflective Property	When light parallel to its axis of symmetry ($x = 0$ or $y = 0$) hits its concave side, the light is reflected to the focus.		For any point P on the ellipse with $a > b$, $PF_1 + PF_2 = 2a$		For any point P on the hyperbola with coeff$(x^2) > 0$, $\lvert PF_1 - PF_2 \rvert = 2a$	

- Polar Form: $x = p$, $x = -p$, $y = p$, or $y = -p$ being the directrix

	Top $r = \frac{ep}{1 + e\sin(\theta)}$	
Left $r = \frac{ep}{1 - e\cos(\theta)}$		Right $r = \frac{ep}{1 + e\cos(\theta)}$
	Bottom $r = \frac{ep}{1 - e\sin(\theta)}$	

Definition

- Major / minor axes $\implies$ lengths of longest and shortest diameters respectively.
- Semi-major / semi-minor $\implies$ half of major / minor axes respectively.
- Focal radius $\implies$ distance from point on conic section to focus.

- Examples:
  - Using the fact that $PF_1 + PF_2 = 2a$ to do simultaneous equations.
  - Converting to polar form (when $e < 1$ so $r \geq 0$) for distances.
  - Congruent/Similar triangles.
  - Classic use of discriminants.
  - Sum and product of roots: Given any polynomial $ax^2 + bx + c$ with the roots α and β,

$$\alpha + \beta = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta = \frac{c}{a}.$$

Functions

1. Horizontal Line Test:
   (a) Fail: Since[a] $y = k$ intersects the graph of $y = f(x)$ more than once, therefore f is not injective.
   (b) Success: Since *any* horizontal line $y = k$ will intersect the graph of $y = g(x)$ *at most once*, so $f(x)$ is one-one.
2. The inverse function, f^{-1}, of a function f exists iff f is one-one.
3. $y = f^{-1}$ is a reflection of $y = f(x)$ about the line $y = x$.
4. The composite function gf exists iff $R_f \subseteq D_g$.
5. $D_{gf} = D_f$ & $R_{gf} = R_g$.
6. Finding the range:
   (a) Graphing method:
   (b) Mapping method, e.g.: $D_f = (0, \infty) \xrightarrow{f} (1, \infty) \xrightarrow{g} (0, \infty) = R_{gf}$

[a] some specific k, e.g. $y = 1/2$

Permutations and Combinations

General Necessities

- Addition and multiplication principles
- ${}^nP_k := \frac{n!}{(n-k)!}$
- Know how to 'bundle' objects together so as to calculate the total no. of permutations.
- There are $\frac{n!}{n_1!n_2!\cdots n_r!}$ number of ways to arrange n objects, of which n_1 are 'similar', n_2 are 'alike', …, n_r are 'the same'.[a]
- Case-wise considerations/calculations (then summing tgt the total no of perms)
- Unordered circular permutations:
  There are $\frac{n!}{n} = (n-1)!$ number of ways of arranging n distinct objects in a circle.[b]
- Complementary Method, i.e. taking no. of arr w/o restriction - no. of arr w/ the opposite of that restriction.[c]
- Insertion Method, place down some of your objects and then insert the rest in the gaps.[d]
- Ordered circular permutations: Add the ordering at the end. Note:
  1. The number of ways is not necessarily just $(n-1)! \cdot n = n!$.
  2. Circular arrangements are not the same as row arrangements.[e]
- $\binom{n}{r} = {}^nC_r := \frac{n!}{(n-r)!r!}$.

[a] Intuition: If there are n_1 objects are non-distinct out of n objects, then there are $n_1!$ ways to arrange these objects that results in 'the same' permutation.

[b] For unordered CPs, we do not care if you rotate the seating arrangement, as long as neighbours are preserved for each object. i.e. $(A,B,C,D) \sim (B,C,D,A)$. As a result, each such collection of n permutations reduces down to one. Thus, explaining the division by n.

[c] E.g. No. of ways 2 girls *cnnt* sit nxt to each other = no. of arr w/o restriction - no. of arr with girls sitting *tgt*.

[d] Boys sit at table first: 2! ways.
From the 3 gaps, choose 2 for the 2 girls to sit at: 3 ways.
The girls can arrange themselves in 2! ways.
So, total no. of ways is $2! \cdot 3 \cdot 2! = 12$.

[e] While A and B are not considered to be seating together in the row arrangement of (A,C,D,E,B), they are seating together in a corresponding row arrangement. Does not always mean no. of row arr < no. of circ arr. It can be <, =, >.

Vectors

Lines	Planes
Equivalent Forms	
1. Vector Equation: $$\ell\colon \mathbf{r} = \mathbf{a} + \lambda\mathbf{m},\ \lambda \in \mathbb{R},$$ 2. Cartesian Equation: $$\frac{x-a_1}{m} = \frac{y-a_2}{m_2} = \frac{z-a_3}{m_3}.$$	1. Vector Equation: $$\Pi\colon \mathbf{r} = \mathbf{a} + \lambda\mathbf{m_1} + \mu\mathbf{m_2} \text{ where } \lambda, \mu \in \mathbf{R},$$ 2. Scalar Product Form: $$\Pi\colon \mathbf{r}\cdot\mathbf{n} = p$$ where the scalar $p := \mathbf{a}\cdot\mathbf{n}$, 3. Cartesian Equation: $$n_1 x + n_2 x + n_3 z = p$$ where the normal vector $\mathbf{n} := \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^t$.
Foot of Perpendicular	
M1: (a) $\overrightarrow{ON} = \mathbf{a} + \lambda\mathbf{m}$, (b) $\overrightarrow{QN}\cdot m = 0$, solve for λ, (c) Substitute λ into (a). M2: (a) $\overrightarrow{AN} = \left(\overrightarrow{AQ}\cdot\hat{\mathbf{m}}\right)\hat{\mathbf{m}}$, (b) $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$.	(a) $\ell_{NQ}\colon \mathbf{r} = \overrightarrow{OQ} + \lambda n$, where $\lambda \in \mathbb{R}$, and $\Pi\colon \mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$, (b) $\left(\overrightarrow{OQ} + \lambda\mathbf{n}\right)\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$, solve for λ, (c) $\overrightarrow{ON} + \overrightarrow{OQ} + \lambda\mathbf{n}$.
Shortest Disance of Point To Line, QN	
M1: $\left\Vert\overrightarrow{AQ}\times\hat{\mathbf{m}}\right\Vert$. M2: (a) $AN = \left\Vert\overrightarrow{AQ}\cdot\hat{\mathbf{m}}\right\Vert$, (b) Pythagoras' Theorem. M3: Using the foot of perpendicular, find distance QN.	M1: $\left\Vert\overrightarrow{AQ}\cdot\hat{\mathbf{n}}\right\Vert$. M2: for distance of plane to *origin*: If $\Pi\colon \mathbf{r}\cdot\mathbf{n} = p$, $\frac{p}{\Vert\mathbf{n}\Vert}$ is the shortest distance from the origin to the plane Π. *Note:* • If $\frac{p}{\Vert\mathbf{n}\Vert} > 0$, then Π 'above' 0. • If $\frac{p}{\Vert\mathbf{n}\Vert} < 0$, then Π 'below' 0. M3: Using the foot of perpendicular, then find distance QN.

Relationship Btw 2 Lines	Relationship Btw Line & Plane
1. Parallel, Non-Intersecting (a) $\mathbf{m_1} \mathbin{/\mkern-5mu/} \mathbf{m_2}$, (b) Solving $\ell_1 = \ell_2$ gives no real solution. 2. Parallel, Coinciding (a) $\mathbf{m_1} \mathbin{/\mkern-5mu/} \mathbf{m_2}$, (b) $\mathbf{a}$ lies in ℓ_1 and ℓ_2. 3. Non-Parallel, Intersecting (a) $\mathbf{m_1}$ not $\mathbin{/\mkern-5mu/}$ $\mathbf{m_2}$, (b) Solve $\ell_1 = \ell_2$ to find intersection. 4. Skew Lines (Non-Parallel, Non-Intersecting) (a) $\mathbf{m_1}$ not $\mathbin{/\mkern-5mu/}$ $\mathbf{m_2}$, (b) Solving $\ell_1 = \ell_2$ gives no real solution.	1. ℓ lies in Π M1: i. $\mathbf{m} \cdot \mathbf{n} = 0$ says $\ell \mathbin{/\mkern-5mu/} \Pi$, ii. Combined with $\mathbf{a} \cdot \mathbf{n} = p$, we conclude ℓ lies in Π. M2: Substitute ℓ into Π and show the system (of lin eqns) is consistent for all λ. 2. $\ell \mathbin{/\mkern-5mu/} \Pi$ but Nonintersecting M1: i. Show $\mathbf{m} \cdot \mathbf{n} = 0$, so $\ell \mathbin{/\mkern-5mu/} \Pi$. ii. Then $\mathbf{a} \cdot \mathbf{n} \neq p$, tells us ℓ and Π are nonintersecting. M2: Substitute ℓ into Π, and show the system (of lin eqns) is inconsistent. 3. Intersect at 1 point M1: $\mathbf{m} \cdot \mathbf{n} \neq 0$. To find point of intersection: For the plane Π and ℓ defined by $\begin{array}{c}\Pi\colon \mathbf{r} \cdot \mathbf{n} = p \\ \ell\colon \mathbf{r} = \mathbf{a} + \lambda \mathbf{m}\end{array}$ Solve for λ using simultaneous equations or G.C.
-	Relationship Btw 2 Planes
	1. Parallel Planes: Show there exists an $\mathbf{a}$ for which (a) $\mathbf{a} \cdot \mathbf{n_1} = p_1$, (b) $\mathbf{a} \cdot \mathbf{n_2} \neq p_2$. 2. Same Plane: Show there exists an $\mathbf{a}$ for which (a) $\mathbf{a} \cdot \mathbf{n_1} = p_1$, (b) $\mathbf{a} \cdot \mathbf{n_2} = p_2$. 3. Intersect in a line ℓ; To find this line: M1: $\mathbf{n_1} \times \mathbf{n_2}$ gives the direction vector. So find a common point with simultaneous equations. M2: Solving system of linear equations, from the *cartesian* form of the planes, using G.C.
Point of Reflection	
1. Find foot of perpendicular, 2. Notice $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AN} = 2\overrightarrow{ON} - \overrightarrow{OA}$.	1. Find the position vector $\overrightarrow{ON}$, 2. Notice $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AN} = 2\overrightarrow{ON} - \overrightarrow{OA}$.

Angle Between		
2 Lines	Line and Plane	2 Planes
$\theta = \cos^{-1}\left\lVert \frac{\mathbf{m_1} \cdot \mathbf{m_2}}{\lVert \mathbf{m_1} \rVert \lVert \mathbf{m_2} \rVert \rVert} \right\rVert.$	$\theta = \sin^{-1}\left\lVert \frac{\mathbf{m} \cdot \mathbf{n}}{\lVert \mathbf{m} \rVert \lVert \mathbf{n} \rVert \rVert} \right\rVert.$	$\theta = \cos^{-1}\left\lVert \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{\lVert \mathbf{n_1} \rVert \lVert \mathbf{n_2} \rVert \rVert} \right\rVert.$

Probability

1. Principle of Inclusion and Exclusion:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

2. Mutually Exclusive Events:

$$\begin{aligned} P(A \cap B) &= 0, \\ P(A \cup B) &= P(A) + P(B). \end{aligned}$$

3. Independent Events:

$$\begin{aligned} P(A \mid B) &= P(A), \\ P(A \cap B) &= P(A)\,P(B). \end{aligned}$$

4. Conditional Probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

5. Use PnC to help compute stuff faster

Example 14.1

There are 6 white balls and 5 black balls. Two are randomly drawn. What is the probability that one is white and the other black?

$$\left(\frac{5}{11}\right)\left(\frac{6}{10}\right) + \left(\frac{6}{11}\right)\left(\frac{5}{10}\right) = \frac{6}{11} \qquad \text{vs} \qquad \frac{\binom{6}{1}\binom{5}{1}}{\binom{11}{2}} = \frac{6}{11}.$$

Differential Equations

15.1 First Order D.E.s

15.1.1 Elementary Solving Techniques

1. Separable Variables:

$$\frac{dy}{dx} = f(y)g(x), \int \frac{1}{f(y)}\, dy = \int g(x)\, dx.$$

2. Integrating Factor:

$$\begin{aligned} \frac{dy}{dx} + P(x)y = Q(x), \quad \text{let I.F.} = e^{\int P(x)\, dx} \\ e^{\int P(x)\, dx}\frac{dy}{dx} + ye^{\int P(x)\, dx}P(x) = Q(x)e^{\int P(x)\, dx}, \\ ye^{\int P(x)\, dx} = \int Q(x)e^{\int P(x)\, dx}\, dx. \end{aligned}$$

15.1.2 Numerical Methods

1. Euler's Method:

$$y_{i+1} + hf(x_i, y_i).$$

Example 15.1

Let (step size) $h = 0.25$ and $f(x, y) = \frac{dy}{dx}$:

$$\begin{aligned} \text{By MF26,} \quad y_2 &= \frac{2}{3} + hf\left(0, \frac{2}{3}\right) \\ &= \frac{13}{18} \\ y_3 &= \frac{13}{18} + hf\left(0.25, \frac{13}{18}\right) \\ &= 0.6701865657. \end{aligned}$$

Therefore, $y(0.5) \approx 0.670$.

2. Improved Euler's Method:

$$i_{i+1} = y_i + hf(x_i, y_i) \quad \& \quad y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, u_{i+1})].$$

3. Error:

   (a) If $\frac{dy}{dx}$ can be shown to be *increasing* from the calculations of $f(x, y)$, then the curve is *concave upwards*, leading to a *underestimate*.

   (b) If $\frac{dy}{dx}$ can be shown to be *decreasing* from the calculations of $f(x, y)$, then the curve is *concave downwards*, leading to a *overestimate*.

Example 15.2

From the computation, *the values of* $\frac{dy}{dx}$ *increases*, i.e. $\frac{d^2y}{dx^2} > 0$, and thus implying the solution curve to be *concave upwards*. Therefore, we have an *underestimation*.

Example 15.3: Misc

It is suggested that the estimation in part (ii)[a] can be further improved by reducing the step size. Sketch the solution curve and hence comment on this suggestion.

The solution curve has a *stationary point at* $x = 1.47$, which is between 1 and 2 and also the gradient of the curve is close to zero for x value beyond this stationary point. Thus, when the step size is reduced, *tangent* at point close to this stationary point becomes *almost parallel* to the curve, making *little improvement* to the estimation due to *little difference in* y.

[a] Given the point (1,1), we estimated the value of y(2) using the Improved Euler's Method

Example 15.4

It is found that the approximation obtained in (i) for the y-coordinate where $x = 0.75$ is an underestimation and has a percentage error of 120.633%. Explain why there is such a substantial error.

From gradient values calculated above, we suspect sharp charges in gradient values within the interval (from negative to positive). Yet *Euler's Method*[a] *simply uses a straight line segment* with gradient[b] -4.6409 to estimate the curve for the first iteration, which could have lead to a significant underestimation of the y-value.

[a] We are explaining what it does
[b] Emphasising negative gradient (Show its value)

Example 15.5

Compare the relative merits of Euler's Method and the Improved Euler's Method.

Euler's Method: It is computationally simpler.

Improved Euler's Method More accurate as it takes the mean of the initial and next gradient.

15.2 Second Order D.E.

Homogenous	
Roots	Solution y_c
$m_1 \neq m_2$	$y = Ae^{m_1 x} + Be^{m_2 x}$
$m := m_1 = m_2$	$y = (Ax + B)e^{mx}$
$m = p \pm qi$	$y = e^{px}(A\cos(qx) + B\sin(qx))$
Non-Homogenous, $c_2\frac{d^2y}{dx^2} + c_1\frac{dy}{dx} + c_0 y = f(x)$	
$y = y_c + y_p$ (C.F. + P.I.)	
$f(x)$	Trial Function for P.I.
Degree n polynomial	$y_p = \sum_{i=0}^{n} a_i x^i$
ke^{ax}	$y_p = ae^{ax}$.
$\alpha\cos(kx) + \beta\sin(kx)$	$y_p = a\cos(kx) + b\sin(kx)$

Note

If y_c and $f(x)$ share some common term, then y_p should be multiplied by x (some least $i \in \mathbb{N}$ times till $x^i y_p$ has no common term with y_c).

Example 15.6

1. If $y_c = A^{-3x}$ and $f(x) = 10e^x$, then $y_p = kxe^x$
2. If $y_c = Ae^x + Be^{-3x}$ and $f(x) = 10e^x$, then $y_p = kxe^x$.
3. If $y_c = Ae^x + Bxe^x + Ce^{-3x}$ and $f(x) = 10e^x$, then $y_p = kx^2e^x$.

15.3 Applications

15.3.1 Exponential Growth

Let k be the *per-capita growth rate*[a] and $P(t)$ be the population at time t. Then we have the model:

$$\frac{dP}{dt} = kP,$$

with the solution

$$P(t) = P_0 e^{kt}.$$

[a] i.e. after accounting for births and deaths.

15.3.2 Logistics Growth

Let k be the *per-capita growth rate*[a], $P(t)$ be the population at time t, and N be the *carrying capacity* of the system. Then we have the model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right).$$

1. Without solving the logistics equation, we can sketch the solution curve by noting the sign of $\frac{dP}{dt}$:
   (a) Equilibrium population values occur at $P = 0$ and $P = N$.
   (b) If, for instance $k > 0$,
   $0 < p < N$: $1 - \frac{P}{N} > 0$ so $\frac{dP}{dt} > 0$,
   $P > N$: $1 - \frac{P}{N} < 0$ so $\frac{dP}{dt} < 0$.

   "As t increases, the population of ______ increases to the stable population of ______."

[a] i.e. after accounting for births and deaths.

Example 15.7: Neat trick of letting $A = \pm$constant

$$
\begin{aligned}
\frac{dP}{dt} &= 3P\left(1 - \frac{P}{200}\right),\\
\int \frac{1}{3P} + \frac{1}{600-3P}\,dP &= \int 1\,dt,\\
\ln\left|\frac{3P}{600-3P}\right| &= 3t + 3c,\\
\frac{3P}{600-3P} &= Ae^{3t}, \text{ where } A = \pm e^{3c},\\
P &= \frac{200A}{A + e^{-3t}}
\end{aligned}
$$

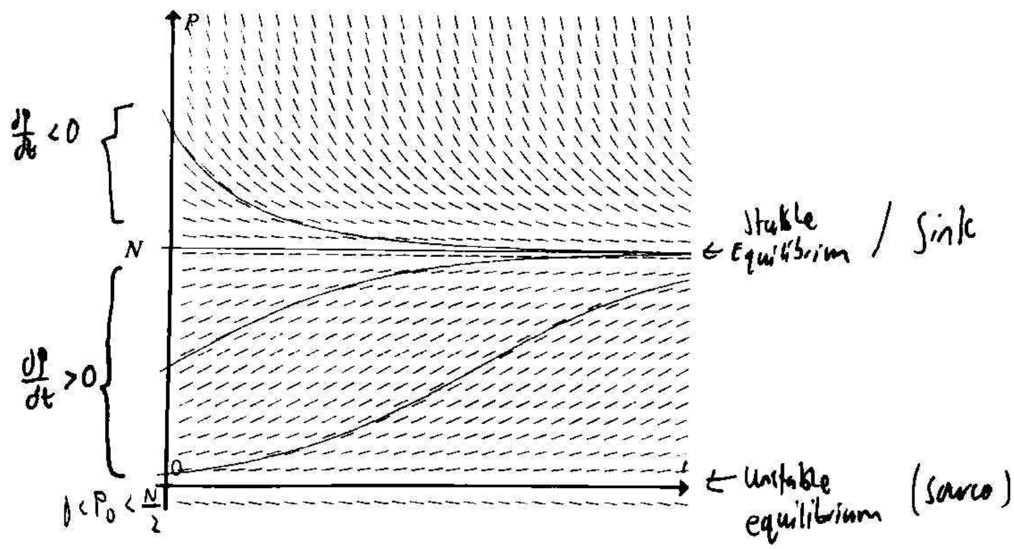


Figure 15.1: Logistics Curve

15.3.3 Harvesting

Let k be the *per-capita growth rate*, $P(t)$ be the population at time t, N be the *carrying capacity* of the system, and H the constant *harvesting rate*. Then we have the model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - H.$$

1. Bifurcation Point
   (a) When $0 \leq H < \frac{kN}{4}$, there are two equilibrium points, $P = \frac{N}{2} \pm \sqrt{\frac{N^2}{4} - \frac{HN}{k}}$.
   (b) When $H = \frac{kN}{4}$, there is one equilibrium point at $P = \frac{N}{2}$ (the bifurcation point).
   (c) When $H > \frac{kN}{4}$, there is no equilibrium point
2. For Non-Extinction:

$$\frac{N^2}{4} - \frac{HN}{k} \geq 0 \qquad \text{and} \qquad P_0 \geq 450 - \sqrt{\frac{N^2}{4} - \frac{HN}{k}}.$$

15.3.4 Physics

MUST rmb the forms.

1. Spring System (where $k > 0$ is the spring constant)

$$\ddot{x} + \frac{k}{m}x = 0.$$

   Solution: use R-formula to convert to $A\cos(\omega t + \phi)$ where angular frequency $\omega = \sqrt{\frac{k}{m}}$. Period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$.

2. Simple Pendulum (where ℓ is its length)

$$\ddot{x} + \frac{g}{\ell}x = 0.$$

   Angular frequency $\omega = \sqrt{\frac{g}{\ell}}$ and period $T = 2\pi\sqrt{\frac{\ell}{g}}$.

3. Spring-Mass-Dashpot System (where $c > 0$ is the damping constant)

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0.$$

   Solution
   (a) Real and Distinct Roots: *Overdamped*
   (b) Identical Real Roots: *Critically Damped*
   (c) Complex Conjugate Roots: *Underdamped*
   "It will oscillate about the equilibrium position with decreasing amplitude."

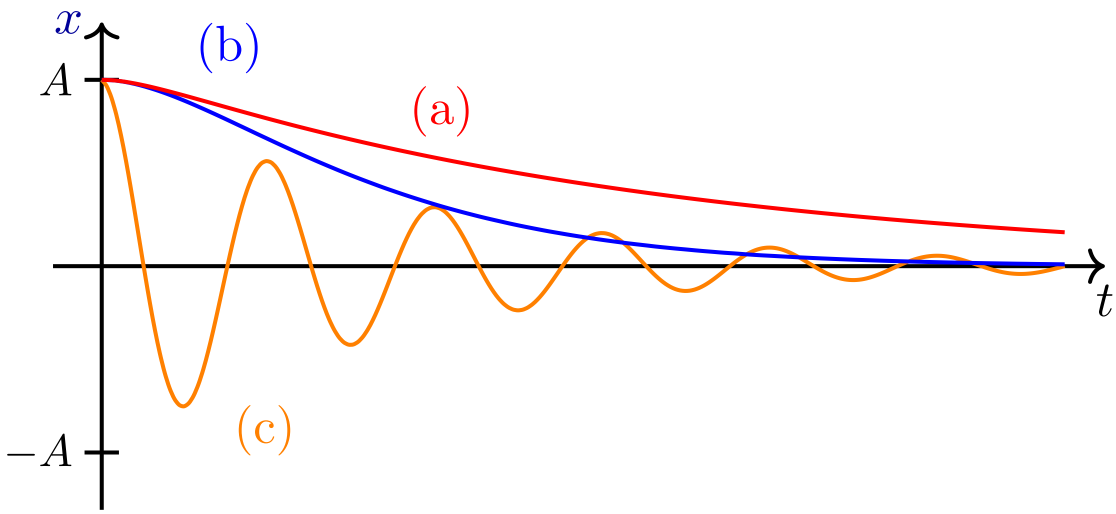


Figure 15.2: Oscillatory Behaviors

Discrete Random Variables

1. Expectation / Mean,
$$\mathrm{E}(X) := \sum_{\text{all } x} x\,\mathrm{P}(X = x).$$
2. Variance
$$\mathrm{Var}(X) := E\big(X^2\big) - [E(X)]^2.$$
3. Standard Deviation
$$\sigma := \sqrt{\mathrm{Var}(X)}.$$
4. Properties for two *independent* random *variables* X and Y; two *independent observations* X_1 and X_2 of X:
   (a) $\mathrm{E}(aX + bY + c) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y) + c$,
   (b) $\mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2) = 2\,\mathrm{E}(X)$.
   (c) $\mathrm{Var}(aX + bY + c) = a\,\mathrm{Var}(X) + b\,\mathrm{Var}(Y)$,
   (d) $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 2\,\mathrm{Var}(X)$.
5. Probability Distribution Table:

x	1	$\cdots$	n
$\mathrm{P}(X = x)$	$\mathrm{P}(X = 1)$	$\cdots$	$\mathrm{P}(X = n)$

Special Discrete Random Variables

Definition 17.1

A probability distribution function of X is a binomial distribution, i.e. $X \sim B(n, p)$ iff

$$P(X = x) := \binom{n}{x} p^x (1-p)^{n-x},$$

for all $0 \leq x \leq n$, where p is probability of success.

If $X \sim \mathrm{B}(n, p)$,

1. Expectation / Mean:

$$\mathrm{E}(X) = np \qquad \text{and} \qquad \mathrm{Var}(X) = np(1-p).$$

Note

We can assume $X \sim \mathrm{B}(n, p)$ whenever

1. The event of a [trial in context] is independent of that of another [trial in context].
2. The probability of each [trial in context] is constant.
3. Each trial has only 2 mutually exclusive outcomes.

Note

Defining random variables: Let X be the number of [event in context], out of [number of trials n in context].

Note

When the question asks for the *most likely* number of [event], it is asking for the *mode*.

G.C. Skills

Finding *mode*:

1. Set $Y_1 = \texttt{binompdf}(n, p, X)$.
2. Go to table.
3. Find the value of X for which the highest value of Y_1 occurs.

G.C. Skills

1. `2nd + Vars + 'A'` $\Longrightarrow$ $\texttt{binompdf}(n, p, x) = \mathrm{P}(X = x)$
2. `2nd + Vars + 'B'` $\Longrightarrow$ $\texttt{binomcdf}(n, p, x) = \mathrm{P}(X \leq x)$

Note

Let X be the random variable such that $X \sim \mathrm{B}(n, p)$. If $P(X = n)$ is the *highest probability* that occurs, $X = n$ is the modal value. So, we solve the two inequalities $P(X = 5) > P(X = 4)$ and $P(X = 5) > P(X = 6)$. This gives the *strictest* range of values that p can take (Fig 17.1).

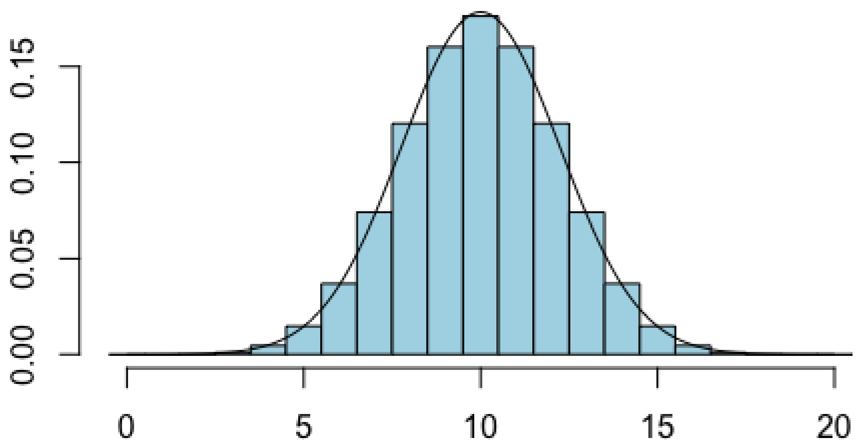


Figure 17.1: In this case, $X = 10$ is the mode.

Example 17.1: 2018 TPJC JC2 H2 MYE P2 8

On average, 3.5% of a certain brand of chocolate turn out misshapen. The chocolates are sold in packets of 25.

(i) State, in context, two assumptions needed for the number of misshapen chocolates in a packet to be well modelled by a binomial distribution.

(ii) Explain why one of the assumptions stated in part (i) may not hold in this context.

Answer:

(i)
1. Each chocolate is *equally likely* (3.) to be misshapen.
2. The event that a chocolate is misshapen is *independent* (2.) of the event that another chocolate is misshapen.

(ii) While on average, the probability that a chocolate is misshapen is 3.5%, it is possible that there are more misshapen chocolates at certain times, possible due to equipment malfunction, which would mean the probability is not constant.

OR

Misshapen chocolates could be the result of equipment used and as the equipment used would not be the same for the same portion of the chocolate produced, whether a chocolate is misshapen may not be independent of another chocolate being misshapen.

[0]Source for Fig 17.1: https://math.oxford.emory.edu/site/math117/normalApproxToBinomial/

Foreword

$>$ Wait isn't the foreword supposed to be... in front?
$> Yes$

18.1 Main Word

Latex is pain,
latex is suffering,
latex is a must.