# *The Quest To Read Lee*

For the basic properties of vector spaces and linear maps, you can consult almost any linear algebra book that treats vector spaces abstractly, such as [FIS03]. Here we just summarize the main points, with emphasis on those aspects that are most important for the study of smooth manifolds.

## Vector Spaces

Let $\mathbb{R}$ denote the field of real numbers. A ***vector space over*** $\mathbb{R}$ (or ***real vector space***) is a set $V$ endowed with two operations: ***vector addition*** $V \times V \to V$, denoted by $(v, w) \mapsto v + w$, and ***scalar multiplication*** $\mathbb{R} \times V \to V$, denoted by $(a, v) \mapsto av$, satisfying the following properties:

(i) $V$ is an abelian group under vector addition.
(ii) Scalar multiplication satisfies the following identities:

$$\begin{aligned} a(bv) &= (ab)v && \text{for all } v \in V \text{ and } a, b \in \mathbb{R}; \\ 1v &= v && \text{for all } v \in V. \end{aligned}$$

(iii) Scalar multiplication and vector addition are related by the following distributive laws:

$$\begin{aligned} (a+b)v &= av + bv && \text{for all } v \in V \text{ and } a, b \in \mathbb{R}; \\ a(v+w) &= av + aw && \text{for all } v, w \in V \text{ and } a \in \mathbb{R}. \end{aligned}$$

This definition can be generalized in two directions. First, replacing $\mathbb{R}$ by an arbitrary field $\mathbb{F}$ everywhere, we obtain the definition of a ***vector space over*** $\mathbb{F}$. In particular, we sometimes have occasion to consider vector spaces over $\mathbb{C}$, called ***complex vector spaces***. Unless we specify otherwise, all vector spaces are assumed to be real.

Second, if $\mathbb{R}$ is replaced by a commutative ring $\mathcal{R}$, this becomes the definition of a ***module over*** $\mathcal{R}$ (or ***$\mathcal{R}$-module***). For example, if $\mathbb{Z}$ denotes the ring of integers,

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