

# Linear Algebra

## Definition 1.1

A vector space (or linear space)  $V$  over a field  $\mathbb{F}$  consists of a set on which two operations (called addition and multiplication respectively here) are defined so that;

- (A) ( $V$  is Closed Under Addition) For all  $\mathbf{x}, \mathbf{y} \in V$ , there exists a unique element  $\mathbf{x} + \mathbf{y} \in V$ .
- (M) ( $V$  is Closed Under Scalar Multiplication) For all elements  $a \in \mathbb{F}$  and elements  $\mathbf{x} \in V$ , there exists a unique element  $a\mathbf{x} \in V$ .

Such that the following properties hold:

- (VS 1) (Commutativity of Addition) For all  $\mathbf{x}, \mathbf{y} \in V$ , we have  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ .
- (VS 2) (Associativity of Addition) For all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ , we have  $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ .
- (VS 3) (Existence of The Zero/Null Vector) There exists an element in  $V$  denoted by  $\mathbf{0}$ , such that  $\mathbf{x} + \mathbf{0} = \mathbf{x}$  for all  $\mathbf{x} \in V$ .
- (VS 4) (Existence of Additive Inverses) For all elements  $\mathbf{x} \in V$ , there exists an element  $\mathbf{y} \in V$  such that  $\mathbf{x} + \mathbf{y} = \mathbf{0}$ .
- (VS 5) (Multiplicative Identity) For all elements  $x \in V$ , we have  $1\mathbf{x} = \mathbf{x}$ , where 1 denotes the multiplicative identity in  $\mathbb{F}$ .
- (VS 6) (Compatibility of Scalar Multiplication with Field Multiplication) For all elements  $a, b \in \mathbb{F}$  and elements  $\mathbf{x} \in V$ , we have  $(ab)\mathbf{x} = a(b\mathbf{x})$ .
- (VS 7) (Distributivity of Scalar Multiplication over Vector Addition) For all elements  $a \in \mathbb{F}$  and elements  $\mathbf{x}, \mathbf{y} \in V$ , we have  $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$ .
- (VS 8) (Distributivity of Scalar Multiplication over Field Addition) For all elements  $a, b \in \mathbb{F}$ , and elements  $\mathbf{x} \in V$ , we have  $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$ .

## General Information

- Let  $V$  be a vector space and  $W$  a subset of  $V$ . Then  $W$  is a subspace of  $V$  iff the following 3 conditions hold for the operations defined in  $V$ .
  - (a)  $\mathbf{0} \in W$
  - (b)  $\mathbf{x} + \mathbf{y} \in W$  whenever  $\mathbf{x} \in W$  and  $\mathbf{y} \in W$ .
  - (c)  $c\mathbf{x} \in W$  whenever  $c \in \mathbb{F}$  and  $\mathbf{x} \in W$ .
- A subset  $S$  of a vector space  $V$  *generates* (or *spans*)  $V$  iff  $\text{span}(S) = V$ . In this case, we also say that the vectors of  $S$  generate (or span)  $V$ .
- Let  $V$  be a vector space and  $S$  a nonempty subset of  $V$ . A vector  $v \in V$  is called a *linear combination* of vectors of  $S$  iff there exists a finite number of vectors  $u_1, u_2, \dots, u_n$  in  $S$  and scalars  $a_1, a_2, \dots, a_n$  in  $\mathbb{F}$  such that

$$v = \sum_{i=1}^n a_i u_i.$$

In this case we also say that  $v$  is a linear combination of  $u_1, u_2, \dots, u_n$  and call  $a_1, a_2, \dots, a_n$  the *coefficients* of the linear combination

- A set subset  $S$  of a vector space  $V$  is called *linearly dependent* iff there exists a finite number of distinct vectors  $u_1, u_2, \dots, u_n$  in  $S$  and scalars  $a_1, a_2, \dots, a_n$  not all zero, such that

$$a_1 u_1 + a_2 u_2 + \dots + a_n u_n = \mathbf{0}.$$

- A *basis*  $\beta$  for a vector space  $V$  is a linearly independent subset of  $V$  that generates  $V$ . If  $\beta$  is a basis for  $V$ , we also say that the vectors of  $\beta$  form a basis for  $V$ .
- The Rank-Nullity Theorem: For any vector spaces  $V$  and  $W$ , and a linear operator  $T: V \rightarrow W$ , it holds that

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

- For any matrix, its row space, column space, and rank are identical.
- A system  $\mathbf{Ax} = \mathbf{b}$  is *homogeneous* iff  $\mathbf{b} = \mathbf{0}$ ; otherwise it is *nonhomogeneous*.
- A system  $\mathbf{Ax} = \mathbf{b}$  of  $m$  linear equations in  $n$  unknowns has a solution space of dimension  $n - \text{rank}(\mathbf{A})$ .
- A system  $\mathbf{Ax} = \mathbf{b}$  of linear equations is *consistent* iff its solution set is nonempty; otherwise it is *inconsistent*.
- The Rouché-Capelli theorem: A system  $\mathbf{Ax} = \mathbf{b}$  is consistent iff  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b})$ .
- A matrix is said to be in *reduced row echelon form* iff

- Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).
- The first nonzero entry in each row is the only nonzero entry in its column.
- The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

- Gaussian elimination.
  - In the forward pass, the augmented matrix is transformed into an upper triangular matrix in which the first nonzero entry of each row is 1 and it occurs in a column to the right of the first nonzero entry of each preceding row.
  - In the backward pass, the upper triangular matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.
- Let  $\mathbf{A}$  be an  $m \times n$  matrix, and  $\mathbf{a}_j$  its  $j$ th column. For any  $\mathbf{x} = (x_1 \ x_2 \ \dots \ x_n)^\top$ ,

$$\mathbf{Ax} = \sum_{j=1}^n x_j \mathbf{a}_j.$$

- Let  $\mathbf{A}$  and  $\mathbf{B}$  be matrices having  $n$  rows. For any matrix  $\mathbf{M}$  with  $n$  columns, we have

$$\mathbf{M}(\mathbf{A}|\mathbf{B}) = (\mathbf{MA}|\mathbf{MB}).$$

- The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if  $\mathbf{A} \in M_{n \times n}(\mathbb{F})$ , then for any integer  $1 \leq i \leq n$ ,

$$\det(\mathbf{A}) = \sum_{j=1}^n (-1)^{i+j} \mathbf{A}_{ij} \cdot \det(\tilde{\mathbf{A}}_{ij}).$$

Here,  $\tilde{\mathbf{A}}_{ij}$  is the  $(n-1) \times (n-1)$  matrix obtained from  $\mathbf{A}$  by deleting its  $i$ th row and  $j$ th column.

- The determinant of a square matrix can also be evaluated by cofactor expansion along any column, since

$$\det(\mathbf{A}) = \det(\mathbf{A}^\top).$$

- A matrix  $\mathbf{A}$  is invertible iff its determinant is nonzero.
- Let  $\mathbf{A}$  be an invertible  $n \times n$  matrix. Then, for some elementary row matrices  $\mathbf{E}_1$  to  $\mathbf{E}_p$ ,

$$\mathbf{E}_p \mathbf{E}_{p-1} \dots \mathbf{E}_1 (\mathbf{A} \mid \mathbf{I}_n) = \mathbf{A}^{-1} (\mathbf{A} \mid \mathbf{I}_n) = (\mathbf{I}_n \mid \mathbf{A}^{-1}).$$

In other words, we can perform Gaussian elimination, so that  $(\mathbf{A} \mid \mathbf{I}_n) \rightarrow (\mathbf{I}_n \mid \mathbf{A}^{-1})$ .

- Alternatively,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A}),$$

where  $\text{adj}(\mathbf{A})$  is the adjugate / classical adjoint of  $\mathbf{A}$ . That is, the matrix whose  $(i, j)$ th entry is the  $(j, i)$ th cofactor  $(-1)^{j+i} \det(\tilde{\mathbf{A}}_{ji})$

# Numerical Methods

## General Information

- The parity of the degree of a real polynomial is the same as that of its number of real roots.
- Let the real polynomial  $p$  given by  $p(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \dots + a_0$  have coefficients  $a_n > 0$  and  $a_0 < 0$ . Then, it has at least one positive and one negative root.
- Suppose we have some function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a root  $\alpha$ , whose value we want to approximate. There are three ways to obtain this approximation.

1. Linear interpolation on an interval  $[a, b]$  containing  $\alpha$ . We let  $x_0 := b$  and

$$x_{i+1} := \frac{a|f(x_i)| + x_i|f(a)|}{|f(a)| + |f(x_i)|}.$$

– *Additional notes.*

2. Fixed-point Iteration. First select a function  $F: \mathbb{R} \rightarrow \mathbb{R}$ , such that  $F(\alpha) = \alpha$ , and choose some initial approximation  $x_0$  to  $\alpha$ . Then, we recursively define  $x_{n+1} := F(x_n)$ . The desired convergence behavior is for  $x_n$  to approach  $\alpha$ .

– *Additional notes.*

## G.C. Skills

Linear interpolation: finding an approximation to a root in  $[a, b]$  up to  $n$  decimal places.

1.  $Y_1 = f(x)$ ,
2.  $a \rightarrow A$  and  $b \rightarrow B$ ,
3.  $\frac{B|Y_1(A)| + A|Y_1(B)|}{|Y_1(A)| + |Y_1(B)|}$ ,
4. Ans  $\rightarrow A$  or  $B$  (choose the one that has the opposite sign to Ans),
5. Repeat steps 4 to 5,
6. Terminate this process when the approximations are consistent up to  $n$  decimal places.