

# Chapter 1

## Linear Algebra

### Definition 1.1

A vector space (or linear space)  $V$  over a field  $\mathbb{F}$  consists of a set on which two operations (called addition and multiplication respectively here) are defined so that;

- (A) ( $V$  is Closed Under Addition) For all  $\mathbf{x}, \mathbf{y} \in V$ , there exists a unique element  $\mathbf{x} + \mathbf{y} \in V$ .
- (M) ( $V$  is Closed Under Scalar Multiplication) For all elements  $a \in \mathbb{F}$  and elements  $\mathbf{x} \in V$ , there exists a unique element  $a\mathbf{x} \in V$ .

Such that the following properties hold:

- (VS 1) (Commutativity of Addition) For all  $\mathbf{x}, \mathbf{y} \in V$ , we have  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ .
- (VS 2) (Associativity of Addition) For all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ , we have  $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ .
- (VS 3) (Existence of The Zero/Null Vector) There exists an element in  $V$  denoted by  $\mathbf{0}$ , such that  $\mathbf{x} + \mathbf{0} = \mathbf{x}$  for all  $\mathbf{x} \in V$ .
- (VS 4) (Existence of Additive Inverses) For all elements  $\mathbf{x} \in V$ , there exists an element  $\mathbf{y} \in V$  such that  $\mathbf{x} + \mathbf{y} = \mathbf{0}$ .
- (VS 5) (Multiplicative Identity) For all elements  $x \in V$ , we have  $1\mathbf{x} = \mathbf{x}$ , where 1 denotes the multiplicative identity in  $\mathbb{F}$ .
- (VS 6) (Compatibility of Scalar Multiplication with Field Multiplication) For all elements  $a, b \in \mathbb{F}$  and elements  $\mathbf{x} \in V$ , we have  $(ab)\mathbf{x} = a(b\mathbf{x})$ .
- (VS 7) (Distributivity of Scalar Multiplication over Vector Addition) For all elements  $a \in \mathbb{F}$  and elements  $\mathbf{x}, \mathbf{y} \in V$ , we have  $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$ .
- (VS 8) (Distributivity of Scalar Multiplication over Field Addition) For all elements  $a, b \in \mathbb{F}$ , and elements  $\mathbf{x} \in V$ , we have  $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$ .

### Theorem 1.2

Let  $V$  be a vector space and  $W$  a subset of  $V$ . Then  $W$  is a subspace of  $V$  iff the following 3 conditions hold for the operations defined in  $V$ .

- (a)  $\mathbf{0} \in W$
- (b)  $\mathbf{x} + \mathbf{y} \in W$  whenever  $\mathbf{x} \in W$  and  $\mathbf{y} \in W$ .
- (c)  $c\mathbf{x} \in W$  whenever  $c \in \mathbb{F}$  and  $\mathbf{x} \in W$ .

**Definition 1.3**

A subset  $S$  of a vector space  $V$  *generates* (or *spans*)  $V$  iff  $\text{span}(S) = V$ . In this case, we also say that the vectors of  $S$  generate (or span)  $V$ .

**Definition 1.4**

Let  $V$  be a vector space and  $S$  a nonempty subset of  $V$ . A vector  $v \in V$  is called a *linear combination* of vectors of  $S$  iff there exists a finite number of vectors  $u_1, u_2, \dots, u_n$  in  $S$  and scalars  $a_1, a_2, \dots, a_n$  in  $\mathbb{F}$  such that

$$v = \sum_{i=1}^n a_i u_i.$$

In this case we also say that  $v$  is a linear combination of  $u_1, u_2, \dots, u_n$  and call  $a_1, a_2, \dots, a_n$  the *coefficients* of the linear combination

**Definition 1.5**

A set subset  $S$  of a vector space  $V$  is called *linearly dependent* iff there exists a finite number of distinct vectors  $u_1, u_2, \dots, u_n$  in  $S$  and scalars  $a_1, a_2, \dots, a_n$  not all zero, such that

$$a_1 u_1 + a_2 u_2 + \dots + a_n u_n = \mathbf{0}.$$

**Definition 1.6**

A *basis*  $\beta$  for a vector space  $V$  is a linearly independent subset of  $V$  that generates  $V$ . If  $\beta$  is a basis for  $V$ , we also say that the vectors of  $\beta$  form a basis for  $V$ .

**Theorem 1.7: The Rank-Nullity Theorem.**

For any vector spaces  $V$  and  $W$ , and a linear operator  $T: V \rightarrow W$ , it holds that

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

**General Information**

- Let  $\mathbf{A}$  be an  $m \times n$  matrix, and  $\mathbf{a}_j$  its  $j$ th column. For any  $\mathbf{x} = (x_1 \ x_2 \ \dots \ x_n)^\top$ ,

$$\mathbf{A}\mathbf{x} = \sum_{j=1}^n x_j \mathbf{a}_j.$$

- Let  $\mathbf{A}$  and  $\mathbf{B}$  be matrices having  $n$  rows. For any matrix  $\mathbf{M}$  with  $n$  columns, we have

$$(\mathbf{A} \mid \mathbf{B}) = (\mathbf{MA} \mid \mathbf{MB}).$$

**Definition 1.8**

A system  $\mathbf{Ax} = \mathbf{b}$  is *homogeneous* iff  $\mathbf{b} = \mathbf{0}$ ; otherwise it is *nonhomogeneous*.

**Theorem 1.9**

For any matrix, its row space, column space, and rank are identical.

**Theorem 1.10**

A system  $\mathbf{Ax} = \mathbf{b}$  of  $m$  linear equations in  $n$  unknowns has a solution space of dimension  $n - \text{rank}(\mathbf{A})$ .

**Definition 1.11**

A system  $\mathbf{Ax} = \mathbf{b}$  of linear equations is *consistent* iff its solution set is nonempty; otherwise it is *inconsistent*.

**Theorem 1.12: The Rouché-Capelli Theorem.**

A system  $\mathbf{Ax} = \mathbf{b}$  is consistent iff  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b})$ .

**Definition 1.13**

A matrix is said to be in *reduced row echelon form* iff

- Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).
- The first nonzero entry in each row is the only nonzero entry in its column.
- The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

- Gaussian elimination.
  - In the forward pass, the augmented matrix is transformed into an upper triangular matrix in which the first nonzero entry of each row is 1 and it occurs in a column to the right of the first nonzero entry of each preceding row.
  - In the backward pass, the upper triangular matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.
- Gaussian elimination always reduces a matrix to its rref form.
- Let  $\mathbf{A}$  be an invertible  $n \times n$  matrix. Then, for some elementary row matrices  $\mathbf{E}_1$  to  $\mathbf{E}_p$ ,

$$\mathbf{E}_p \mathbf{E}_{p-1} \dots \mathbf{E}_1 (\mathbf{A} | \mathbf{I}_n) = \mathbf{A}^{-1} (\mathbf{A} | \mathbf{I}_n) = (\mathbf{I}_n | \mathbf{A}^{-1}).$$

In other words, we can perform Gaussian elimination, so that  $(\mathbf{A} | \mathbf{I}_n) \rightarrow (\mathbf{I}_n | \mathbf{A}^{-1})$ .

- Let  $\mathbf{A} := (\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n)$  be  $m \times n$  matrix, and  $\mathbf{A}' := (\mathbf{a}'_1 \ \mathbf{a}'_2 \ \dots \ \mathbf{a}'_n)$  its rref. Then,  $\{\mathbf{a}_{k_1}, \mathbf{a}_{k_2}, \dots, \mathbf{a}_{k_m}\}$  is linearly independent iff  $\{\mathbf{a}'_{k_1}, \mathbf{a}'_{k_2}, \dots, \mathbf{a}'_{k_m}\}$  is. Moreover, the row space of  $\mathbf{A}$  and  $\mathbf{A}'$  are clearly identical.
- Finding a basis for an intersection of subspaces. Let  $V$  and  $W$  be subspaces of  $\mathbb{F}^n$  generated by the columns of the  $n \times m$  matrix  $\mathbf{A}$  and  $n \times k$  matrix  $\mathbf{B}$ , respectively. Find a basis for the subspace  $V \cap W$ .

1. First notice that  $\mathbf{v} \in V \cap W$  iff

$$\mathbf{v} = \mathbf{Ax}_1 = \mathbf{Bx}_2$$

for some  $\mathbf{x}_1 \in \mathbb{F}^m$  and  $\mathbf{x}_2 \in \mathbb{F}^k$ . That is,

$$(\mathbf{A} \ \mathbf{B}) \begin{pmatrix} \mathbf{x}_1 \\ -\mathbf{x}_2 \end{pmatrix} = \mathbf{0}.$$

So, equivalently, we write

$$(\mathbf{A} \ \mathbf{B}) \mathbf{y} = \mathbf{0}.$$

for some  $\mathbf{y} \in \mathbb{F}^{m+k}$ . As such, by row reducing  $(\mathbf{A} \ \mathbf{B})$ , we find a basis

$$\beta := \left\{ \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}'_1 \end{pmatrix}, \begin{pmatrix} \mathbf{u}_2 \\ \mathbf{u}'_2 \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}'_r \end{pmatrix} \right\},$$

where  $\mathbf{u}_i \in \mathbb{F}^m$  and  $\mathbf{u}_i \in \mathbb{F}^k$ . Now, a generating set for  $V \cap W$  is

$$\Gamma := \{\mathbf{Au}_1, \mathbf{Au}_2, \dots, \mathbf{Au}_r\}.$$

Alternatively, another generating set for  $V \cap W$  is

$$\Delta := \{\mathbf{Bu}'_1, \mathbf{Bu}'_2, \dots, \mathbf{Bu}'_r\}.$$

From here, it is simple to choose bases  $\gamma \subseteq \Gamma$  and  $\delta \subseteq \Delta$  for  $V \cap W$ .

(Naturally, it holds that  $\mathbf{Au}_i + \mathbf{Bu}'_i = \mathbf{0}$ .)

2. An alternative method. By row reduction, we can calculate

$$\begin{aligned} r &:= \dim(V \cap W) = \dim(U) + \dim(V) - \dim(U + V), \\ &= \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}) - \text{rank}(\mathbf{A} \ \mathbf{B}), \\ &= \text{rank}(\mathbf{A}^\top) + \text{rank}(\mathbf{B}^\top) - \text{rank}\begin{pmatrix} \mathbf{A}^\top \\ \mathbf{B}^\top \end{pmatrix}. \end{aligned}$$

Then, a basis for  $V \cap W$  can be formed by choosing  $r$  linearly independent vectors in  $V \cap W$ .

3. [Another alternative](#), probably the best option! Skip the row reduction of  $\mathbf{A}$  and  $\mathbf{B}$  in the above method. We just reduce

$$(\mathbf{A} \ \mathbf{B}) \rightarrow (\mathbf{A}' \ \mathbf{B}').$$

Let  $\mathbf{c}_i$  and  $\mathbf{c}'_i$  be the  $i$ th column of  $(\mathbf{A} \ \mathbf{B})$  and  $(\mathbf{A}' \ \mathbf{B}')$ , respectively. We compare the columns of  $A'$  and  $B'$  to find (with relative ease) a basis  $\beta' := \{\mathbf{c}'_{i_1}, \mathbf{c}'_{i_2}, \dots, \mathbf{c}'_{i_r}\}$  for the intersection of the column spaces of  $A'$  and  $B'$ . Then,  $\beta := \{\mathbf{c}_{i_1}, \mathbf{c}_{i_2}, \dots, \mathbf{c}_{i_r}\}$  is a basis for  $V \cap W$  (the intersection of the column spaces of  $A$  and  $B$ ).

4. [A fourth method](#) for when I learn about orthogonal complements.

#### Definition 1.14

Let  $\mathbf{A} \in M_{n \times n}(\mathbb{F})$ . If  $n = 1$ , so that  $A = (a_{11})$ , we define  $\det(\mathbf{A}) := a_{11}$ . For  $n \geq 2$ , we define  $\det(\mathbf{A})$  recursively as

$$\det(\mathbf{A}) := \sum_{j=1}^n (-1)^{1+j} \mathbf{A}_{1j} \cdot \det(\tilde{\mathbf{A}}_{1j}).$$

The scalar  $\det(\mathbf{A})$  is called the *determinant* of  $\mathbf{A}$  and is also denoted by  $|\mathbf{A}|$ . The scalar

$$(-1)^{i+j} \det(\tilde{\mathbf{A}}_{ij})$$

is called the cofactor of the entry of  $\mathbf{A}$  in row  $i$ , column  $j$ .

- A matrix  $\mathbf{A}$  is invertible iff its determinant is nonzero.

#### Theorem 1.15

The determinant  $\det: M_{n \times n}(\mathbb{F}) \rightarrow \mathbb{F}$  is an alternating  $n$ -linear function. The former (alternating) means that for  $\mathbf{A} \in M_{n \times n}(\mathbb{F})$  and any  $\mathbf{B}$  obtained from  $\mathbf{A}$  by interchanging any two rows of  $\mathbf{A}$ ,

$$\det(\mathbf{B}) = -\det(\mathbf{A}).$$

The latter ( $n$ -linearity) means that, for any scalar  $k \in \mathbb{F}$  and vectors  $\mathbf{u}, \mathbf{v}, \mathbf{a}_i \in \mathbb{F}^n$ ,

$$\det \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_{r-1} \\ \mathbf{u} + k\mathbf{v} \\ \mathbf{a}_{r+1} \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = \det \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_{r-1} \\ \mathbf{u} \\ \mathbf{a}_{r+1} \\ \vdots \\ \mathbf{a}_n \end{pmatrix} + k \det \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_{r-1} \\ \mathbf{v} \\ \mathbf{a}_{r+1} \\ \vdots \\ \mathbf{a}_n \end{pmatrix}.$$

(In fact, it can be shown that  $\det: M_{n \times n}(\mathbb{F}) \rightarrow \mathbb{F}$  is the *unique* alternating  $n$ -linear function, such that  $\det(\mathbf{I}) = 1$ .)

**Corollary 1.16**

Let  $\mathbf{A} \in M_{n \times n}(\mathbb{F})$ . Then, for any matrix  $\mathbf{B}$  obtained by adding a scalar multiple of one row/column of  $\mathbf{A}$  to another,  $\det(\mathbf{B}) = \det(\mathbf{A})$ .

**Theorem 1.17**

The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if  $\mathbf{A} \in M_{n \times n}(\mathbb{F})$ , then for any integer  $1 \leq i \leq n$ ,

$$\det(\mathbf{A}) = \sum_{j=1}^n (-1)^{i+j} \mathbf{A}_{ij} \cdot \det(\tilde{\mathbf{A}}_{ij}).$$

Here,  $\tilde{\mathbf{A}}_{ij}$  is the  $(n-1) \times (n-1)$  matrix obtained from  $\mathbf{A}$  by deleting its  $i$ th row and  $j$ th column.

**Corollary 1.18**

The determinant of any triangular matrix is the product of its diagonals.

**Theorem 1.19**

Let  $\mathbf{A}$  be an  $n \times n$  matrix. Then,

$$\det(\mathbf{A}) = \det(\mathbf{A}^\top).$$

So, the determinant of a square matrix can also be evaluated by cofactor expansion along any column.

**Theorem 1.20**

Let  $\mathbf{A}$  be an invertible  $n \times n$  matrix. Then,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A}),$$

where  $\text{adj}(\mathbf{A})$  is the adjugate/classical adjoint of  $\mathbf{A}$ . That is, the matrix whose  $(i, j)$ th entry is the  $(j, i)$ th cofactor  $(-1)^{j+i} \det(\tilde{\mathbf{A}}_{ji})$ .

**Theorem 1.21**

For any  $\mathbf{A}, \mathbf{B} \in M_{n \times n}(\mathbb{F})$ , we have  $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$ .

**Definition 1.22**

A linear operator  $T$  on a finite-dimensional vector space  $V$  is called *diagonalisable* iff there is an ordered basis  $\beta$  for  $V$  such that  $[T]_\beta$  is a diagonal matrix. A square matrix  $\mathbf{A}$  is called diagonalisable iff  $L_{\mathbf{A}}$  is diagonalisable.

**Definition 1.23**

Let  $T$  be a linear operator on a vector space  $V$ . A nonzero vector  $\mathbf{v} \in V$  is called an *eigenvector* of  $T$  iff there exists a scalar  $\lambda$  such that  $T(\mathbf{v}) = \lambda \mathbf{v}$ . The scalar  $\lambda$  is called the *eigenvalue* corresponding to the eigenvector  $\mathbf{v}$ .

Let  $\mathbf{A}$  be in  $M_{n \times n}(\mathbb{F})$ . A nonzero vector  $v \in \mathbb{F}^n$  is called an *eigenvector* of  $\mathbf{A}$  iff  $v$  is an eigenvector of  $L_{\mathbf{A}}$ ; that is, iff  $\mathbf{A}v = \lambda v$  for some scalar  $\lambda$ . The scalar  $\lambda$  is called the eigenvalue of  $\mathbf{A}$  corresponding to the eigenvector  $v$ .

**Definition 1.24**

Let  $\mathbf{A} \in M_{n \times n}(\mathbb{F})$ . The polynomial  $f(t) = \det(\mathbf{A} - t\mathbf{I}_n)$  is called the *characteristic polynomial* of  $\mathbf{A}$ .

- A matrix  $\mathbf{A} \in M_{n \times n}(\mathbb{F})$  is diagonalizable iff there exists an ordered basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  for  $\mathbb{F}^n$  consisting of eigenvectors of  $\mathbf{A}$ , i.e. a eigenbasis. Furthermore, if  $\mathbf{Q}$  is the  $n \times n$  matrix whose  $j$ th column is  $\mathbf{v}_j$ , then  $\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}\mathbf{Q}$  is a diagonal matrix such that  $d_{jj}$  is the eigenvalue of  $\mathbf{A}$  corresponding to  $\mathbf{v}_j$ . The matrix  $\mathbf{Q}$  is said to *diagonalise*  $\mathbf{A}$ .
- Hence, we obtain the following procedure to diagonalise a  $3 \times 3$  matrix  $\mathbf{A}$  with three distinct eigenvalues.
  1. Find the eigenvalues  $\lambda_1, \lambda_2$ , and  $\lambda_3$  of  $\mathbf{A}$ . They are just the roots of the characteristic polynomial of  $\mathbf{A}$ . [This can be done using the GC.](#)
  2. Find an eigenvector  $\mathbf{v}_j$  corresponding to each eigenvalue  $\lambda_j$  by finding the nullspace of  $\mathbf{A} - \lambda_j\mathbf{I}$ .
  3. Let  $\mathbf{Q} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ . Then,

$$\mathbf{D} := \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$$

is a diagonal matrix.

#### Note

Let  $\mathbf{A}$  be a  $3 \times 3$  real matrix with the eigenvalue  $\lambda$ . Then, the cross product of two nonzero rows/columns of  $\mathbf{A} - \lambda\mathbf{I}$  is an eigenvector of  $\mathbf{A}$ .

#### Theorem 1.25: The Cayley-Hamilton Theorem.

Let  $T$  be a linear operator on a finite dimensional vector space  $V$ , and let  $f(t)$  be the characteristic polynomial of  $T$ . Then  $f(T) = T_0$ , the zero transformation. That is,  $T$  “satisfies” its characteristic equation.

#### Corollary 1.26: The Cayley-Hamilton Theorem for Matrices.

Let  $A$  be an  $n \times n$  matrix, and let  $f(t)$  be the characteristic polynomial of  $A$ . Then,  $f(A) = O$ , the  $n \times n$  zero matrix.

#### G.C. Skills

Finding eigenvalues of a matrix  $\mathbf{A}$  using the GC.

1. `2nd  $\Rightarrow$   $x^{-1}$  (matrix)  $\Rightarrow$  Key in the matrix  $\mathbf{A} - t\mathbf{I}$ , e.g. into  $[A]$ .`
2. `Plot  $Y_1 = \det([A])$ .`
3. `2nd  $\Rightarrow$  trace  $\Rightarrow$  2:zero  $\Rightarrow$  Find the roots.`

## 1.1 Conics

Given a conic section defined by  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ , we may wish to find its lines of symmetry, center, radii, etc. The core idea is simple: complete the square, to express  $Ax^2 + Bxy + Cy^2$  in the form  $a(x')^2 + b(y')^2$ , for some linear combinations  $x'$  and  $y'$  of  $x$  and  $y$ . Then, the initial equation becomes  $a(x')^2 + b(y')^2 + d(x') + e(y') + F = 0$ , which is easily reduced to the standard form for conics. Before that, we need to develop some machinery.

### Definition 1.27

Let  $\mathbb{F}$  be a field not of characteristic two. A function  $K: \mathbb{F}^n \rightarrow \mathbb{F}$  is called a *quadratic form on  $\mathbb{F}^n$*  if there exists a symmetric matrix  $A \in M_{n \times n}(\mathbb{F})$ , such that  $K(\mathbf{x}) = \mathbf{x}^\top \mathbf{A} \mathbf{x}$  for all  $\mathbf{x} \in \mathbb{F}^n$ .

### Note

Let  $\mathbb{F}$  be a field not of characteristic two and scalars  $a_i \in \mathbb{F}$ . The polynomial  $f: \mathbb{F}^n \rightarrow \mathbb{F}$  given by  $f(t_1, t_2, \dots, t_n) = \sum_{i \leq j} a_{ij} t_i t_j$  is a quadratic form. In fact, the matrix  $\mathbf{A}$  with

$$\mathbf{A}_{ij} = \begin{cases} a_{ii} & \text{if } i = j \\ a_{ij}/2 & \text{if } i \neq j \end{cases}$$

gives us our desired quadratic form  $K(\mathbf{x}) = \mathbf{x}^\top \mathbf{A} \mathbf{x}$ .

### Theorem 1.28

Let  $K(\mathbf{x}) = \mathbf{x}^\top \mathbf{A} \mathbf{x}$  be a quadratic form on a finite-dimensional real inner product space  $V$ . There exists an orthonormal eigenbasis  $\beta := \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  for  $\mathbf{A}$  and eigenvalues  $\lambda_i$ , such that  $K(\sum_{i=1}^n t_i \mathbf{v}_i) = \sum_{i=1}^n \lambda_i t_i^2$  for all  $t_i \in \mathbb{R}$ .

We now return to our initial problem on conics. We first diagonalise the matrix

$$\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$$

to find its eigenvalues  $\lambda$  and  $\mu$ , then the corresponding unit eigenvectors  $\mathbf{u} = (\alpha \ \beta)^\top$  and  $\mathbf{v} = (\gamma \ \delta)^\top$ . Then,

$$\begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{u} \ \mathbf{v}) \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \alpha t_1 + \gamma t_2 \\ \beta t_1 + \delta t_2 \end{pmatrix}.$$

Furthermore,  $Ax^2 + Bxy + Cy^2 = \lambda t_1^2 + \mu t_2^2$  by 1.28. Therefore,

$$\begin{aligned} & \lambda t_1^2 + \mu t_2^2 + D(\alpha t_1 + \gamma t_2) + E(\beta t_1 + \delta t_2) + F = 0 \\ \lambda \left( t_1 + \frac{D\alpha + E\beta}{2\lambda} \right)^2 + \mu \left( t_2 + \frac{D\gamma + E\delta}{2\mu} \right)^2 - \frac{(D\alpha + E\beta)^2}{4\lambda} - \frac{(D\gamma + E\delta)^2}{4\mu} + F = 0 \quad (\text{if } \lambda, \mu \neq 0) \end{aligned}$$

gives an equivalent form for our conic section.

Since our basis  $\{\mathbf{u}, \mathbf{v}\}$  is orthonormal, the above change of basis from the standard ordered basis to  $\{\mathbf{u}, \mathbf{v}\}$  is an isometry (that maps  $\mathbf{0}$  to itself). It is obtained via a rotation and/or reflection. Hence, the radii are  $\sqrt{\lambda/k}$  and  $\sqrt{\mu/k}$ , for  $4k = (D\alpha + E\beta)^2/\lambda + (D\gamma + E\delta)^2/\mu - F$ ; the center is  $\left( -\frac{D\alpha + E\beta}{2\lambda}, -\frac{D\gamma + E\delta}{2\mu} \right)$ . Notice that  $t_1 = 0$  and  $t_2 = 0$  are parallel to the axes of symmetry. i.e.  $\mathbf{u}$  and  $\mathbf{v}$  are parallel to the axes of symmetry of our conic. As such, the lines of symmetry are

$$y + \frac{D\gamma + E\delta}{2\mu} = \frac{\beta}{\alpha} \left( x + \frac{D\alpha + E\beta}{2\lambda} \right) \quad \text{and} \quad y + \frac{D\gamma + E\delta}{2\mu} = \frac{\delta}{\gamma} \left( x + \frac{D\alpha + E\beta}{2\lambda} \right).$$

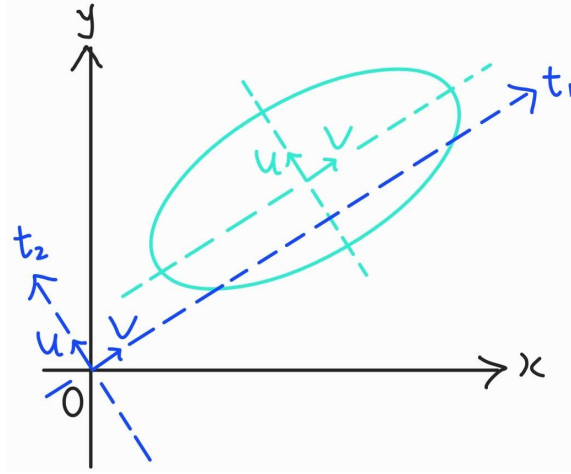


Figure 1.1: Rotated and translated conic.

**Note**

Without orthonormality, i.e. an isometric change of coordinates, even if we were successful in reducing our conic to the form  $a(x')^2 + b(y')^2 = f$ , it may not prove to be a useful form. Consider the ellipse  $2x^2 + 2xy + y^2 = 1$ . Clearly,  $x^2 + (x + y)^2 = 1$ . But, this gives a circle in  $(x, x + y)$  coordinates; we can't deduce much about our initial conic. So, little meaning is found in such a factorisation.

**Corollary 1.29**

Consider a parabola  $x^2 = 4py$  rotated about the  $x$ -axis by an angle  $\theta$  anti-clockwise, about the origin. Then,  $y$  remains a function of  $x$  iff  $\theta = 0$ .

*Proof.* From our preceding work, we see that such a rotated parabola, will have equation  $\square$

## 1.2 Miscellaneous

An asterisk denotes the conjugate transpose.

**Theorem 1.30**

Let  $\mathbf{M} \in M_{n \times n}(\mathbb{K})$  be Hermitian (i.e.  $\mathbf{M}^* = \mathbf{M}$ ), with eigenvectors  $\mathbf{u}$  and  $\mathbf{v}$  that correspond to the eigenvalues  $\lambda$  and  $\mu$ . Then,  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal with respect to the standard inner product, if  $\lambda \neq \mu^*$ .

*Proof.* Let  $\mathbf{u}$  and  $\mathbf{v}$  be eigenvectors of  $\mathbf{M}$ . Then,

$$\langle \mathbf{u}, \mu \mathbf{v} \rangle = (\mathbf{M} \mathbf{v})^* \mathbf{u} = (\mathbf{v}^* \mathbf{M}^*) \mathbf{u} = \mathbf{v}^* (\mathbf{M}^* \mathbf{u}) = \mathbf{v}^* (\lambda \mathbf{u}) = \langle \lambda \mathbf{u}, \mathbf{v} \rangle.$$

As such,  $(\lambda - \mu^*) \langle \mathbf{u}, \mathbf{v} \rangle = 0$ . Hence,  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ .  $\square$

Consider the equation  $x^2 + y = 0$  and the matrix  $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ . Then,  $x' = x$  and  $y' = x + y$ . However, even though  $1^2 - 1 = 0$ , we have  $1^2 + (1 - 1) = 1 \neq 0$ .



## Chapter 2

# Numerical Methods

### General Information

- The parity of the degree of a real polynomial is the same as that of its number of real roots.
- Let the real polynomial  $p$  given by  $p(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \dots + a_0$  have coefficients  $a_n > 0$  and  $a_0 < 0$ . Then, it has at least one positive and one negative root.
- To show that there a continuous function  $f$  attains a root in an interval  $[a, b]$ , we find two values  $x < y$  in the interval (e.g.  $a < b$ ) such that  $f(a)f(b) < 0$ . i.e. show that  $f$  changes sign in  $[a, b]$ . Then, *by continuity*, a root of  $f$  must lie in  $[a, b]$ .
- To further show that the root is *unique* in  $[a, b]$ , it suffices to prove that  $f$  is *strictly* monotone on  $[a, b]$ .
- Suppose we have some function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a root  $\alpha$ , whose value we want to approximate. There are three ways to obtain this approximation.

1. Linear interpolation on an interval  $[a, b]$  containing  $\alpha$ . Our approximation is

$$\frac{a|f(b)| + b|f(a)|}{|f(a)| + |f(b)|}.$$

- The sequence  $\{x_n\}$  of approximations *always* converges to  $\alpha$ .
- The smaller  $|f''(x)|$  is (i.e. the slower the gradient  $f'(x)$  changes) near  $\alpha$ , the faster the rate of convergence.
- Error:

Concave/Gradient	Positive	Negative
Upwards $\cup$	underestimation	overestimation
Downwards $\cap$	overestimation	underestimation

**Table 2.1:** Approximation errors when using linear interpolation.

- See Figure 2.2 for an illustration.

Screw trying to make nice diagonal cells. Pain. Suffering.

### Note

At every iteration of linear interpolation, we must ensure that  $\alpha \in [a, x_n]$ . Otherwise  $x_n$  may not approximate  $\alpha$ . If  $\alpha \notin [a, x_n]$ , simply consider  $\alpha \in [x_n, b]$  (or any other suitable interval) instead.

**Note**

It is important to show which interval we are interpolating on, not just the iteratively obtained values. We can present our working using the table below.

$a$	$f(a)$	$b$	$f(b)$	$\frac{a f(b)  + b f(a) }{ f(a)  +  f(b) }$
$a$	$f(a) > 0$	$b$	$f(b) < 0$	$x_1$
$x_1$	$f(x_1) > 0$	$b$	$f(b) < 0$	$x_2$
$x_1$	$f(x_1) > 0$	$x_2$	$f(x_2) < 0$	$x_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

**Table 2.2:** Required working for linear interpolation.

2. Fixed-point Iteration. First select a function  $F: \mathbb{R} \rightarrow \mathbb{R}$ , such that  $F(\alpha) = \alpha$ , and choose some initial approximation  $x_0$  to  $\alpha$ . Then, we recursively define  $x_{n+1} := F(x_n)$ . We want  $x_n \rightarrow \alpha$ .

– Convergence behavior

Behavior of $ F'(x) $	Converges?	Rate of convergence
$ F'(x)  < 1$ and is small near $\alpha$	✓	fast
$ F'(x)  < 1$ but is close to 1 near $\alpha$	✓	slow
$ F'(x)  \geq 1$ near $\alpha$	✗	-

**Table 2.3:** Convergence behavior of fixed-point iterations.

– See Figure 2.3 for an illustration.

**Note**

We must write out *all* iterations, not just the final two. The working below is sufficient.

Let  $x_0 = \underline{\hspace{1cm}}$  and  $x_{n+1} = F(x_n)$ ,  $x \geq 0$ .

$$\begin{aligned}
 x_1 &= \underline{\hspace{1cm}} \\
 x_2 &= \underline{\hspace{1cm}} \\
 &\vdots \\
 x_{m-1} &= \underline{\hspace{1cm}} \\
 x_m &= \underline{\hspace{1cm}}
 \end{aligned}$$

Therefore,  $\alpha = x_m$  ( $k$  d.p.), since  $f(x_m - 0.0 \dots 05)f(x_m + 0.0 \dots 05) = \underline{\hspace{1cm}} < 0$ .

3. The Newton-Raphson Method. Let  $\alpha$  be a root of the function  $f: \mathbb{R} \rightarrow \mathbb{R}$ . The Newton-Raphson formula is

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}.$$

– The Newton-Raphson method fails in the following cases.

- The gradient at  $x_0$  is too gentle.
- The gradient changes too rapidly.
- The initial approximation  $x_0$  is too far from the root  $\alpha$ .
- There is a turning point between the initial approximation  $x_0$  and the root  $\alpha$ .

(e) There is a point of inflection — where the concavity changes/the sign of  $f''(x)$  changes.

– Error:

Concave/Gradient	Positive	Negative
Upwards $\cup$	overestimation	underestimation
Downwards $\cap$	underestimation	overestimation

**Table 2.4:** Approximation errors when using the Newton-Raphson method.

– See Figure 2.4 for an illustration.

### Note

We must write out *all* iterations, not just the final two. One way to present our working is as follows.

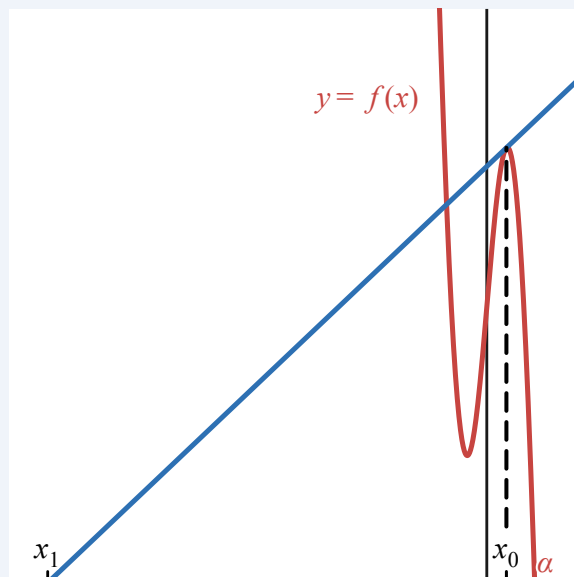
Let  $x_0 = \underline{\hspace{1cm}}$  and  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \underline{\hspace{1cm}}$ ,  $x \geq 0$ .

$$\begin{aligned} x_1 &= \underline{\hspace{1cm}} \\ x_2 &= \underline{\hspace{1cm}} \\ &\vdots \\ x_{m-1} &= \underline{\hspace{1cm}} \\ x_m &= \underline{\hspace{1cm}} \end{aligned}$$

Therefore,  $\alpha = x_m$  ( $k$  d.p.), since  $f(x_m - 0.0 \dots 05)f(x_m + 0.0 \dots 05) = \underline{\hspace{1cm}} < 0$ .

### Note

Explain whether  $x_0 = \underline{\hspace{1cm}}$  is a suitable starting value for using the Newton-Raphson method to find an approximation to  $\alpha$ .



**Figure 2.1:** (Desmos)

1. Since  $x_0 = \_\_\_$  is very close to the stationary point, the tangent to the curve  $y = f(x)$  has a very gentle gradient. Thus, it cuts the  $x$ -axis far away from the initial approximation.
2. Furthermore, as  $x_0 = \_\_\_$  is to the left/right of the minimum/maximum point  $x = \_\_\_$ , the values of the gradient  $f'(x_n)$  will be negative/positive for all  $n \geq 0$ . Hence,  $x_n$  converges to the root  $\beta$  instead  $\alpha$ .

(The second point may be omitted if it is irrelevant.)

### Note

Suppose a question asks for the approximation of a root to  $k$  significant figures/ $k$  decimal places. Then:

1. We leave our iterative approximations  $x_n$  to at least  $k + 2$  significant figures/ $k + 2$  decimal places.
2. We continue the iterative process till two consecutive ones agree up to  $k$  significant figures/ $k$  decimal places.

### Note

Perform \_\_\_\_\_ (e.g. linear interpolation) to obtain an approximation for  $\alpha$ , correct to two decimal places. Justify whether this approximation is sufficiently accurate.

Suppose our approximation is some  $a = 1.00$ , then we note the sign of  $f$  at  $a \pm 0.005$ . (For an arbitrary number of s.f. or d.p., simply adjust the value 0.005 accordingly. E.g. for 3 d.p. we instead use 0.0005). Our working should look similar to the following:

Since  $f(0.995) = \_\_\_ < 0$  and  $f(1.005) = \_\_\_ > 0$ , we conclude that 1.00 is a sufficiently accurate approximation, at 2 d.p..

### Note

The *error obtained* when using an approximation should be the *absolute* difference of the true value and the approximation.

### Note

Use the results of part (i) and the differential equation  $dy/dx = \sin(xy)$  to estimate the  $x$ -coordinate  $x_P$  of  $P$ .

The maximum point occurs when  $dy/dx = \sin(xy) = 0$ , i.e.  $xy = k\pi$  where  $k \in \mathbb{Z}$ . From (i),  $\frac{dy}{dx}|_{x=5/3} \approx 0.643 > 0$  so  $y_P > y(5/3) \approx 2.0468$ . Hence,  $x_P \approx \pi/2.04679402 = 1.53$  (3.s.f.).

### G.C. Skills

Linear interpolation: finding an approximation to a root in  $[a, b]$  up to  $n$  decimal places.

1.  $Y_1 = f(x)$ ,
2.  $a \rightarrow A$  and  $b \rightarrow B$ ,
3.  $\frac{B|Y_1(A)| + A|Y_1(B)|}{|Y_1(A)| + |Y_1(B)|}$ ,
4. Ans  $\rightarrow A$  or  $B$  (choose the one that has the opposite sign to Ans),
5. Repeat steps 4 to 5,

6. Terminate this process when the approximations are consistent up to  $n$  decimal places.

You can freely enter any function and shift the initial values in the Desmos graphs below!

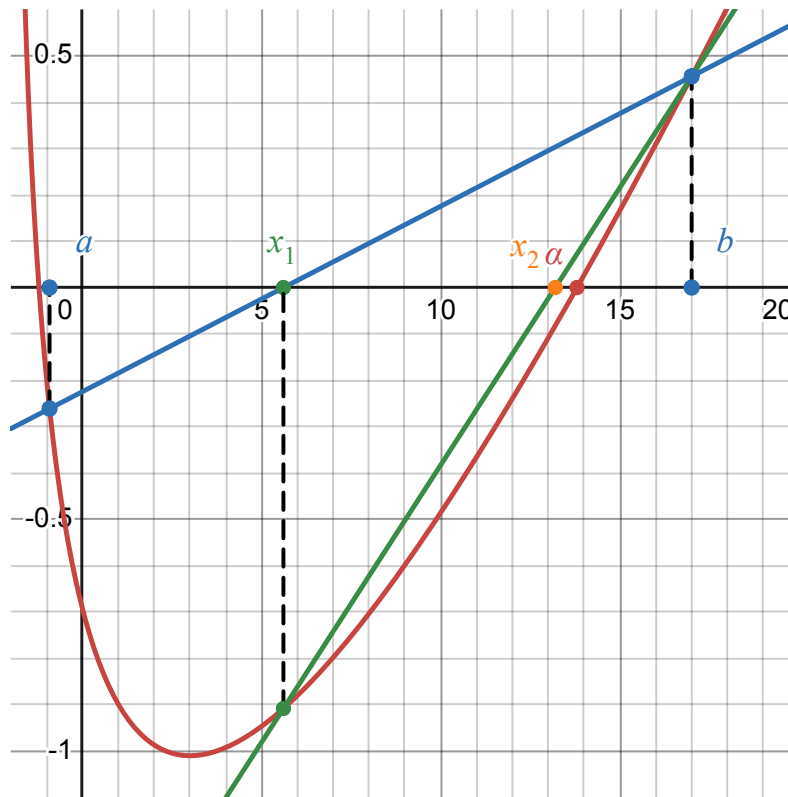


Figure 2.2: An illustration of linear interpolation ([Desmos](#)).

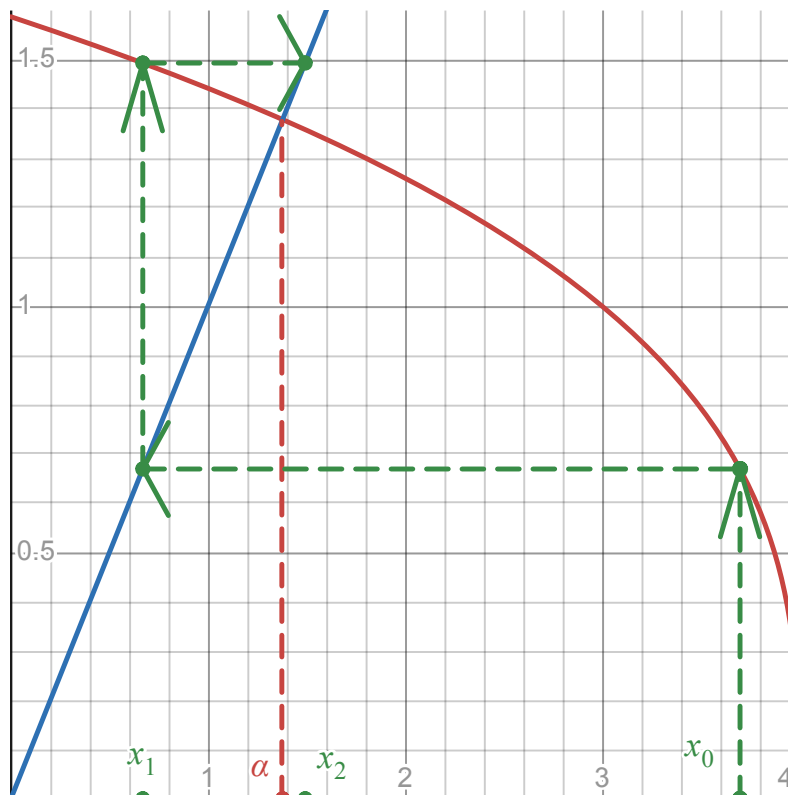


Figure 2.3: An illustration of fixed-point iteration ([Desmos](#)).

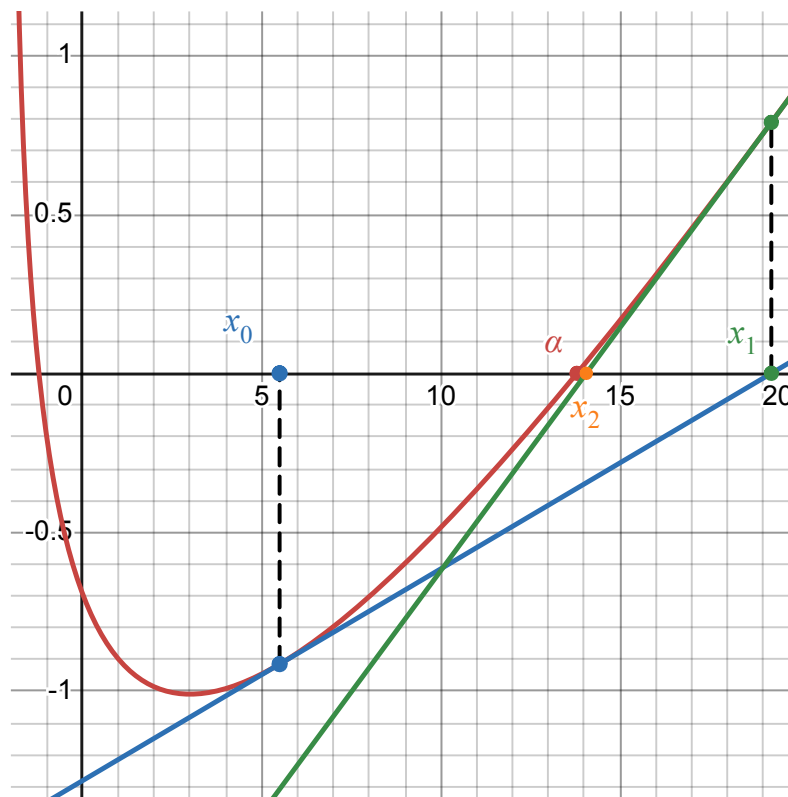


Figure 2.4: An illustration of Newton's Method ([Desmos](#)).