Linear Algebra

Definition 1.1

A vector space (or linear space) V over a field $\mathbb{F}$ consists of a set on which two operations (called addition and multiplication respectively here) are defined so that;

(A) (V is Closed Under Addition) For all $\mathbf{x}, \mathbf{y} \in V$, there exists a unique element $\mathbf{x} + \mathbf{y} \in V$.

(M) (V is Closed Under Scalar Multiplication) For all elements $a \in \mathbb{F}$ and elements $\mathbf{x} \in V$, there exists a unique element $a\mathbf{x} \in V$.

Such that the following properties hold:

(VS 1) (Commutativity of Addition) For all $\mathbf{x}, \mathbf{y} \in V$, we have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

(VS 2) (Associativity of Addition) For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

(VS 3) (Existance of The Zero/Null Vector) There exists an element in V denoted by $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$.

(VS 4) (Existance of Additive Inverses) For all elements $\mathbf{x} \in V$, there exists an element $\mathbf{y} \in V$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$.

(VS 5) (Multiplicative Identity) For all elements $x \in V$, we have $1\mathbf{x} = \mathbf{x}$, where 1 denotes the multiplicative identity in $\mathbb{F}$.

(VS 6) (Compatibility of Scalar Multiplication with Field Multiplication) For all elements $a, b \in \mathbb{F}$ and elements $\mathbf{x} \in V$, we have $(ab)\mathbf{x} = a(b\mathbf{x})$.

(VS 7) (Distributivity of Scalar Multiplication over Vector Addition) For all elements $a \in \mathbb{F}$ and elements $\mathbf{x}, \mathbf{y} \in V$, we have $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

(VS 8) (Distributivity of Scalar Multiplication over Field Addition) For all elements $a, b \in \mathbb{F}$, and elements $\mathbf{x} \in V$, we have $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

General Information

- Let V be a vector space and W a subset of V. Then W is a subspace of V iff the following 3 conditions hold for the operations defined in V.

  (a) $\mathbf{0} \in W$

  (b) $\mathbf{x} + \mathbf{y} \in W$ whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$.

  (c) $c\mathbf{x} \in W$ whenever $c \in \mathbb{F}$ and $\mathbf{x} \in W$.

- A subset S of a vector space V *generates* (or *spans*) V iff $\text{span}(S) = V$. In this case, we also say that the vectors of S generate (or span) V.

- Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a *linear combination* of vectors of S iff there exists a finite number of vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$ in $\mathbb{F}$ such that

  $$v = \sum_{i=1}^{n} a_i u_i.$$

  In this case we also say that v is a linear combination of $u_1, u_2, \ldots, u_n$ and call $a_1, a_2, \ldots, a_n$ the *coefficients* of the linear combination

- A set subset S of a vector space V is called *linearly dependent* iff there exists a finite number of distinct vectors $u_1, u_2, \dots, u_n$in S and scalars $a_1, a_2, \dots, a_n$ not all zero, such that

$$a_1 u_1 + a_2 u_2 + a_n u_n = \mathbf{0}.$$

- A *basis* β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.
- The Rank-Nullity Theorem: For any vector spaces V and W, and a linear operator $T\colon V \to W$, it holds that

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$

- For any matrix, its row space, column space, and rank are identical.
- A system $\mathbf{Ax} = \mathbf{b}$ is *homogeneous* iff $\mathbf{b} = 0$; otherwise it is *nonhomogeneous*.
- A system $\mathbf{Ax} = \mathbf{b}$ of m linear equations in n unknowns has a solution space of dimension $n - \operatorname{rank}(A)$.
- A system $\mathbf{Ax} = \mathbf{b}$ of linear equations is *consistent* iff its solution set is nonempty; otherwise it is *inconsistent*.
- The Rouché-Capelli theorem: A system $\mathbf{Ax} = \mathbf{b}$ is consistent iff $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b})$.
- A matrix is said to be in *reduced row echelon form* iff
  - Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).
  - The first nonzero entry in each row is the only nonzero entry in its column.
  - The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.
- Gaussian elimination.
  - In the forward pass, the augmented matrix is transformed into an upper triangular matrix in which the first nonzero entry of each row is 1 and it occurs in a column to the right of the first nonzero entry of each preceding row.
  - In the backward pass, the upper triangular matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.
- Let $\mathbf{A}$ be an $m \times n$ matrix, and $\mathbf{a}_j$ its jth column. For any $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top$,

$$\mathbf{Ax} = \sum_{j=1}^{n} x_j \mathbf{a}_j.$$

- Let $\mathbf{A}$ and $\mathbf{B}$ be matrices having n rows. For any matrix $\mathbf{M}$ with n columns, we have

$$\left(\mathbf{A} \;\middle|\; \mathbf{B}\right) = \left(\mathbf{MA} \;\middle|\; \mathbf{MB}\right).$$

- Let V and W be subspaces of $\mathbb{F}^n$ generated by the columns of the $n \times m$ matrix $\mathbf{A}$ and $n \times k$ matrix $\mathbf{B}$, respectively. To find a basis for the subspace $V \cap W$, first notice that $\mathbf{v} \in V \cap W$ iff

$$\mathbf{v} = \mathbf{Ax}_1 = \mathbf{Bx}_2$$

  for some $\mathbf{x}_2 \in \mathbb{F}^m$ and $\mathbf{x}_2 \in \mathbb{F}^k$. That is,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{x_1} \\ -\mathbf{x_2} \end{pmatrix} = \mathbf{0}.$$

So, equivalently, we write

$$\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}\mathbf{y} = \mathbf{0}.$$

for some $\mathbf{y} \in \mathbb{F}^{m+k}$. Furthermore, a basis $\beta := \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ for the solution space can be found by reducing $\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}$. Now, let $\mathsf{u_i} \in \mathbb{F}^m$ be vector obtained by deleting all but the first m entries of $\mathbf{u}_i$. Then, a basis for $V \cap W$ is

$$\gamma := \{\mathbf{A}\mathsf{u_1}, \mathbf{A}\mathsf{u_2}, \ldots, \mathbf{A}\mathsf{u_r}\}.$$

Alternatively, letting $\mathsf{u_i'} \in \mathbb{F}^m$ be vector obtained by deleting all but the last k entries of $\mathbf{u}_i$, another basis for $V \cap W$ is

$$\delta := \{\mathbf{B}\mathsf{u_1'}, \mathbf{B}\mathsf{u_2'}, \ldots, \mathbf{B}\mathsf{u_r'}\}.$$

(Naturally, it holds that $\mathbf{A}\mathsf{u_i} + \mathbf{B}\mathsf{u_i'} = 0$.)

- The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if $\mathbf{A} \in \mathrm{M}_{n\times n}(\mathbb{F})$, then for any integer $1 \leq i \leq n$,

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{i+j} \mathbf{A}_{ij} \cdot \det(\widetilde{\mathbf{A}}_{ij}).$$

  Here, $\widetilde{\mathbf{A}}_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained from $\mathbf{A}$ by deleting its ith row and jth column.

- The determinant of a square matrix can also be evaluated by cofactor expansion along any column, since

$$\det(\mathbf{A}) = \det(\mathbf{A}^\top).$$

- A matrix $\mathbf{A}$ is invertible iff its determinant is nonzero.

- Let $\mathbf{A}$ be an invertible $n \times n$ matrix. Then, for some elementary row matrices $\mathbf{E}_1$ to $\mathbf{E}_p$,

$$\mathbf{E}_p\mathbf{E}_{p-1}\ldots\mathbf{E}_1(\mathbf{A} \,|\, \mathbf{I}_n) = \mathbf{A}^{-1}(\mathbf{A} \,|\, \mathbf{I}_n) = (\mathbf{I}_n \,|\, \mathbf{A}^{-1}).$$

  In other words, we can perform Gaussian elimination, so that $(\mathbf{A} \,|\, \mathbf{I}_n) \to (\mathbf{I}_n \,|\, \mathbf{A}^{-1})$.

- Alternatively,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\,\mathrm{adj}(A),$$

  where $\mathrm{adj}(\mathbf{A})$ is the adjugate / classical adjoint of $\mathbf{A}$. That is, the matrix whose (i,j)th entry is the (j,i)th cofactor $(-1)^{j+i}\det(\widetilde{\mathbf{A}}_{ji})$

Numerical Methods

General Information

- The parity of the degree of a real polynomial is the same as that of its number of real roots.
- Let the real polynomial p given by $p(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_0$ have coefficients $a_n > 0$ and $a_0 < 0$. Then, it has at least one positive and one negative root.
- Suppose we have some function $f\colon \mathbb{R} \to \mathbb{R}$ with a root α, whose value we want to approximate. There are three ways to obtain this approximation.
  1. Linear interpolation on an interval $[a, b]$ containing α. We let $x_0 \coloneqq b$ and
  $$x_{i+1} \coloneqq \frac{a|f(x_i)| + x_i|f(a)|}{|f(a)| + |f(x_i)|}.$$
     - *Additional notes.*
  2. Fixed-point Iteration. First select a function $F\colon \mathbb{R} \to \mathbb{R}$, such that $F(\alpha) = \alpha$, and choose some initial approximation x_0 to α. Then, we recursively define $x_{n+1} \coloneqq F(x_n)$. The desired convergence behavior is for x_n to approach α.
     - *Additional notes.*

G.C. Skills

Linear interpolation: finding an approximation to a root in $[a, b]$ up to n decimal places.

1. $Y_1 = f(x)$,
2. $a \to A$ and $b \to B$,
3. $\dfrac{B|Y_1(A)| + A|Y_1(B)|}{|Y_1(A)| + |Y_1(B)|}$,
4. Ans $\to A$ or B (choose the one that has the opposite sign to Ans),
5. Repeat steps 4 to 5,
6. Terminate this process when the approximations are consistent up to n decimal places.