A-Levels Math Notes

Grass

Contents

A1: Inequalities and Equations

1.1 Solving Inequalities

1.1.1 Rational Inequalities

General Methods

1. Quadratic formula for factorisation / finding roots (of polynomial).
2. Completing the square.
3. Discriminant/Completing the Sq to eliminate* factors which are *always* positive or negative (e.g. removing $x^2 - 3x + 4$). *Note to include coefficient of x^2 in argument.*
4. GC (include sketch).
5. *Rational Functions*[a]: Move everything to one side (+,−), then use number line.
6. Number line (more complicated functions).

[a] Fractions of Polynomials

Important Notes

- ☐ Eliminating Factors — *only*[a] works for $c = 0$ in $f(x) \geq c$ or $f(x) \leq c$.
- ☐ Discriminant — include coefficient of x^2 in argument.
- ☐ Rational functions — exclude the values that causes division by zero to occur.
- ☐ With inequalities, be really careful about multiplication! If $x > y$ *and* $z > 0$, then $xz > yz$.
- ☐ Cross multiplication preserves order for $\frac{x}{y} < \frac{x'}{y'}$ iff y and y' are *both* positive or negative.[b]
- ☐ Squaring preserves order for $x < y$ iff x and y are *both* positive or negative.

[a] Counterexample: $P(x) = x(3x^2 - 9x + 10) \leq 2$ iff $x \leq 2$ is false. E.g.: $P(1.8) = 6.336 \nleq 2$.
[b] Otherwise, note the counterexample $\frac{1}{2} < \frac{1}{-3}$.

1.2 Modulus Inequalities

Fact

Given $x \in \mathbb{R}$, we have that

- $|x| \geq 0$,
- $|x^2| = |x|^2 = x^2$,
- $\sqrt{x^2} = |x|$.

And as long as $x \in \mathbb{R}^+$,

- $\sqrt{x}^2 = |x|$.

Useful Properties

For every $x, k \in \mathbb{R}$:

(a) $|x| < k$ iff[a] $-k < x < k$.

(b) $|x| > k$ iff $x < -k$ or $x > k$.

(of course, similarly applies for the non-strict ordering $\leq$)

[a] Notice that $k > 0$ here since $0 \leq |x| < k$.

Important Notes

- Note that when solving for $|x| = y$, $|x| < y$, etc, y must be greater than or equal to 0. In other words, there may be solutions we will need to reject.
  (For $<$, equality is of course not allowed.)

Important Notes

- △ Carelessness: Look at the question carefully! If they ask for a *set* of values, then rmb to give it as a *set*!
- △ Exponentiation and Logarithms: Simply use ln and avoid $\log_c$ for $c < 1$.[a]

[a] Order is *Preserved* under exponentiation/logarithms if the base is *larger than* one. Otherwise, when it is *less than* one, the order is *reversed*. https://www.desmos.com/calculator/gd8z5fa0bg

1.3 System of Linear Equations

Things

- χ For more complicated real-world-context qns, try playing around with the values (e.g. use simult eqns) first. It may work out nicer than expected.

1.4 Summary

G.C. Skills

1. Plotting curves $y = f(x)$ in G.C.
2. How to use simultaneous equation solver.

Important Notes

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- ☐ Rational functions — exclude the values that causes division by zero to occur.
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- ☐ Carelessness: Look at the question carefully! If they ask for a *set* of values, then rmb to give it as a *set*!
- ☐ Exponentiation and Logarithms: Simply use ln and avoid $\log_c$ for $c < 1$.[c]
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A5.1: Differentiation

2.1 Limits

2.2 Geometrical Results of the Derivatives

Definition

(i) A function f is called (strictly) increasing on an interval I iff $f'(x) > 0$ for all $x \in I$.

(ii) A function f is called monotonically increasing on an interval I iff $f'(x) \geq 0$ for any $x \in I$.

Things To Know

1. How to sketch the graph of the integral or[a] derivative of a function f.
2. Relationship btw. a function f and its derivative, f':

$y = f(x)$	$y = f'(x)$
Vertical asymptote at $x = a$	Vertical asymptote at $x = a$.
Horizontal asymptote at $y = a$	Horizontal asymptote $y = 0$.

[a] Of course, provided that f is integrable/differentiable.

B1(A): Graphing Techniques (Part I)

3.1 Graphing 'Familiar' Functions and Asymptotic bois

Definition

1. **Lines of Symmetry**: A *line of symmetry* of a function is a line, such that the function is a reflection of itself about that line.
2. **Horizontal Asymptotes**: A (horizontal) line $g(x) = c$ is the *horizontal asymptote* of the curve $f(x)$ iff $\lim_{x\to\infty} f(x) = c$ (or with $-\infty$ instead of ∞).[a]
3. **Vertical Asymptotes**: A (vertical) line $x = c$ is a *vertical asymptote* of the curve $f(x)$ iff $\lim_{x\to c} f(x) = \infty$ or $-\infty$.
4. **Oblique Asymptotes**: A line $g(x) = mx + c$ — where $m \neq 0$ — is an *oblique asymptote* of the curve $f(x)$ iff $\lim_{x\to\infty}[f(x) - g(x)] = 0$ (or with $-\infty$ instead of ∞).

[a]Otherwise notated by $f(x) \to c$ as $x \to \infty$.

Curve Sketching (Rational Funcs)

S Stationary points

I Intersection with axes

A Asymptotes

i Know how to sketch the graphs of $y = \frac{ax+b}{cx+d}$ and $y = \frac{ax^2+bx+c}{dx+e}$.

ii Rectangular Hyperbolas $\left(\text{of the form } y = \frac{ax+b}{cx+d}\right)$:

- *Two* asymptotes, namely $x = -\frac{d}{c}$ and $y = \frac{a}{c}$.
- *Two* lines of symmetry with gradients ± 1 *and* pass through the intersection point of the aforementioned two asymptotes.

iii *If* $n = \deg P = \deg Q$, then

- $y = R(x)$ is the *horizontal* asymptote of $\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}$.
- Equivalently, $y = \frac{\text{coeff}_P(x^n)}{\text{coeff}_Q(x^n)}$ is a *horizontal* asymptote.[a]

iv If $\deg P = \deg Q + 1$, then $R(x)$ is an *oblique* asymptote of $\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}$.

v Write down asymptotes and lines of symmetry.[b] If none are present indicate with "No lines of symmetry."

[a]E.g.: $y = \frac{1}{15}$ is a horizontal asymptote of $y = \frac{1x^2+2x-3}{(5x+1)(3x+2)}$.

[b]E.g.:

Asymptotes: $x = 4$, $y = 20$.

Lines of Symmetry: $y = x + 16$, $y = -x + 24$.

Important Notes

— *Can* explicitly write out the asymptotes and lines of symmetry (or if they are not present) to be safe.

— Using the discriminant intelligently can result in nice answers.

— Know how to use the G.C. Transfrm app.[a][b]

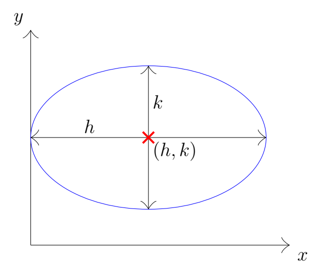
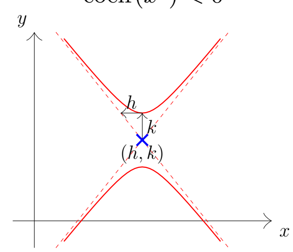
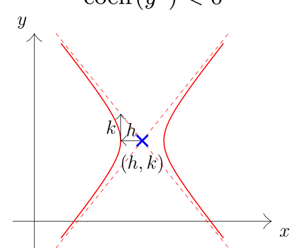
[a] Thingy that allows you to vary the value of some parameter A for a function $f(Ax)$.

[b] E.g.: Solve for the values of k being a positive *integer*. We can use the app to visually see where the curves intersect.

3.2 Conics

3.2.1 Ellipses

"Tikz is pain, PGFPlots is suffering" — Wise Man.

	Ellipses	Hyperbolas
Standard Forms	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
General Equation	$ax^2 + by^2 + cx^2 + dx + e = 0$, where $\operatorname{sgn}(a) = \operatorname{sgn} b$.	$ax^2 + by^2 + cx^2 + dex + e = 0$, where $\operatorname{sgn}(a) \neq \operatorname{sgn} b$.
Center	(h, k)	
Vertical 'Radius' (variables here from *standard form*!)	b	
Horizontal 'Radius' (variables here from *standard form*!)	a	
Vertical Vertices (variables here from *standard form*!)	$(h, k \pm b)$	
Horizontal Vertices (variables here from *standard form*!)	$(h \pm a, k)$	
Shape		coeff$(x^2) < 0$  coeff$(y^2) < 0$ 
Asymptotes (No need to rmb!)		$y = k \pm \frac{b(x-h)}{a}$
Lines of Symmetry	$x = h$, $y = k$	

General Info

$\mathscr{H}$ To find asymptote of hyperbolas, just solve

$$\frac{(x-h)^2}{a^2} = \frac{(y-k)^2}{b^2}.$$

$\mathscr{H}$ Label vertices *or* radii, together with the center and asymptotes.

3.3 Parametric Equations

Important Notes

- ⋆ Check the qns for any *restrictions* on the parameter! And modify that of the G.C.'s accordingly (Tmin & Tmax).
- ⋆ Vary the $t - step$ or resolution (of cartesian form) as necessary (when the graph is oddly jagged)0.
- ⋆ Think carefully for tricker qns[a].

[a]E.g.: Simult eqns can be useful in converting frm parametric to cartesian form.

3.4 Summary

G.C. Skills

1. Plot conics with the two ways.
2. Know G.C. functions like finding axial intercepts.

Important Notes

-☐- *Can* explicitly write out the asymptotes and lines of symmetry (or if they are not present) to be safe.

-☐- Using the discriminant intelligently can result in nice answers.

-☐- Know how to use the G.C. Transfrm app.[a b]

-☐- Check the qns for any *restrictions* on the parameter! And modify that of the G.C.'s accordingly (Tmin & Tmax).

-☐- Vary the $t - step$ or resolution (of cartesian form) as necessary (when the graph is oddly jagged)0.

-☐- Think carefully for tricker qns[c].

[a]Thingy that allows you to vary the value of some parameter A for a function $f(Ax)$.
[b]E.g.: Solve for the values of k being a positive *integer*. We can use the app to visually see where the curves intersect.
[c]E.g.: Simult eqns can be useful in converting frm parametric to cartesian form.

Chapter B1(B): Graphing Techniques (Part II)

4.1 Scaling, Translations, and Reflections

Playing With x		
Function	x is replaced with	(Horizontal) Transformation
$f(x+a)$	$x+a$	Translate a units in the positive $(a \leq 1)$ O/R negative x-direction $(a \geq 1)$.
$f(-x)$	$-x$	Reflect about the y-axis
$f(ax)$	ax	Scale parallel to the x-axis by a scale factor of $\frac{1}{a}$ if[1] $a \geq 1$.
Playing With $f(x)$		
Function / Change to $f(x)$		(Vertical) Transformation
$f(x)+a$		Translate a units in the positive $(a \geq 1)$ O/R negative y-direction $(a \leq 1)$.
$-f(x)$		Reflect about the x-axis.
$af(x)$		Scale parallel to the y-axis by scale factor a.

Important Notes

Transform x

Translation ⇨ Scaling / Reflection

$\Downarrow$

Transform y

Scaling / Reflection ⇨ Translation

4.2 $|f(x)|$ and $f(|x|)$

Basics

➻ $f(|x|)$ ➢ Graph of 'negative side' is a reflection of the 'positive side' (across the y-axis).

4.3 $y = \frac{1}{f(x)}$

Conditions	Results
$f(x) > 0$	$\frac{1}{f(x)} > 0$
$f(x) < 0$	$\frac{1}{f(x)} < 0$
Vertical Asymptote at $x = c$	$\frac{1}{f(x)}$ *tends* to 0 * $\frac{1}{f(x)}$ is undefined at $x = c$
$\frac{df}{dx} = -\frac{d}{dx}\left(\frac{1}{f(x)}\right)$ i.e. when $f(x)$ increases, $\frac{1}{f(x)}$ decreases.	
(a, b) is a *minimum* pt	$\left(a, \frac{1}{b}\right)$ is a *maximum* pt
(a, b) is a *maximum* pt	$\left(a, \frac{1}{b}\right)$ is a *minimum* pt

4.4 Summary

Statistics 1: Permutations and Combinations

5.1 The Addition and Multiplication Principles

Example 1: The Addition Principle

There are three distinct cups of black sugar bubble tea and five unique cups of zero sugar bubbles tea available, I am buying *exactly* one of them. How many choices do I have? Answer: $3 + 5 = 8$.

Example 2: The Multiplication Principle

A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible. Answer: $3 \cdot 4 \cdot 5 \cdot 2 = 120$.

5.2 Permutation

Definition 1: ${}^{n}P_k$

$${}^{n}P_k := \frac{n!}{(n-k)!}$$

General Necessities

- Know how to 'group' objects together so as to calculate the total no. of permutations.