A-Levels Math Notes

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Contents

A1: Inequalities and Equations

1.1 Solving Inequalities

1.1.1 Rational Inequalities

General Methods

1. Quadratic formula for factorisation / finding roots (of polynomial).
2. Completing the square.
3. Discriminant/Completing the Sq to eliminate* factors which are *always* positive or negative (e.g. removing $x^2 - 3x + 4$). *Note to include coefficient of x^2 in argument.*
4. GC (include sketch).
5. *Rational Functions*[a]: Move everything to one side (+,−), then use number line.
6. Number line (more complicated functions).

[a] Fractions of Polynomials

Important Notes

- ☐ Eliminating Factors — *only*[a] works for $c = 0$ in $f(x) \geq c$ or $f(x) \leq c$.
- ☐ Discriminant — include coefficient of x^2 in argument.
- ☐ Rational functions — exclude the values that causes division by zero to occur.
- ☐ Cross multiplication preserves order for $\frac{x}{y} < \frac{x'}{y'}$ iff y and y' are *both* positive or negative.[b]
- ☐ Squaring preserves order for $x < y$ iff x and y are *both* positive or negative.

[a] Counterexample: $P(x) = x(3x^2 - 9x + 10) \leq 2$ iff $x \leq 2$ is false. E.g.: $P(1.8) = 6.336 \not\leq 2$.
[b] Otherwise, note the counterexample $\frac{1}{2} < \frac{1}{-3}$.

1.2 Modulus Inequalities

Fact

Given $x \in \mathbb{R}$, we have that

- $|x| \geq 0$,
- $|x^2| = |x|^2 = x^2$,
- $\sqrt{x^2} = |x|$.

And as long as $x \in \mathbb{R}^+$,

- $\sqrt{x}^2 = |x|$.

Useful Properties

For every $x, k \in \mathbb{R}$:

(a) $|x| < k$ iff[a] $-k < x < k$.

(b) $|x| > k$ iff $x < -k$ or $x > k$.

(of course, similarly applies for the non-strict ordering $\leq$)

[a] Notice that $k > 0$ here since $0 \leq |x| < k$.

Important Notes

- ○ Note that when solving for $|x| = y$, $|x| < y$, etc, y must be greater than or equal to 0. In other words, there may be solutions we will need to reject.
  (For $<$, equality is of course not allowed.)

Important Notes

- △ Carelessness: Look at the question carefully! If they ask for a *set* of values, then rmb to give it as a *set*!
- △ Exponentiation and Logarithms: Simply use ln and avoid $\log_c$ for $c < 1$.[a]

[a] Order is *Preserved* under exponentiation/logarithms if the base is *larger than* one. Otherwise, when it is *less than* one, the order is *reversed*. `https://www.desmos.com/calculator/gd8z5fa0bg`

B1(A): Graphing Techniques (Part I)

General Definitions

1. **Lines of Symmetry**: A *line of symmetry* of a function is a line, such that the function is a reflection of itself about that line.
2. **Horizontal Asymptotes**: A (horizontal) line $g(x) = c$ is the *horizontal asymptote* of the curve $f(x)$ iff $\lim_{x\to\infty} f(x) = c$ (or with $-\infty$ instead of ∞).[a]
3. **Vertical Asymptotes**: A (vertical) line $x = c$ is a *vertical asymptote* of the curve $f(x)$ iff $\lim_{x\to c} f(x) = \infty \text{ or } -\infty$.
4. **Oblique Asymptotes**: A line $g(x) = mx + c$ — where $m \neq 0$ — is an *oblique asymptote* of the curve $f(x)$ iff $\lim_{x\to\infty}[f(x) - g(x)] = 0$ (or with $-\infty$ instead of ∞).

[a]Otherwise notated by $f(x) \to c$ as $x \to \infty$.

Curve Sketching (Rational Funcs)

S Stationary points

I Intersection with axes

A Asymptotes

i Know how to sketch the graphs of $y = \frac{ax+b}{cx+d}$ and $y = \frac{ax^2+bx+c}{dx+e}$.

ii Rectangular Hyperbolas (of the form $y = \frac{ax+b}{cx+d}$):

- *Two* asymptotes, namely $x = -\frac{d}{c}$ and $y = \frac{a}{c}$.
- *Two* lines of symmetry with gradients ± 1 *and* pass through the intersection point of the aforementioned two asymptotes.

iii *If* $n = \deg P = \deg Q$, then

- $y = R(x)$ is the *horizontal* asymptote of $\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}$.
- Equivalently, $y = \frac{\text{coeff}_P(x^n)}{\text{coeff}_Q(x^n)}$ is a *horizontal* asymptote.[a]

iv If $\deg P = \deg Q + 1$, then $R(x)$ is an *oblique* asymptote of $\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}$.

v Write down asymptotes and lines of symmetry.[b] If none are present indicate with “No lines of symmetry.”

[a]E.g.: $y = \frac{1}{15}$ is a horizontal asymptote of $y = \frac{1x^2+2x-3}{(5x+1)(3x+2)}$.

[b]E.g.:

Asymptotes: $x = 4$, $y = 20$.

Lines of Symmetry: $y = x + 16$, $y = -x + 24$.