A-Levels Math Notes

Shao Hong

Contents

A1: Inequalities and Equations

1.1 Solving Inequalities

1.1.1 Rational Inequalities

General Methods

1. Quadratic formula for factorisation / finding roots (of polynomial).
2. Completing the square.
3. Discriminant/Completing the Sq to eliminate* factors which are *always* positive or negative (e.g. removing $x^2 - 3x + 4$). *Note to include coefficient of x^2 in argument.*
4. GC (include sketch).
5. *Rational Functions*[a]: Move everything to one side (+,−), then use number line.
6. Number line (more complicated functions).

[a] Fractions of Polynomials

Important Notes

- ☐ Eliminating Factors — *only*[a] works for $c = 0$ in $f(x) \geq c$ or $f(x) \leq c$.
- ☐ Discriminant — include coefficient of x^2 in argument.
- ☐ Rational functions — exclude the values that causes division by zero to occur.
- ☐ With inequalities, be really careful about multiplication! If $x > y$ *and* $z > 0$, then $xz > yz$.
- ☐ Cross multiplication preserves order for $\frac{x}{y} < \frac{x'}{y'}$ iff y and y' are *both* positive or negative.[b]
- ☐ Squaring preserves order for $x < y$ iff x and y are *both* positive or negative.

[a] Counterexample: $P(x) = x(3x^2 - 9x + 10) \leq 2$ iff $x \leq 2$ is false. E.g.: $P(1.8) = 6.336 \not\leq 2$.
[b] Otherwise, note the counterexample $\frac{1}{2} < \frac{1}{-3}$.

1.2 Modulus Inequalities

Fact

Given $x \in \mathbb{R}$, we have that

- $|x| \geq 0$,
- $|x^2| = |x|^2 = x^2$,
- $\sqrt{x^2} = |x|$.

And as long as $x \in \mathbb{R}^+$,

- $\sqrt{x}^2 = |x|$.

Useful Properties

For every $x, k \in \mathbb{R}$:

(a) $|x| < k$ iff[a] $-k < x < k$.

(b) $|x| > k$ iff $x < -k$ or $x > k$.

(of course, similarly applies for the non-strict ordering $\leq$)

[a] Notice that $k > 0$ here since $0 \leq |x| < k$.

Important Notes

- Note that when solving for $|x| = y$, $|x| < y$, etc, y must be greater than or equal to 0. In other words, there may be solutions we will need to reject.
  (For $<$, equality is of course not allowed.)

Important Notes

- △ Carelessness: Look at the question carefully! If they ask for a *set* of values, then rmb to give it as a *set*!
- △ Exponentiation and Logarithms: Simply use ln and avoid $\log_c$ for $c < 1$.[a]

[a] Order is *Preserved* under exponentiation/logarithms if the base is *larger than* one. Otherwise, when it is *less than* one, the order is *reversed*. https://www.desmos.com/calculator/gd8z5fa0bg

1.3 System of Linear Equations

Things

- χ For more complicated real-world-context qns, try playing around with the values (e.g. use simult eqns) first. It may work out nicer than expected.

1.4 Summary

G.C. Skills

1. Plotting curves $y = f(x)$ in G.C.
2. How to use simultaneous equation solver.

Important Notes

- ☐ Eliminating Factors — *only*[a] works for $c = 0$ in $f(x) \geq c$ or $f(x) \leq c$.
- ☐ Discriminant — include coefficient of x^2 in argument.
- ☐ Rational functions — exclude the values that causes division by zero to occur.
- ☐ With inequalities, be really careful about multiplication! If $x > y$ *and* $z > 0$, then $xz > yz$.
- ☐ Cross multiplication preserves order for $\frac{x}{y} < \frac{x'}{y'}$ iff y and y' are *both* positive or negative.[b]
- ☐ Squaring preserves order for $x < y$ iff x and y are *both* positive or negative.
- ☐ Note that when solving for $|x| = y$, $|x| < y$, etc, y must be greater than or equal to 0. In other words, there may be solutions we will need to reject.
  (For $<$, equality is of course not allowed.)
- ☐ Carelessness: Look at the question carefully! If they ask for a *set* of values, then rmb to give it as a *set*!
- ☐ Exponentiation and Logarithms: Simply use ln and avoid $\log_c$ for $c < 1$.[c]
- ☐ For more complicated real-world-context qns, try playing around with the values (e.g. use simult eqns) first. It may work out nicer than expected.

---

[a] Counterexample: $P(x) = x(3x^2 - 9x + 10) \leq 2$ iff $x \leq 2$ is false. E.g.: $P(1.8) = 6.336 \not\leq 2$.

[b] Otherwise, note the counterexample $\frac{1}{2} < \frac{1}{-3}$.

[c] Order is *Preserved* under exponentiation/logarithms if the base is *larger than* one. Otherwise, when it is *less than* one, the order is *reversed*. `https://www.desmos.com/calculator/gd8z5fa0bg`

A5.1: Differentiation

2.1 Limits

2.2 Geometrical Results of the Derivatives

Definition

(i) A function f is called (strictly) increasing on an interval I iff $f'(x) > 0$ for all $x \in I$.

(ii) A function f is called monotonically increasing on an interval I iff $f'(x) \geq 0$ for any $x \in I$.

Things To Know

1. How to sketch the graph of the integral or[a] derivative of a function f.
2. Relationship btw. a function f and its derivative, f':

$y = f(x)$	$y = f'(x)$
Vertical asymptote at $x = a$	Vertical asymptote at $x = a$.
Horizontal asymptote at $y = a$	Horizontal asymptote $y = 0$.

[a]Of course, provided that f is integrable/differentiable.

B1(A): Graphing Techniques (Part I)

3.1 Graphing 'Familiar' Functions and Asymptotic bois

Definition

1. **Lines of Symmetry**: A *line of symmetry* of a function is a line, such that the function is a reflection of itself about that line.
2. **Horizontal Asymptotes**: A (horizontal) line $g(x) = c$ is the *horizontal asymptote* of the curve $f(x)$ iff $\lim_{x\to\infty} f(x) = c$ (or with $-\infty$ instead of ∞).[a]
3. **Vertical Asymptotes**: A (vertical) line $x = c$ is a *vertical asymptote* of the curve $f(x)$ iff $\lim_{x\to c} f(x) = \infty$ or $-\infty$.
4. **Oblique Asymptotes**: A line $g(x) = mx + c$ — where $m \neq 0$ — is an *oblique asymptote* of the curve $f(x)$ iff $\lim_{x\to\infty}[f(x) - g(x)] = 0$ (or with $-\infty$ instead of ∞).

[a] Otherwise notated by $f(x) \to c$ as $x \to \infty$.

Curve Sketching (Rational Funcs)

S Stationary points

I Intersection with axes

A Asymptotes

i Know how to sketch the graphs of $y = \frac{ax+b}{cx+d}$ and $y = \frac{ax^2+bx+c}{dx+e}$.

ii Rectangular Hyperbolas $\left(\text{of the form } y = \frac{ax+b}{cx+d}\right)$:

- *Two* asymptotes, namely $x = -\frac{d}{c}$ and $y = \frac{a}{c}$.
- *Two* lines of symmetry with gradients ± 1 *and* pass through the intersection point of the aforementioned two asymptotes.

iii *If* $n = \deg P = \deg Q$, then

- $y = R(x)$ is the *horizontal* asymptote of $\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}$.
- Equivalently, $y = \frac{\text{coeff}_P(x^n)}{\text{coeff}_Q(x^n)}$ is a *horizontal* asymptote.[a]

iv If $\deg P = \deg Q + 1$, then $R(x)$ is an *oblique* asymptote of $\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}$.

v Write down asymptotes and lines of symmetry.[b] If none are present indicate with "No lines of symmetry."

[a] E.g.: $y = \frac{1}{15}$ is a horizontal asymptote of $y = \frac{1x^2+2x-3}{(5x+1)(3x+2)}$.

[b] E.g.:

Asymptotes: $x = 4$, $y = 20$.

Lines of Symmetry: $y = x + 16$, $y = -x + 24$.

Important Notes

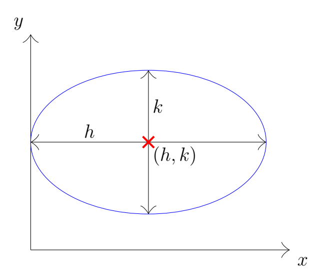
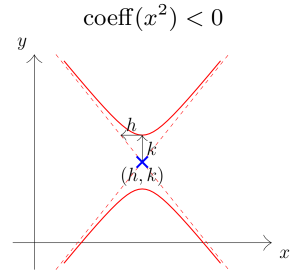
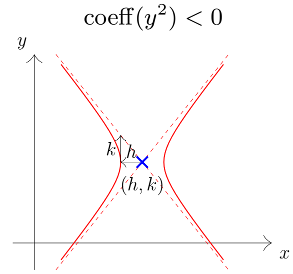
— *Can* explicitly write out the asymptotes and lines of symmetry (or if they are not present) to be safe.

— Using the discriminant intelligently can result in nice answers.

3.2 Conics

3.2.1 Ellipses

"Tikz is pain, PGFPlots is suffering" — Wise Man.

	Ellipses	Hyperbolas
Standard Forms	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
General Equation	$ax^2 + by^2 + cx^2 + dx + e = 0$, where $\operatorname{sgn}(a) = \operatorname{sgn} b$.	$ax^2 + by^2 + cx^2 + dex + e = 0$, where $\operatorname{sgn}(a) \neq \operatorname{sgn} b$.
Center	(h, k)	(h, k)
Vertical 'Radius' (variables here from *standard form*!)	b	b
Horizontal 'Radius' (variables here from *standard form*!)	a	a
Vertical Vertices (variables here from *standard form*!)	$(h, k \pm b)$	$(h, k \pm b)$
Horizontal Vertices (variables here from *standard form*!)	$(h \pm a, k)$	$(h \pm a, k)$
Shape	 	coeff$(x^2) < 0$  coeff$(y^2) < 0$ 
Asymptotes (No need to rmb!)	$y = k \pm \frac{b(x-h)}{a}$	$y = k \pm \frac{b(x-h)}{a}$
Lines of Symmetry	$x = h,\ y = k$	$x = h,\ y = k$

General Info

$\mathscr{H}$ To find asymptote of hyperbolas, just solve

$$\frac{(x-h)^2}{a^2} = \frac{(y-k)^2}{b^2}.$$

$\mathscr{H}$ Label vertices *or* radii, together with the center and asymptotes.

3.3 Parametric Equations

Important Notes

⋆ Check the qns for any *restrictions* on the parameter! And modify that of the G.C.'s accordingly (Tmin & Tmax).

3.4 Summary

G.C. Skills

1. Plot conics with the two ways.
2. Know G.C. functions like finding axial intercepts.

Important Notes

-☐- *Can* explicitly write out the asymptotes and lines of symmetry (or if they are not present) to be safe.

-☐- Using the discriminant intelligently can result in nice answers.

-☐- Check the qns for any *restrictions* on the parameter! And modify that of the G.C.'s accordingly (Tmin & Tmax).

Statistics 1: Permutations and Combinations

4.1 The Addition and Multiplication Principles

Example 1: The Addition Principle

There are three distinct cups of black sugar bubble tea and five unique cups of zero sugar bubbles tea available, I am buying *exactly* one of them. How many choices do I have? Answer: $3 + 5 = 8$.

Example 2: The Multiplication Principle

A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible. Answer: $3 \cdot 4 \cdot 5 \cdot 2 = 120$.

4.2 Permutation

Definition 1: ${}^{n}P_k$

$$ {}^{n}P_k := \frac{n!}{(n-k)!} $$