The Single Cop Single Robber Graph Pursuit Evasion Game and Significance of Variations of Conditions

1st Shi
Harvard University

I. INTRODUCTION

Robotics applications in the past have inspired a lot of research at the intersection of traditional disciplines, and recently researchers have begin to study the application of robotic systems to search missions and pursuit-evasion games. [1]

A pursuit-evasion game is one in which one or more pursuers try to capture one or more evaders who, in turn, try to avoid capture. In robotics, pursuit-evasion games can be used to study motion planning problems, such as catching burglars, playing hide-and-seek, etc. Additionally, pursuit-evasion games can be simulated and analyzed to obtain results on the worst-case performance of robotic systems, for example, a search and rescue robots system.

Pursuit-evasion games are also applicable to numerous other fields such as network security, modelling animal behavior in game theory, etc. They provide valuable insight into real life applications and occurrences, and therefore studying them as models with various conditions to mimic real life occurrences are analytically significant. There exist many different versions of the pursuit evasion games. Variations include considerations of the limitations of the environment, the information available to the players, controllable of the players motion, and the meaning of the capture. In this paper, I will be exploring some of these options. [1]

I limit the focus to graph pursuit-evasion games, specifically the single cop Cop and Robber game, and the variations of this game by the parameters described above. In this game, the cops (pursuers) try to capture a robber (evader) by moving along the vertices of a graph. The players move in turns along the edges. The cops win the game if they can move onto the robbers vertex. The problem that immediately arises is to answer whether or not with the conditions given, the cop can capture the robber. Thus in this paper we will be exploring algorithms to determine if a game with various conditions listed above as given is a cop win or a robber win. That is, whether the robber will eventually be caught as defined, or if the game will never end and the robber will always evade capture, respectively. Exploring this problem and some variations of it give us insights and analyses that are applicable to robotics and other fields as mentioned above. [1]

This falls under the CS 286 topic of Differential Games, which are defined as a group of problems related to the modeling and analysis of conflict in the context of a dynamical system. [7]

II. LITERATURE REVIEW

The baseline for this project aligns with the paper ”Search and pursuit-evasion in mobile robotics” by Chung et al [1]. In the paper they also address the (unsolved) problem of specifically trying to determine the minimum number of cops necessary to capture the robber regardless of the initial locations of the players, but I will be studying the single cop algorithm and addressing the problem of whether the cop will catch the robber. There exist many variations of the problems presented in this game, as well as many variations of the game itself, all of which are studied and analyzed for their possible insights and applications to robotics.

The cops and robbers game with the case of a single cop is initially brought up by Nowakowski and Winkler in ”Vertex-to-vertex pursuit in a graph”[2]. In this game, the cops (pursuers) try to capture a robber (evader) by moving along the vertices of a graph. The players move in turns along the edges. The cops win the game if they can move onto the robbers vertex. The immediate question raised is whether the cop can catch the robber, and in how much time [2].

There is also the question explored about the number of cops required for the environment, and then Aigner and Fromme in ”A game of cops and robbers” show that the cop number of planar graphs is at most three by limiting the structure of the graph itself from the general k cops problem. This game explores how many cops are required to capture a robber for the general case, given initial locations [4].

The k cops game given initial locations is also studied in Goldstein and Reingold ”The complexity of pursuit on a graph”[3]. However, the problem remains that determining whether k cops with specified initial locations can capture a robber on some given undirected graph is EXPTIME-complete

[1] . Therefore, it is not likely that significantly more efficient algorithms exist for the general case for the k cops.

Beyond the environment, there are other versions of this game depending on the variations of the different factors of the game. For example, in regards to player motion controllability, a different version of this game where players cannot observe each other unless they are on the same vertex is studied by Adler et al. in ”Randomized pursuit-evasion in graphs” [5]. This is an extension known as the hunter and rabbit game, which is so named for intuitive reasons. They showed that in the no-visibility version, a single pursuer can catch the evader in $O(n \log n)$ expected time on any graph, and that this is a tight analysis. [1]

Isler et al. in ”The role of information in the cop and robber game” studied the case where the evader only has local visibility with some range of distance [6]. They studied the variation in which players move simultaneously, and they introduced the notion of i-visibility. In this instance, a player with i-visibility can only view another player if and only if they are within a i distance of visibility radius [5].

Again, these are variations of the cop and robber problem which have been explored for insights and applications to robotics, but I will be exploring the problem of the single cop single robber instance and determining whether it is possible for the cop to win.

III. Baseline Simulation Results

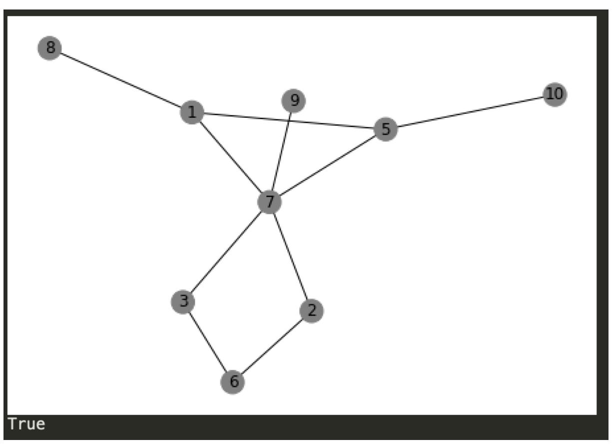

Fig. 1. Any graph that all states are eventually marked is a cop win. It can be seen here that any node can be reached by any other node, and therefore, and therefore eventually there will be a state of the game in which the cop and robber end up in the same position.

The baseline algorithm I implemented is the one given by Chung et al. [1] Namely, let us have a graph $G = (V, E)$ serve as the environment of this game. Then for each vertex $v \in V$ there will be a neighborhood $N(v)$ such that $N(v) = \{u : (u, v) \in E, u, v \in V\}$. Specifically I will simulate multiple iterations for the case of undirected graphs with 10 nodes and 14 edges that are randomly generated, and then I will simulate the cases for more edges.

I then employ a bottom up dynamic program algorithm in which determines if a game state, as defined by (c, r) cop

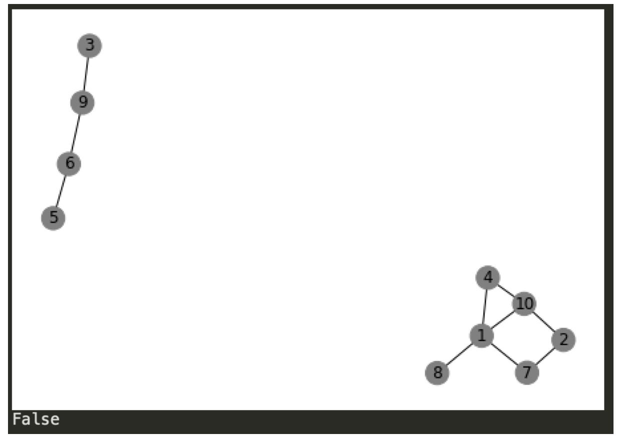

Fig. 2. A graph in which there remains unmarked states means a robber win. It can be seen that because the graph is not connected, that means that the robber can simply choose an initial position on a different component than the cop and therefore the cop will never catch it.

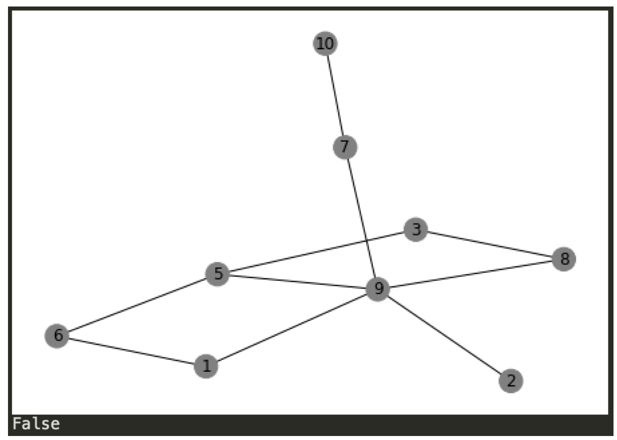

Fig. 3. A graph in which there remains unmarked states means a robber win, but it does not necessarily have to be unconnected. In this case, there is simply a cycle in which .

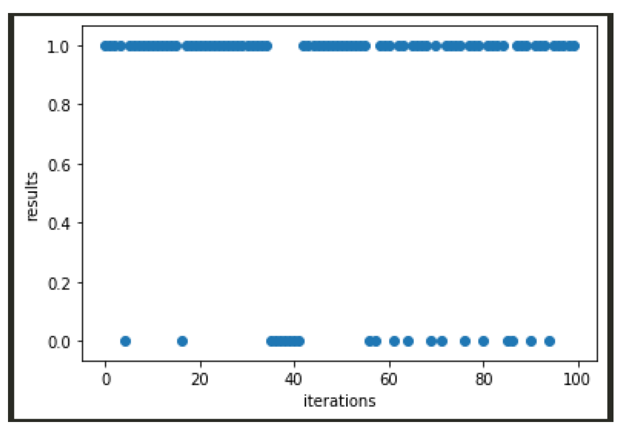

Fig. 4. Most randomly generated graphs with the given edges and nodes will be cop wins.

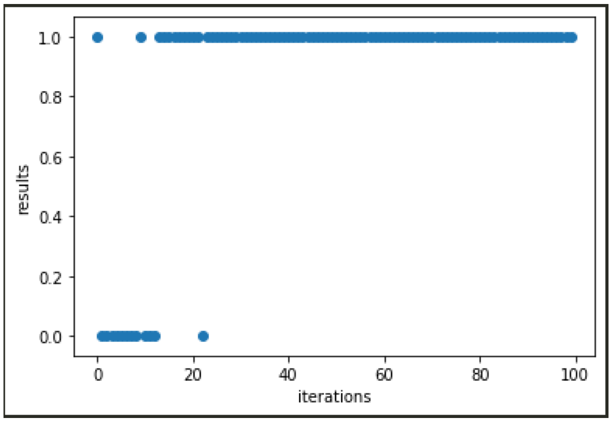

Fig. 5. In varying edges, most of the games become cop wins after the number of edges exceed the number of nodes

position, robber position respectively, can eventually reach a state (c, r) such that $c = r$. This is done by first marking all states such that $c = r$ and the robber is caught by our basic definition, and repeating for all unmarked states such that if it is in the neighbor of a marked state, it is marked. Then it can be proved by induction that if a state is marked, it is able to reach a game state in which the cop can catch the robber from those initial positions. This is repeated until there are no further markings possible. If there are no unmarked states remaining, then that means a cop can catch the robber from any starting state and therefore a cop-win is eventually possible by induction on the marking order. This is shown by results in Fig. 1.

Alternatively, if not all states are marked, that means there is some state such that the robber wins the game. This follows by the reasoning that for some unmarked (c, r) there must be a vertex $r' \in N(r)$ (the neighbors of r) such that if the robber moves to r', there is necessarily a c' such that (c', r') is unmarked by the construction of an unmarked state (otherwise (c, r) would be marked). Then the robber can therefore force the game to a halt and the cop will never catch the robber and it will be a robber win if the robber goes to that position. By our definition of a robber win, if a cop can be shown to never catch the robber and therefore the game would continue forever, the robber wins . This is shown by results in Fig. 2 and Fig. 3.

When we have the graph as given, a randomly generated undirected graph with 10 nodes and 14 edges, most of the games end up being cop wins as in Fig. 4. These numbers were chosen to demonstrate an approximate graph that has a good change of being well connected but not too well connected. Then in the case of varying edges from 1 to 100, it can be seen that after the number of edges exceeds the number of nodes, most graphs will be cop wins in Fig. 5, which is consistent with our analysis in Fig. 4.

What is missing from this algorithm is that it only considers the optimal and rational conditions under which the cop and the robber operate. That is, it says that if there is a way for the player to win, either by eventually capturing the robber or forcing the game to a halt, the cop or robber will win respectively. I believe that further insight into the problem can be given when we consider the limitations of the players themselves. This is also more applicable to a real life situation, be it the irrational and subjective decisions of the cop or robber in evasion, or the speed and visibility limitations of a robotic system or network. Therefore, I believe this necessitates certain extensions of the game.

IV. Extensions

The extensions I will explore will still pertain to the single cop single robber game. That is, I will be exploring the same problem of whether the game with conditions as given will be a cop or robber game, or if it is indeterminable. However I will introduce some considerations I believe have not already been considered on the subject that are real considerations when considering the applications of the game.

A. Pursuit with Limited Visibility

In the basic cops and robbers game, it is assumed that the players know each other's positions at all times. There are some settings where this is plausible, but in most practical applications to robotics or anything of pursuit evasion games, graph pursuit-evasion games, and the cop robber game in particular, this unrealistic. [1]

However, another factor to consider is that sometimes the control also does not absolutely know all player positions at all times. For example, in the case of someone controlling a robot system, or observing animal behavior, there is always an error probability that the positions of the robber and the cop as reported are incorrect. This will inevitably affect the conclusions and analyses of whether the game as it is given is going to be a cop win or robber win.

Specifically, limited visibility is a problem explored in the form of limiting the range of view (a cop has a limited visibility range of i and a robber has a limited visibility range of j). This is explored in the hunter-rabbit game [5]. However, I will explore the effects of adding noise to the knowledge of the robber and cop positions themselves. That is, for any state (c, r) there is some error $\epsilon > 0$ that confounds the true state. Then I will calculate the conclusion of whether the game is a cop win or robber win as it is set up, and discuss the error of the game conclusion. I hypothesize that the correctness metric will increase with visibility.

Regarding implementation, we will include some visibility constraint at each node. That is, for each node of the graph, there will be some probability $P(v) \in [0, 1]$ such that there is a probability $P(v)$ chance that the player is actually on that node.

Then when the marking algorithm is implemented, each time a state is marked, we mark the probability that it is actually the true state of the game, namely $P(c)P(r)$ for the state (c, r). Then the cumulative sum of all those probabilities is divided by the total number of possible game states to give

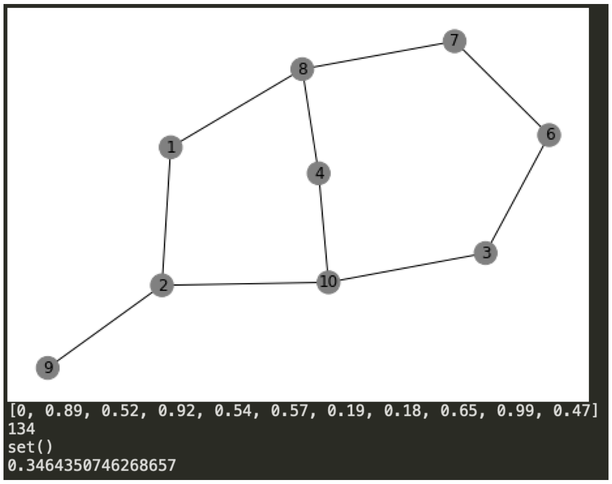

Fig. 6. A graph as shown with the visibilities for each node in order. Then when each state is marked, it will return a correctness metric that accounts for the correctness that all those states are marked adjusted for visibility.

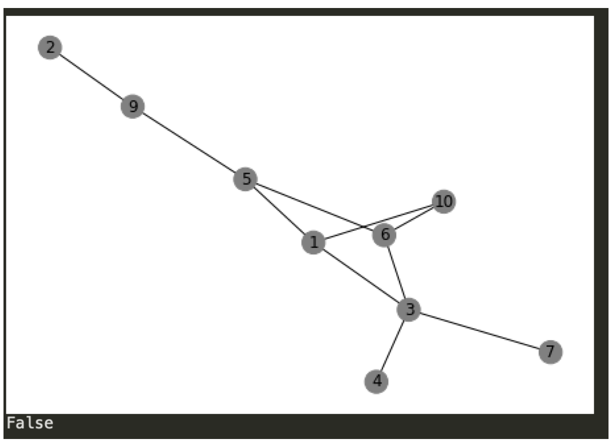

Fig. 7. A graph as shown with the visibilities for each node in order. If a state is unmarked, the algorithm will still return False under the assumption that any unmarked state is considered unmarked with probability 1.

us a resulting metric, as in Fig. 6. If there remain unmarked states we still return False as in Fig. 7.

This metric serves as the percentage of correctness of our conclusion. The validity of this metric follows from that if there was full visibility, and all probabilities were 1, this metric would become the total number of marked states divided by the total number of states. If the graph were to be a cop win, this metric would be 1, and < 1 otherwise if it was a robber win. The probability of a state being less than instead of 1 means that we are accounting that for each state we mark, there is a certain possibility that it should not be marked (the cop and robber actually are not there). Therefore, this can be thought of as measuring the probabilistic number of unmarked states and the metric serves to measure the relative correctness of this conclusion, namely, how sure the cop win is.

Note that the algorithm will return False and give a robber win if there remains an unmarked state because the probability that an unmarked state should be marked is low to none because it would mean its unmarked neighbor should be marked, and its neighbor, and so on and so forth. Therefore for simplicity, we assume that the unmarked state is certainly unmarked. This assumption intuitively means that the impacted visibility would benefit the robber, which makes sense because they could force the game into a halt, or as the players believe to be a halt.

It can be seen in Fig. 8 that as the visibilities increase – as it becomes more confident for each node that the state of the game is actually (c, r) for the node positions c and r– so does the correctness metric.

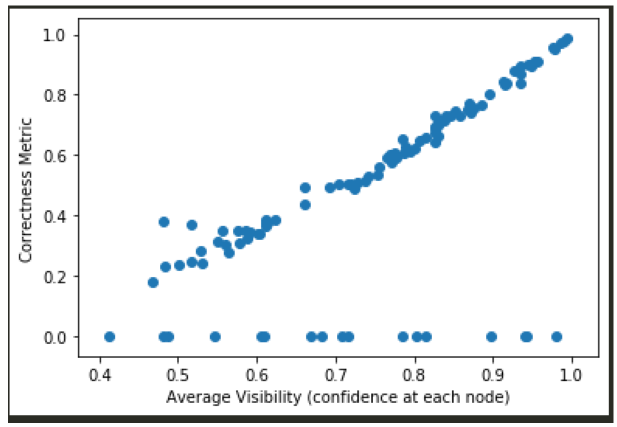

Fig. 8. As the average visibility probability increases, the correctness probability also increases proportionately.

Namely, this indicates that there is a linear and directly proportional relationship between the values of the visibility probabilities and the resulting correctness metric we have defined. There are some exceptions where the correctness metric is 0, because there remained unmarked states which we have assumed to be for sure unmarked. The significance of this relationship being directly proportional is that we can determine a threshold $p \in [0, 1]$ such that if the final correctness metric is $> p$, it is considered most likely a cop win and it is considered most likely a robber win otherwise. For example, it is statistically reasonable to use 0.5 for p since it acts as a reasonable divider in the middle. However, it should be noted that different values of p could be explored for various levels of strictness. There could be a range in which the game is indeterminate.

But as predicted, that higher error values (lower visibility) will lead to lower correctness. The direct proportionality signifies that with a few outliers, it is almost certain that any average visibility below 0.5 will mean that the correctness metric will be below 0.5, and therefore mean a most likely robber win. This can be extended to the general p case.

This aligns with the intuition that low visibility means that in the case of ambiguity, the worst case should be assumed and the robber can be assumed to be able to halt the game and never be caught. This is applicable to real life situations and practically speaking it is more advantageous to assume towards the worse case so that the best case conditions can be tightly analyzed.

B. Pursuit Speed

When we consider the classic Cop and Robber problem explored in the baseline, we consider the cop and robber to have the same speeds when evaluating the connections of the game states. While there are some settings in which this is plausible, in most practical applications to robotics and in game theory, this is unrealistic.

Namely, in a more realistic setting the cop and robber might have different speeds. For example, in the case of a police drone trying to find a human escaped convict, it is much more likely that the drone can move much faster than the human evader. Same situation applies with a human cop in a car and an escaping robber on foot.

Specifically, another extension I will pursue is the condition of the cop having faster speeds relative to the robber, and whether that causes some robber win games to become cop win. I predict that having a faster cop means that some games that were initially robber win games with the given conditions will become cop win games.

Note that in addition to being less realistic and applicable, the case of the robber being faster than the cop can be thought of as the symmetric case, so we will only simulate the case of the cop being faster than the robber and assume the symmetric case for the robber being faster than the cop.

In implementing this, we will have some relative cop speed s_c capacity represented as a whole number of the ratio of the cop speed to the robber speed when the robber speed is 1. Namely, $s_c = s_c : s_r$ and $s_r = 1$, so for each 1 robber step the cop can choose to take any of $[1, s_c]$ steps. This speed limit seems more practical to applications than the alternative option where the cop can only take s_c steps. an a I chose to only evaluate the discrete case and not consider any fractional ratio such as 3 : 2 both for simplicity and because the general trend can be found with only the discrete whole number cases.

Then we will have that for each state which should be marked, (c, r), the states which have the cop positions as the set of all neighbor nodes C within $s_c$degrees of c and the same value of r should be also marked because the cop can choose to move as many steps within that as they want, for the same r position. This is done with a recursive program that will recurse through s_c degrees of neighbors of each c in each state (c, r) to be marked.

We have plotted in Fig. 11 the relative cop speed to the robber speed in discrete numbers against the change from going from 1:1 cop to robber speed to the new increased cop to robber speed ratio. We find that for some graph configurations, increased cop speed will indeed change the game from a robber win to a cop win and the robber is no longer able to force the game to a halt as in Fig. 10. In other games, even a greatly increased relative cop to robber speed ratio is unable to make the game a cop win, as is the case in Fig. 9. Additionally in the 0 change figures of the data, however are the graphs which are already cop-wins and have not changed and remain cop-wins. Further extensions could be made towards what the effects on the speed of capture is. Note that since there are no negative numbers, the data suggests that increasing the cop

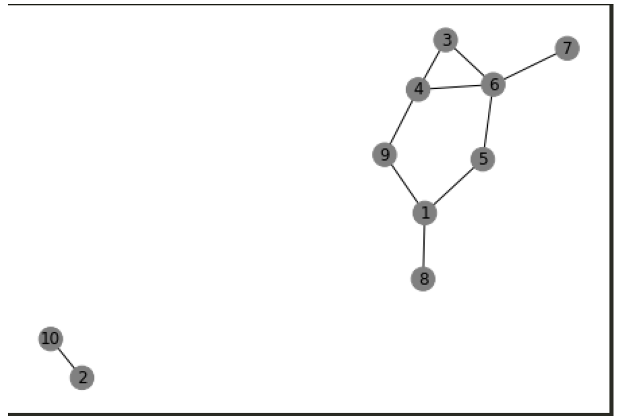

Fig. 9. This graph even with the cop speed limit at 14 will still be a robber win because the graph is not connected.

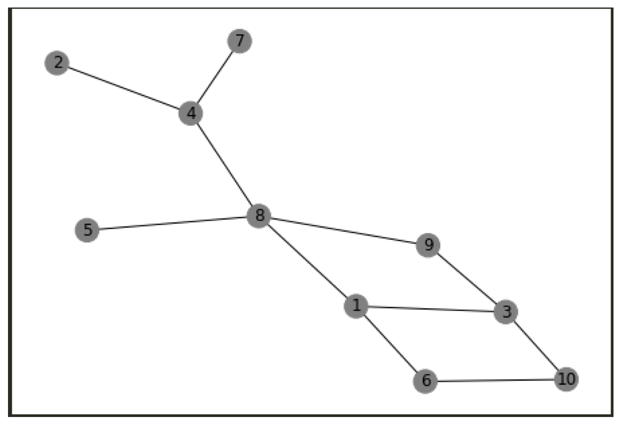

Fig. 10. For this graph, the cop was unable to catch the robber at a 1:1 speed but is able to catch the robber at a 12:1 speed

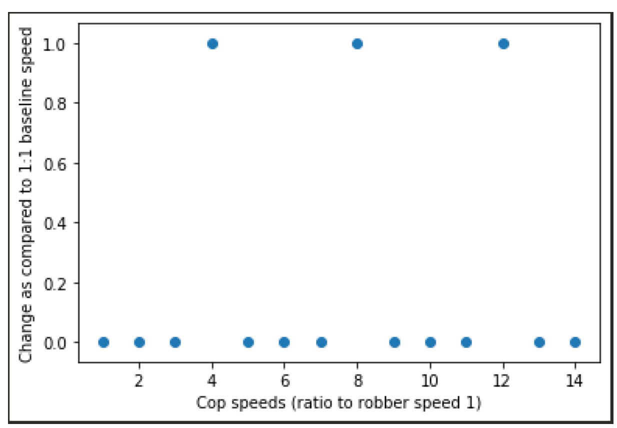

Fig. 11. It is evident that for some graphs, having greater cop speed compared to robber speed will actually change the game from a robber to a cop win. For others where there is no change, even relatively great speeds will not ensure a cop win, or it was already a cop win so there is no change. There is no negative result, which means that increasing cop speed only increases the number of game configurations that are cop wins and never decreases.

speed can only change the games in the cop's favor, and not the robbers' , which aligns with our initial hypothesis.

The significance of these results also apply to the symmetric robber case. That is, we can also predict that a faster robber to cop speed would indicate that the game would favor the robber win more than a 1:1 speed ratio. This also suggests that which ever side has a greater speed, individual games will either be unaffected in result, or be more likely to be a win for the player with greater relative speed. Additionally, over many discrete games, the results will trend more towards the faster player than in the 1:1 case. This could also mean that in application, speed is a worthwhile investment for most robot systems and also provides only advantage in the pursuit-evasion game.

C. Pursuit on Directed Graph

In the base case of basic Cops and Robbers, there are 2 way roads in the environment which makes it an undirected graph. However, this is unrealistic to practical applications of this game to certain instances of robotics and other fields. Take for example a system with autonomously driving vehicles that serve as self driving police cars, which must follow roads. Some roads only have 1 way lanes, others 2 way, but its not guaranteed to be 2 way. In this case, the map can be represented as a directed graph which allows bi directions, representing 2 way roads, and parallel edges , representing multiple ways to get from one location (node) to another.

I will implement the graph as a directed graph that allows parallel edges as well as bidirectional edges, and then I will run the same marking algorithm as before which goes through all the states and marks them if they are able to reach a cop win state. I will analyze the effect that changing it to a directed graph will affect the distribution of cop to robber win in a randomly generated graph. I hypothesized that there will not be a huge difference between the cop and robber wins between the cop and robber wins, but it can be seen in the results from Fig. 14 that for a graph with 10 nodes as in the baseline, there are not many occurrences of cop wins until the edges reach the 20 to 40 range, increasing significantly over that range. This is in contrast to the baseline results Fig. 5 where the cop wins are immediately dense after it becomes possible for a cop win (ie. when there are more edges than nodes in the graph and the case is non-trivial). This is not what I predicted and suggests that cop wins are occurring much less frequently for the same graphs in directed graphs than undirected.

It is clear from Fig. 14 that the number of edges in a directed graph correlates directly with the frequency of cop wins and inversely with the robber wins. With directed edges, it appears as if it takes twice the number of undirected edges to achieve the same increase in frequency of cop edges for given graph, with a few outliers. Examples can be seen in Fig. 12 and Fig. 13 that graphs with fewer edges will have game states where the robber is able to force the game to a halt, and similarly graphs with more edges will necessarily mean an eventual cop win from any starting points in the graph.

These results are significant in that directed graphs significantly reduce the frequency of a cop win from undirected

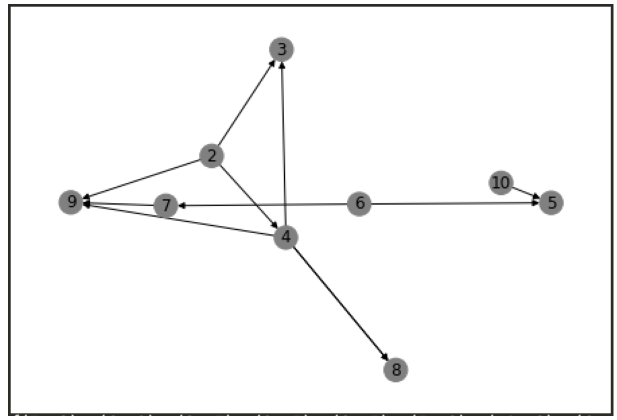

Fig. 12. For this graph, the cop was unable to catch the robber with 9 edges in the graph.

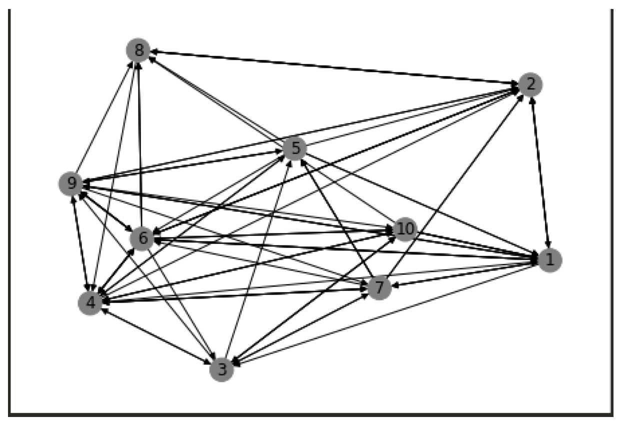

Fig. 13. For this graph, the cop was able to catch the robber with 98 edges in the graph.

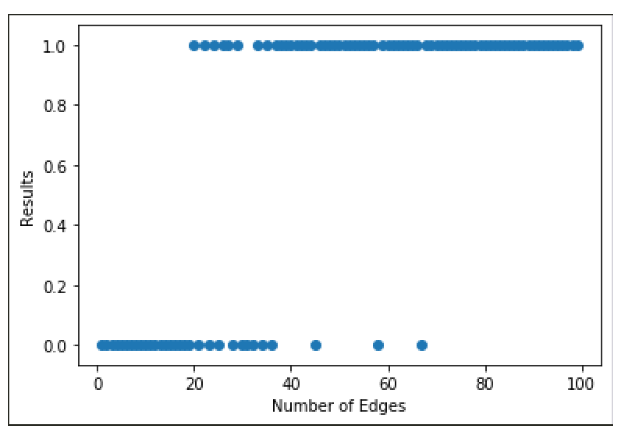

Fig. 14. It is evident that as the number of edges increase, it trends more and more towards that a graph will be a cop-win. The greater the number of edges, the higher the density of cop wins, and the lesser the number of edges, the higher the density of robber wins.

graphs when they are the given environment and requires more connectivity to increase the frequency of cop wins. In application, this means that in most cases, having all bidirectional pathways will more likely benefit the evader/robber than the pursuer/cop, and that increasing the number of edges helps to ameliorate these effects but requires approximately a twofold increase.

V. Conclusion

From the extension study, I am able to conclude that the visibility of the positions is directly proportional to the correctness with which an analysis can be made about the result of a given cop-robber game. This has applications to fields such as animal behavior analysis, or some remote robotic pursuit system because with the assumption that the lack of visibility benefits the evader, then we are able to get a tighter analysis of the success of the system with some metric and the resulting system which is tested and developed by such metrics will be more robust to failure. More studies can be done without the assumption of unmarked states remaining unmarked in order to test the effect of visibility from the robber's perspective.

I am also able to conclude having greater relative speed can only benefit or do nothing for a player for a given game, and will trend towards benefiting the player. This is consistent with predictions that greater speed benefits a player, and it is applicable to real world applications of pursuit evader games to various fields as well, most likely decisions about the effectiveness and marginal benefits of investing in greater speed for players. More research can be done with statistical analysis of these results to look into whether there is a certain zone of effectiveness that optimizes the cost of investing in more speed and the marginal benefit of having a speed advantage.

From the final extension, we conclude that in a directed graph, the number of edges is directly correlated with the frequency of cop wins (and inversely robber wins), and that it requires approximately twice as many edges to achieve the same frequency of cop wins as in the undirected graph. This has significance to applications of higher stakes pursuit and evasion games because it demonstrates that the better connected a graph is, the more it benefits the pursuer. More research into this field could be done into the best paths for the pursuer/cop to choose that would maximize the connectivity of the graph depending on where the evader/robber is at any moment.

The baseline algorithm did not fail to analyze the base case with optimal conditions in that it was able analyze whether it was a cop win or robber win depending on the structure of the graph. However, the purpose of this problem is to provide insight and analysis into real life applications of the game model, so while the base line algorithm is sufficient in analyzing cop wins or robber wins in a hypothetical situation with the only varying condition being the structure of the environment itself, real life applications would need to consider other conditions such as visibility, player movement control, accessibility, which we have explored in this paper, and many other conditions as well. By exploring these extensions and isolating the conditions, we are able to gain more insight into the effects of the conditions on the game and of the real life applications that it describes.

The extensions were able to demonstrate a more accurate consideration of more real world conditions so that the analyses can be more applicable to real world applications. They demonstrated favorable performance because in an mathematical model, which is what a differential game is fundamentally, the conditions must be altered and the game made more complex to simulate applications and provide information for real life situations.

Ideas for future work would include the future considerations given above, as well as exploring more variations of the conditions such as considerations of the limitations of the environment, the information available to the players, controllable of the players motion, and the meaning of the capture. The meaning of the capture could be very significant in that capture requirements would by construction be different for different problems, and may very well change the outcomes and the effects other conditions have on the outcomes.

One of the main things I learned from this is that pursuers in general seem to benefit more from the environment's "power" so to speak. Namely, the number of edges of the graph, the number of ways a player can traverse the graph, the connectivity and therefore the speed of travelling the nodes of the graph, the visibility, etc. This impacts my understanding of pursuit evasion games significantly because it seems intuitively that an environment that allows for greater player movement control would benefit both players equally and not actually affect the outcomes of the game, so this definitely changes the narrative and increases the importance of the environment itself in real applications of these games to various fields of robotics, game theory, animal behavior, etc, as is the purpose of these mathematical model games.

References

[1] "Search and pursuit – evasion in mobile robotics, a survey" by Timothy Chung, Geoffrey Hollinger, and Volkan Isler in Autonomous Robots 2011

[2] Nowakowski, R., Winkler, P. (1983). Vertex-to-vertex pursuit in a graph. Discrete Mathematics, 43(2–3), 235–239.

[3] Goldstein, A. S., Reingold, E. M. (1995). The complexity of pursuit on a graph. Theoretical Computer Science, 143(1), 93–112.

[4] Aigner, M., Fromme, M. (1984). A game of cops and robbers. Discrete Applied Mathematics, 8(1), 1–12.

[5] Adler, M., Räcke, H., Sivadasan, N., Sohler, C., Vöcking, B. (2003). Randomized pursuit-evasion in graphs. Combinatorics, Probabil- ity Computing, 12(3), 225–244.

[6] Isler, V., Karnad, N. (2008). The role of information in the cop- robber game. Theoretical Computer Science, 3(399), 179–190 Special issue on graph searching.

[7] Tembine, Hamidou (2017-12-06). "Mean-field-type games". AIMS Mathematics. 2 (4): 706–735. doi:10.3934/Math.2017.4.706.