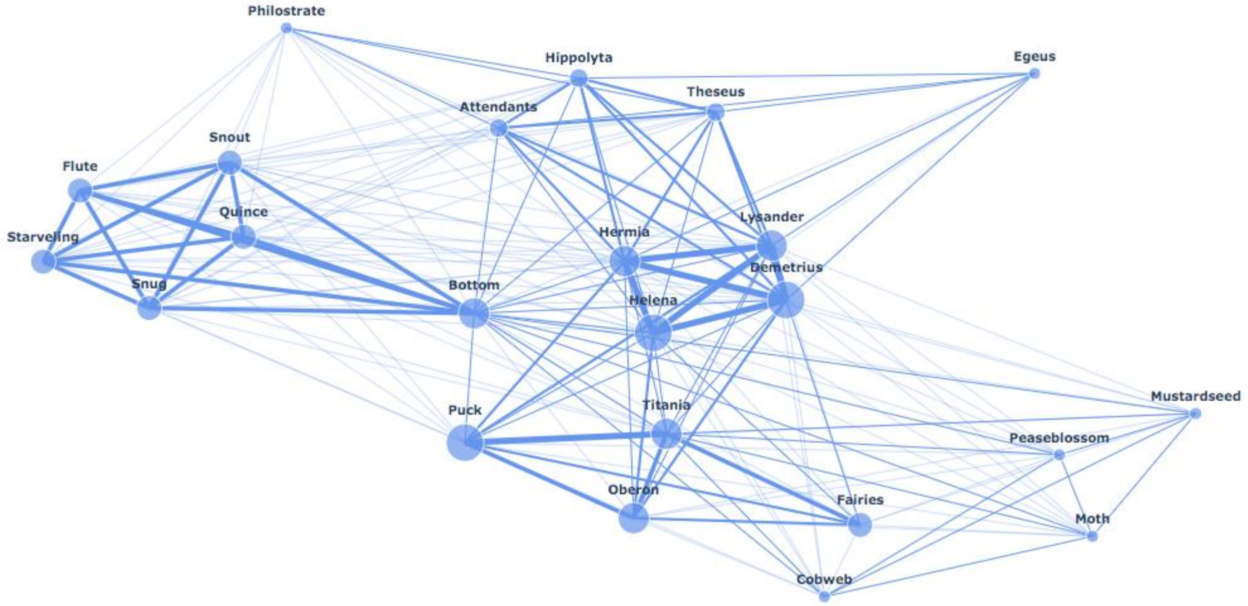
2025

TdP

Intro to Graphs

Theoretical foundations

Giuseppe Averta
Carlo Masone
Francesca Pistilli
Davide Buoso
Gaetano Falco



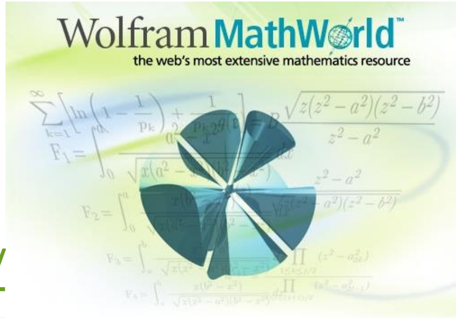
Introduction to Graphs

DEFINITION: GRAPH

Definition: **Graph**

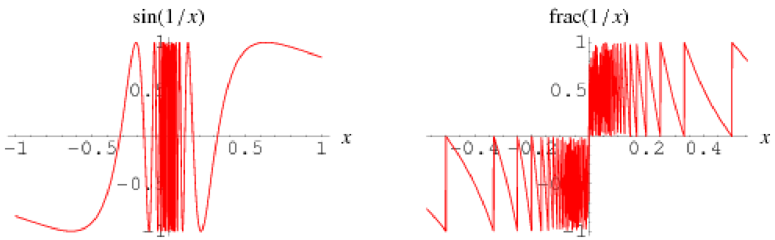
- A **graph** is a collection of **points** and **lines** connecting some (eventually empty) subset of them.
- The points of a graph are tipically known as **graph vertices**, but may also be called “nodes” or simply “points.”
- The lines connecting the vertices of a graph are called **graph edges**, but may also be called “arcs” or “lines.”

http://mathworld.wolfram.com/

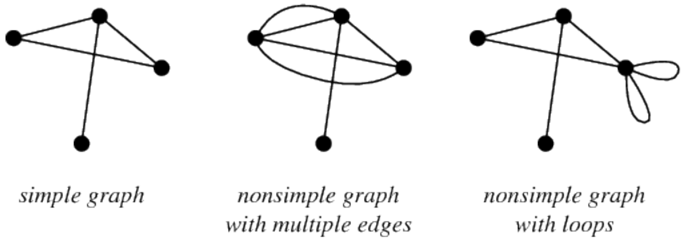


Big warning: Graph ≠ Graph ≠ Graph

- Graph (plot)
- (italiano: grafico)

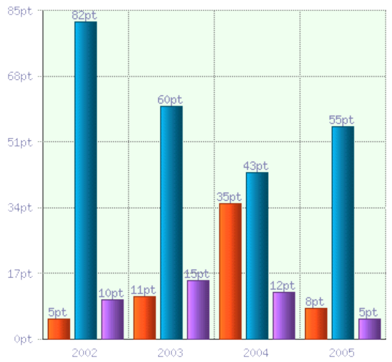


- Graph (maths)
- *(italiano: grafo)*



≠

Graph (chart)
(italiano: grafico)



History

- The study of graphs is known as **graph theory**, and was first systematically investigated by D. König in the 1930s
- Euler's proof about the *walk across all seven bridges of Königsberg* (1736), now known as the *Königsberg bridge* problem, is a famous precursor to graph theory.
- Indeed, the study of various sorts of paths in graphs has many applications in real-world problems.

Königsberg Bridge Problem

- Can the 7 bridges the of the city of Königsberg over the river Preger all be traversed in a single trip without doubling back, with the additional requirement that the trip ends in the same place it began?

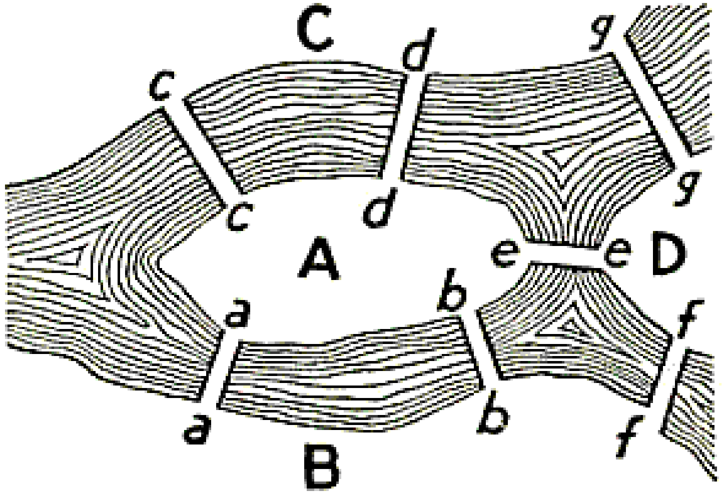
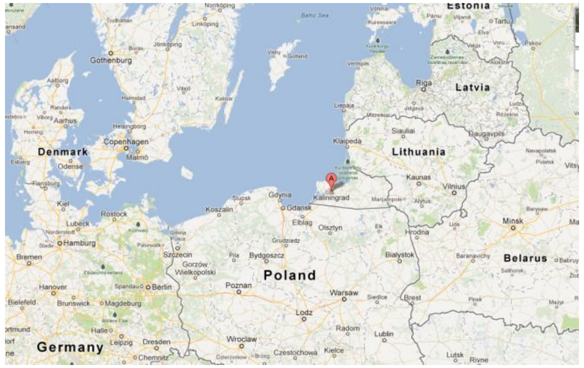


FIGURE 98. *Geographic Map: The Königsberg Bridges.*



Today: Kaliningrad, Russia

Königsberg Bridge Problem

- Can the 7 bridges the of the city of Königsberg over the river Preger all be traversed in a single trip without doubling back, with th quirement that the trip ends ace it began?

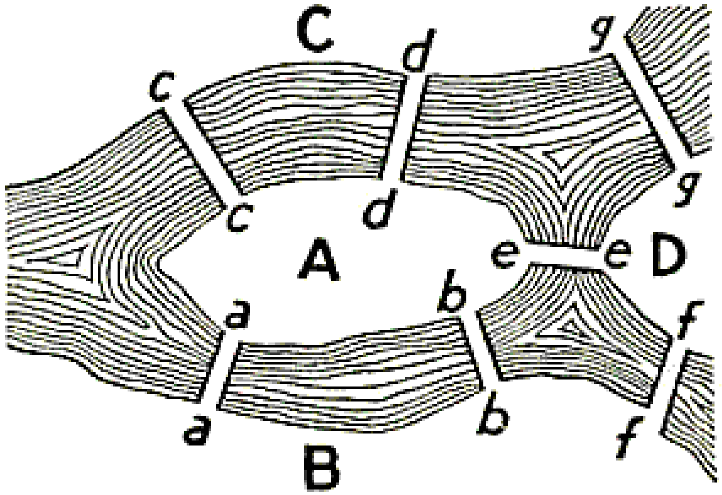
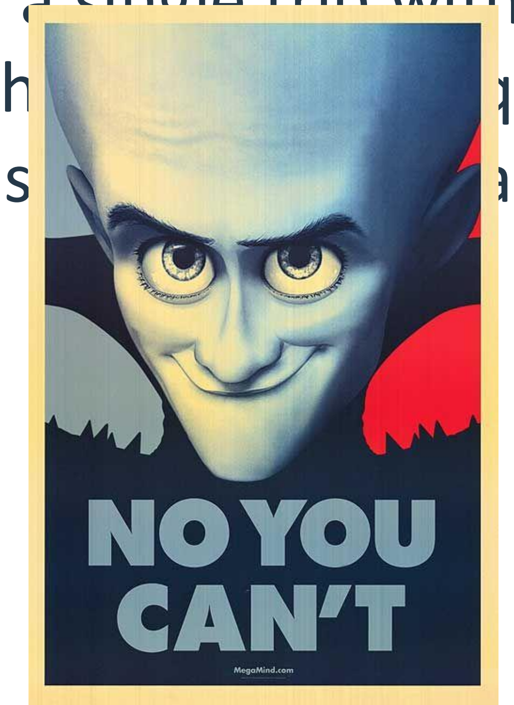
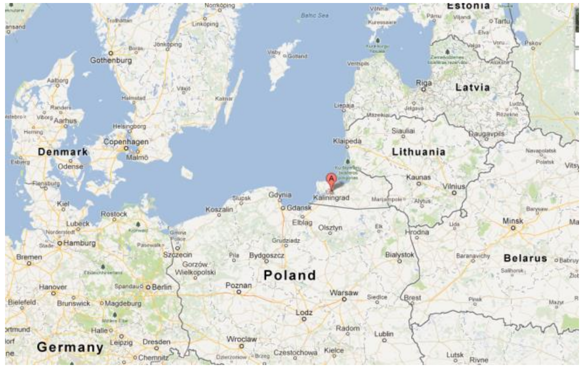


FIGURE 98. *Geographic Map: The Königsberg Bridges.*

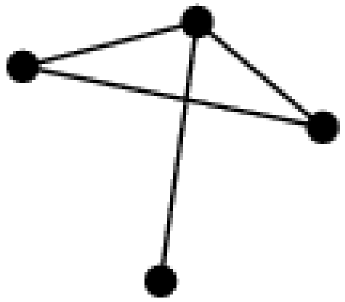




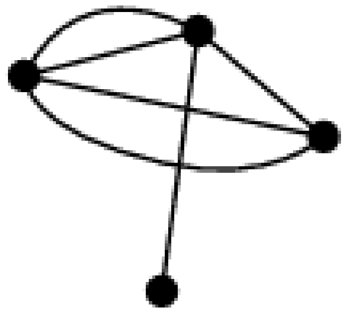
Today: Kaliningrad, Russia

Types of graphs: edge cardinality

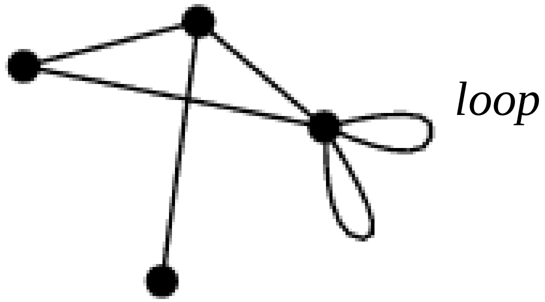
- Simple graph:
  - At most one edge (i.e., either one edge or no edges) may connect any two vertices
- Multigraph:
  - Multiple edges are allowed between vertices
- Loops:
  - Edge between a vertex and itself
- Pseudograph:
  - Multigraph with loops



simple graph



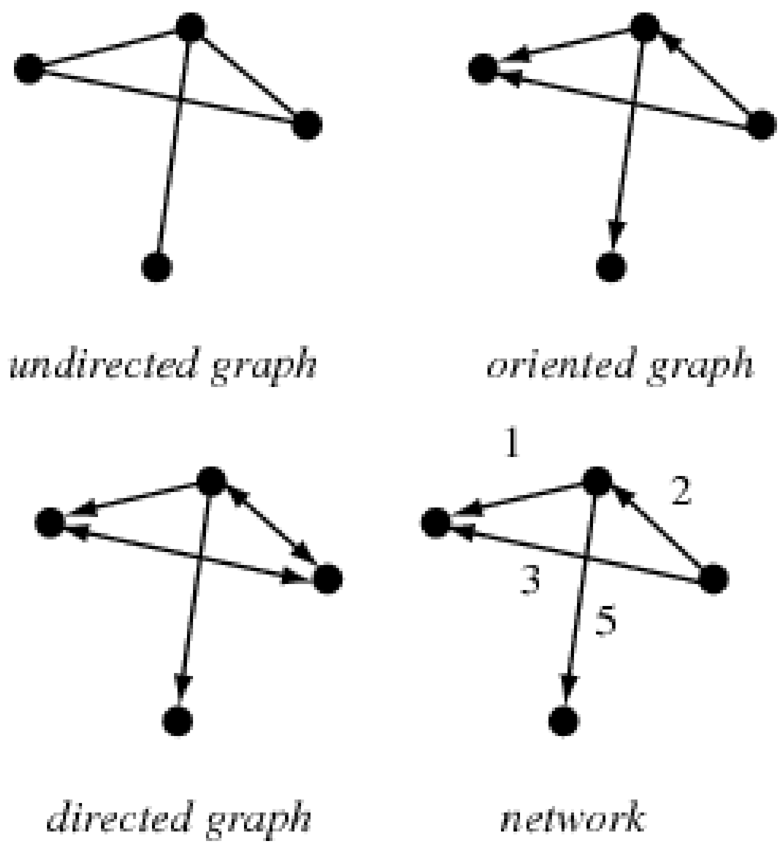
multigraph



pseudograph

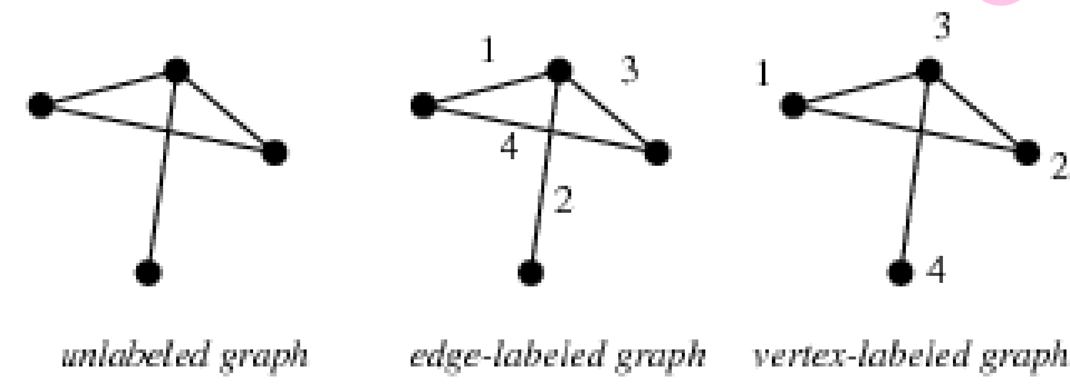
Types of graphs: edge direction

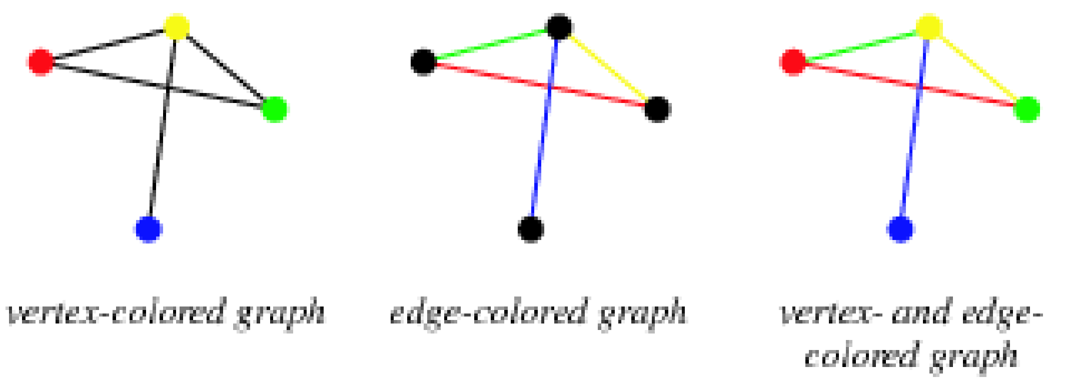
- Undirected
- Oriented
  - Edges have **one** direction (indicated by arrow)
- Directed
  - Edges may have **one or two** directions
- Network
  - Oriented graph with weighted edges



Types of graphs: labeling

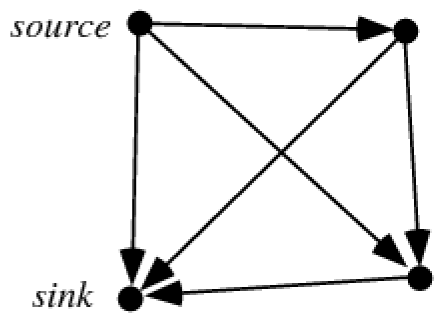
- Labels
  - None
  - On Vertices
  - On Edges
- Groups (=colors)
  - Of Vertices
    - no edge connects two identically colored vertices
  - Of Edges
    - adjacent edges must receive different colors
  - Of both



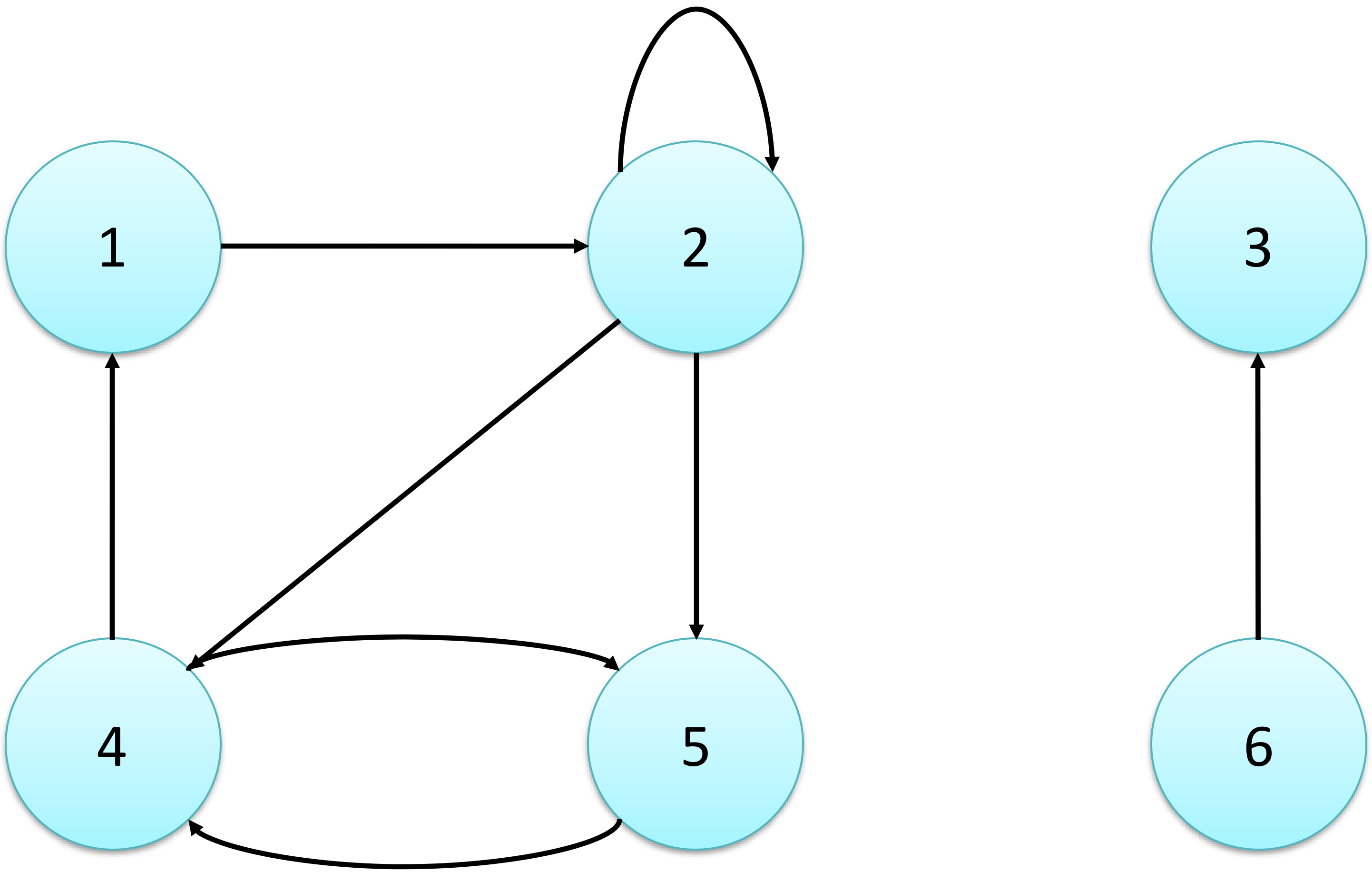


Directed and Oriented graphs

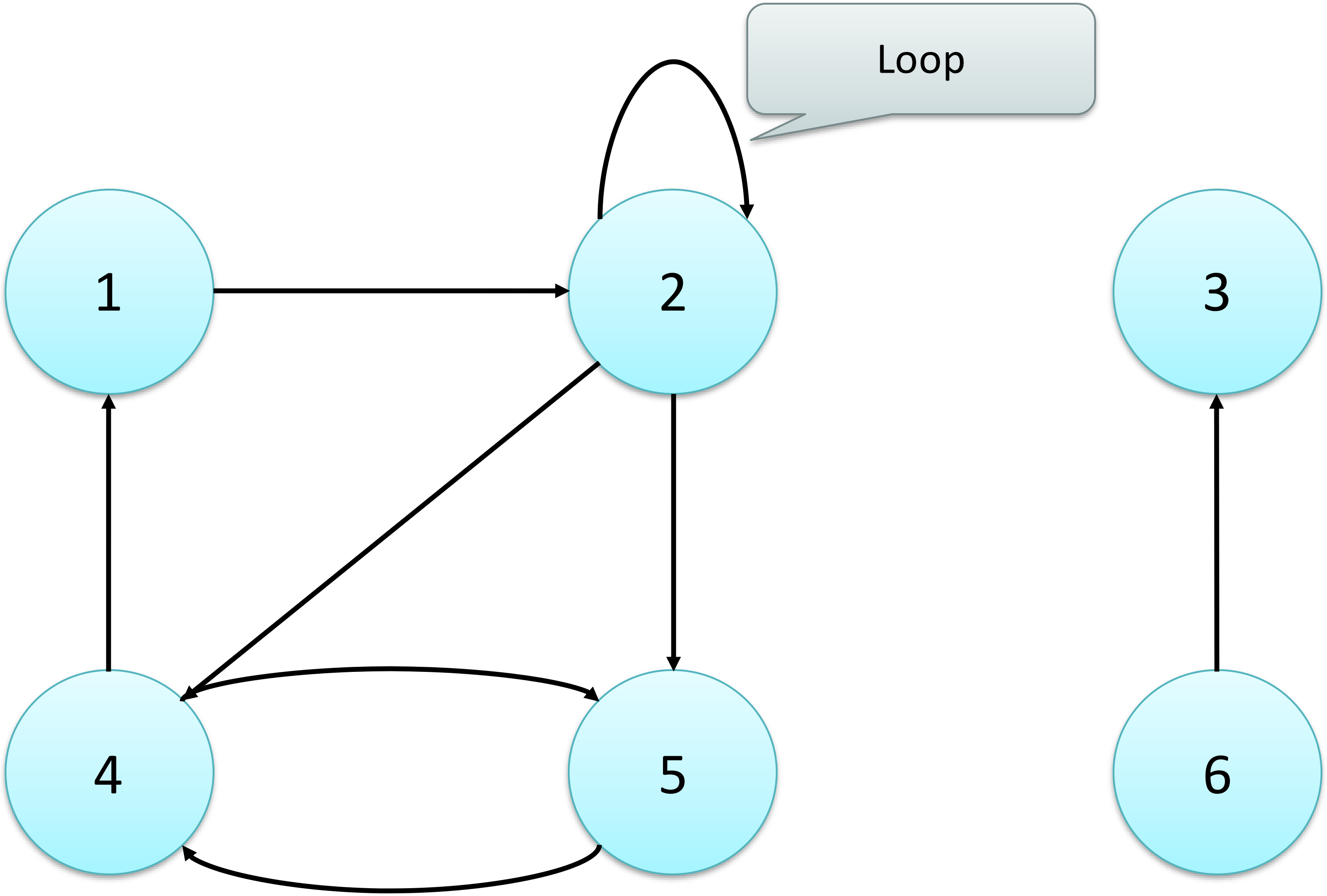
- A Directed Graph (*di-graph*) G is a pair (V,E), where
  - V is a (finite) set of *vertices*
  - E is a (finite) set of *edges*, that identify a binary relationship over V
    - $E \subseteq V \times V$



Example



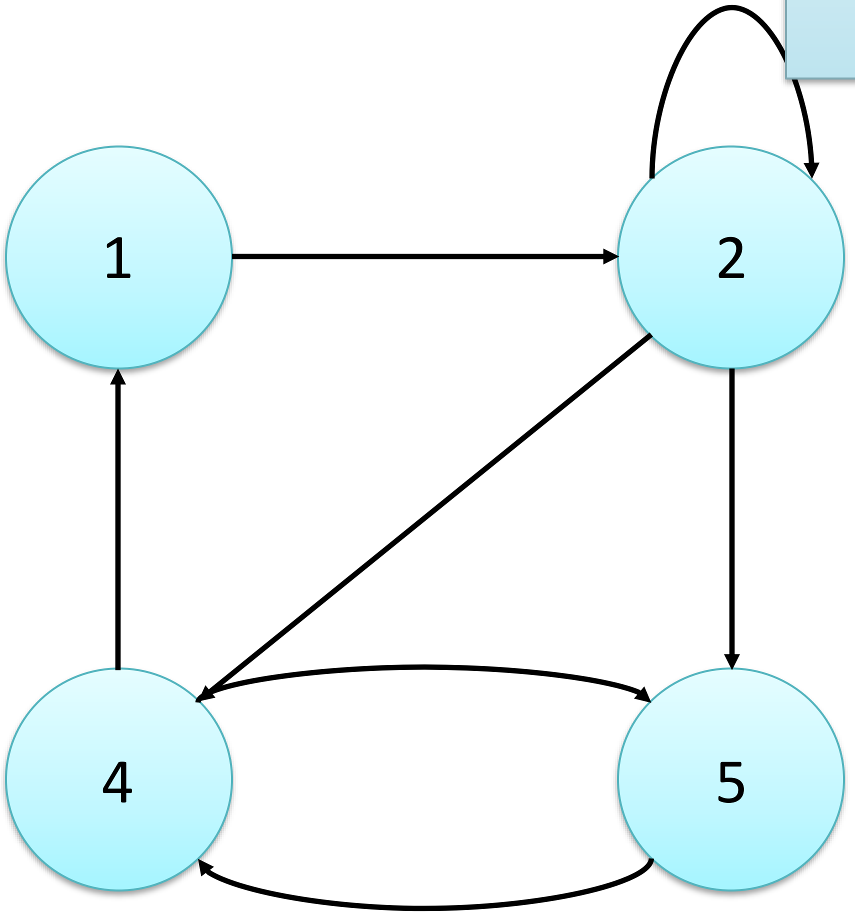
Example



Example

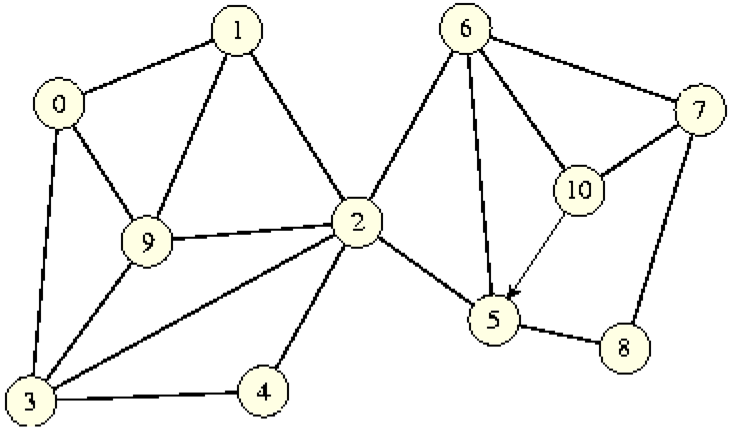
V = {1,2,3,4,5,6}

E = { (1,2), (2,2), (2,5), (5,4), (4,5), (4,1), (2,4), (6,3) }



Undirected graph

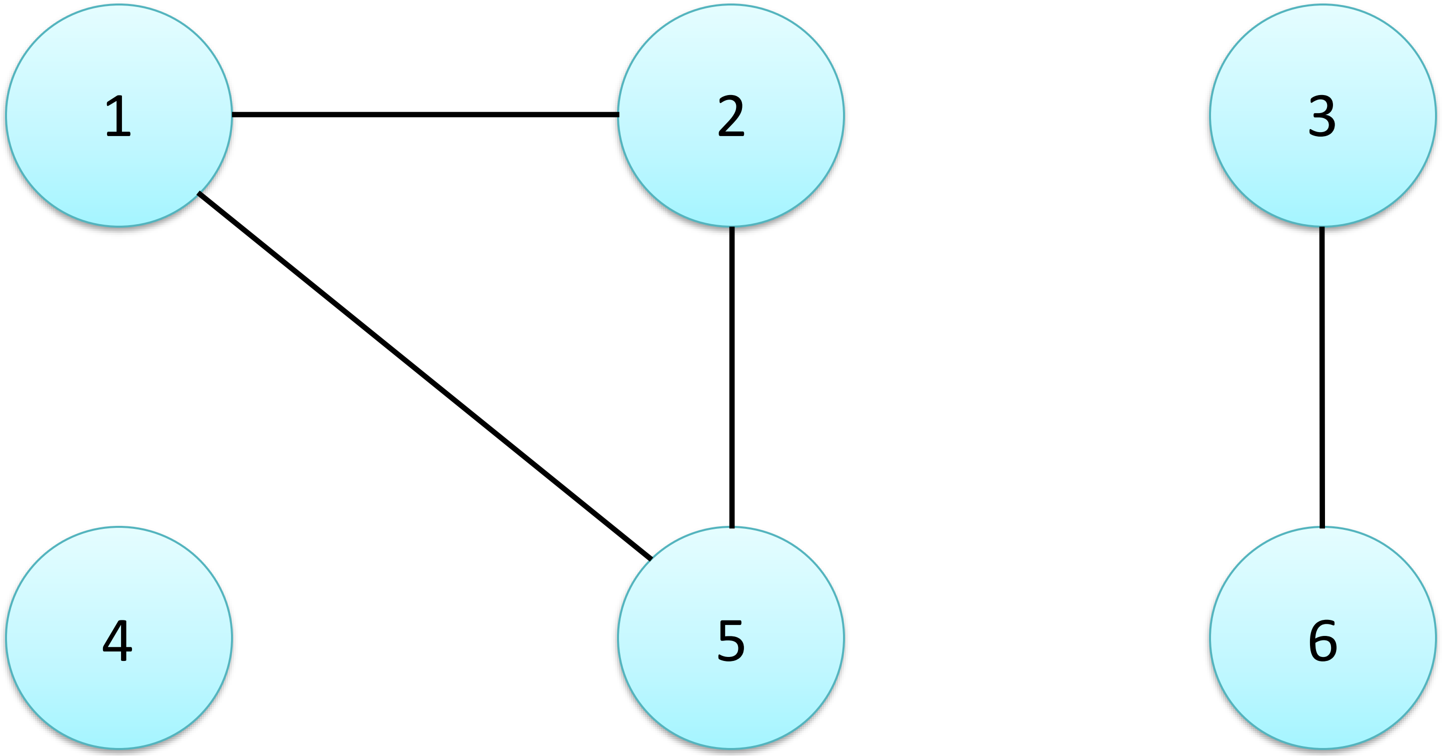
- An **Undirected** Graph is still represented as a touple G=(V,E), but the set E is made of **non-ordered pairs** of vertices



Example

V = { 1, 2, 3, 4, 5, 6 }

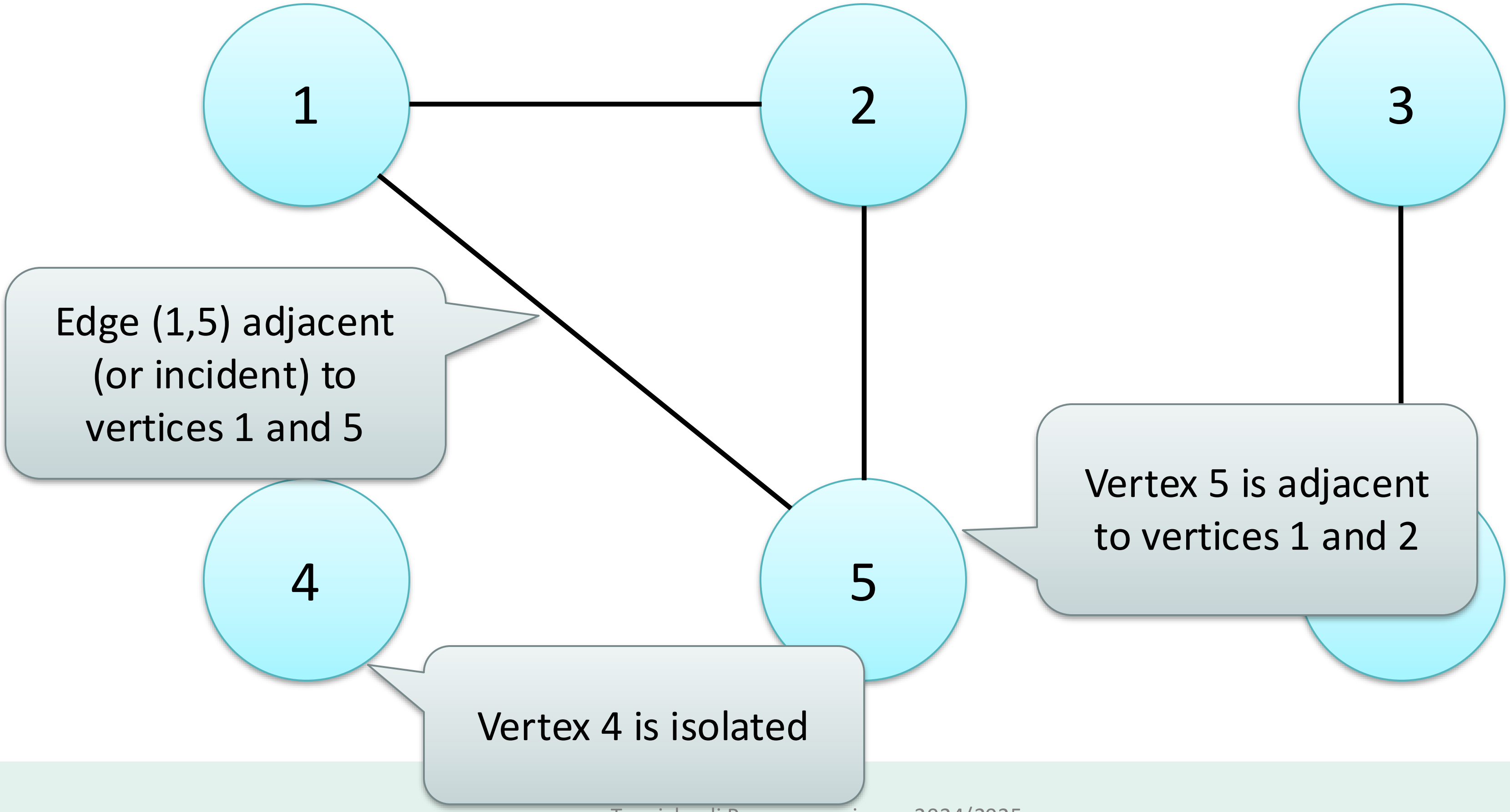
E = { {1,2}, {2,5}, {5,1}, {6,3} }



Example

V = { 1, 2, 3, 4, 5, 6 }

E = { {1,2}, {2,5}, {5,1}, {6,3} }

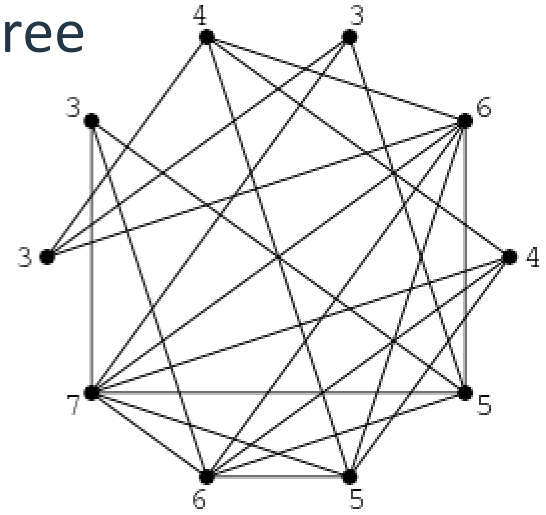


Introduction to Graphs

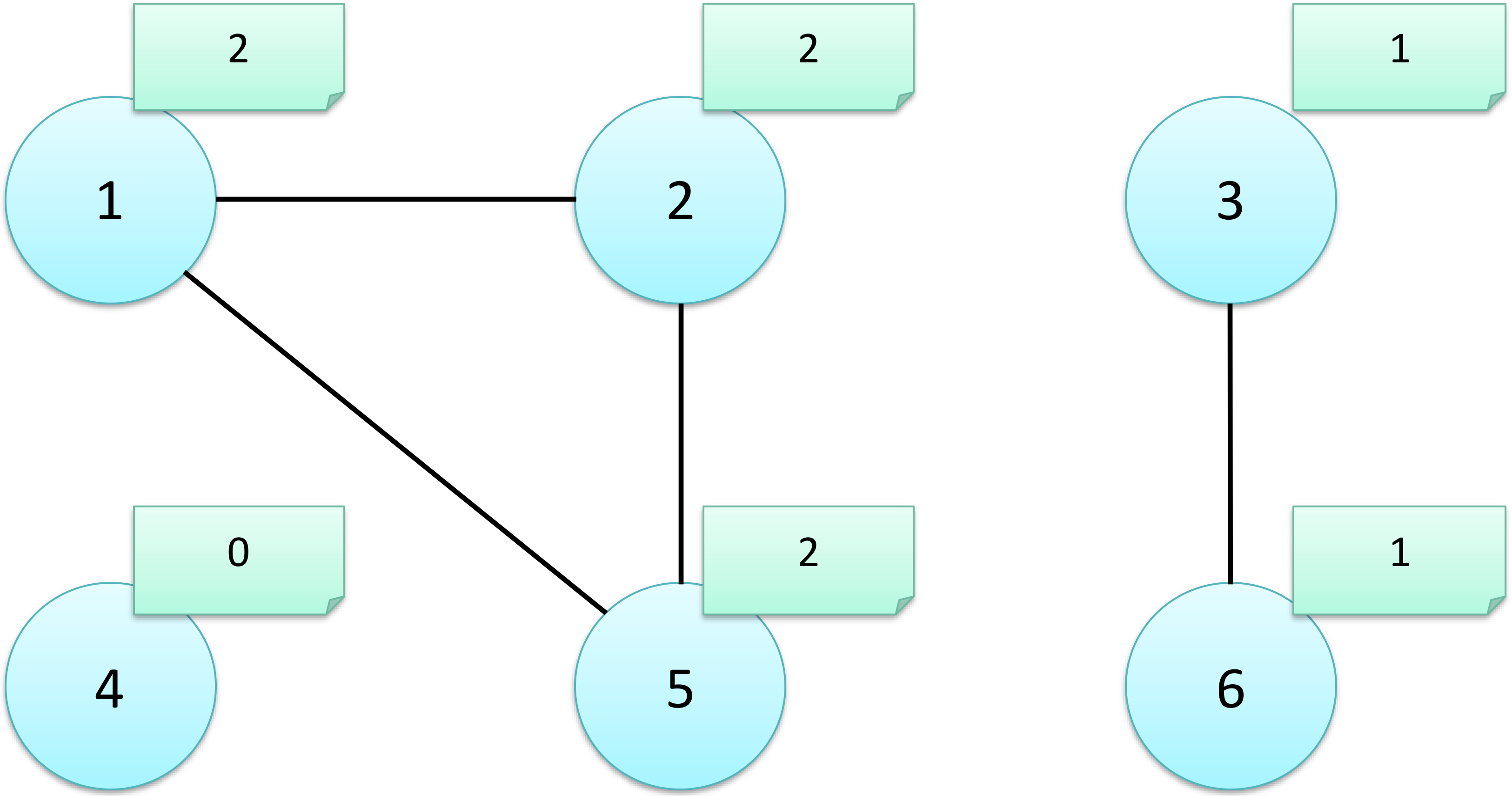
RELATED DEFINITIONS

Degree

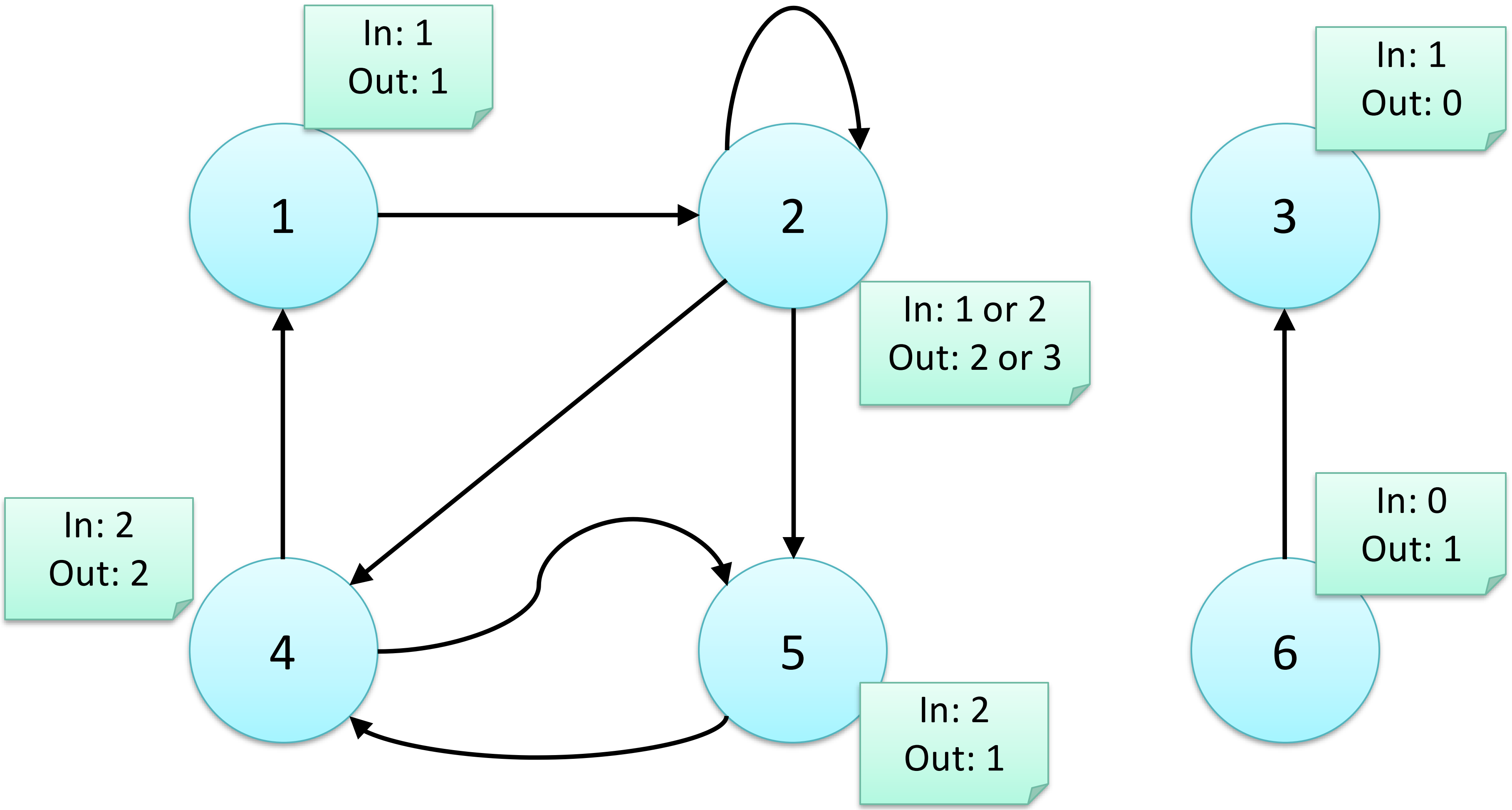
- In an *undirected* graph,
  - the **degree** of a vertex is the number of incident edges
- In a *directed* graph
  - The **in-degree** is the number of incoming edges
  - The **out-degree** is the number of departing edges
  - The **degree** is the sum of in-degree and out-degree
- A vertex with degree 0 is **isolated**



Degree



Degree



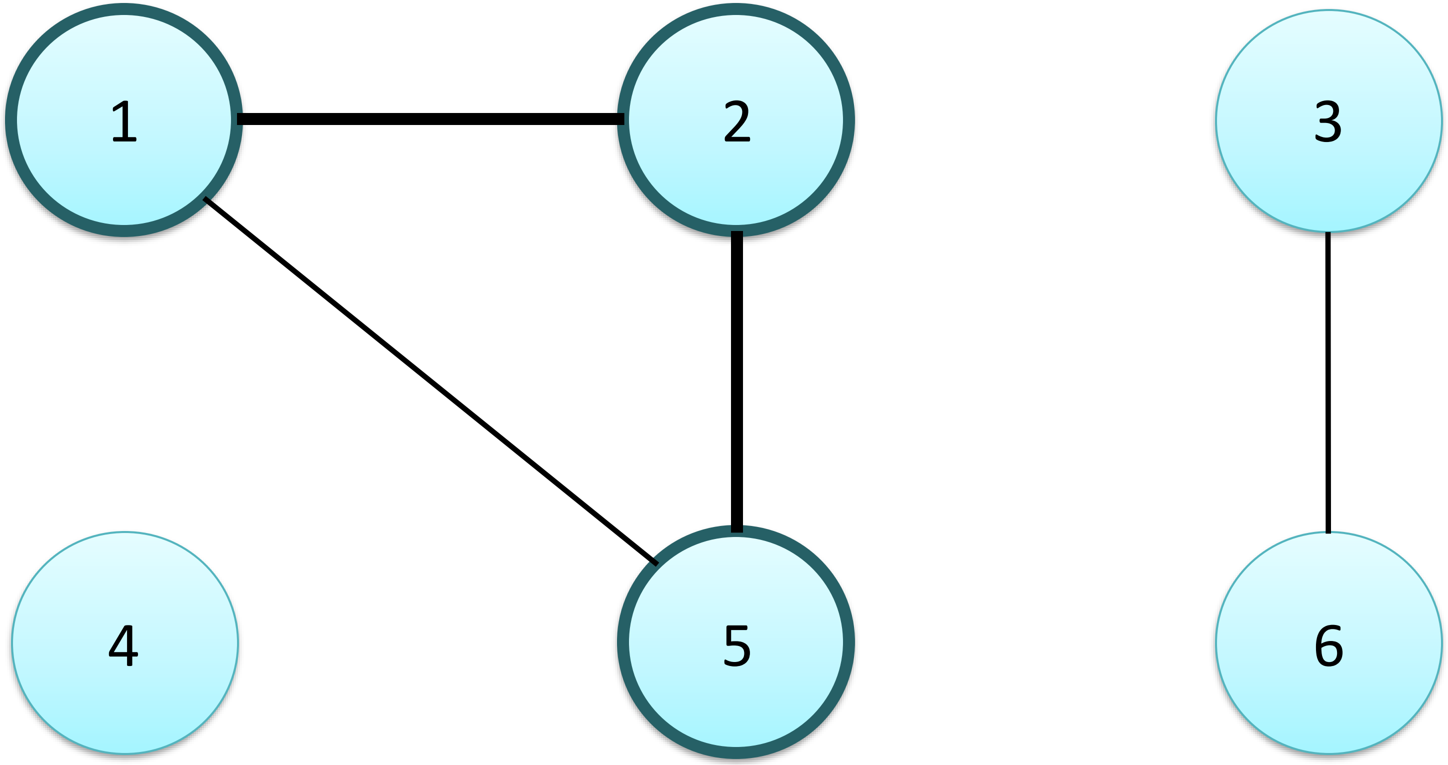
In: 1
Out: 1
1
2
In: 1 or 2
Out: 2 or 3
3
In: 1
Out: 0
In: 2
Out: 2
4
5
In: 2
Out: 1
6
In: 0
Out: 1

Paths

- A **path** on a graph G=(V,E), also called a trail, is a sequence $\{v_1, v_2, \ldots, v_n\}$ such that:
  - $v_1, \ldots, v_n$ are vertices: $v_i \in V$
  - $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1},v_n)$ are graph edges: $(v_{i-1},v_i) \in E$
  - v_i are distinct (for "simple" paths).
- The **length** of a path is the number of edges (n-1)
- If there exist a path between v_A and v_B we say that v_B is **reachable** from v_A

Example

Path = (1, 2, 5)
Length = 2

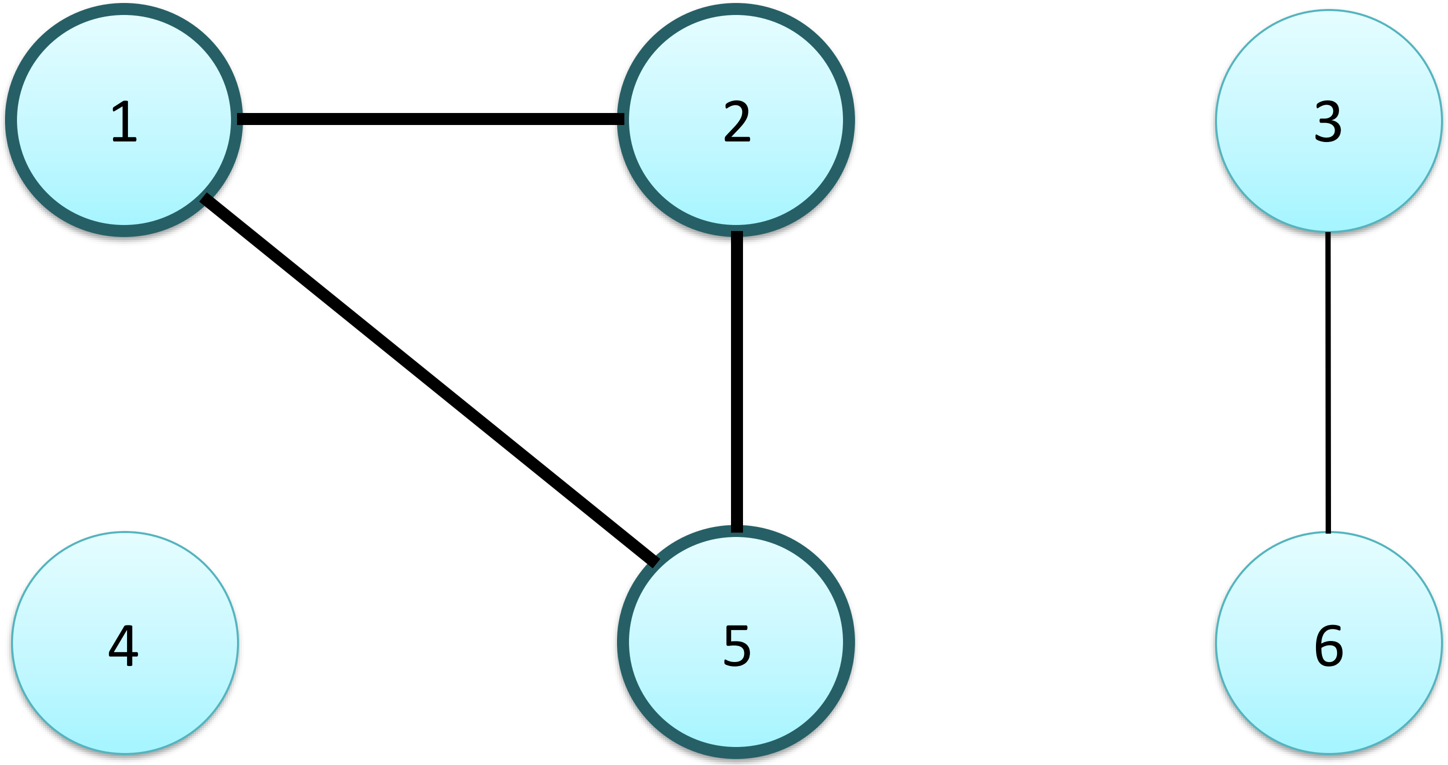


Cycles

- A cycle is a path where $v_1 = v_n$
- A graph with no cycles is said acyclic

Example

Path = (1, 2, 5, 1)
Length = 3

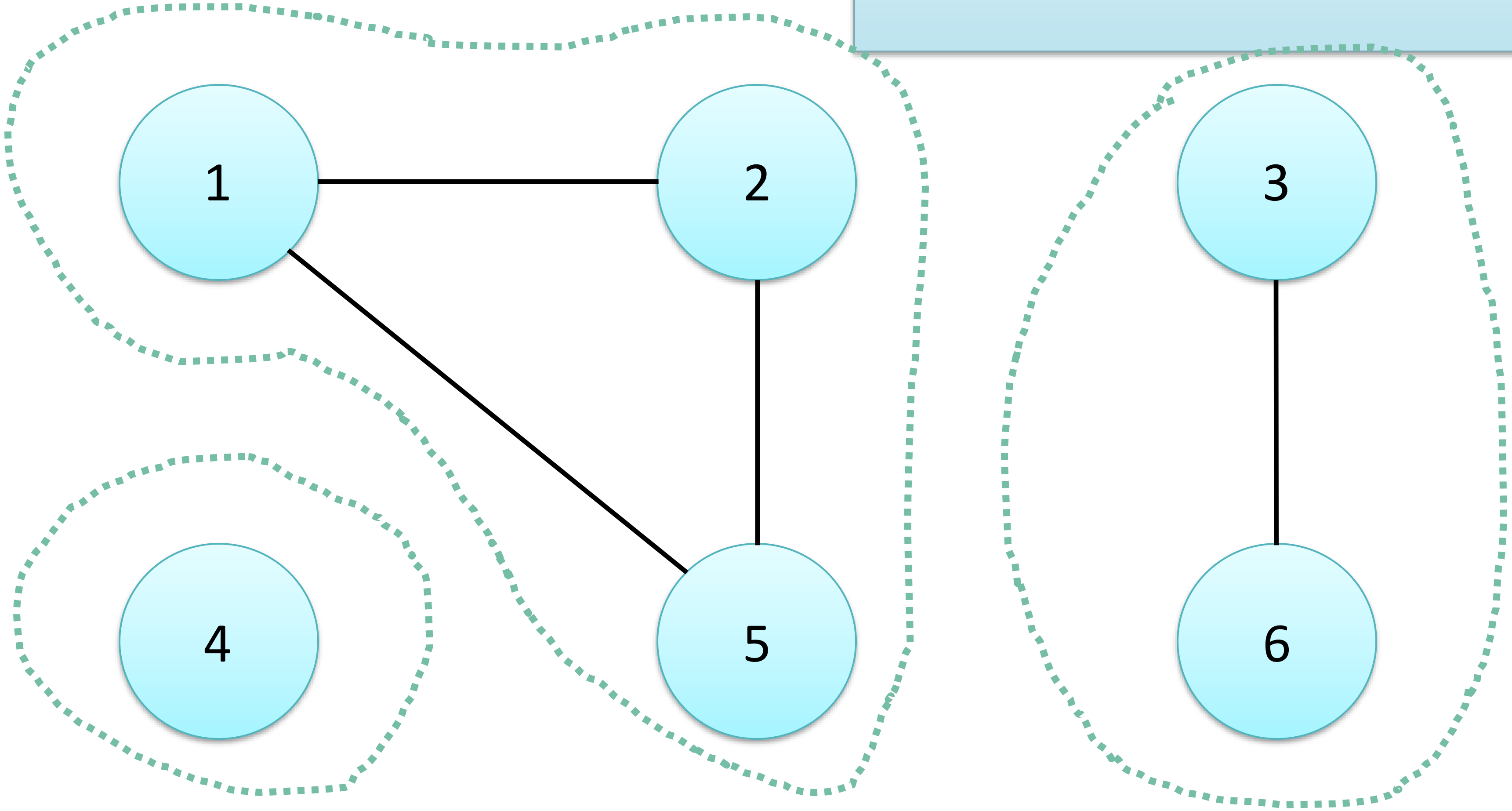


Reachability (Undirected)

- An undirected graph is **connected** if, for every couple of vertices, there is a path connecting them
- The connected sub-graphs of maximum size are called **connected components**
- A connected graph has exactly one connected component

Connected components

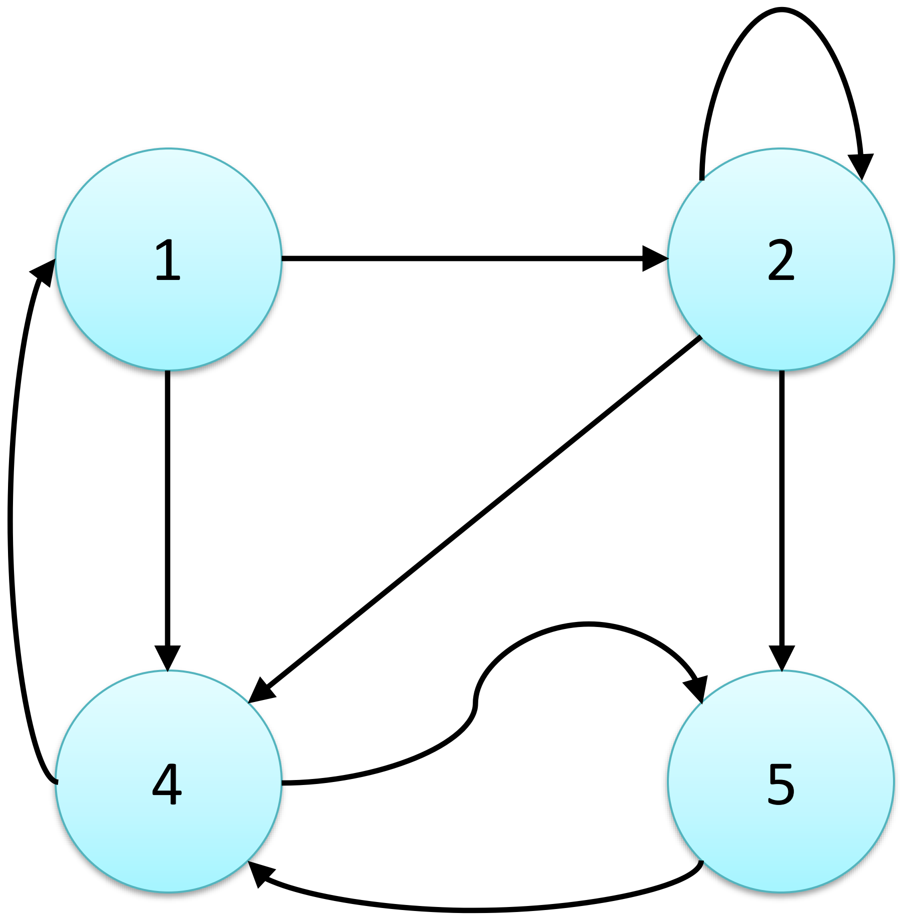
The graph is **not** connected.
Connected components = 3
{ 4 } , { 1, 2, 5 }, { 3, 6 }



Reachability (Directed)

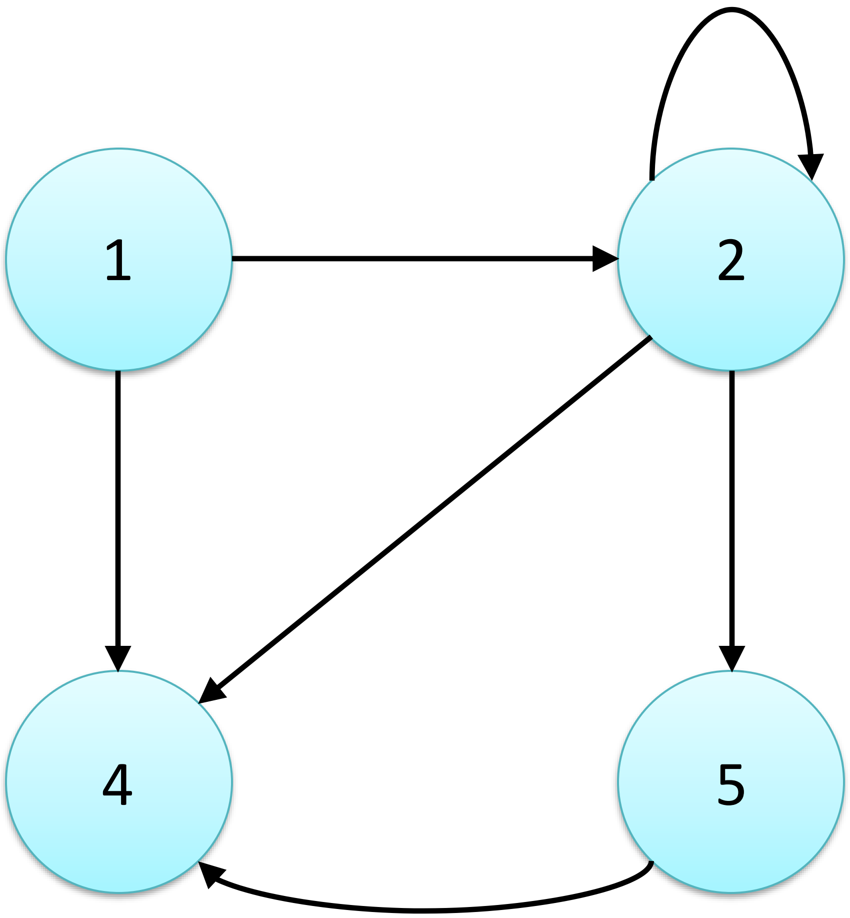
- A directed graph is **strongly connected** if, for <u>every</u> ordered pair of vertices (v, v’), there exists at least one path connecting v to v’

Example



The graph is **strongly connected**

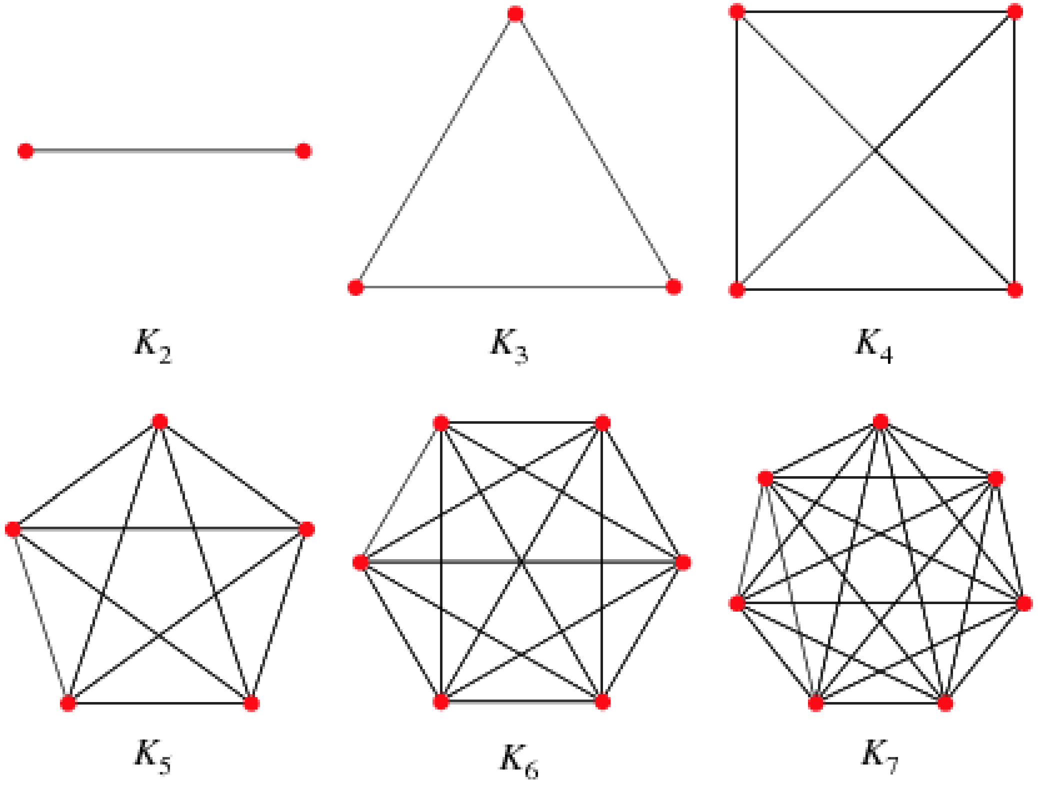
Example



The graph is **not** strongly connected

Complete graph

- A graph is complete if, for every pair of vertices, there is an edge connecting them (they are adjacent)
- Symbol: K_n



Complete graph: edges

- In a **complete** graph with n vertices, the number of **edges** is

	Directed	Undirected
No self loops	$n(n-1)$	$\frac{n(n-1)}{2}$
With self loops	n^2	$\frac{n(n+1)}{2}$

Density

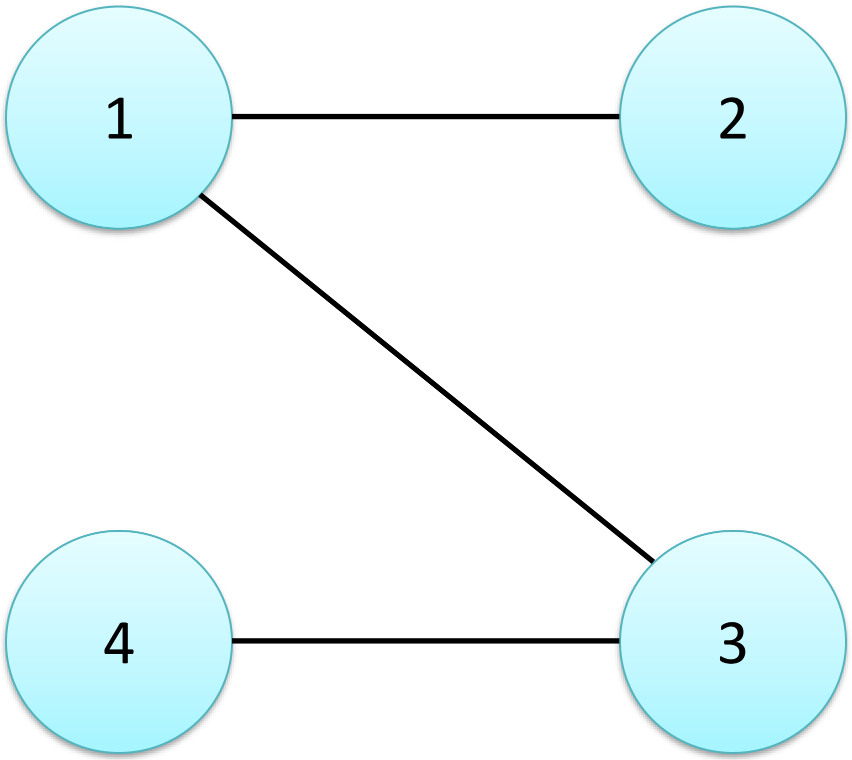
- The density of a graph G=(V,E) is the ratio of the number of edges to the total number of possible edges

$$d = \frac{|E(G)|}{\left|E\left(K_{|V(G)|}\right)\right|}$$

Example

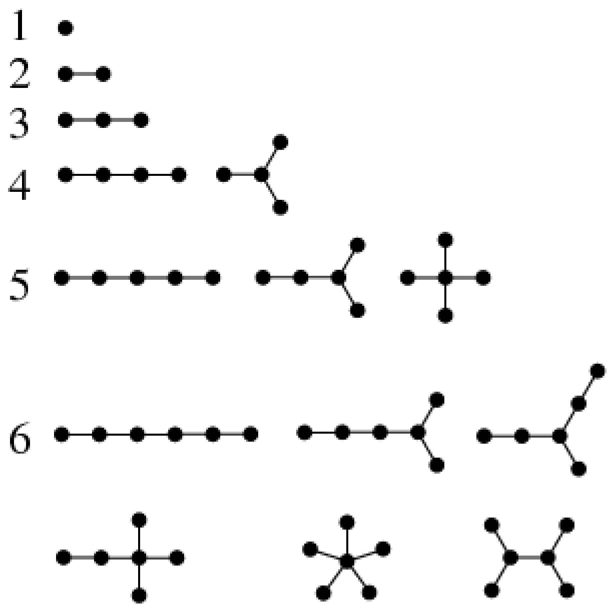
Density = 0.5

Existing: 3 edges
Total: 6 possible edges



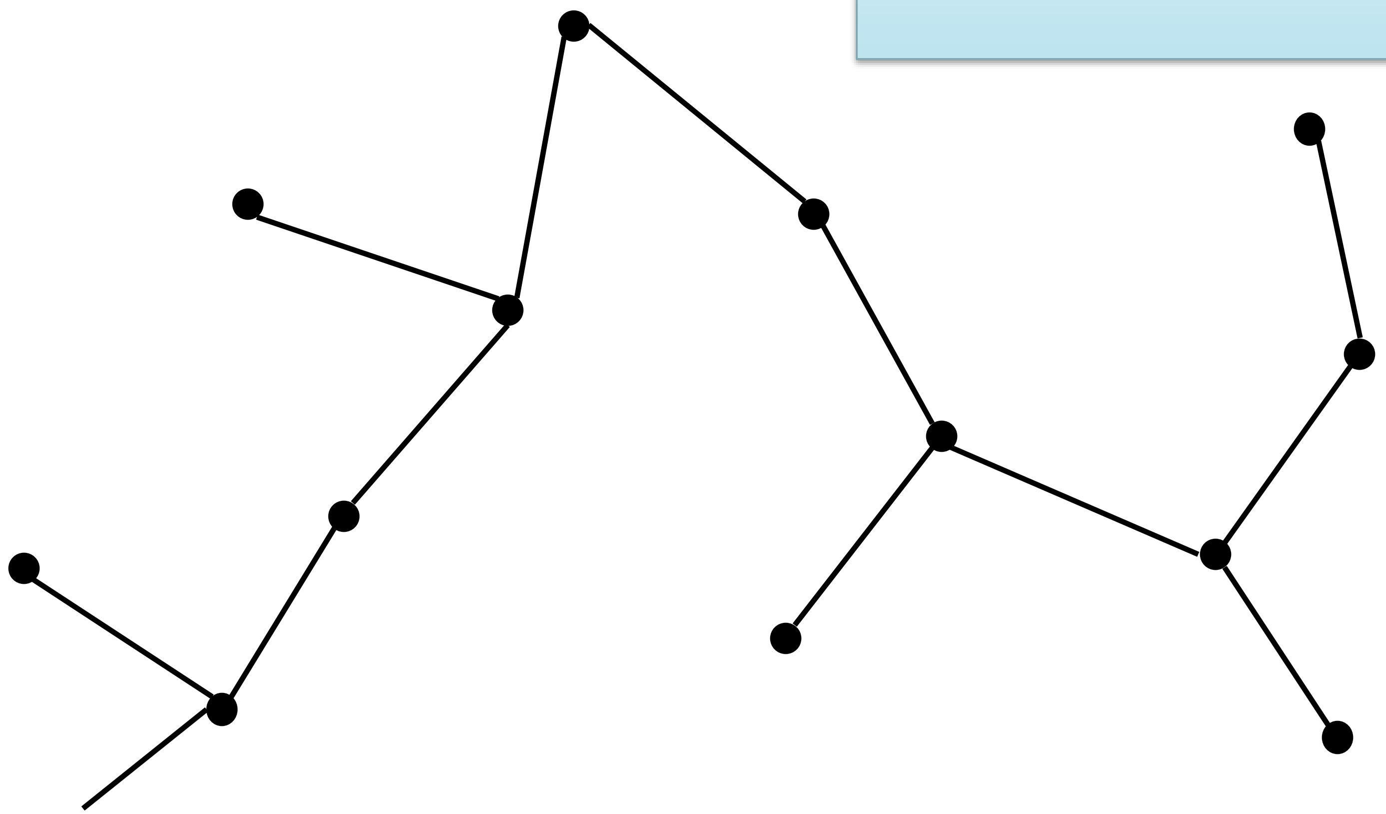
Trees and Forests

- An undirected acyclic graph is called **forest**
- An undirected acyclic connected graph is called **tree**



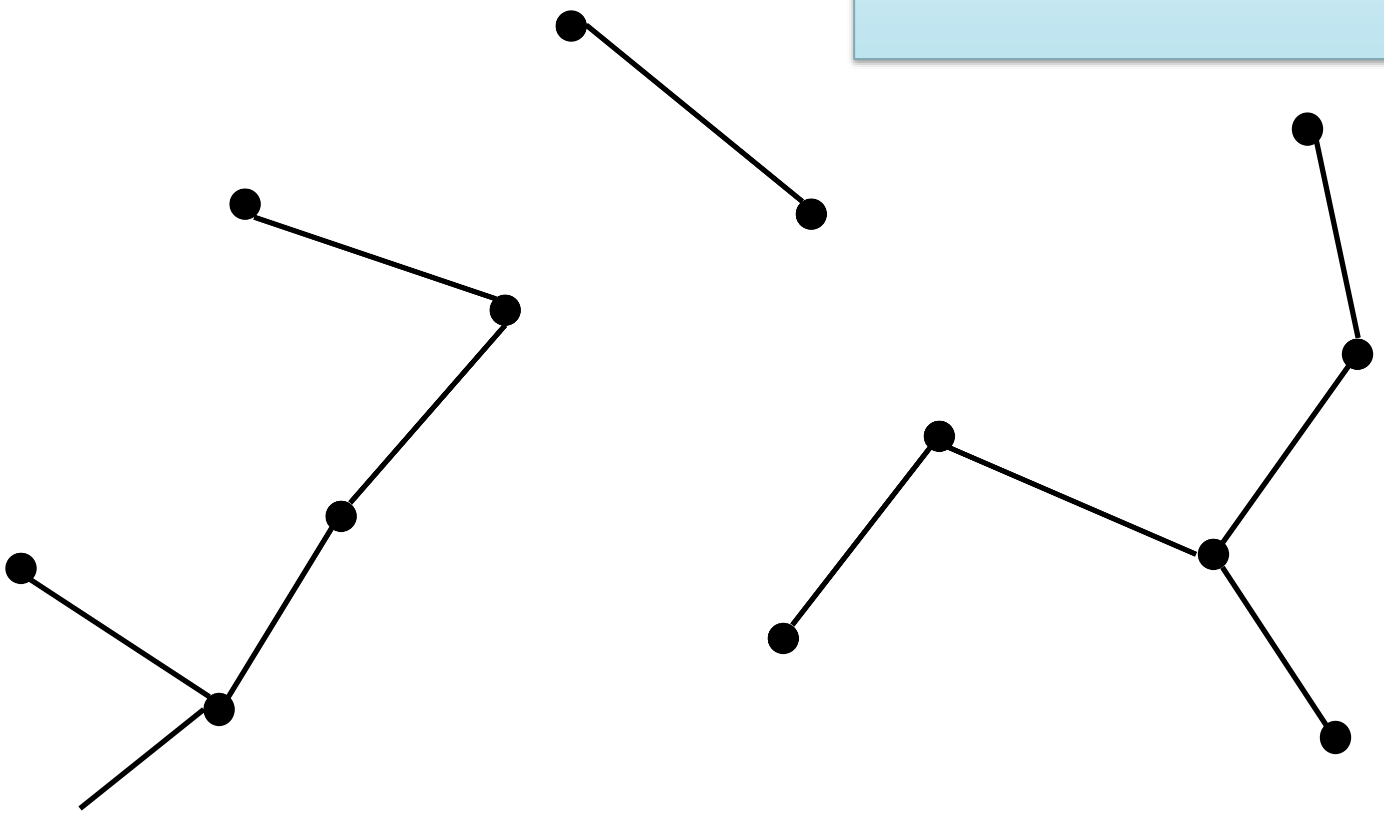
Example

Tree



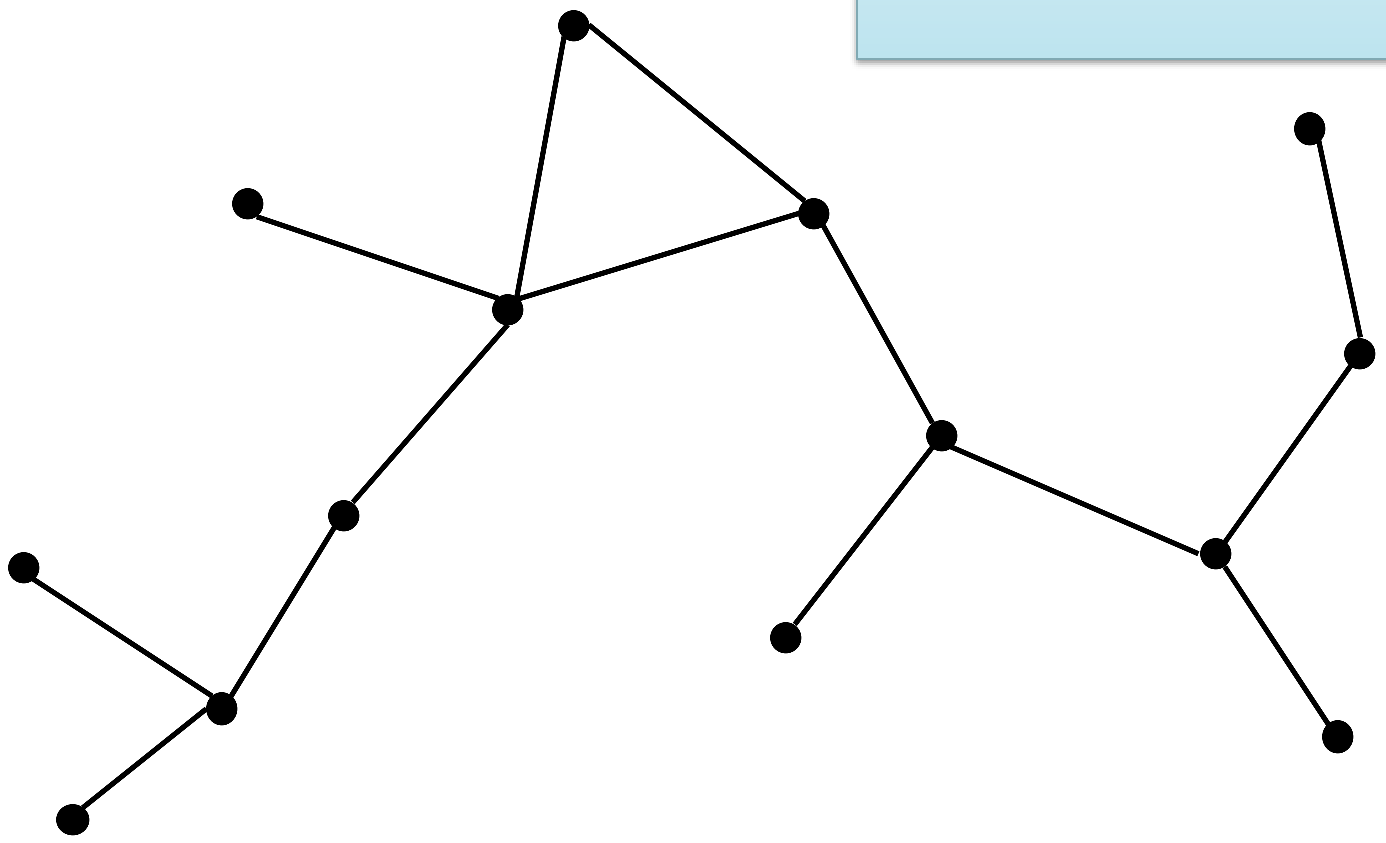
Example

Forest



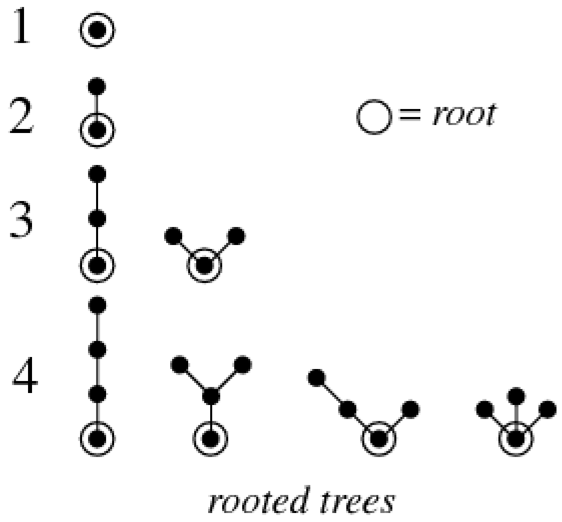
Example

This is not a tree nor a forest
(it contains a cycle)



Rooted trees

- In a tree, a special node may be singled out
- This node is called the “**root**” of the tree
- Any node of a tree can be the root

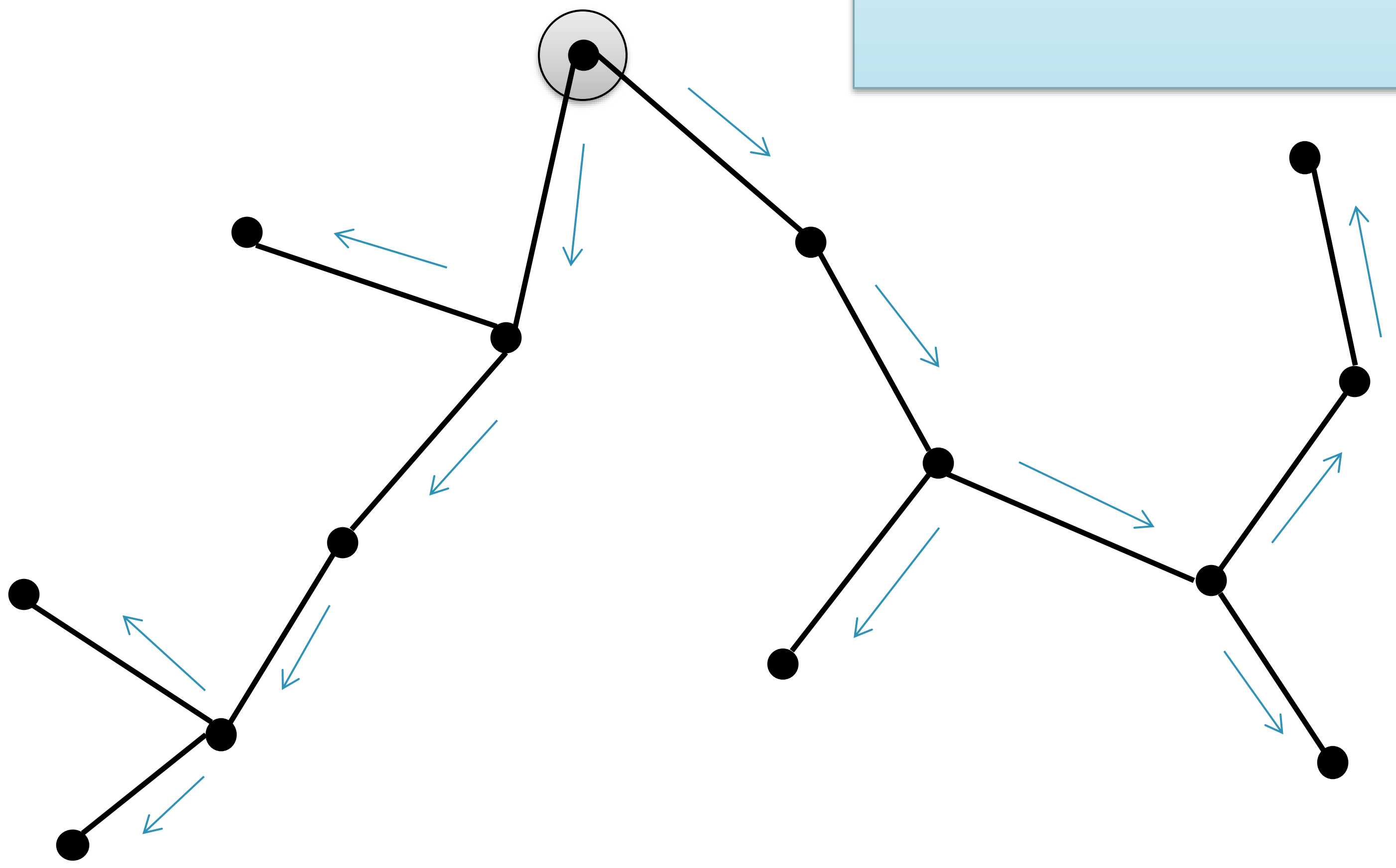


Tree (implicit) ordering

- The root node of a tree **induces an ordering** of the nodes
- The root is the “ancestor” of all other nodes/vertices
  - “children” are “away from the root”
  - “parents” are “towards the root”
- The root is the only node without parents
- All other nodes have exactly one parent
- The furthermost (children-of-children-of-children…) nodes are “leaves”

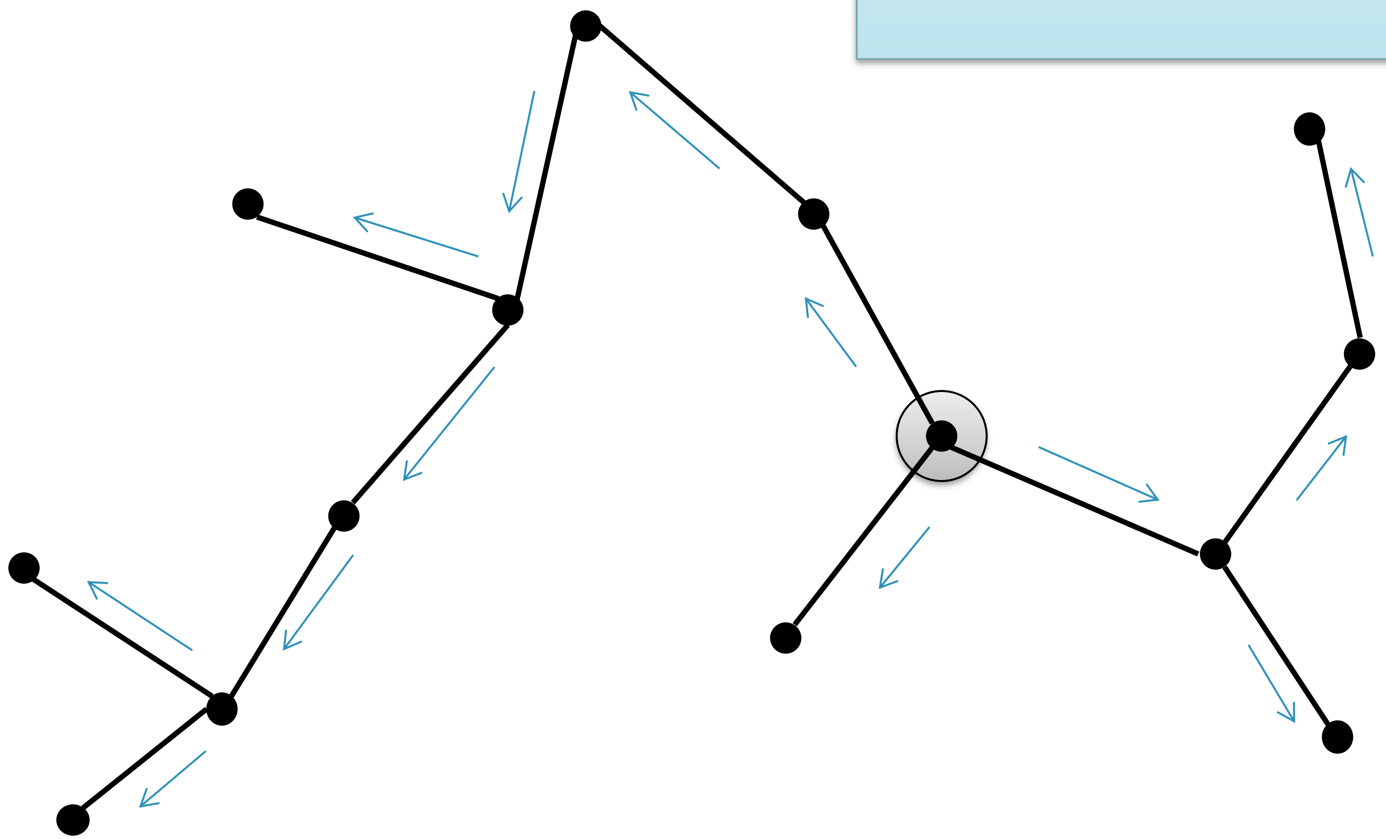
Example

Rooted Tree



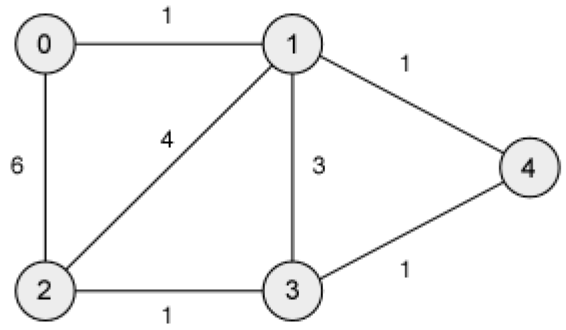
Example

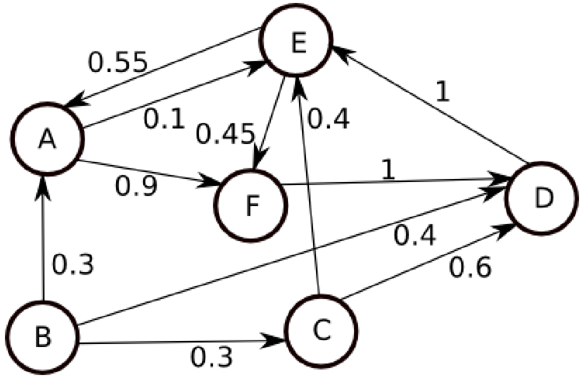
Rooted Tree



Weighted graphs

- A weighted graph is a graph in which each branch (edge) is given a numerical weight.
- A weighted graph is therefore a special type of labeled graph in which the labels are numbers (which are usually taken to be positive).





License