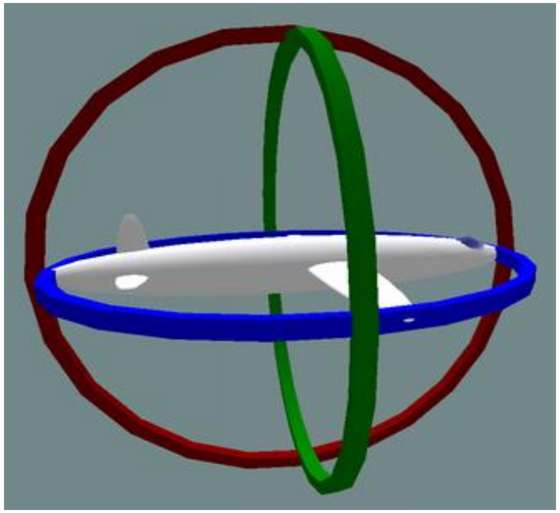
Direct Kinematics - Quaternions



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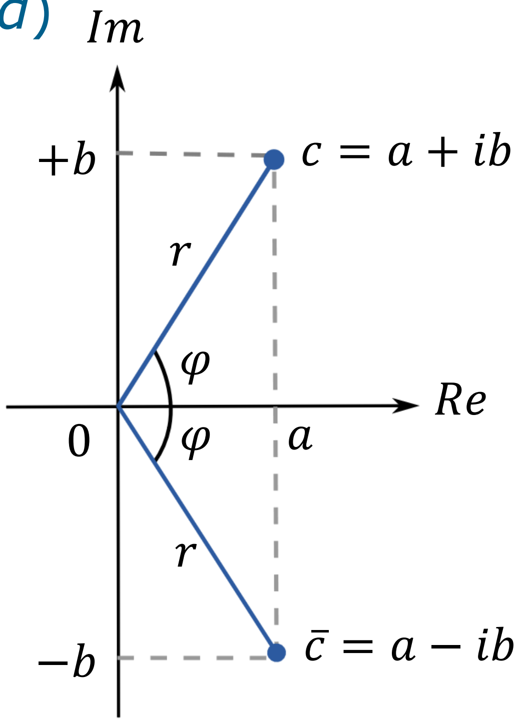
Contents:

Complex Numbers

- **Form of Complex number**: $a + ib$
  - a,b are real numbers,
  - i is the **imaginary unit** with $i^2 = -1$, $i = \sqrt{-1}$, $(\pm i)^2 = -1$
- **Adding:** $(a + ib) + (c + id) = (a + c) + i(b + d)$
- **Subtracting:** $(a + ib) - (c + id) =$
  $(a - c) + i(b - d)$
- **Multiplying:** $(a + ib) \times (c + id) =$
  $(ac - bd) + i(ad + bc)$
- **Complex conjugate**: $a - ib$.
- **Dividing:** $1 / (a + ib) = (a - ib) / (a^2+b^2)$

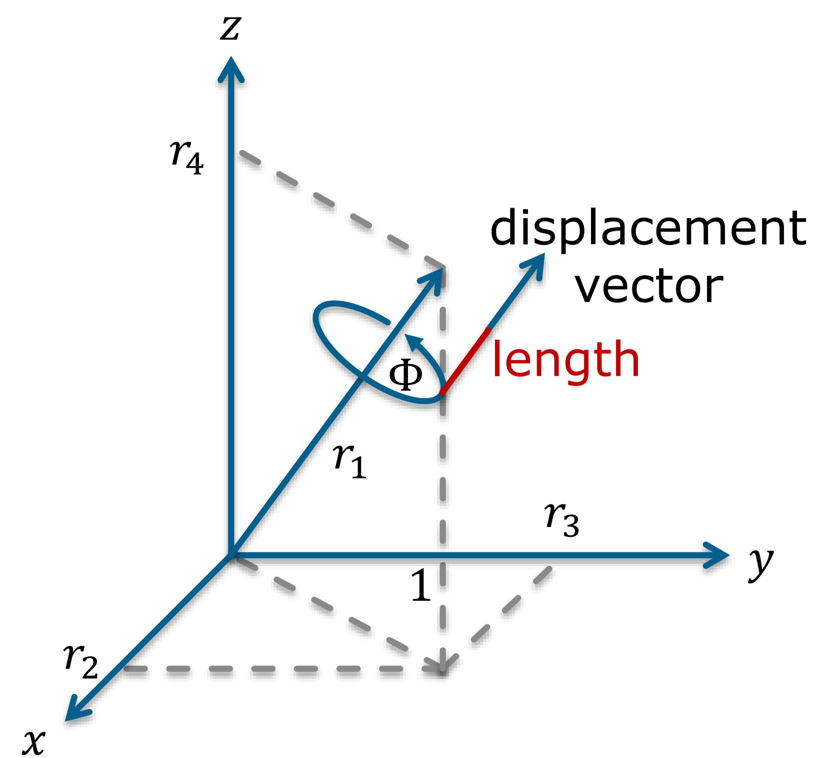


Historical Aspects

- Complex number $a + ib$ viewable as point in a plane
- Transfer to space leads to Quaternions
- Quaternion from lat. Word *quaternio, -ionis f.* „Foursome“)
- 1843, Hamilton representation of vector in space through complex numbers
- Multiplication not possible with triples but with quadruples
- Basic rules of multiplication $i^2 = j^2 = k^2 = ijk = -1$
- $Q = r_1 + i \cdot r_2 + j \cdot r_3 + k \cdot r_4$

Quaternions

- Problems of (homogeneous) rotation matrix
  - High redundancy
  - Many arithmetic operations at concatenation
  - Singularities
- Orientation of a rigid body
  - Quaternion: Rotation axis (3 dim. Vector $\vec{g}$) and angle θ sufficient
  - Reduction of the required computational effort



Real Quaternions

- Quaternions $Q = (r_1, r_2, r_3, r_4)$ with $r_1, r_2, r_3, r_4 \in \mathbb{R}$
- Often represented as a linear vector space over $\mathbb{R}$
  - $Q = r_1 + i \cdot r_2 + j \cdot r_3 + k \cdot r_4$ (extension of $\mathbb{C}$)
- For basic elements $1, i, j, k$ the following multiplicative linkage table (not commutative) applies:

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Real Quaternions

- Scalar part: r_1 (angle of rotation)
- Vector part: $i \cdot r_2 + j \cdot r_3 + k \cdot r_4$ (axis of rotation)
- Quaternions can be used to display all rotations in which the axis of rotation passes through the origin of the reference system
  - Conjugated: $\bar{Q} = r_1 - i \cdot r_2 - j \cdot r_3 - k \cdot r_4$
  - Magnitude: $|Q| = \sqrt{Q \cdot \bar{Q}}$
  - Inverse: $Q^{-1} = \frac{\bar{Q}}{|Q|^2} \rightarrow Q \cdot Q^{-1} = Q^{-1} \cdot Q = 1$

Real Quaternions - Example

- Let us consider: $Q_1 = (3,2,-4,1)$ and $Q_2 = (4,-3,1,-5)$
- It holds:
  - $Q_1 + Q_2 = (7,-1,-3,-4)$
  - $Q_1 \cdot Q_2 = (3 + 2i - 4j + k)(4 - 3i + j - 5k) =$
    $12 - 9i + 3j - 15k + 8i - 6i^2 + 2ij - 10ik - 16j + 12ji - 4j^2 +$
    $20jk + 4k - 3ki + kj - 5k^2 =$
    $12 - 9i + 3j - 15k + 8i + 6 + 2k + 10j - 16j - 12k + 4 + 20i +$
    $4k - 3j - i + 5 = (12 + 6 + 4 + 5) + i(-9 + 8 + 20 - 1) +$
    $j(3 + 10 - 16 - 3) + k(-15 + 2 - 12 + 4) = (27,18,-6,-21)$
  - $Q_2 \cdot Q_1 = (27,-20,-20,-1)$
  - $Q_1^{-1} = \frac{(3-2i+4j-k)}{30}$

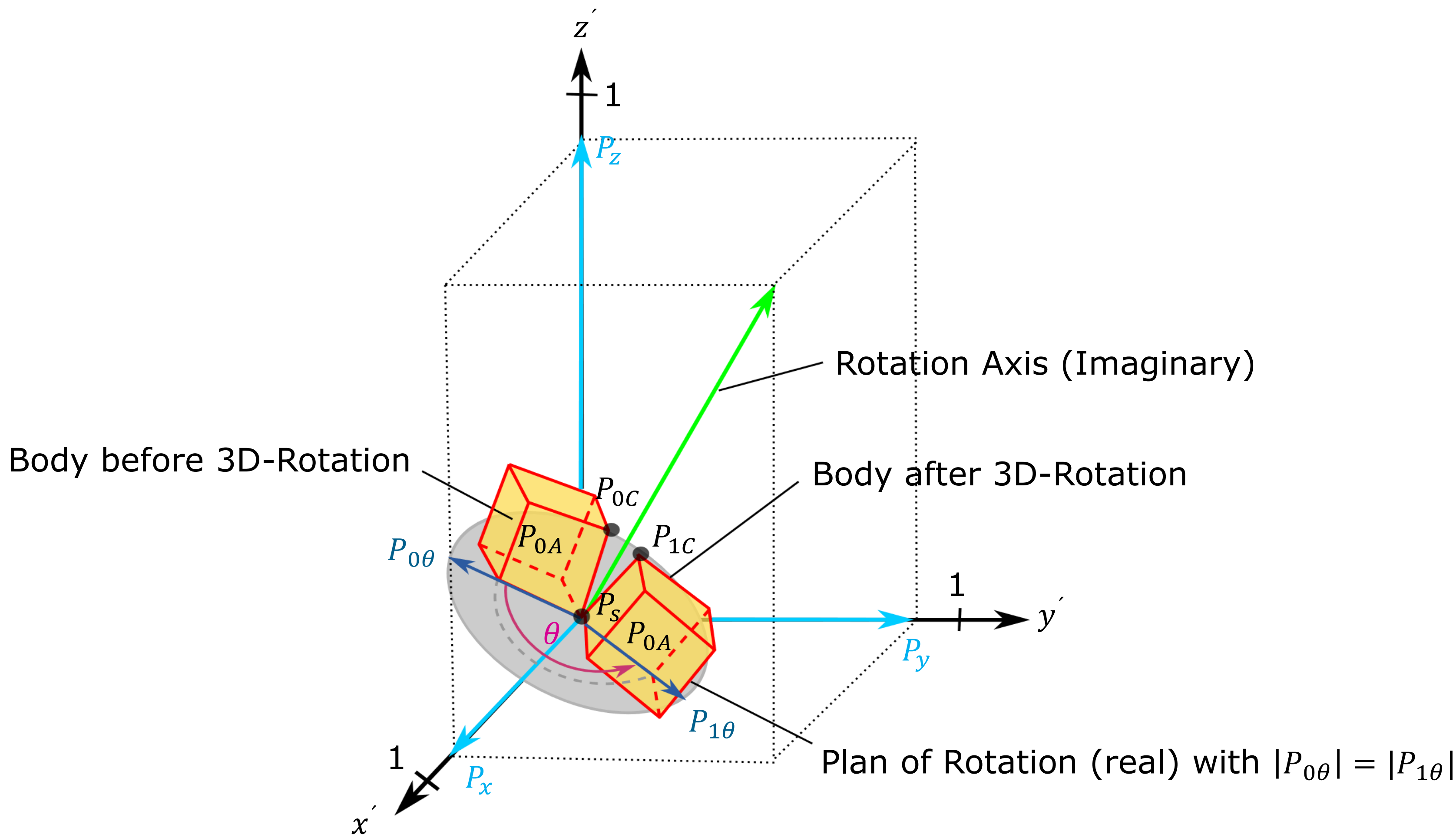
Rotation of Points by Means of Quaternions

Unit quaternion : $|Q| = 1 \Rightarrow Q^{-1} = \bar{Q}$,
since $|Q|^2 = 1 \rightarrow$ Simple forward / backward calculation

Rotation of point $\vec{p}$ at axis $\vec{g}$ with angle θ

1. Create a unique quaternion from $\vec{g}$ and θ
   (1) Standardization of $\vec{g}$ to 1
   (2) $Q = \left[\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\vec{g}\right]$, since $\cos^2(\theta) + \sin^2(\theta) = 1$
2. Represent point $\vec{p}$ as quaternion
   $P = [0, \vec{p}]$
3. Final rotation:
   $P' = Q \cdot P \cdot Q^{-1} = Q \cdot P \cdot \bar{Q}$

Rotation of Points by Means of Quaternions



Converting Quaternion/Rotation matrix

- Rotation quaternion $Q = \left(s, (x, y, z)\right)$

- From rotation by means of unit quaternion $|Q| = 1 \Rightarrow Q^{-1} = \bar{Q}$ follows the rotation matrix R:
-

$$R = \begin{bmatrix} 1 - 2(y^2 + z^2) & 2xy - 2sz & 2sy + 2xz \\ 2xy + 2sz & 1 - 2(x^2 + z^2) & -2sx + 2yz \\ -2sy + 2xz & 2sx - 2yz & 1 - 2(x^2 + y^2) \end{bmatrix}$$

Converting Quaternion/Rotation matrix

- From R with the entries $r_{ij}, i, j \in \{1,2,3\}$, the corresponding rotation quaternion is calculated $Q = \left(s, (x, y, z)\right)$ as follows:
  - $s = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$
  - $x = \frac{r_{32} - r_{23}}{4s}$
  - $y = \frac{r_{13} - r_{31}}{4s}$
  - $z = \frac{r_{21} - r_{12}}{4s}$

Dual Quaternions

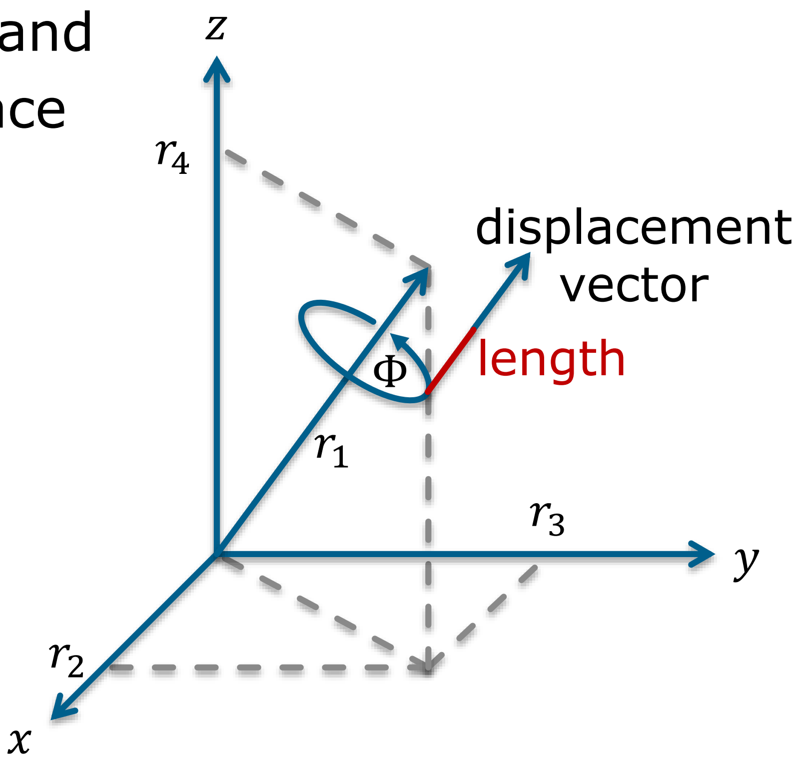
- Real quaternions are suitable for the description of the orientation, but not the position of an object
- Position and orientation can be expressed by quaternions
- Real numbers are replaced by complex numbers
  - $\mathrm{Q} = (q_1, q_2, q_3, q_4)$
  - $q_i = q_{r_i} + \varepsilon \cdot q_{d_i}$
  - $\varepsilon^2 = 0$

  Denote the real part by $q_r = \left(q_{r_1}, q_{r_2}, q_{r_3}, q_{r_4}\right)$,

  and the dual part by $q_d = \left(q_d, q_{d_2}, q_{d_3}, q_{d_4}\right)$. Then,
  - $Q = [q_r, q_d]$

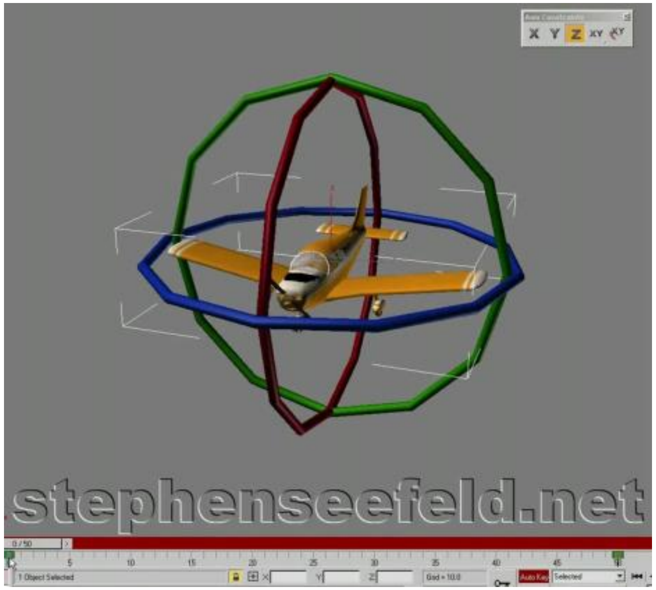
Dual Quaternions

- d_1: Angle value and displacement length
- d_2, d_3, d_4: Description of a directed straight line in space in which the rotation and translation take place



Properties of Dual Quaternions

- Dual quaternions suitable for location description
- Operations on dual quaternions allow all needed transformations
- Low redundancy, as only 8 characteristics
- Gimbal lock does not exist
- Weaknesses
  - Difficulty for the user to describe a location by specifying a dual quaternion
  - Complex processing rules (e.g. multiplication)



Gimbal Lock

- Loss of one degree of freedom in a three-dimensional three-gimbal mechanism
- Occurs when the axes of two of the three gimbals are driven into a parallel configuration, "locking" the system into rotation in a degenerate two-dimensional space.
- The word *lock* is misleading:
  - no gimbal is restrained.
- All three gimbals can still rotate freely about their respective axes of suspension.
- Parallel orientation of two of the gimbals' axes

  → There is no gimbal available to accommodate rotation about one axis.

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Dual-Quaternion Arithmetic Operations

- The elementary arithmetic operations for use to use dual-quaternions are:

  - Scalar multiplication: $sq = sq_r + sq_d\varepsilon$

  - Addition: $q_1 + q_2 = q_{r1} + q_{r2} + (q_{d1} + q_{d2})\varepsilon$

  - Multiplication: $q_1 q_2 = q_{r1} q_{r2} + (q_{r1} q_{d2} + q_{d1} q_{r2})\varepsilon$
    (Multiplication of quaternions)

  - Conjugate: $q^* = q_r^* + q_d^*\varepsilon$

Dual-Quaternion Arithmetic Operations

- Magnitude: $\|q\| = qq^*$
- Unit condition: $\|q\| = 1$

$$q_r^* q_d + q_d^* q_r = 0$$

- The unit dual-quaternion is our key concern as it can represent any rigid rotational and translational transformations.

Dual-Quaternions representation

- Rotation around $\vec{g}$ with angle ϕ

  Then,

  - $$q_{rot} = \left[cos\left(\frac{\phi}{2}\right), g_1\, sin\left(\frac{\phi}{2}\right), g_2\, sin\left(\frac{\phi}{2}\right), g_3\, sin\left(\frac{\phi}{2}\right), 0,0,0,0\right]$$

- Transformation by $\vec{h}$ with no rotation

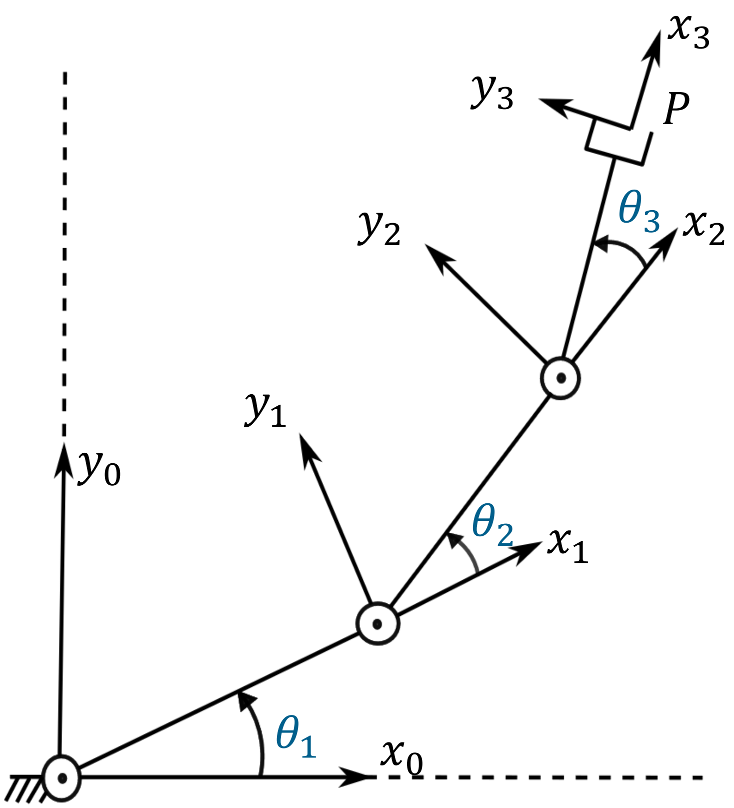
  - $$q_{trans} = \left[1,0,0,0,0, \frac{h_1}{2}, \frac{h_2}{2}, \frac{h_3}{2}\right]$$

- $$q = q_{trans} \times q_{rot}$$

Direct Kinematics: Example 1

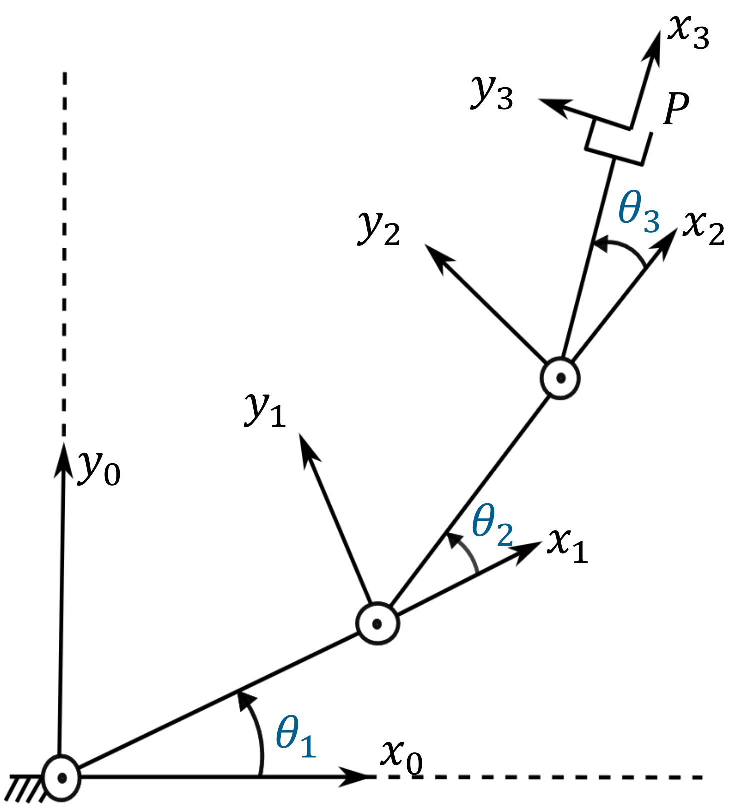
- ${}^{i-1}_{i}q = {}^{i-1}_{i}q_{trans}\,{}^{i-1}_{i}q_{rot}$
- ${}^{i-1}_{i}q_{trans} = [1,0,0,0,0,\frac{a_i c_i}{2},\frac{a_i s_i}{2},0]$
- ${}^{i-1}_{i}q_{rot} = \left[\cos\left(\frac{\theta_i}{2}\right),0,0,\sin\left(\frac{\theta_i}{2}\right),1,0,0,0\right]$
- Define: $\cos\left(\frac{\theta_i}{2}\right) =: c_{\frac{i}{2}}$
- It follows: ${}^{i-1}_{i}q = [c_{\frac{i}{2}},0,0,s_{\frac{i}{2}},$
  $0,\frac{a_i c_{\frac{i}{2}}}{2},\frac{a_i s_{\frac{i}{2}}}{2},0]$

Joint	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3



Direct Kinematics: Example 1

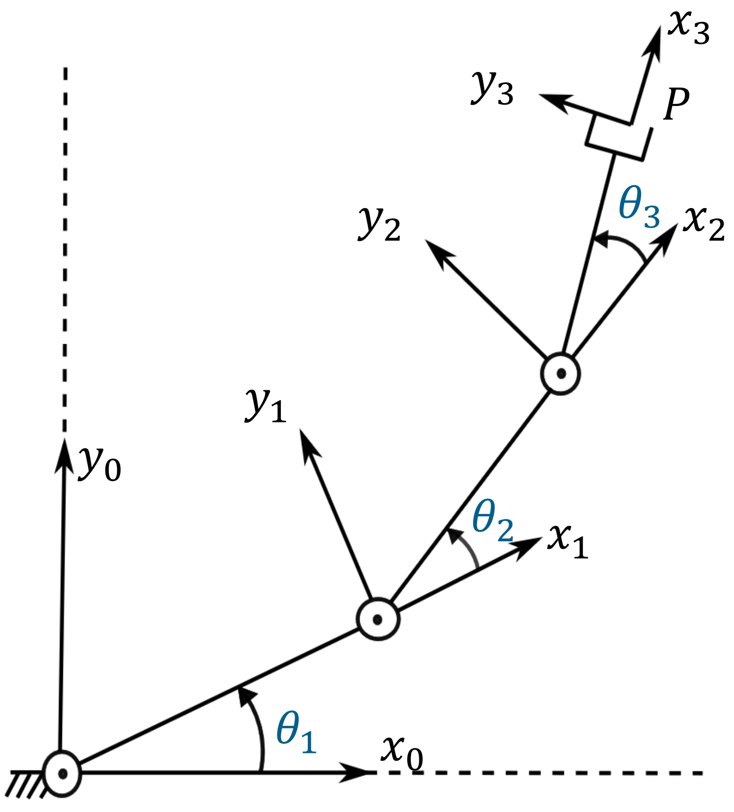
- Then:
- $${}^3_0q = {}^3_2q\,{}^2_1q\,{}^1_0q = \left[c_{\frac{123}{2}}, 0, 0, s_{\frac{123}{2}}, 0, \frac{a_3 c_{\frac{3-2-1}{2}} + a_2 c_{\frac{2-1-3}{2}} + a_1 c_{\frac{1-2-3}{2}}}{2}, \frac{a_3 s_{\frac{3-2-1}{2}} + a_2 s_{\frac{2-1-3}{2}} + a_1 s_{\frac{1-2-3}{2}}}{2}, 0\right]$$



Joint	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

Direct Kinematics: Example 1

- And for a vector $\vec{x}$ it holds:
- ${}^{3}_{0}A\vec{x} \triangleq {}^{3}_{0}qX{}^{3}_{0}\overline{q}$



Joint	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

Coming up next…

Exponential Coordinates and Screws