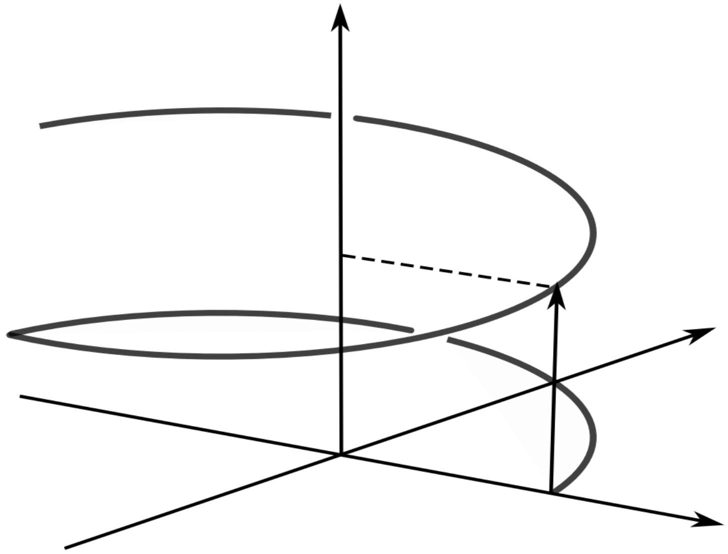
# Direct Kinematics - Exponential Coordinates and Screws



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# Contents:

# Reminder: Euler Theorem from chapter 3

Every rotation matrix $R_3$ is equivalent to a rotation around a fixed axis

$$\vec{g} \in \mathbb{R}^3, ||\vec{g}|| = 1,$$

And a rotation angle

$$\theta \in [0,2\pi).$$

# Screws

- A screw $S = S(h, \theta, \vec{g}, \vec{P})$ is defined by:
  - a normalized screw axis $\vec{g}$
  - a twist angle $\theta$
  - a translation $h$
  - a location $\vec{P}$

# Screw types

- $p = \frac{h}{\theta}$ is called the pitch of screw $S = S\left(h, \theta, \vec{g}, \vec{P}\right)$

- If $p > 0$, $S$ is called right handed
- If $p < 0$, $S$ is called left handed

- If $\vec{P} = 0$, then we write $S(h, \theta, \vec{g}) := S(h, \theta, \vec{g}, 0)$, and $S$ is called a central screw.

# Chasles Theorem

- For all homogenous Matrices

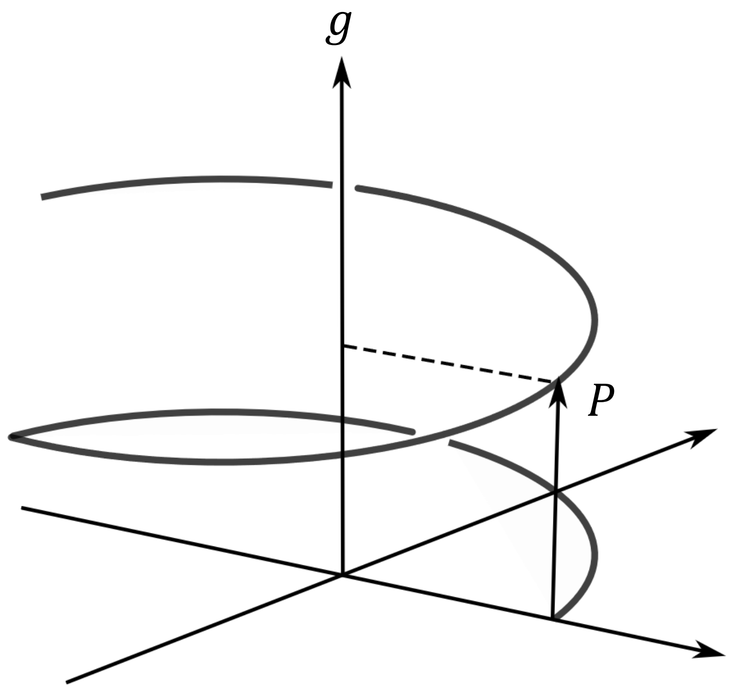
$$\begin{bmatrix} R & \vec{u} \\ 0 & 1 \end{bmatrix}$$

- There exists $\vec{g}$ and $\theta$ such that:

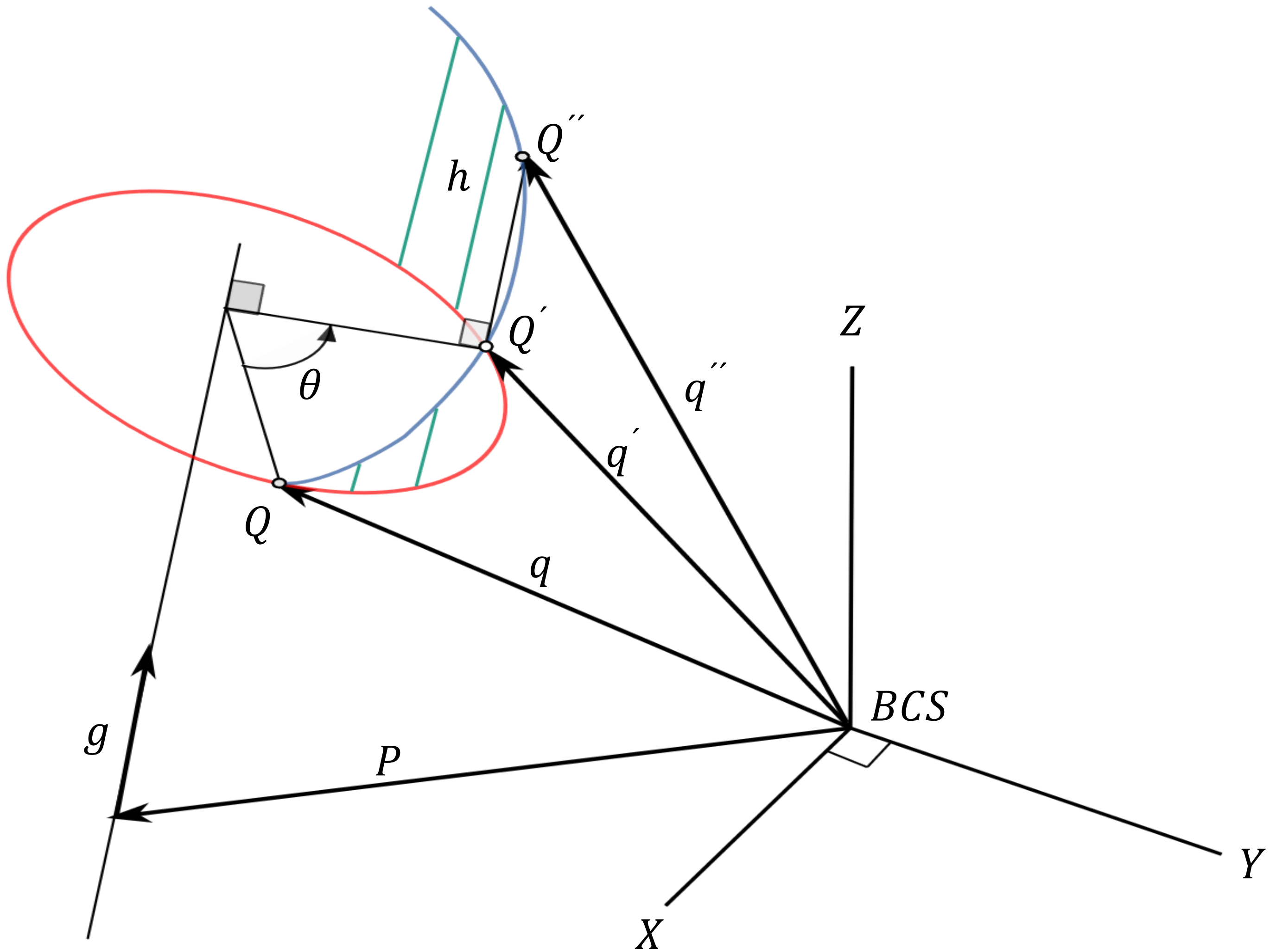
$\begin{bmatrix} R & \vec{u} \\ 0 & 1 \end{bmatrix}$ can be described as

$$\begin{bmatrix} R_{\vec{g}}(\theta) & \vec{g} \\ 0 & 1 \end{bmatrix}$$

- $\vec{g}$ is called the screw axis
- $\theta$ is called the twist angle
  - Direction of $\vec{g}$ is given by Rodrigues formula, but needs to be scaled.



# Screw Motion of a rigid body



# From central Screws to Homogenous Matrices

- If we have $S(h, \theta, \vec{g})$ then,

$$R_{\vec{g}}(\theta) = \begin{bmatrix} g_1^2\eta\theta + C\theta & g_1 g_2 \eta_\theta - g_3 S\theta & g_1 g_3 \eta\theta + g_2 S\theta \\ g_1 g_2 \eta\theta + g_3 S\theta & g_2^2 \eta\theta + C\theta & g_2 g_3 \eta\theta - g_1 S\theta \\ g_1 g_3 \eta\theta - g_2 S\theta & g_2 g_3 \eta\theta + g_1 S\theta & g_3^2 \eta\theta + C\theta \end{bmatrix}$$

and hence,

$$A_S(h, \theta, \vec{g}) = \begin{bmatrix} R_{\vec{g}}(\theta) & h\vec{g} \\ 0 & 1 \end{bmatrix}$$

$$A_s\left(h, \theta, \vec{g}, \vec{P}\right) = \begin{bmatrix} C\theta I_3 + \vec{g}\vec{g}^T\eta\theta + \hat{g}S\theta & \left((I - \vec{g}\vec{g}^T)\eta\theta - \hat{g}S\theta\right)\vec{\hat{P}} + h\vec{g} \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{S}\theta = \sin\theta,\ \ C\theta = \cos\theta,\ \ \eta\theta = 1 - \cos\theta\ \ , \vec{g} = (g_1, g_2, g_3)^T = \frac{1}{2S\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

# From Homogenous Matrices to Screws

- By Rodrigues formula we get
  - $\theta = \cos^{-1}\left(\frac{tr(R)-1}{2}\right) \in [0, \pi]$, and
  - $\vec{g} = \frac{1}{2S_\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$
- $\vec{P}$ is any point on the screw.
- $h$ and $\vec{P}$ are still missing.

# From Homogenous Matrices to Screws

- Goal:

  Find $\vec{P} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$ and $h$, such that:

$$\begin{bmatrix} R & \vec{u} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\vec{g}}(\theta) & h\vec{g} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} R - R_{\vec{g}}(\theta) & \vec{u} - h\vec{g} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \left(R - R_{\vec{g}}(\theta)\right) \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} + \vec{u} - h\vec{g} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

# From Homogenous Matrices to Screws

$$\Leftrightarrow \quad \vec{u} = h\vec{g} - \left(R - R_{\vec{g}}(\theta)\right) \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

- W.l.o.g., $P_1 = 0$, since any point on the screw is fine

$$\Longrightarrow \vec{u} = \left[\vec{g},\ \left(R - R_{\vec{g}}(\theta)\right)_2,\quad \left(R - R_{\vec{g}}(\theta)\right)_3\right] \begin{bmatrix} h \\ P_2 \\ P_3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} h \\ P_2 \\ P_3 \end{bmatrix} = \left[\vec{g},\ \left(R - R_{\vec{g}}(\theta)\right)_2,\quad \left(R - R_{\vec{g}}(\theta)\right)_3\right]^{-1} \vec{u}$$

- Here $(\cdot)_i$ denotes the $i$–th column

# Principle Screws

- For a central screw we have

$$A_S(h,\phi,\vec{g}) = \begin{bmatrix} R_{\vec{g}}(\theta) & h\vec{g} \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

Example for D-H

- If $\vec{g} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, i.e. $\vec{g} = \vec{x}_{i-1}$ (D-H- axis)

$$\Longrightarrow A(h,\phi,\vec{x}_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & h \\ 0 & C\theta & -S\theta & 0 \\ 0 & S\theta & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Compound Screws

- Similar for homogeneous Matrices, screws can be combined:

$$ {}^{k-2}_{k}S\left(h, \phi, \vec{g}, \vec{P}\right) = {}^{k-2}_{k-1}S\left(h_1, \phi_1, \vec{g}_1, \vec{P}_1\right) \odot {}^{k-1}_{k}S(h, 0{,}0{,}0) $$

- General solution is lengthy, but not needed.

# D-H with Screws

- We have $\theta_i, d_i, a_i$ and $\alpha_i$ from D-H for ${}^{i-1}_{i}A$.
  Then,
  $${}^{i-1}_{i}A = S(d_i, \theta_i, z_{i-1}) \odot S(a_i, \alpha_i, x_i)$$
  $$= \begin{bmatrix} C\theta_i & -S\theta_i C\theta_i & S\theta_i S\alpha_i & \alpha_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\theta_i & \alpha_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# D-H with Screws-compound

- From this, we can compute the compound $S(h, \phi, \vec{g}, \vec{P})$:
  - $\cos\phi = \frac{1}{2}\left(tr\left({}^{G}R_B\right) - 1\right)$
    $= \frac{1}{2}(\cos\theta_i + \cos\theta_i \cos\alpha_i + \cos\alpha_i - 1)$
    - $\vec{g} = \frac{1}{2S\phi}\begin{bmatrix} \sin\alpha_i + \cos\theta_i \sin\alpha_i \\ \sin\theta_i \sin\alpha_i \\ \sin\theta_i + \cos\alpha_i \sin\theta_i \end{bmatrix}$

based on theorem of Rodrigues

$S\theta = \sin\theta, \;\; C\theta = \cos\theta, \;\; \eta\theta = 1 - \cos\theta \;, \vec{g} = (g_1, g_2, g_3)^T = \frac{1}{2S\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$

# D-H with Screws-compound

- Screw with $x = 0$: (intersection with y-z plane)
- $$\begin{bmatrix} h \\ y \\ z \end{bmatrix} = \begin{bmatrix} g_1 & -r_{12} & -r_{13} \\ g_2 & 1 - r_{22} & -r_{23} \\ g_3 & -r_{32} & 1 - r_{33} \end{bmatrix}^{-1} \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix}$$

$$= \frac{1}{2S\phi} \begin{bmatrix} S\alpha_i + C\theta_i S\alpha_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i \\ S\theta_i S\alpha_i & 1 - C\theta_i C\alpha_i & -C\theta_i S\alpha_i \\ S\theta_i + C\alpha_i S\theta_i & S\alpha_i & C\alpha_i \end{bmatrix}^{-1} \begin{bmatrix} a_i C\theta_i \\ a_i S\theta_i \\ d \end{bmatrix}$$

$$\vec{P} = (0, y, z)^T$$

# D-H with Screws-compound

- Screw with $y = 0$: (intersection with x-z plane)

- $$\begin{bmatrix} h \\ x \\ z \end{bmatrix} = \begin{bmatrix} g_1 & 1 - r_{11} & -r_{13} \\ g_2 & -r_{21} & -r_{23} \\ g_3 & -r_{31} & 1 - r_{33} \end{bmatrix}^{-1} \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix}$$

$$= \frac{1}{2S\phi} \begin{bmatrix} S\alpha_i + C\theta_i S\alpha_i & 1 - C\theta_i & S\theta_i S\alpha_i \\ S\theta_i S\alpha_i & S\theta_i & -C\theta_i S\alpha_i \\ S\theta_i + C\alpha_i S\theta_i & 0 & C\alpha_i \end{bmatrix}^{-1} \begin{bmatrix} a_i C\theta_i \\ a_i S\theta_i \\ d \end{bmatrix}$$

$$\vec{P} = (x, 0, z)^T$$

# D-H with Screws-compound

- Screw with $z = 0$: (intersection with x-y plane)

- $$\begin{bmatrix} h \\ x \\ y \end{bmatrix} = \begin{bmatrix} g_1 & 1 - r_{11} & -r_{12} \\ g_2 & -r_{21} & 1 - r_{22} \\ g_3 & -r_{31} & -r_{32} \end{bmatrix}^{-1} \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix}$$

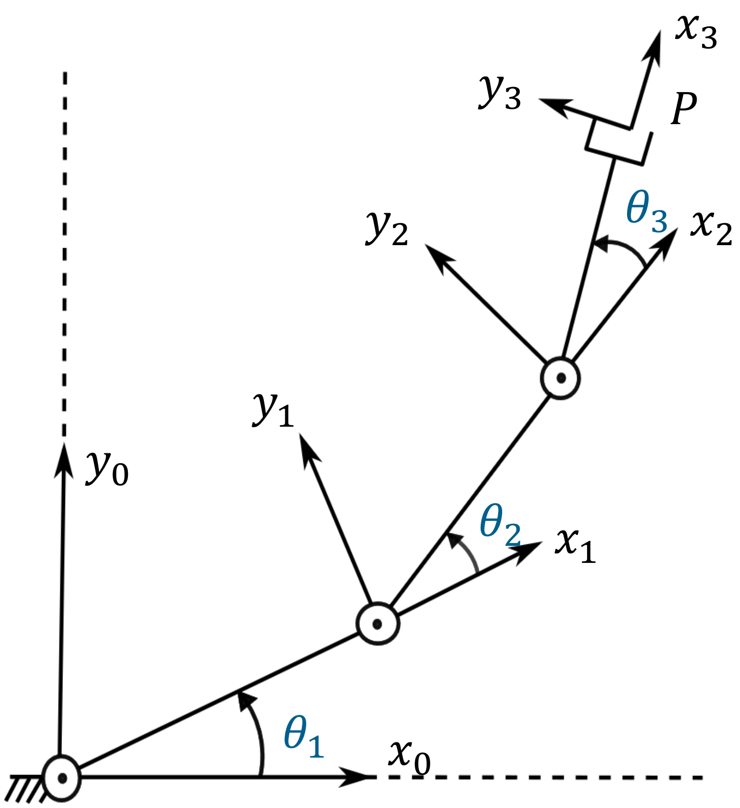
$$= \frac{1}{2S\phi} \begin{bmatrix} S\alpha_i + C\theta_i S\alpha_i & 1 - C\theta_i & -S\theta_i S\alpha_i \\ S\theta_i S\alpha_i & S\theta_i & 1 - C\theta_i S\alpha_i \\ S\theta_i + C\alpha_i S\theta_i & 0 & C\alpha_i \end{bmatrix}^{-1} \begin{bmatrix} a_i C\theta_i \\ a_i S\theta_i \\ d \end{bmatrix}$$

$$\vec{P} = (x, y, 0)^T$$

# Direct Kinematics: Example 1

- $\phi_i = \theta_i$
- $\hat{g}_i = (0,0,1)^T$
- $h_i = 0$
- $\vec{P}_i = \left(\frac{a(2C\theta-1)}{4S\theta}, \frac{a(2C\theta-1)}{4C\theta-1}, 0\right)^T$
- ${}^0_3S\left(h, \phi, \vec{g}, \vec{P}\right) = {}^0_1S\left(h_1, \phi_1, \vec{g}_1, \vec{P}_1\right)$
- ${}^1_2S\left(h_2, \phi_2, \vec{g}_2, \vec{P}_2\right)$
- ${}^2_3S\left(h_3, \phi_3, \vec{g}_3, \vec{P}_3\right)$

| Joint | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | $\theta_1$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2$ |
| 3 | $a_3$ | 0 | 0 | $\theta_3$ |



# Exponential coordinates: Motivation

- Rotation axis $\vec{g}$, rotation angle $\theta$
- Motivation:
  Rotate a point $\vec{q}$ with a velocity of 1 around an axis $\vec{g}$

$$\Rightarrow \dot{\vec{q}}(t) = \vec{g} \times \vec{q}(t) =: \hat{g}\vec{q}(t)$$

With:

$$\hat{g} = \begin{bmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{bmatrix}$$

# Exponential coordinates

- Rotate by angle $\theta$ with integral equation:

$$\int_0^{\theta} \hat{g}\,\vec{q}(t)dt = e^{\hat{g}\theta}\vec{q}(0)$$

Where $e^{\hat{g}\theta}$ is exponential of a matrix.

$$\begin{aligned} e^{\hat{g}\theta} &:= 1 + \hat{g}\theta + \frac{(\hat{g}\theta)^2}{2!} + \frac{(\hat{g}\theta)^3}{3!} + \ldots \\ &= 1 + \hat{g}\sin\theta + \hat{g}^2(1 - \cos\theta) \end{aligned}$$

# Exponential Representation of a Screw

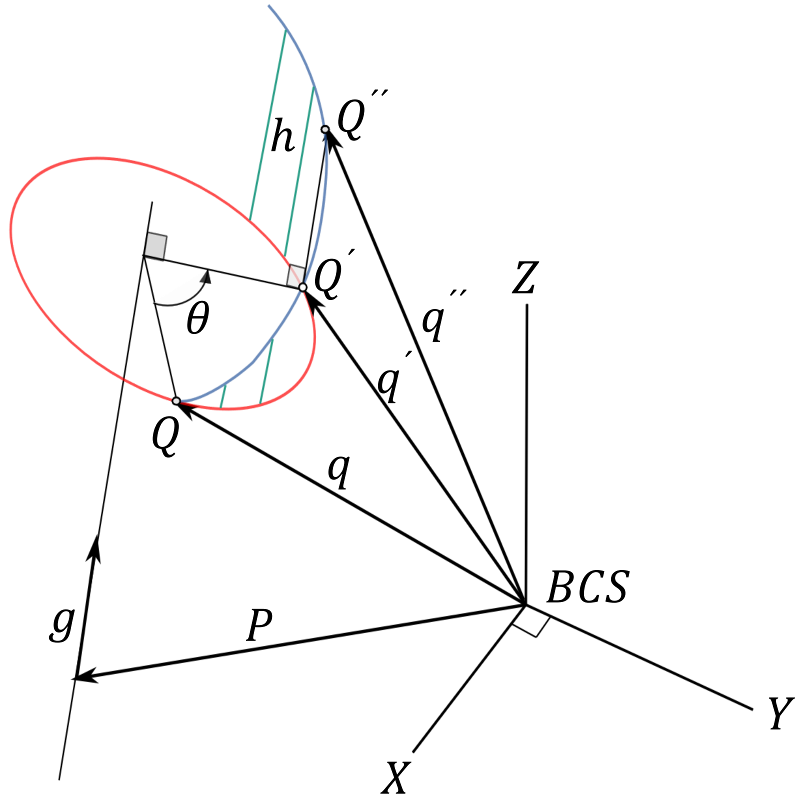
- Similar to Exponential coordinates for a rotation, a transformation can be rotated.

As in image,

- $$Q'' = P + e^{\phi \hat{g}(q-P)} + h\vec{g} =: [T]$$

Where,

- $$[T] = \begin{bmatrix} e^{\phi \vec{g}} & \left(I - e^{\phi \vec{g}}\right)P + h\vec{g} \\ 0 & 1 \end{bmatrix}$$



## Exponential representation of central Screws

- For central screws ($\vec{P}$=0) the same holds.
  And:
- $$[T] = \begin{bmatrix} e^{\phi\hat{\vec{g}}} & h\vec{g} \\ 0 & 1 \end{bmatrix}$$

Note that:

$$e^{\begin{bmatrix} \phi\hat{g} & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} e^{\phi\hat{g}} & 0 \\ 0 & 1 \end{bmatrix}$$ Rotation

and,

$$e^{\begin{bmatrix} 0 & h\vec{g} \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} I_3 & h\vec{g} \\ 0 & 1 \end{bmatrix}$$ Translation

# Exponential representation of D&H

- Therefore:

$$[T] = e^{\begin{bmatrix} \phi\hat{g} & h\vec{g} \\ 0 & 0 \end{bmatrix}} =: e^{\xi(h,\phi,\vec{g})}$$

- For D-H this means:

$$\begin{aligned} {}^{i-1}_{i}A &= S(d_i, \theta, z_{i-1}) \odot S(a_i, \alpha_i, x_{i-1}) \\ &= e^{\xi(d_i,\theta_i,z_{i-1})} e^{\xi(a_i,\alpha_i,x_i)} \end{aligned}$$

# Exponential Representation of Example 1

- $${}^{i-1}_{i}A = e^{\xi(0,\theta_i,z_{i-1})} e^{\xi(a_i,0,x_i)}$$

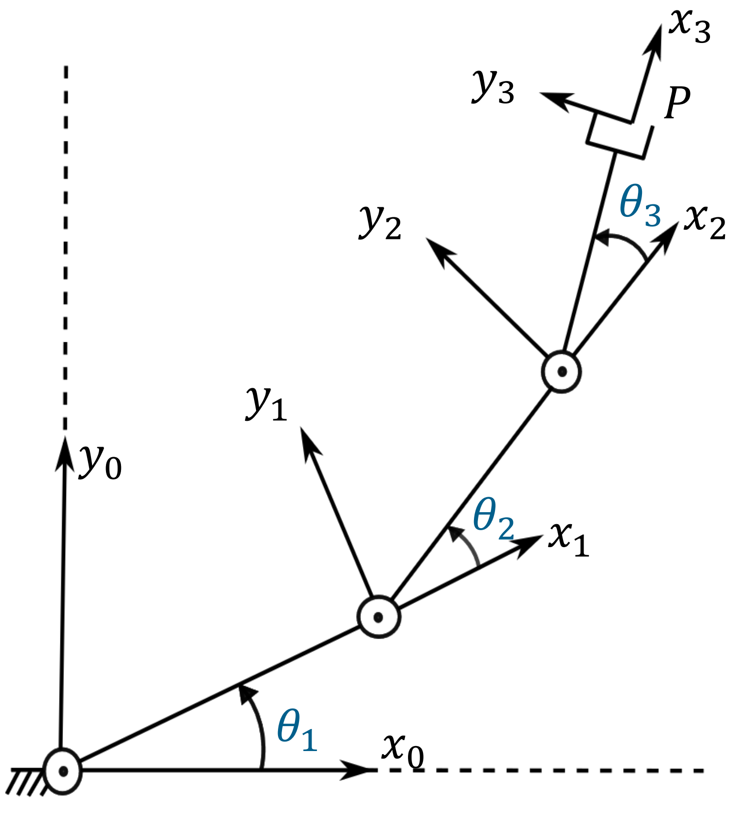
$${}^{i-1}_{i}A = e^{\begin{bmatrix}\theta_i \hat{z}_{i-1} & 0 \\ 0 & 0\end{bmatrix}} e^{\begin{bmatrix}0 & a_i x_i \\ 0 & 0\end{bmatrix}}$$

$$= e^{\begin{bmatrix}\theta_i \hat{z}_{i-1} & a_i x_i \\ 0 & 0\end{bmatrix}}$$

$${}^{0}_{3}A = e^{\begin{bmatrix}\theta_1 \hat{z}_0 & a_1 x_1 \\ 0 & 0\end{bmatrix}} e^{\begin{bmatrix}\theta_2 \hat{z}_1 & a_2 x_2 \\ 0 & 0\end{bmatrix}} e^{\begin{bmatrix}\theta_3 \hat{z}_2 & a_3 x_3 \\ 0 & 0\end{bmatrix}}$$

$$= e^{\begin{bmatrix}\theta_1 \hat{z}_0 + \theta_2 \hat{z}_1 + \theta_3 \hat{z}_2 & a_1 x_1 + a_2 x_2 + a_3 x_3 \\ 0 & 0\end{bmatrix}}$$

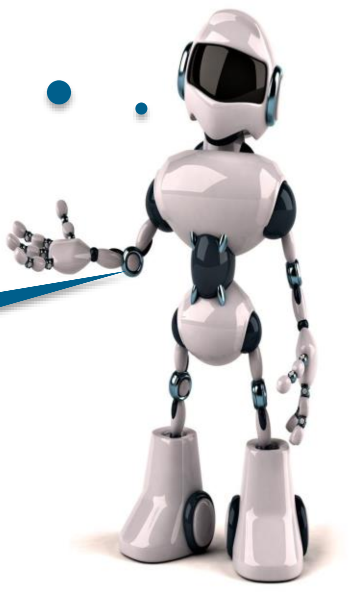
| Joint | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | $\theta_1$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2$ |
| 3 | $a_3$ | 0 | 0 | $\theta_3$ |



# Coming up next ...

## *Inverse Kinematics Problem*

How should I move my hand there?



Find the joint angles