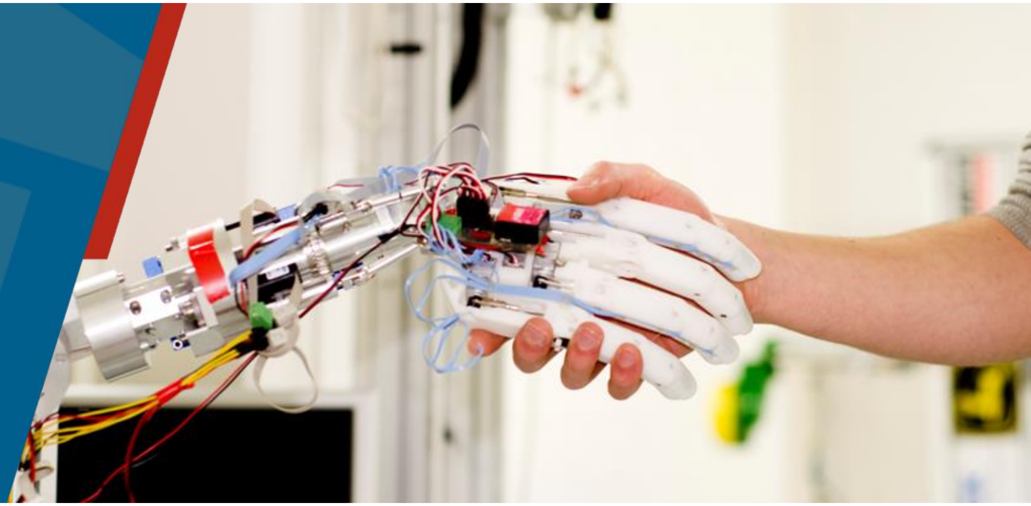
THE ROBOTICS RESEARCH LAB

Direct Kinematics



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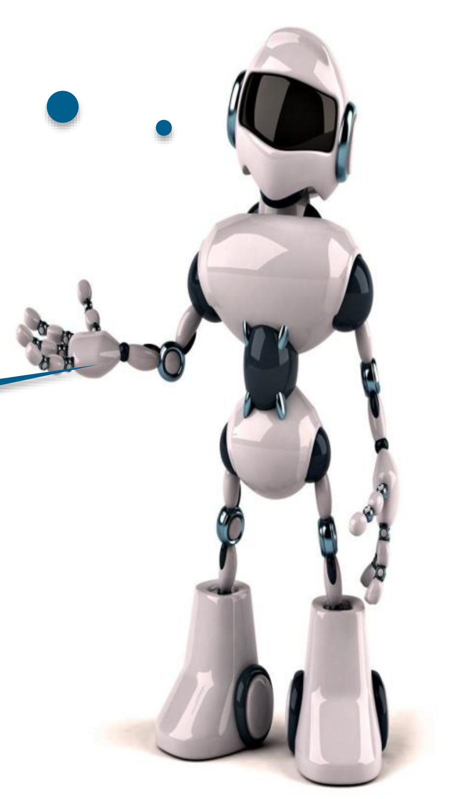
Contents

Direct Kinematic Problem

Where is my hand?

Find pose of TCP



Degree of Freedom (DoF) f of an Object in E_3

- Number of possible independent movements in relation to the BCS
  - Minimal number of translations and rotations for complete description of the object´s pose

- For objects with unconstrained movement in $3D$-space $f = 6$
  - 3 translations
  - 3 rotations

Kinematic Degrees of Freedom of a Robot

- Degrees of freedom of a rotational joint: $F_R \leq 3$
  - Hinge joint
  - Cardan joint
  - Linear joint
  - Spherical joint
- DoF of a translational joint: $F_T = 1$
- Number of joints of robot: n (in general $n \geq 6$)
- Kinematic DoFs: $F = \sum_{i=1}^{n}\left(F_{R_i} + F_{T_i}\right)$

Relation between f and F

- The relationship holds: $F \geq f$
- Examples
  - 8-axis robot: DoF $f = 6$, kinematic DoF $F = 8$
  - Human hand: $f = 6,\ F = 22$
  - Human arm including shoulder: $f = 6,\ F = 12$

- In order to reach a DoF $f = 6$ for a robot‘s effector, it requires at least $F = 6$ axes of movement

Models: Terminology

- Geometry: Mathematical description of robot form
  - Displays bodies graphically
  - Basis of movement calculation
  - Identification of acting forces and moments
  - Starting point of distance and collision measurement
- Kinematics: Geometric and analytic description of mechanical systems` states of movement
- Dynamics: Investigates movement of objects based on the forces and moments acting upon them

Kinematic Model

- Describes the pose (position and orientation) of bodies in space with the help of the geometric model

- Kinematic chain: Several bodies, kinematically connected via joints (e.g. robot arm)
  - Closed kinematic chains
  - Open kinematic chains

- Purpose of kinematic model
  - Determining the relation between joint values and poses
  - Reachability analysis

Dynamic Model

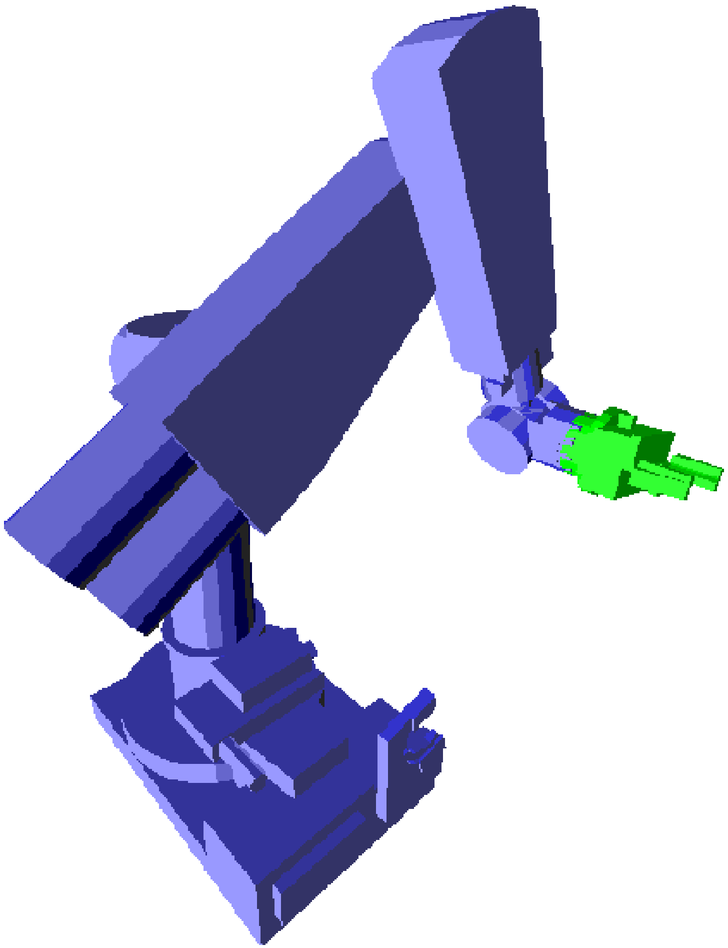
- Describes forces and moments acting in a mechanical multi-body system

- Purpose of dynamic model
  - Dimensioning of driving mechanism
  - Optimization of construction (light weight)
  - Consideration of bending and stiffness
  - Support for controller design

Geometric Model

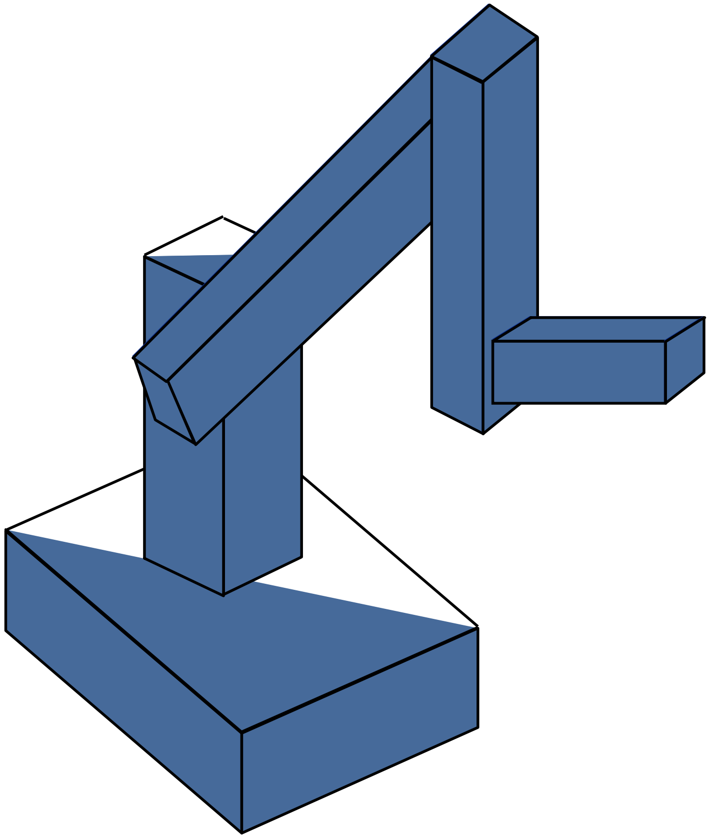
Classification:

- 2D-model
- 2,5D-model
- 3D-model
- Edge or wire-frame models
- Surface models
- Volumetric models



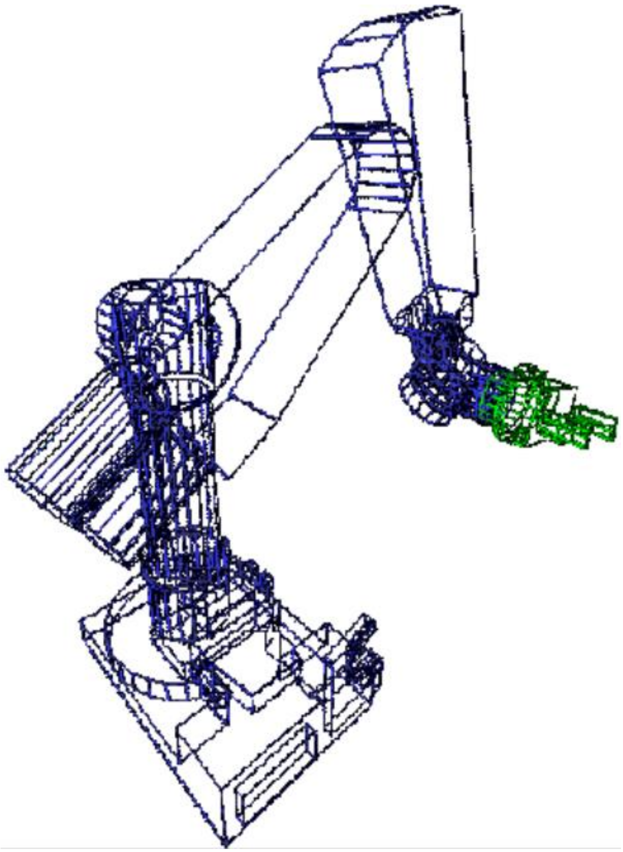
Geometric Model: Block World

- Bodies represented by enveloping cuboids (bounding boxes)
- Easy calculation with regard to collision avoidance



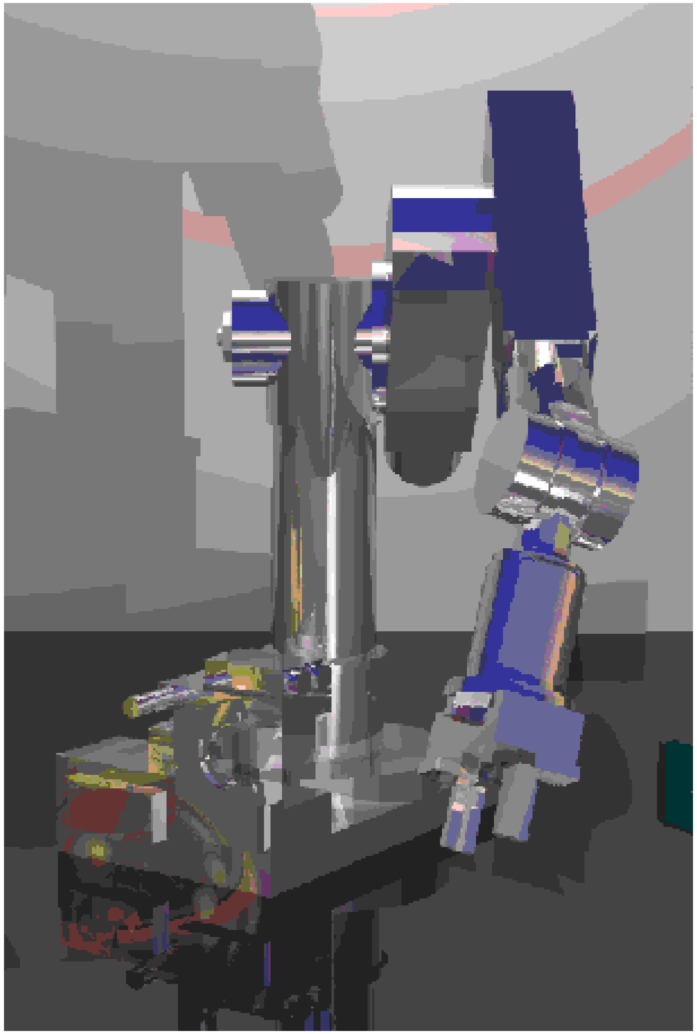
Geometric Model: Edge Model

- Bodies represented by polygons (edges)
- Quick visualization

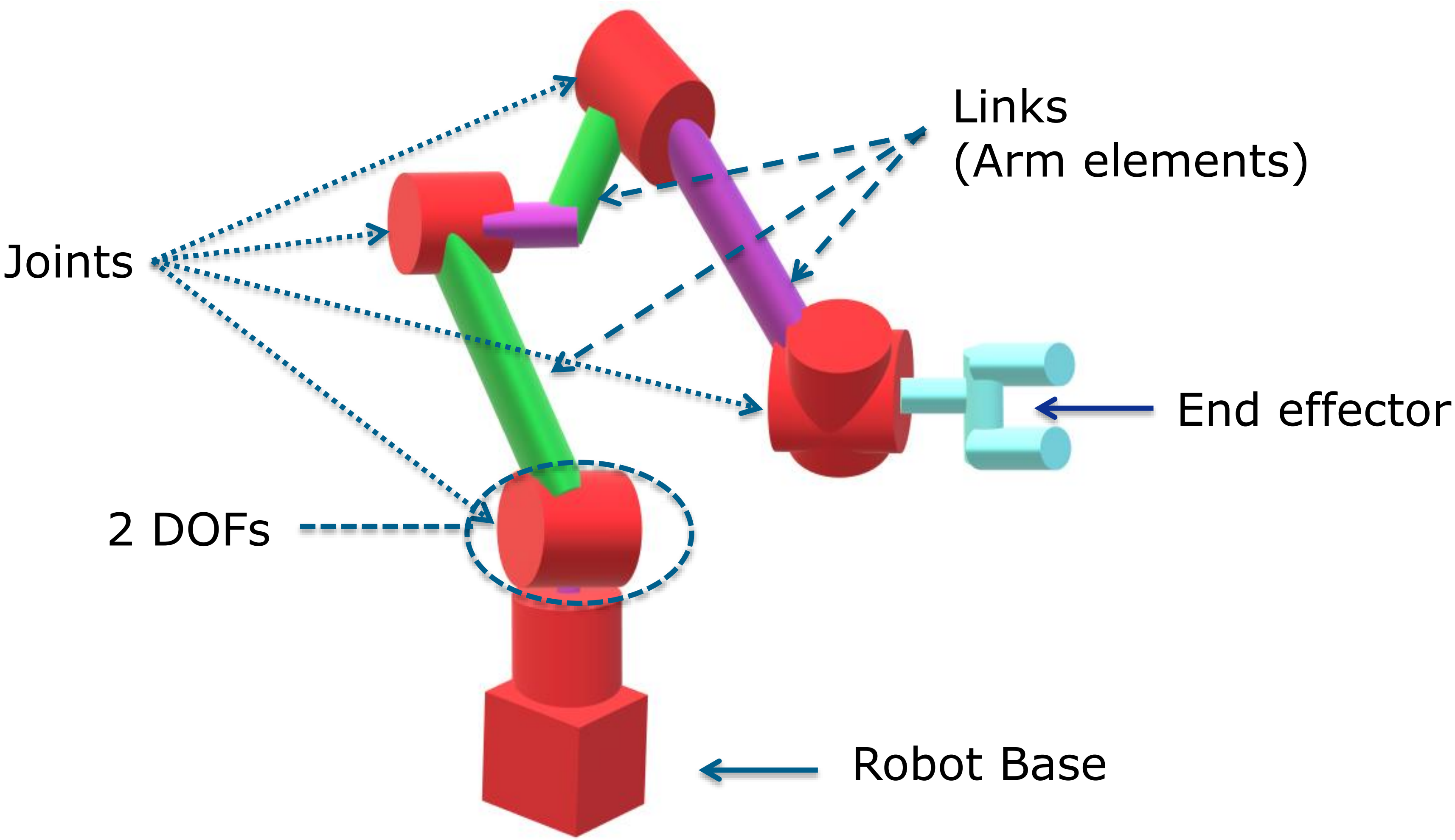


Geometric Models: Volumetric Models

- Precise representation of bodies
- Exact computation of contact points for collision avoidance
- Representation with animations



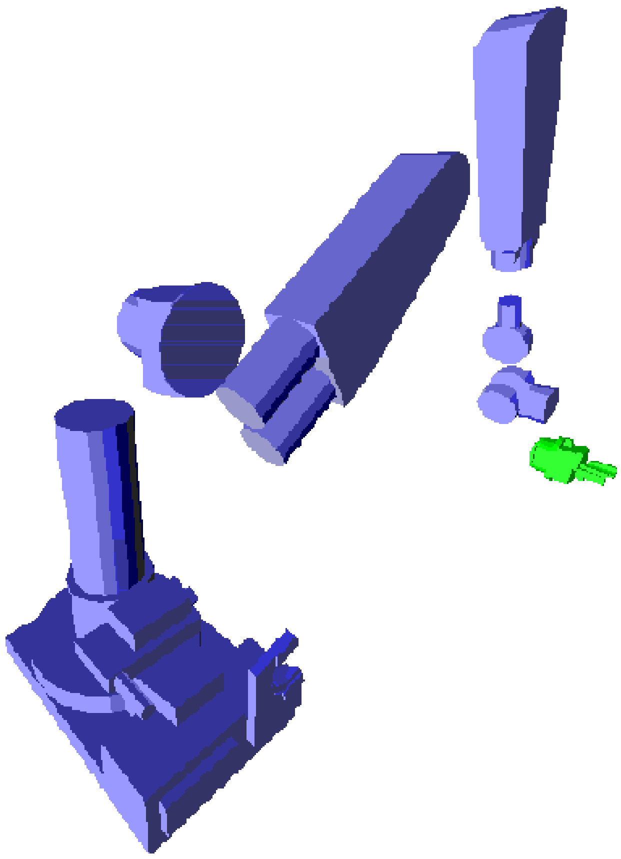
Kinematic Models: Links and Joints



Kinematic Model: Puma 260

Volumetric model

- Every arm element corresponds to a rigid body
- Every arm element is joined to the next via a linear or rotational joint
- Each joint has only one DoF (rot. or transl.)
- Kinematic pair = joint + joined arm elements



Puma 260

- 6-axis robot
- Basis and 6 arm elements (links)

Kinematic Model: Coordinate Systems

In order to describe the kinematics of a robot (kinematic chain), it is necessary to define the links�still poses in relation to a reference coordinate system.

- Each link receives a fixed local CS
- Origin of each CS in joint responsible for moving given link
- A transformation matrix, relating the local CS to the reference system, needs to be determined for every link
- Transformation of local CS to reference CS via description vector or 4×4 homogenous transformation matrix.

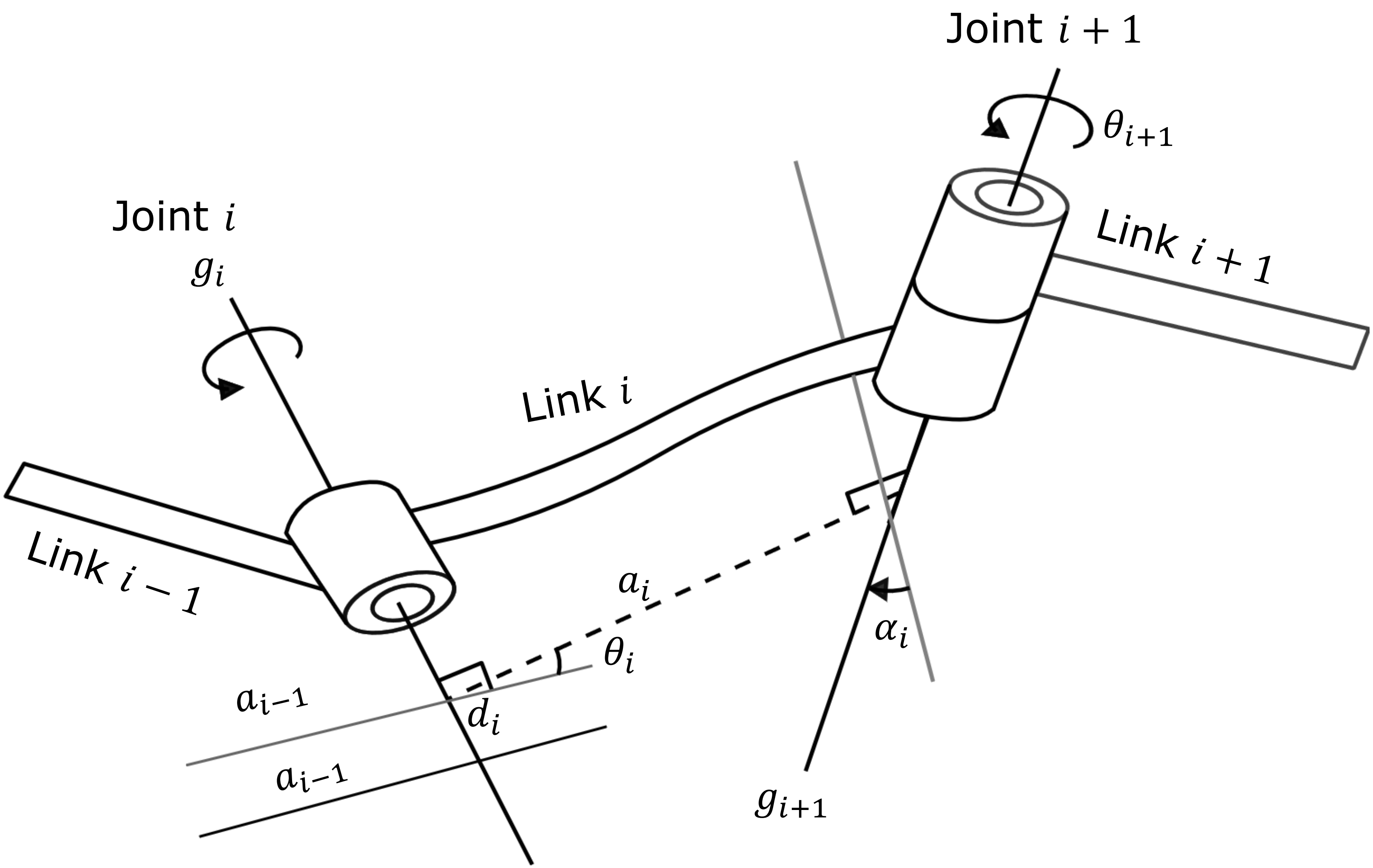
Link Parameters

- Every link i is connected through 2 confining joints i and $i+1$
- Let g_i and g_{i+1} be movement axes of joints (skewed to each other)
- Let a_i be the normal between g_i and g_{i+1}
- The distance of intersections of a_{i-1} and a_i with g_i is referred to as joint offset d_i
- Angle θ_i between a_{i-1} und a_i is referred to as joint angle
- Length of a_i (shortest distance between g_i and g_{i+1}) is called link length
- Angle α_i between g_i und g_{i+1} is called link twist

Parameters of Joints

Parameter	Symbol	Rotational joint	Prismatic joint
Link length	a	Invariant	Invariant
Link twist	α	Invariant	Invariant
Joint distance	d	Invariant	Variable
Joint angle	θ	Variable	Invariant

Derivation of Joint Distance and Angle

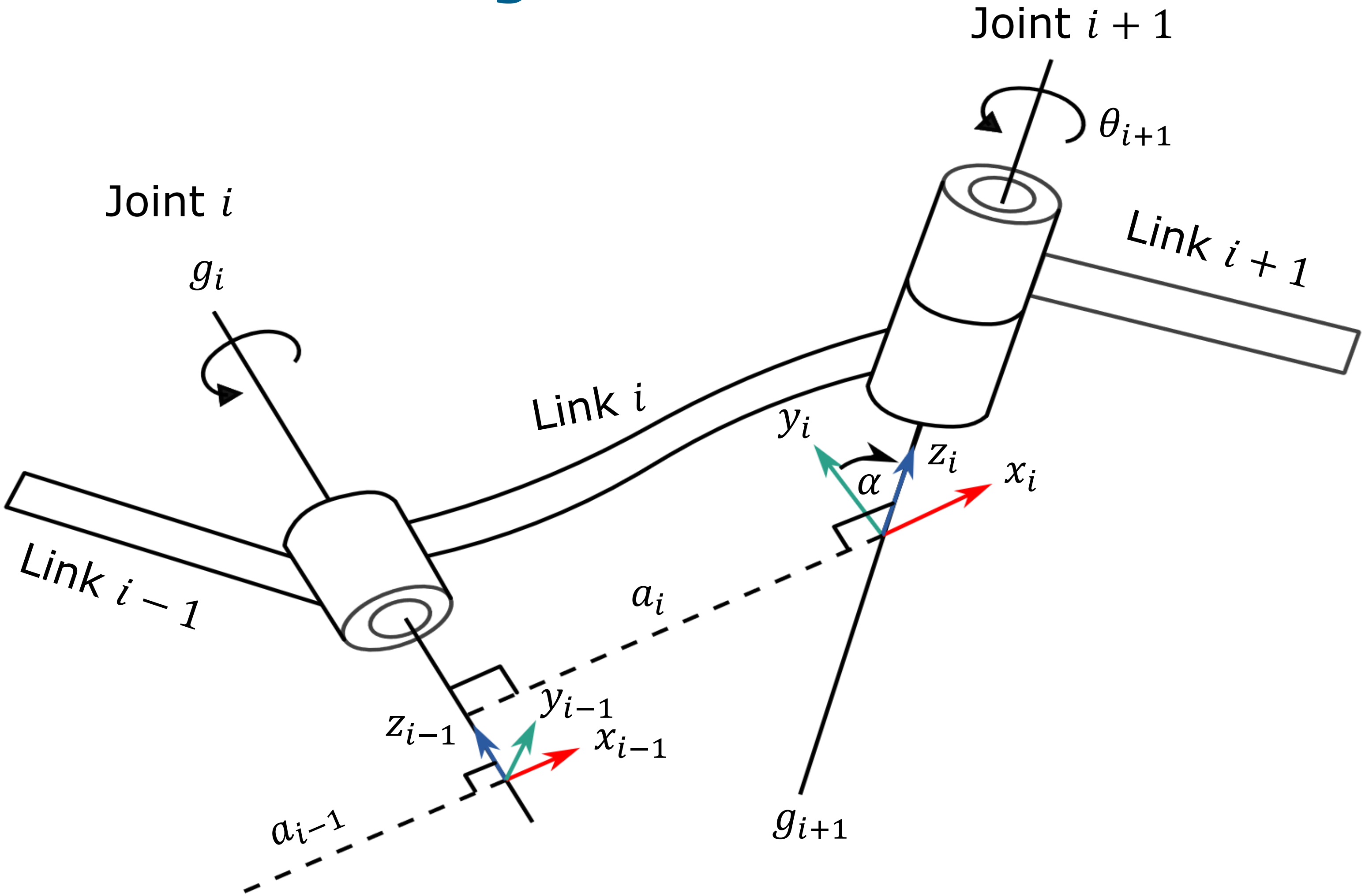


Denavit-Hartenberg-Convention

Definition of coordinate systems for every joint:

- Origin of CS i lies in intersection of a_i and g_{i+1}
- Axis z_i lies along the joint axis g_{i+1}
- Axis x_i follows as extension of normal a_i
- Complemented by axis y_i, so that a clockwise CS follows
- Origin of CS 0 can be freely placed along g_1
- The last coordinate system lies in the end effector

Denavit-Hartenberg-Convention



Denavit-Hartenberg-Convention

- Transformation from OCS of link i to OCS of link $i-1$ via Denavit-Hartenberg-Transformation

- Requirements
  - All CS follow the Denavit-Hartenberg-Convention
  - Parameters of link i are known

Literature: Denavit, Hartenberg: „A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices“, Journal of Applied Mechanics, vol 77, pp 215-221

Denavit-Hartenberg-Transformation

Transformation from OCS_i to OCS_{i-1}

(1) Rotation θ_i around z_{i-1}-axis so that x_{i-1}-axis lies parallel to x_i-axis

$$R_{z_{i-1}}(\theta_i) = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2) Translation d_i along z_{i-1}-axis to the intersection point of z_{i-1} and x_i

$$T_{z_{i-1}}(d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denavit-Hartenberg-Transformation

Transformation from OCS_i to OCS_{i-1}

(3) Translation a_i along x_i-axis in order to make the origins of the coordinate systems congruent

$$T_{x_i}(a_i) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(4) Rotation α_i around x_i-axis, to transform the z_{i-1}-axis into z_i

$$R_{x_i}(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) & 0 \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denavit-Hartenberg-Transformation

In matrix notation, the D-H-Transformation of the coordinate system of link $i-1$ to i looks like:

$$^{i-1}_{i}A = R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(a_i) \cdot R_{x_i}(\alpha_i)$$

$$= \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) \cdot \cos(\alpha_i) & \sin(\theta_i) \cdot \sin(\alpha_i) & a_i \cdot \cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \cdot \cos(\alpha_i) & -\cos(\theta_i) \cdot \sin(\alpha_i) & a_i \cdot \sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denavit-Hartenberg-Transformation (Inverse)

The inverse of transformation matrix ${}^{i-1}_{i}A$ corresponds to the transformation from CS_{i-1} to CS_i:

$$
{}^{i-1}_{i}A^{-1} = {}^{i}_{i-1}A
$$

$$
= \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) & 0 & -a_i \\ -\cos\alpha_i \sin(\theta_i) & \cos(\theta_i)\cdot\cos(\alpha_i) & \sin(\alpha_i) & -d_i \sin(\alpha_i) \\ \sin(\theta_i)\cdot\sin(\alpha_i) & -\sin(\alpha_i)\cos(\theta_i) & \cos(\alpha_i) & -d_i\cos(\alpha_i) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Denavit-Hartenberg-Transformation (Inverse)

To determine the pose of the manipulator (Tool Center Point, TCP) in relation to the BCS, the D-H-matrices are multiplied in order of the corresponding links:

$$ {}^{\mathrm{Basis}}_{\mathrm{TCP}}A = {}^{\mathrm{Basis}}_{1}A(\theta_1) \cdot {}^{1}_{2}A(\theta_2) \cdots {}^{n-2}_{n-1}A(\theta_{n-1}) \cdot {}^{n-1}_{n}A(\theta_n) $$

Approach with D-H

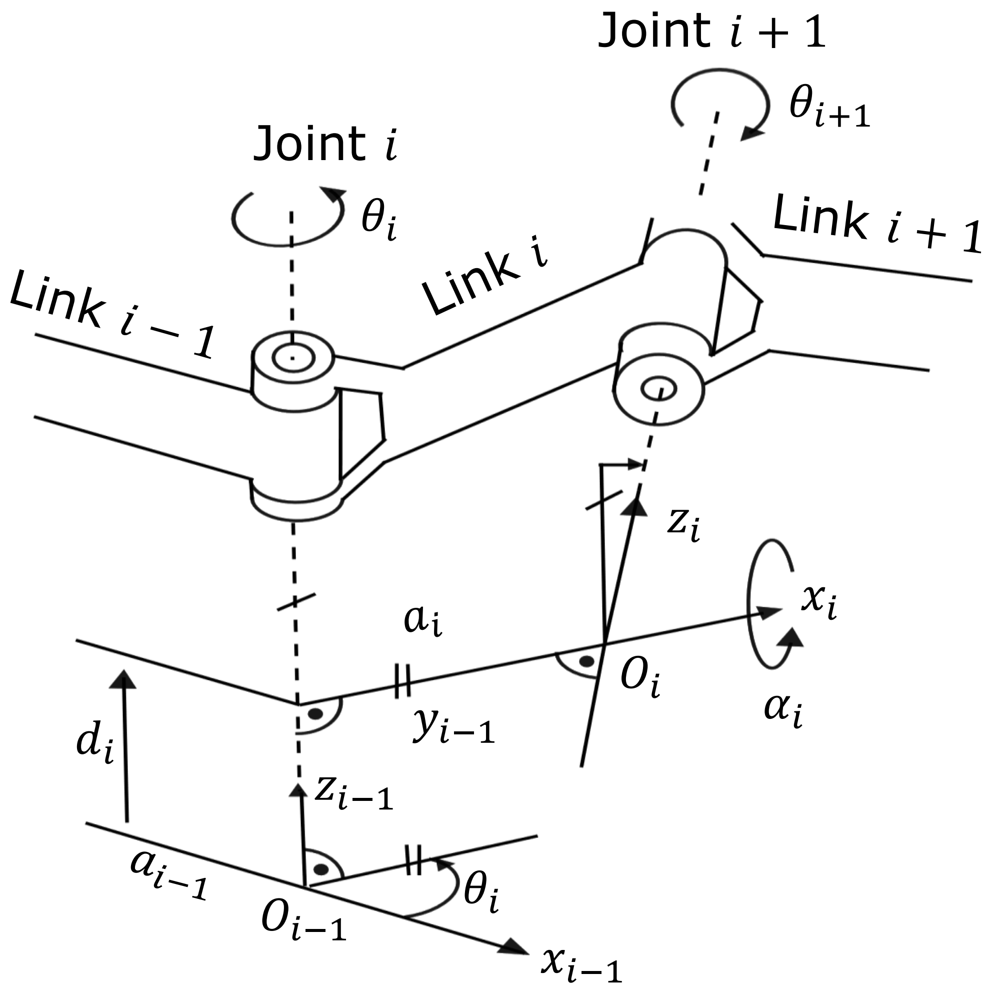
(1) Finding the normal a_i

- Joint axes g_i
- a_i points from g_i to g_{i+1}

(2) Defining the CS

- Origin O_i in the intersection of a_i with g_{i+1}
- x_i lies on normal a_i and points in the same direction
- z_i lies on g_{i+1}, in the direction allowing joint rotation or translation in mathematically positive sense
- y_i completes the clockwise CS; at TCP, $y_i$indicates width of opening

Denavit-Hartenberg-Convention



Approach with D-H

Special cases of $(1), (2)$

- g_i and g_{i+1} intersect
  - Direction of x_i not defined
  - x_i arises from x_{i-1} through smallest possible rotation around z_{i-1}
- g_i and g_{i+1} are parallel or collinear
  - Intersection of a_i and g_{i+1} not well defined
  - Determine the normals by backstepping (recursively)
  - Starting at next uniquely identifiable origin O_j with $j > i$
  - At last joint, the origin is in the center of TCP

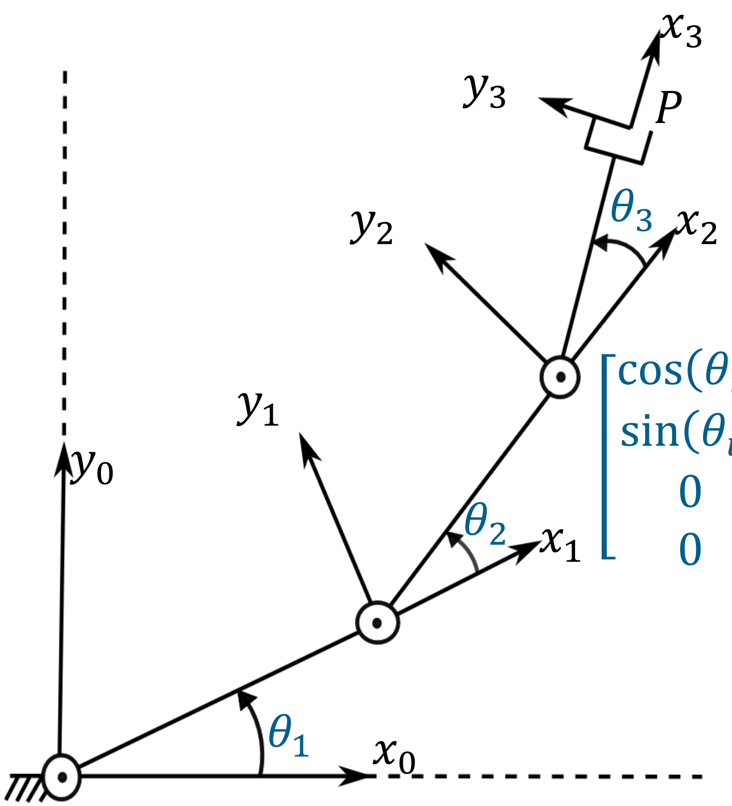
Direct Kinematics: Approach with D-H

(3) Determining the transformation matrix ${}^{i+1}_{i}A$

- Rotation of CS of joint i around z_{i-1} with joint angle θ_i
  → x'_{i-1} is parallel to x_i
- Translation by d_i along z_{i-1}
  → Origin lies in intersection of z_{i-1} and x_i
- Translation by $|a_i|$ along x_i
  → Origins are congruent
- Rotation around x_i with twist α_i
  → Z'_{i-1} is parallel to z_i
- Combination of all these results:

$$ {}^{i-1}_{i}A = R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(|a_i|) \cdot R_{x_i}(\alpha_i) $$

Direct Kinematics: Example 1

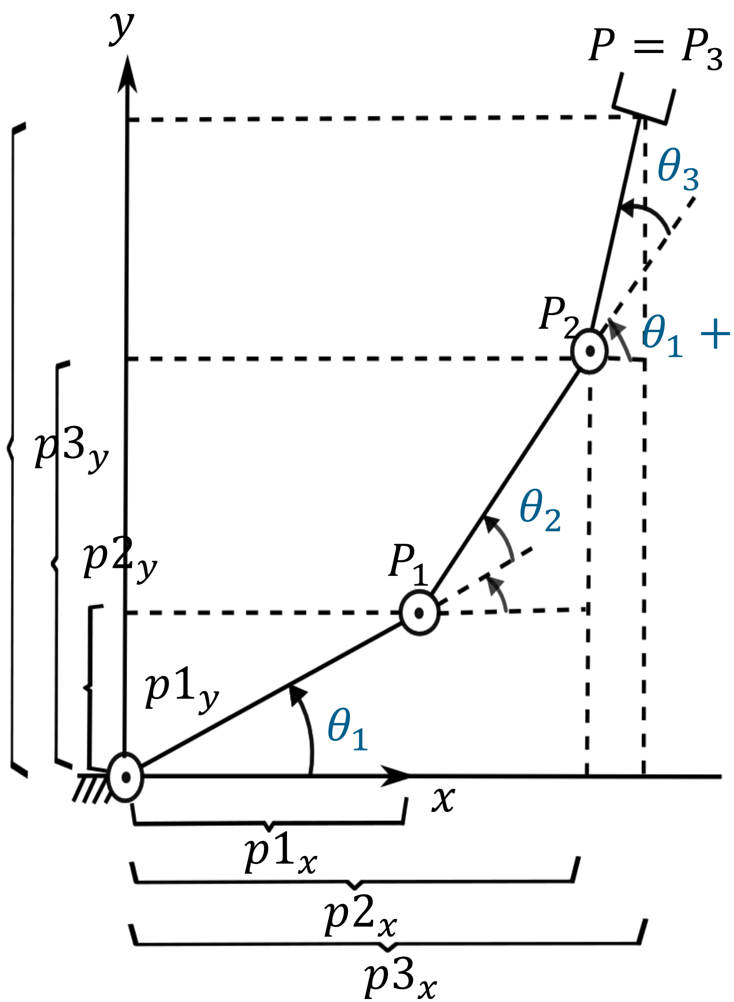


Joint	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

$$\begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cdot\cos(\alpha_i) & \sin(\theta_i)\cdot\sin(\alpha_i) & a_i\cdot\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cdot\cos(\alpha_i) & -\cos(\theta_i)\cdot\sin(\alpha_i) & a_i\cdot\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow {}_{i}^{i-1}A = \begin{bmatrix} c_i & -s_i & 0 & a_i\cdot c_i \\ s_i & c_i & 0 & a_i\cdot s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Direct Kinematics: Example 1



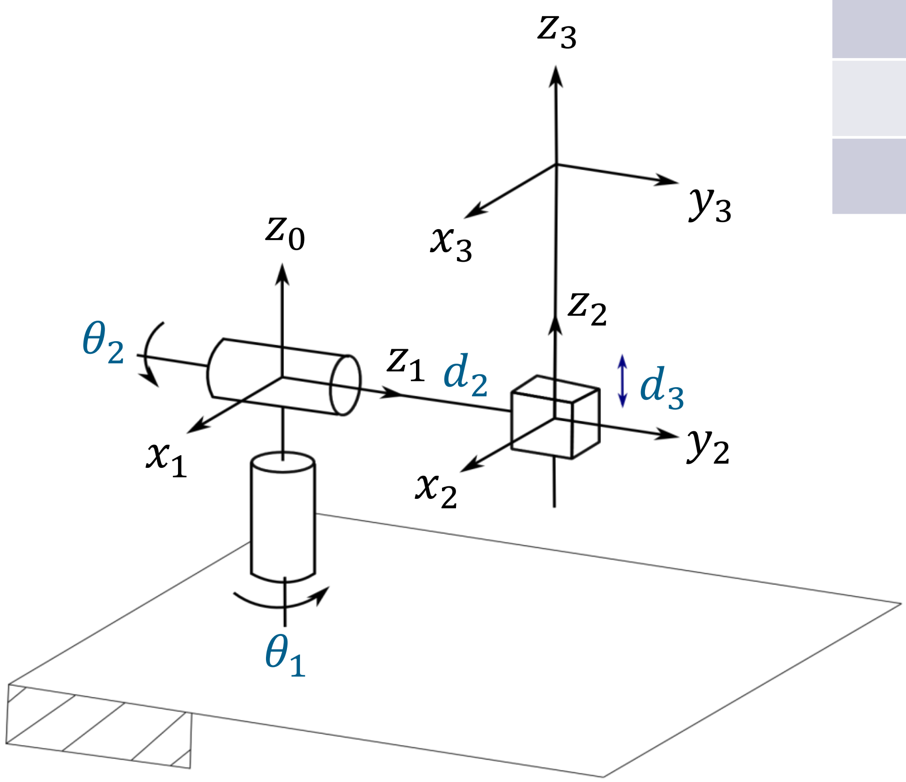
Result:

$$
{}^0_3A = {}^0_1A \cdot {}^1_2A \cdot {}^2_3A = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 \cdot c_1 + a_2 \cdot c_{12} + a_3 \cdot c_{123} \\ s_{123} & c_{123} & 0 & a_1 \cdot s_1 + a_2 \cdot s_{12} + a_3 \cdot s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

With:

$s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$ and
$c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$

Direct Kinematics: Example 2



Joint	a_i	α_i	d_i	θ_i
1	0	$-90°$	0	θ_1
2	0	$90°$	d_2	θ_2
3	0	0	d_3	0

$$ {}^0_1A = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2A = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

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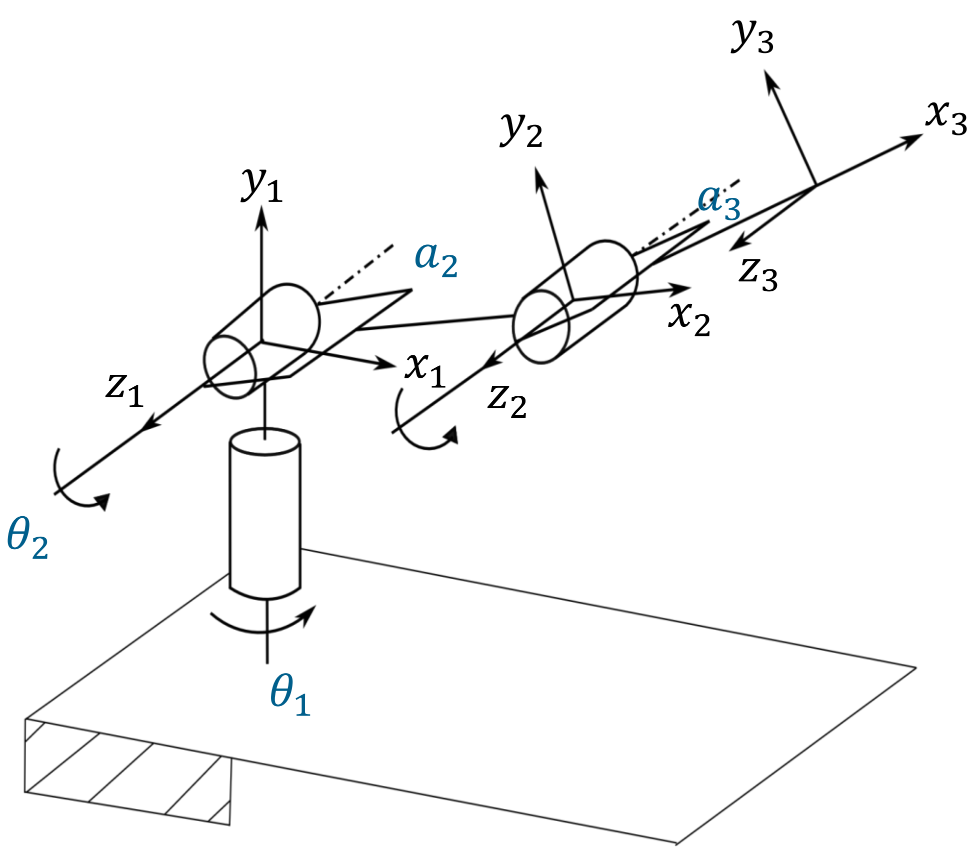
Direct Kinematics: Example 2

$$
{}^{2}_{3}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

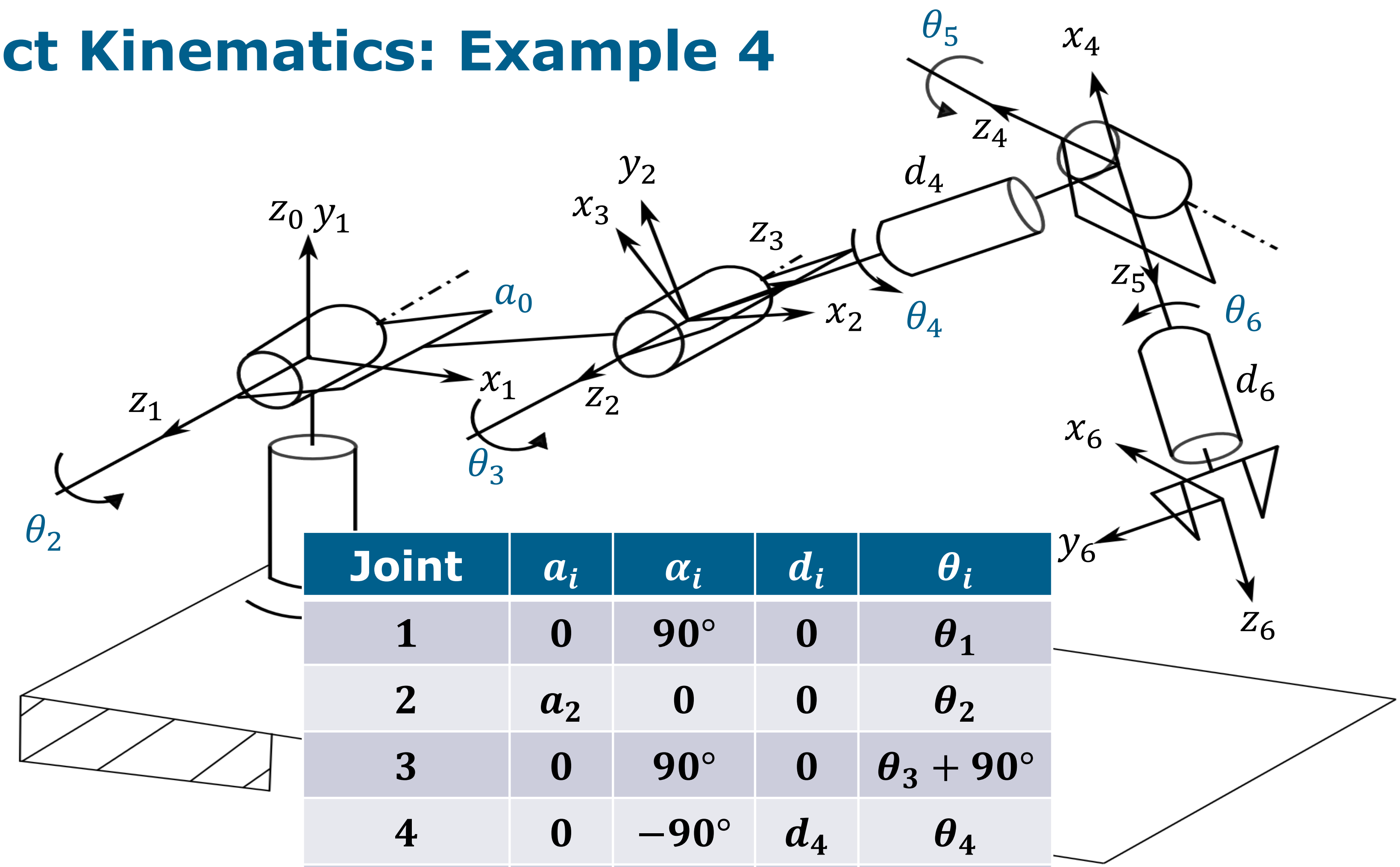
$$
\Rightarrow {}^{3}_{0}A = {}^{0}_{1}A \cdot {}^{1}_{2}A \cdot {}^{2}_{3}A = \begin{bmatrix} c_1 \cdot c_2 & -s_1 & c_1 \cdot s_2 & c_1 \cdot s_2 \cdot d_3 - s_1 \cdot d_2 \\ s_1 \cdot c_2 & c_1 & s_1 \cdot s_2 & s_1 \cdot s_2 \cdot d_3 + c_1 \cdot d_2 \\ -s_2 & 0 & c_2 & c_2 \cdot d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Direct Kinematics: Example 3



Joint	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

Direct Kinematics: Example 4



Joint	a_i	α_i	d_i	θ_i
1	0	$90°$	0	θ_1
2	a_2	0	0	θ_2
3	0	$90°$	0	$\theta_3 + 90°$
4	0	$-90°$	d_4	θ_4
5	0	$90°$	0	$\theta_5 - 90°$
6	0	0	d_6	$\theta_6 + 90°$

Robot Kinematics

- Describes relations between joint angle space and the end effector‘s pose space in world coordinates
  - Joint angle space:
    Robot coordinates, configuration space
  - EE: Abbreviation for end effector
- Direct kinematic problem (forward kinematics)
- Inverse kinematic problem (backward kinematics)

Direct Kinematics Problem (Forward Kinematics)

- The manipulator‘s pose is to be determined from D-H-parameters and joint
- TCP pose in relation to the BCS (basis)

$$ {}^{\mathrm{BASIS}}_{\mathrm{TCP}}A = {}^{\mathrm{BASIS}}_{1}A(\theta_1) \cdot {}^{1}_{2}A(\theta_2) \cdots {}^{n-2}_{n-1}A(\theta_{n-1}) \cdot {}^{n-1}_{n}A(\theta_n) $$

- $\vec{\theta}$ given

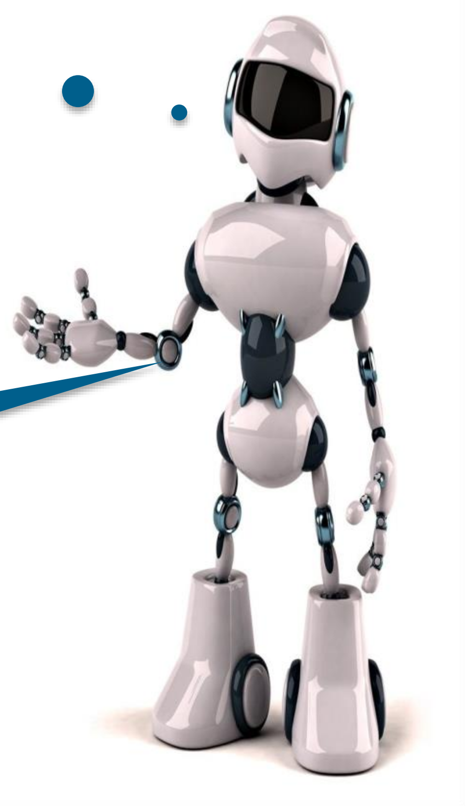
Direct Kinematics Summary

- Sketch of the manipulator
- Enumerate links: Basis $= 0$, last link $= n$
- Identify and enumerate the joints
- Draw axes z_i for every joint i
- Determine parameters a_i between z_{i-1} and z_i
- Draw x_{i-1}-axes
- Determine parameter α_{i-1} (twist around x_{i-1}-axes)
- Determine parameter d_i (offset)
- Determine angle θ_i around z_i-axes
- Joint-transformation matrices $A_{i-1,1}$

Inverse Kinematic Problem (Chapter 7)

How should I move my hand there?

Find the joint angles



Coming up next …

Real and Dual Quaternions