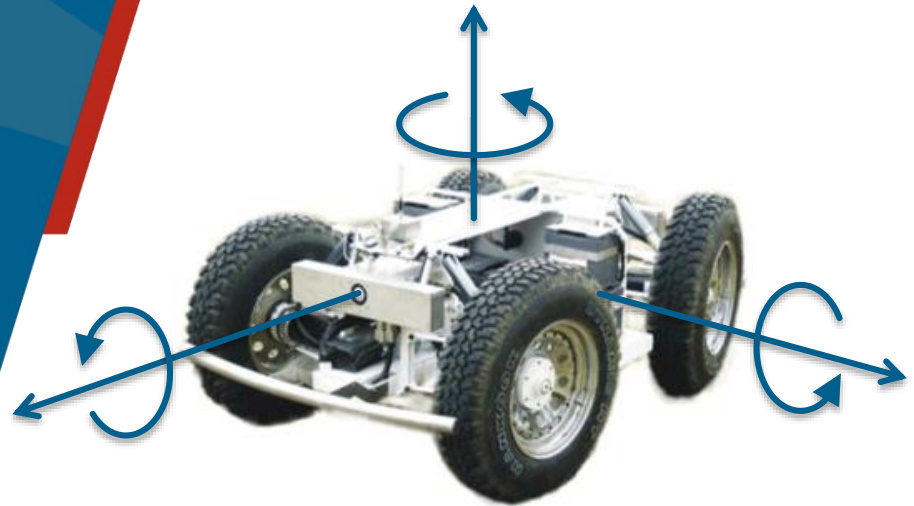


Spatial Kinematics II



Prof. Karsten Berns

Robotics Research Lab

Department of Computer Science

University of Kaiserslautern, Germany

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Object Pose in a 3D Euclidian Space (E_3)

Orientation of a Rigid Body

- Every orientation of a rigid body in E_3 is reachable by 3 rotations around the axes of the coordinate system
- Every rotation around a axis can be constructed as a 3×3 rotation matrix
- Composition of rotations by multiplying the corresponding matrices
- Matrix multiplication has no commutative but an associative property
- Interpret $R_1 \cdot R_2 \cdots R_n$ from left to right
 - Rotation of R_i according to the coordinate system defined by $R_1 \cdot \dots \cdot R_{i-1}$
 - R_1 rotates the BCS
- Euler angles: $R_S = R_Z(\alpha) \cdot R_{y'}(\beta) \cdot R_{z''}(\gamma)$

6-Dimensional Description Vector

- The pose of an object in E_3 can be described by a 6-tuple $(x, y, z, \alpha, \beta, \gamma) \in \mathbb{R}^6$
$${}^B\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + {}^B_O R(\alpha, \beta, \gamma) {}^O\vec{u}$$
- x, y, z : Coordinates of the origin of OCS relative to the BCS (describing position)
- α, β, γ : Angle of rotation, corresponding to the axes of rotation (describing orientation)
- Position vector and rotation matrix are intuitive and mainly used for the pose description of objects and end effectors
- Cons: Vector and matrix operators are separate

Homogeneous Coordinate Transformation Matrix

- Replacement of translations and rotations operators through homogeneous matrices
-> Transformation from Cartesian to homogeneous coordinates
- Let P be a point with Cartesian coordinates (p_x, p_y, p_z) ,
let $s \in \mathbb{R}$ then $P' = (sp_x, sp_y, sp_z, s) \in \mathbb{R}^4$ is the representation of P in homogeneous coordinates
- For each P there is an infinite number of homogeneous points P'

Homogeneous Coordinates

- Cartesian coordinates of a homogeneous point

$$P(x, y, z, s) \text{ are } \left(\frac{x}{s}, \frac{y}{s}, \frac{z}{s}\right)$$

- Homogeneous coordinates in E_3 allows to create 4×4 transformation matrices containing:
rotation, translation, scaling and perspective transformation
- In homogeneous coordinates, transformations of n -dimensional physical vectors into an $n+1$ -dimensional space.
- In robotics, $s = 1$

Object Pose in Homogeneous Coordinates

- Scaling factor s
- Perspective transformation P
 - Here $(0,0,0)^T$
- Translation vector $u = (u_x, u_y, u_z)$
 - Position: Origin of OCS relative to BCS
- Rotations $R = (n_x, n_y, n_z), (o_x, o_y, o_z), (a_x, a_y, a_z)$
 - Orientation: 3 unit vectors in x -, y - and z -direction of the OCS

$$A = \begin{bmatrix} R & \vec{u} \\ \vec{P} & s \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & u_x \\ R_{21} & R_{22} & R_{23} & u_y \\ R_{31} & R_{32} & R_{33} & u_z \\ P_1 & P_2 & P_3 & s \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & u_x \\ n_y & o_y & a_y & u_y \\ n_z & o_z & a_z & u_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Translation Matrix T

- Let p, p' be position vectors in homogeneous coordinates
- Let a be a homogeneous translation vector where $a = (a_x, a_y, a_z, 1)^T$
- A Cartesian translation $p = a + p'$ can be represented by the translation matrix T :

$$p = T(a_x, a_y, a_z)p' \text{ with } T = \begin{bmatrix} 1 & 0 & 0 & a_x \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow T(a_x, a_y, a_z)^{-1} = \begin{bmatrix} 1 & 0 & 0 & -a_x \\ 0 & 1 & 0 & -a_y \\ 0 & 0 & 1 & -a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Local and Global Scaling

- Local (anisotropic) scaling

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} ax \\ by \\ cz \end{bmatrix}$$

- Global (isotropic) scaling

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ s \end{bmatrix} \rightarrow \begin{bmatrix} x \\ s \\ y \\ s \\ z \\ s \end{bmatrix}$$

- $s > 1$ down scaling
- $s < 1$ up scaling

Homogeneous Rotation Matrices around Elementary Axis

- $R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- $R_y(\beta) = \begin{pmatrix} \cos \beta & 0 & \sin \beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \beta & 0 & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- $R_z(\gamma) = \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 & 0 \\ \sin \gamma & \cos \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Inverse Homogeneous Rotation Matrix R^{-1}

Let R_3 be a 3×3 rotation matrix, then:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow R^{-1} = \begin{bmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Homogeneous Matrix A

$$A = T(a_x, a_y, a_z)R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & a_x \\ r_{21} & r_{22} & r_{23} & a_y \\ r_{31} & r_{32} & r_{33} & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

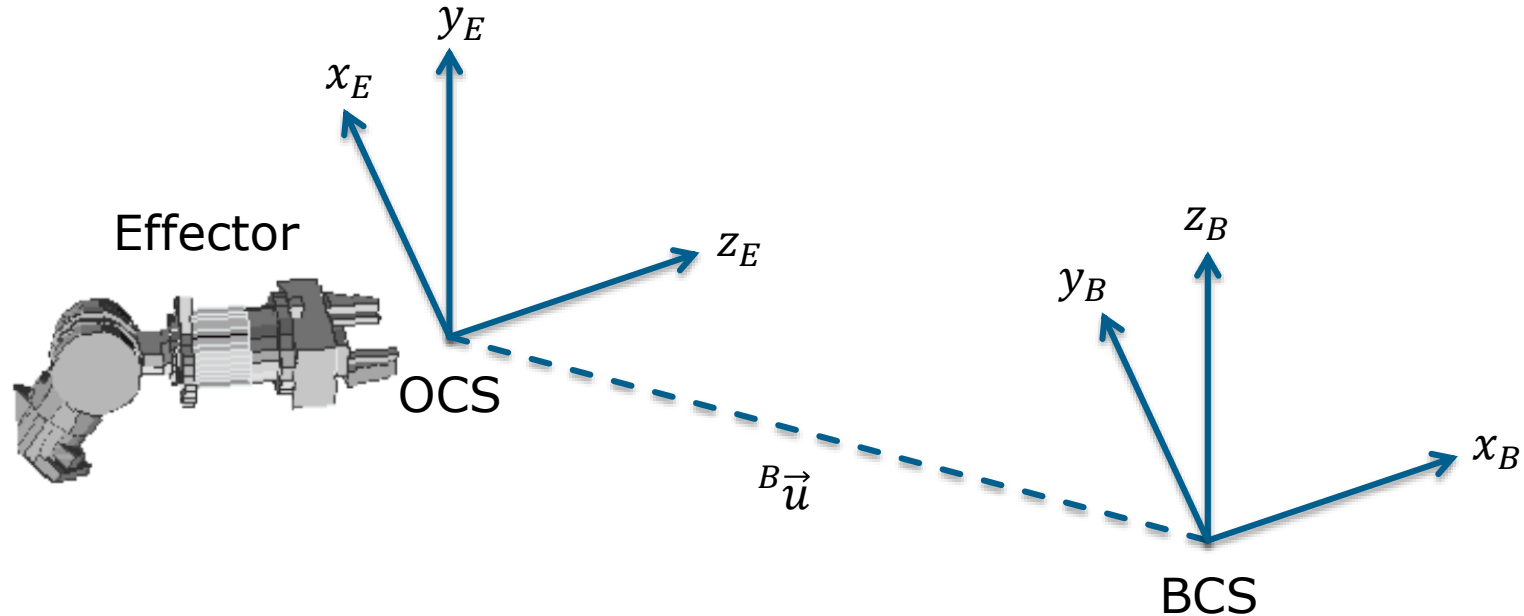
$$\Rightarrow A^{-1} = R^{-1}T^{-1} = \begin{bmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -a_x \\ 0 & 1 & 0 & -a_y \\ 0 & 0 & 1 & -a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{21} & r_{31} & -r_{11}a_x - r_{21}a_y - r_{31}a_z \\ r_{12} & r_{22} & r_{32} & -r_{12}a_x - r_{22}a_y - r_{32}a_z \\ r_{13} & r_{23} & r_{33} & -r_{13}a_x - r_{23}a_y - r_{33}a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Properties of Homogeneous Matrices

- In a homogeneous 4×4 matrix there are 12 $(\vec{n}, \vec{o}, \vec{a}, \vec{u})$ nontrivial parameters, but only 6 $(x, y, z, \alpha, \beta, \gamma)$ are necessary
- Redundancy because of orthogonality

Calculation of a Homog. Transformation Matrix

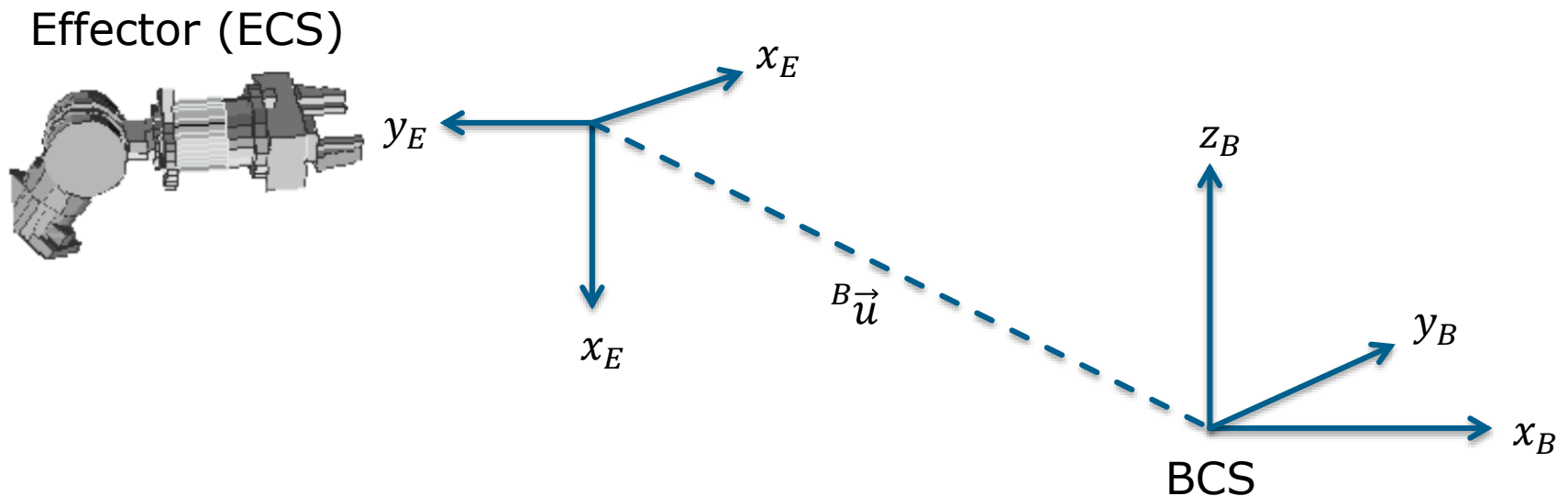


$$({}^E x, {}^E y, {}^E z, R) = \begin{array}{ccc|c} \text{Orientation} & & & \text{Position} \\ \hline 0 & 0 & 1 & {}^B u_x \\ 1 & 0 & 0 & {}^B u_y \\ 0 & 1 & 0 & {}^B u_z \\ \hline 0 & 0 & 0 & 1 \end{array}$$

Calculation of a Homog. Transformation Matrix

Given: Description vector: $(-7, 0, 8, 0^\circ, 90^\circ, 90^\circ)$, roll-pitch-yaw

Wanted: Homogeneous representation of the pose as a matrix A



Calculation of a Homog. Transformation Matrix

- Compute $R = R_z(90^\circ)R_y(90^\circ)R_x(0^\circ)$

- $$R_z(90^\circ) = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $$R_y(90^\circ) = \begin{bmatrix} \cos 90^\circ & 0 & \sin 90^\circ \\ 0 & 1 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

- $$R_x(0^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 0^\circ & -\sin 0^\circ \\ 0 & \sin 0^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculation of a Homog. Transformation Matrix

$$R = R_z(90^\circ)R_y(90^\circ)R_x(0^\circ)$$

$$\begin{aligned} R_s &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow A = \begin{bmatrix} 0 & -1 & 0 & -7 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Calculation of the Description Vector

Given: Homogeneous matrix: A

Wanted: Description vector: $\vec{v} = (x, y, z, \alpha, \beta, \gamma)$

$$A = \begin{bmatrix} 0 & 0 & 1 & -7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Position: $x = -7, y = 0, z = 8$
- Orientation: Calculated by matrix equation:
 - $R_s = R_1 \cdot R_2 \cdot R_3$
 - 3×3 -orientation part of homogeneous matrix
 - Product of rotation matrices corresponding to axes of rotation (Euler, roll-pitch-yaw)

Reformulating Matrix Equations

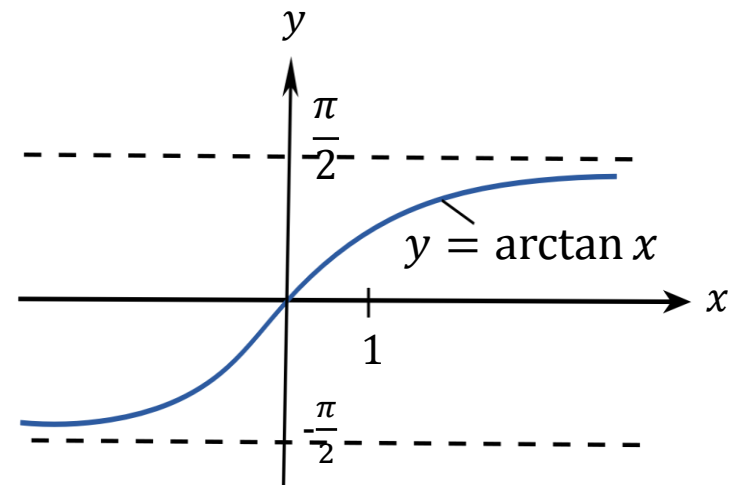
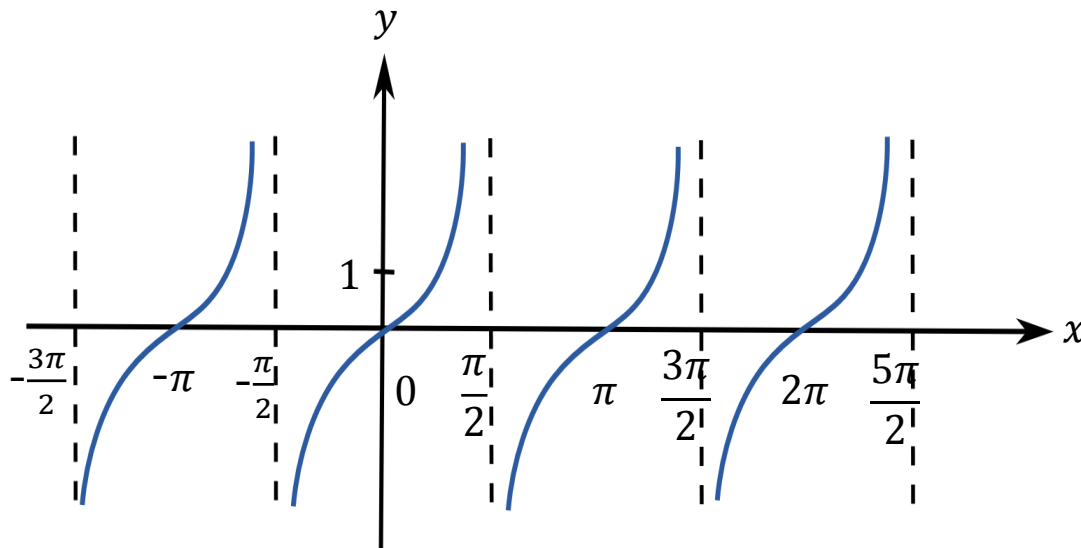
- The matrix equations can be represented as 16 individual equations, where 12 are non trivial
- The matrix equation can be solved by multiplying with the inverse matrix R_1^{-1} or R_3^{-1}
 - $R_1^{-1} \cdot R_s = R_2 \cdot R_3$
 - $R_s \cdot R_3^{-1} = R_1 \cdot R_2$

Form of the Equations

- After these reformulations there exists for each α, β, γ :
 - (1) An equation of the form $b \cdot \sin \alpha - a \cdot \cos \alpha = 0$
 - (2) Or a set of equations of the form $\sin \alpha = a, \cos \alpha = b$
- Approaches to determine the angle
 - An unknown angle should/must not be calculated with \sin or \cos because of its ambiguity
 - Approach: Use \arctan (arcus tangent) and projection of the angle to the correct quadrant

Solving the Equations

- With (1) or (2) one can compute α , since from (1), and from (2) it follows that $\alpha = \arctan \frac{a}{b}$
- Problem: $\arctan x$ is ambiguous
 - Restriction to angles between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$

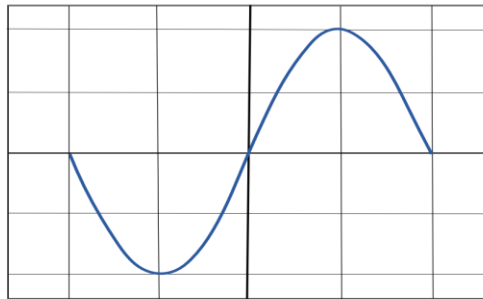


ATAN2

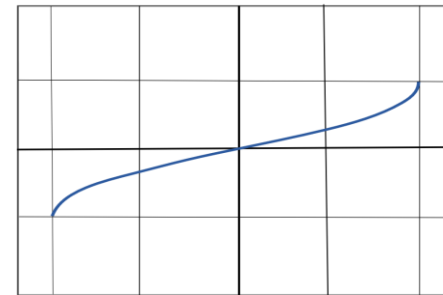
- To calculate α we do not use \arctan , but the 2-argument arctangent ATAN2
- Starting point: Equations of the form $\sin \alpha = a, \cos \alpha = b$
- Are only solvable for α ,
if $a^2 + b^2 = 1$, then $\sin^2 \alpha + \cos^2 \alpha = 1$
- Especially a and b can not be zero simultaneously

Arc-Functions of Sine and Cosine

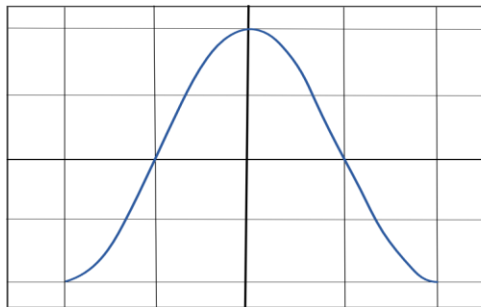
(3) $\arcsin a = \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
 $\sin x$



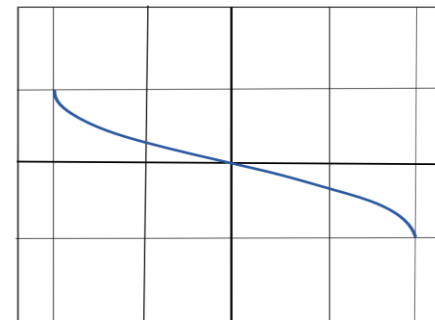
$\arcsin x$



(4) $\arccos b = \alpha$ where $\alpha \in [0, \pi]$
 $\cos x$



$\arccos x$



(5) Tangent $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$

Definition ATAN2

- $\text{ATAN2}(a, b)$, where $a = \sin$ and $b = \cos$ due to (5)
- $$\text{ATAN2}(a, b) = \begin{cases} \arccos b, & \text{if } a \geq 0 \\ -\arccos b, & \text{if } a < 0 \end{cases}$$

Due to (3) and (4)
- Can also be computed for $\text{ATAN2}(a, b) = \tan^{-1} \left(\frac{a}{b} \right)$
 - Choose quadrant with the signs of a and b
 - Example: $\text{ATAN2}(-2, -2) = -135^\circ$ (third quadrant)
 - Example: $\text{ATAN2}(2, 2) = -45^\circ$ (first quadrant)

Computing Angles with ATAN2

- $\sin \theta = a \Rightarrow \theta = \pm \text{ATAN2}(\sqrt{1 - a^2}, a)$
- $\cos \theta = b \Rightarrow \theta = \pm \text{ATAN2}(b, \pm \sqrt{1 - b^2})$
- $a \cdot \cos \theta + b \cdot \sin \theta = 0$ has two solutions
 - $\theta = \text{ATAN2}(a, -b)$
 - $\theta = \text{ATAN2}(-a, b)$
- $a \cdot \cos \theta + b \cdot \sin \theta = c \Rightarrow \theta = \text{ATAN2}(b, a) \pm \text{ATAN2}(\sqrt{a^2 + b^2 + c^2}, c)$
- $a \cdot \cos \theta - b \cdot \sin \theta = c$ and $a \cdot \sin \theta + b \cdot \cos \theta = d$
 $\Rightarrow \theta = \text{ATAN2}(ad - bc, ac + bd)$

Computation of Euler Angles

Computation of Euler angles for a general orientation matrix

- Matrix equation $R = R_Z(\alpha) \cdot R_{y'}(\beta) \cdot R_{z''}(\gamma) = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$
- Multiplying from the left with $R_Z(\alpha)^{-1}$:

$$R_Z(\alpha)^{-1} \cdot R_Z(\alpha) \cdot R_{y'}(\beta) \cdot R_{z''}(\gamma) = R_Z(\alpha)^{-1} \cdot R$$
- For orthogonal matrices it holds that $A^{-1} = A^T$ therefore

$$R_{y'}(\beta) \cdot R_{z''}(\gamma) = R_Z(\alpha)^{-1} \cdot R = R_Z(\alpha)^T \cdot R$$

$$\begin{bmatrix} C\beta \cdot C\gamma & -C\beta \cdot S\gamma & S\beta \\ S\gamma & C\gamma & 0 \\ -S\beta \cdot C\gamma & S\beta \cdot S\gamma & C\beta \end{bmatrix} = \begin{bmatrix} C\alpha n_x + S\alpha n_y & C\alpha o_x + S\alpha o_y & C\alpha a_x + S\alpha a_y \\ -S\alpha n_x + C\alpha n_y & -S\alpha o_x + C\alpha o_y & -S\alpha a_x + C\alpha a_y \\ n_z & o_z & a_z \end{bmatrix}$$

Computation of Euler Angles - an Example

The following equations are given:

(row.column)

$$(1.1) \quad C\beta \cdot C\gamma = C\alpha \cdot n_x + S\alpha \cdot n_y$$

$$(1.2) \quad -C\beta \cdot S\gamma = C\alpha \cdot o_x + S\alpha \cdot o_y$$

$$(1.3) \quad S\beta = C\alpha \cdot a_x + S\alpha \cdot a_y$$

$$(2.1) \quad S\gamma = -S\alpha \cdot n_x + C\alpha \cdot n_y$$

$$(2.2) \quad C\gamma = -S\alpha \cdot o_x + C\alpha \cdot o_y$$

$$(2.3) \quad 0 = -S\alpha \cdot a_x + C\alpha \cdot a_y$$

$$(3.1) \quad -S\beta \cdot C\gamma = n_z$$

$$(3.2) \quad S\beta \cdot S\gamma = o_z$$

$$(3.3) \quad C\beta = a_z$$

Computation of Euler Angles - an Example

- Angle α : From (2.3) it follows that:
 - $S\alpha \cdot a_x = C\alpha \cdot a_y \Leftrightarrow \frac{S\alpha}{C\alpha} = \tan\alpha = \frac{a_y}{a_x}$
 - Therefore $\alpha = \text{ATAN2}(a_y, a_x)$
- Angle β : From (1.3), (3.3) it follows that:
 - $\beta = \text{ATAN2}(C\alpha a_x + S\alpha a_y, a_z)$
- Angle γ : From (2.1), (2.2) it follows that
 - $\gamma = \text{ATAN2}(C\alpha n_x + S\alpha n_y, -S\alpha o_x + C\alpha o_y)$
- Note: α is present in the solutions of β, γ

Concatenated Poses

- Poses are most of the time not given relative to the BCS, but relative to a more suitable CS (relative definition)
- Transformation between different coordinate system (e.g. BCS) necessary
- Pros of the relative pose definition
 - Less effort needed to track an objects motion
 - Individual coordinates only cover small distances

Rotation/Translation of Poses

- Let ${}^{BCS}_A H_{obj} = (4 \times 4)$ be the pose of an object in frame A relative to BCS
- Let ${}^A_B H_{obj} = (4 \times 4)$ be the pose of an object in frame B relative to the OCS of A
- Let ${}^{BCS}_B H_{obj} = (4 \times 4)$ be the pose of an object in frame B relative to BCS
 - It holds: ${}^{BCS}_B H_{obj} = {}^{BCS}_A H_{obj} \cdot {}^A_B H_{obj}$
- Notation is more compact, compared to the Cartesian notation:

$$\begin{aligned} {}^{BCS}R + {}^{BCS}\vec{v} &= {}^{BCS}_A R_1 \cdot ({}^A_B R_2 + {}^A\vec{v}_B) + {}^{BCS}\vec{v}_A \\ &= {}^{BCS}_A R_1 \cdot {}^A_B R_2 + ({}^{BCS}_A R_1 \cdot {}^A\vec{v}_B + {}^{BCS}\vec{v}_A) \end{aligned}$$

Concatenated Poses - an Example

- Pose of object in frame 1 relative to BCS: ${}^{BCS}_1H_{obj}$
- Pose of object in frame 2 relative to frame 1: ${}_1^2H_{obj}$
- Pose of object in frame 3 relative to frame 2 : ${}_2^3H_{obj}$
- Pose of object in frame 3 relative to BCS : ${}^{BCS}_3H_{obj}$
 - ${}^{BCS}_3H_{obj} = {}_1^{BCS}H_{obj} \cdot {}_2^1H_{obj} \cdot {}_3^2H_{obj}$
- In order to use concatenated poses each matrix must be defined relative to the frame defined by its left matrix
 - $\prod_{i=1}^n {}^{i-1}_iH$ with $1 \leq i \leq n$ and ${}^0H = BCS$

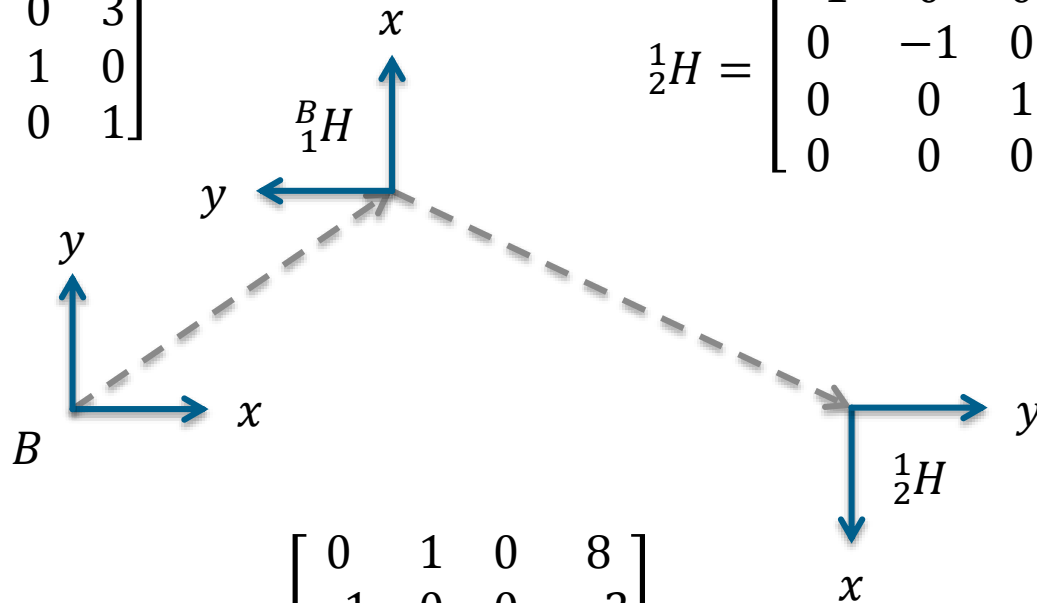
Concatenated Poses - an Example

- System of object H_1 , is generated by a transformation $((3,3,0)^T, R_z(90^\circ))$ of an arbitrary coordinate system B : B_1H
- System of object H_2 , is generated by a transformation $((-5,-5,0)^T, R_z(-180^\circ))$ of the system of object H_1 : 1_2H

Concatenated Poses - an Example

$${}^{BCS}_1H = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2H = \begin{bmatrix} -1 & 0 & 0 & -5 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$${}^{BCS}_2H = {}^{BCS}_1H \cdot {}^1_2H = \begin{bmatrix} 0 & 1 & 0 & 8 \\ -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Vector z is the same, so is omitted

Coming up next ...

Direct Kinematics

