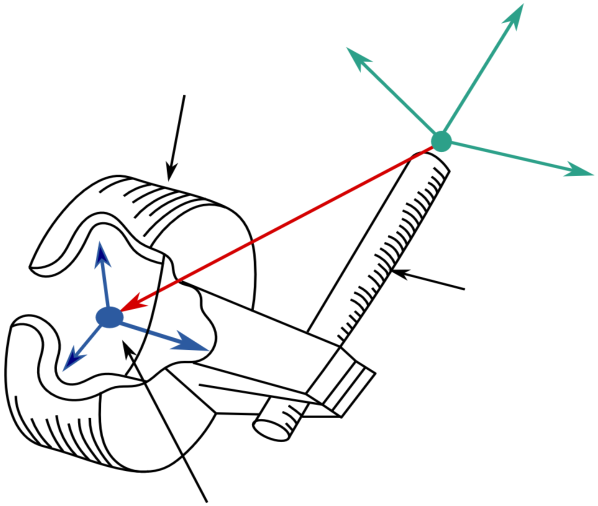
RRLAB THE ROBOTICS RESEARCH LAB

Velocity



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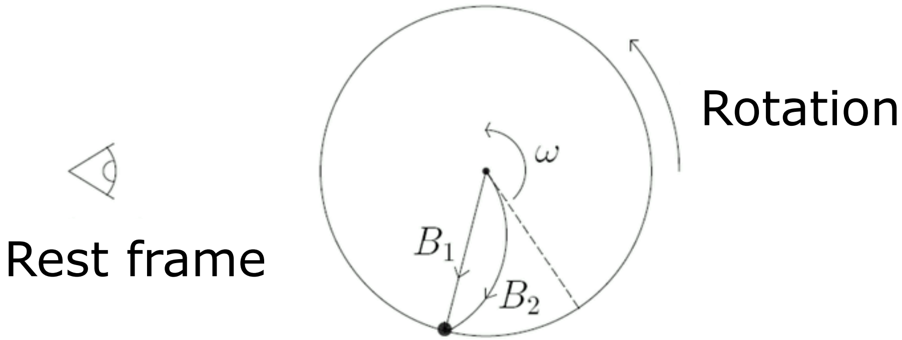
Contents

Reminder: Physics Background

Forces

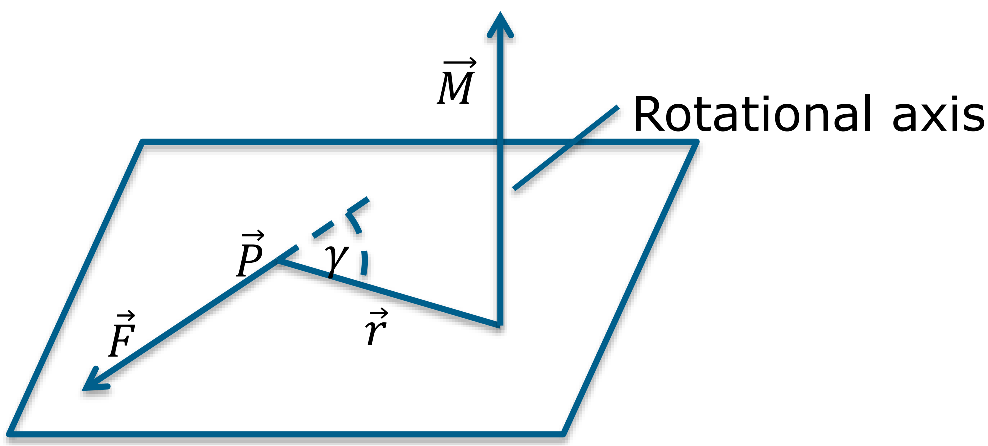
- Newton‘s second law: $\vec{F} = m \cdot \vec{a}$ (Basic equation of mechanics)
- Weight $G = m \cdot g$ $(1\text{kg} \approx 9{,}81\text{N})$
- Centrifugal force $\vec{F_z} = -m \cdot \overrightarrow{a_r} = -m \cdot \vec{r} \cdot \omega^2$
- Coriolis force: Deflects radial moving bodies in a rotating frame of reference
  - Straight path B_1 with respect to a rest frame
  - Curved path B_2 with respect to a rotating frame
  - Therefore a force needs to be applied to keep a straight path



Reminder: Physics Background

Torque

- Torque $\vec{M} = \vec{r} \times \vec{F}$ on a body with lever arm $\vec{r}$ and force $\vec{F}$
  - Distance r between point of mass and axis
- Equation for magnitude of torque $M = F \cdot r \cdot \sin\gamma$



Reminder: Physics Background

Moment of Inertia

- Moment of inertia $dJ = r^2 dm$ for mass point with mass dm
- Moment of inertia $J = \int_{\text{Volume}} r^2 dm$ for a body
  - With mass distribution J relative to rotational axis
- Tensor: Inertia w.r.t x-y-z-System in homogeneous coordinates
- $$M = \int \vec{r} \cdot \vec{r}^T dm = \begin{bmatrix} \int x^2 dm & \int yx\, dm & \int xz\, dm & \int x\, dm \\ \int xy\, dm & \int y^2 dm & \int yz\, dm & \int y\, dm \\ \int xz\, dm & \int yz\, dm & \int z^2 dm & \int z\, dm \\ \int x\, dm & \int y\, dm & \int z\, dm & \int dm \end{bmatrix}$$

Velocity Vector

- Free vector (no starting point; only magnitude and direction)
  - Only rotation is considered

- Derivation of a position vector with respect to time:

$$^{B}\vec{v}_q = \frac{d}{dt}\,^{B}\vec{q}$$

- Conversion into rotated CS:

$$^{A}\vec{v}_q = {}^{A}_{B}R \cdot {}^{B}\vec{v}_q$$

- Only rotation matrix considered here, not the displacement vector.

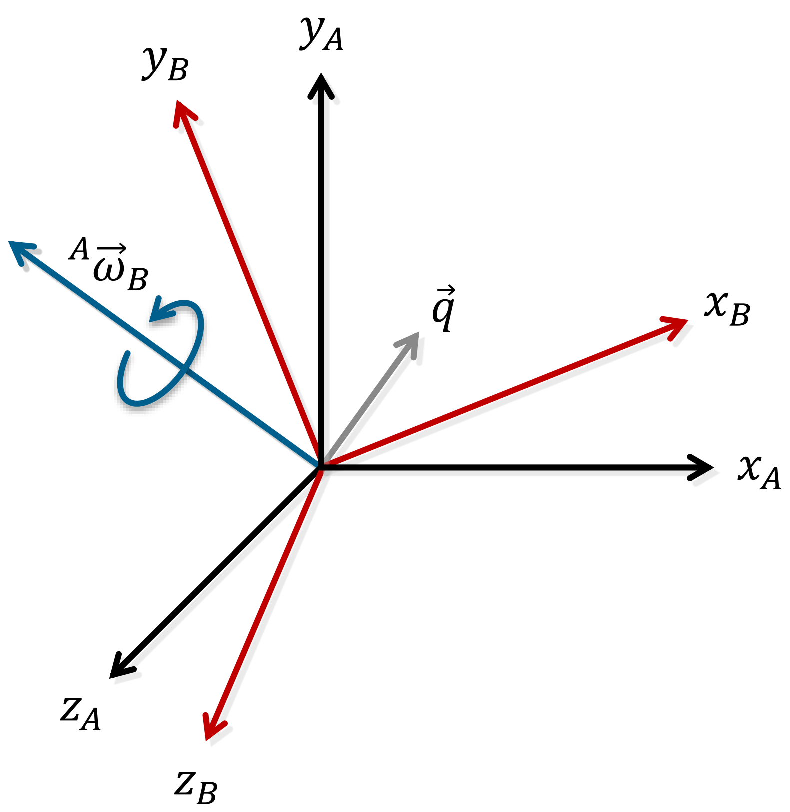
Linear Velocity

- Origin OB of the system B moves with a linear velocity ${}^{A}\vec{v}_{OB}$ relative to system A
- Point ${}^{B}\vec{q}$ represented in system B moves with a linear velocity ${}^{B}\vec{v}_{q}$
- System B was created from system A by rotation ${}^{A}_{B}R$
- Linear velocity of the point ${}^{B}\vec{q}$ relative to system A:

$$ {}^{A}\vec{v}_{q} = {}^{A}\vec{v}_{OB} + {}^{A}_{B}R \cdot {}^{B}\vec{v}_{q} \tag{8.1} $$

Rotational Velocity

- System A and system B share a common origin
- Linear velocity between the systems is 0: ${}^{A}\vec{v}_{OB} = 0$
- ${}^{B}\vec{q}$ is represented in system B
- System B rotates about an axis through the common origin of A and B at a rotational speed ${}^{A}\vec{\omega}_{B}$



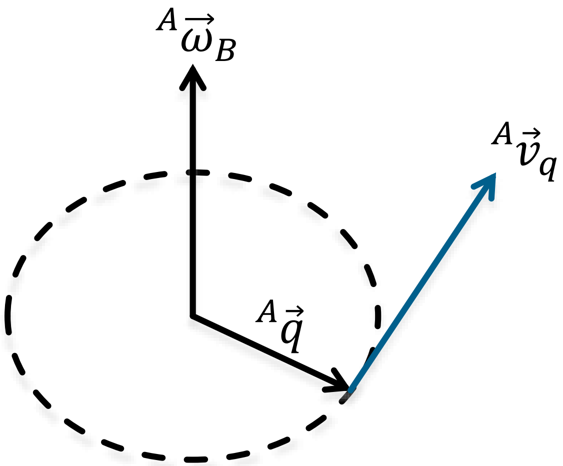
Rotational Velocity

- Speed of the point $\vec{q}$: ${}^{A}\vec{v}_q = {}^{A}\vec{\omega}_B \times {}^{A}\vec{q}$
- Considering the linear velocity

$$ {}^{A}\vec{v}_q = {}^{A}_{B}R \cdot {}^{B}\vec{v}_q + {}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} \qquad (8.2) $$

- Linear and rotational velocity

$$ {}^{A}\vec{v}_q = {}^{A}\vec{v}_{OB} + {}^{A}_{B}R \cdot {}^{B}\vec{v}_q + {}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} $$



Point Velocity in Another Reference System

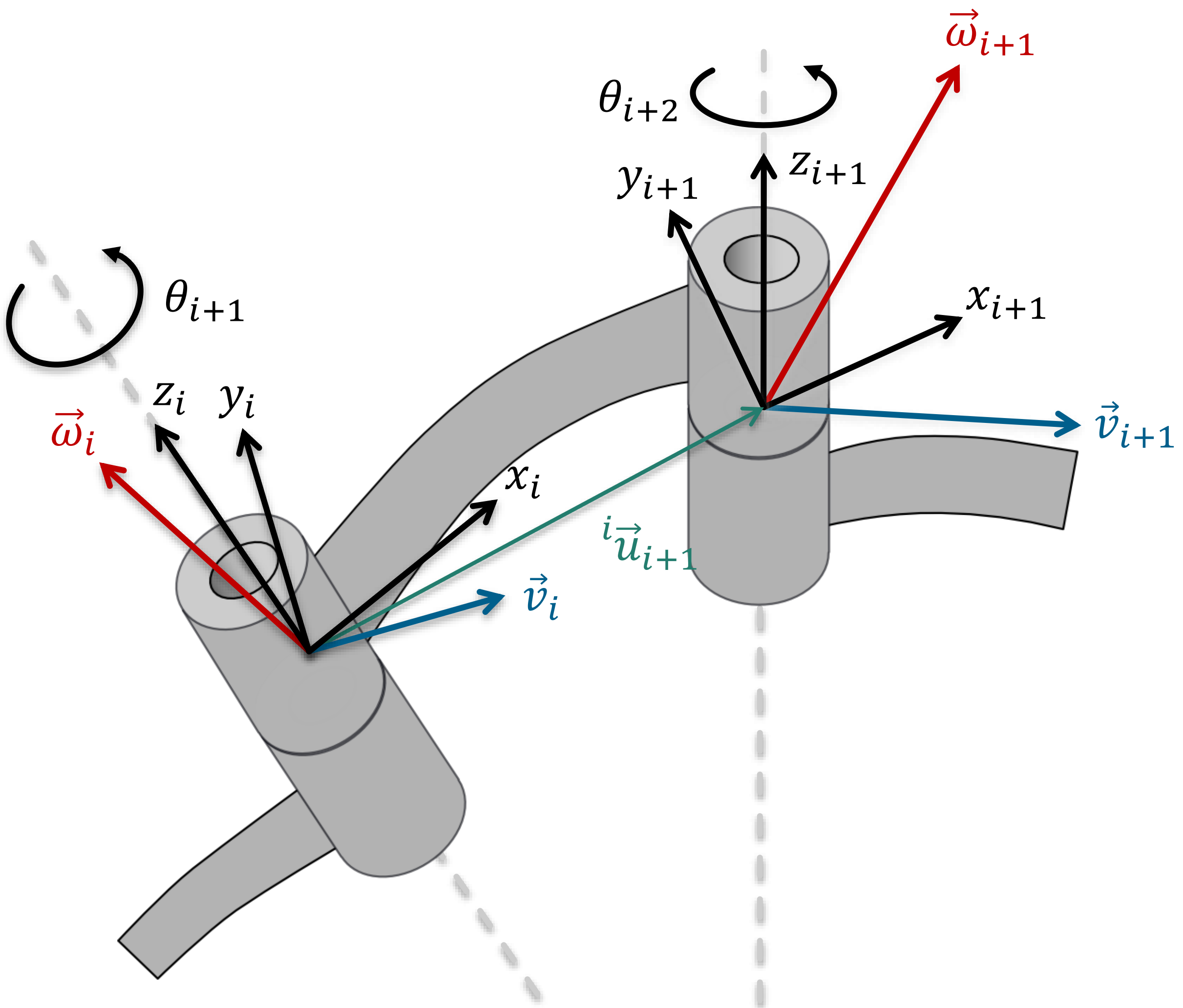
$$^{A}\vec{v}_q = {}^{A}\vec{v}_{OB} + {}^{A}_{B}R \cdot {}^{B}\vec{v}_q + {}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q}$$

- $^{A}\vec{v}_{OB}$: Translational velocity of origin OB in system A
- $^{A}_{B}R \cdot {}^{B}\vec{v}_q$: Translational velocity of the point $^{B}\vec{q}$ in the system B transformed to the reference system A
- $^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q}$: Translational point velocity due to the rotation of the system B compared to A

Velocity of the Robot Parts

- Velocity of the end effector of a robot with n joints is calculated from the kinematic structure and all the members involved in the movement
- Velocity of a part consists of the velocity of its fixed CS and rotational and translational velocity of the part
- Velocity of the end effector in the base coordinate system is determined by successive calculation of the velocities of the parts from the base
- Velocity of the part $i + 1$ is the sum of the velocity of member i and the component resulting from relative motion between i and $i + 1$
    - Attention: Both summands must be in the same coordinate system!

Coordinate System and Identifiers



Rotational Velocity at Rotational Joints

- Let joint $i+1$ be a rotational join with degree of freedom θ_{i+1}
- *Rotational velocity of link* i+1:

$$ {}^{i}\vec{\omega}_{i+1} = {}^{i}\vec{\omega}_{i} + \dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i} $$

  - ${}^{i}\vec{\omega}_{i}$: Rotational velocity of the part i
  - $\dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i}$: Component by rotation of joint i + 1
  - $\dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i} = (0 \quad 0 \quad \dot{\theta}_{i+1})^T$
- Transformation of ${}^{i}\vec{\omega}_{i+1}$ in the system $i+1$ by multiplying with ${}^{i+1}_{\ \ i}R$:

$$ {}^{i+1}\vec{\omega}_{i+1} = {}^{i+1}_{\ \ i}R \cdot \left({}^{i}\vec{\omega}_{i} + \dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i} \right) \qquad (8.3) $$

Linear Velocity at Rotational Joints

- For the translational speed of the origin of coordinate system $i+1$ represented in system i it holds:

$$ {}^{i}\vec{v}_{i+1} = {}^{i}\vec{v}_{i} + {}^{i}\vec{\omega}_{i+1} \times {}^{i}\vec{u}_{i+1} $$

- Represented in system $i+1$

$$ {}^{i+1}\vec{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\vec{v}_{i} + {}^{i}\vec{\omega}_{i+1} \times {}^{i}\vec{u}_{i+1} \right) $$

Velocity of Linear Joints

- Let joint i be a translational joint with degree of freedom d_i

- Rotational velocity:

$$^{i+1}\vec{\omega}_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\vec{\omega}_{i}$$

- Translational velocity:

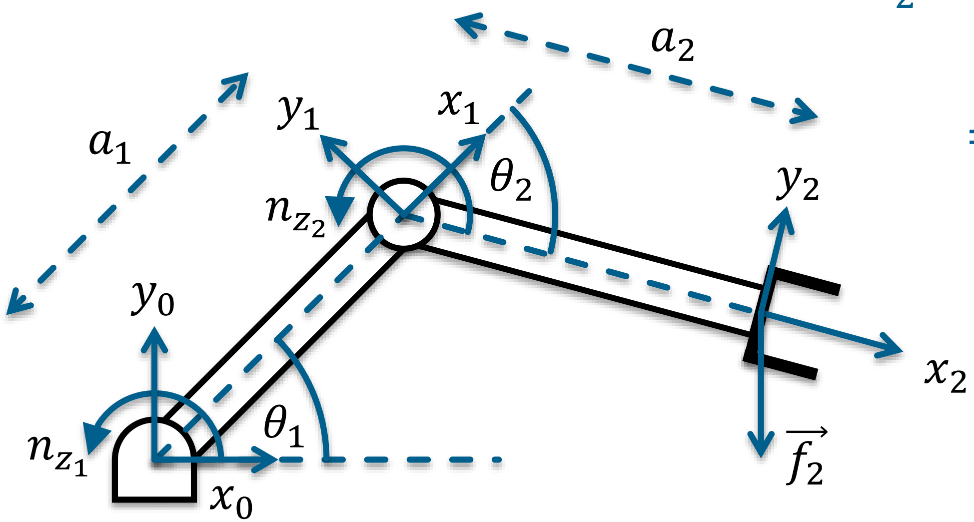
$$^{i+1}\vec{v}_{i+1} = {}^{i+1}_{i}R \cdot \left({}^{i}\vec{v}_{i} + {}^{i}\vec{\omega}_{i+1} \times {}^{i}\vec{u}_{i+1} + \dot{d}_{i+1}\, {}^{i}\vec{e}_{z_i} \right) \qquad (8.4)$$

Example: Planar Robot Arm

- Calculation of the rotation matrices required for the velocity ${}^{i+1}_{i}R = {}^{i}_{i+1}R^T$
- Rotations and translations separately

$$
{}^{0}_{1}A = T_{z_0}(0) \cdot R_{z_0}(\theta_1) \cdot R_{x_1}(0°) \cdot T_{x_1}(a_1)
= \begin{bmatrix} c_1 & -s_1 & 0 & c_1 a_1 \\ s_1 & c_1 & 0 & s_1 a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}A = T_{z_1}(0) \cdot R_{z_1}(\theta_2) \cdot R_{x_2}(0°) \cdot T_{x_2}(a_2)
= \begin{bmatrix} c_2 & -s_2 & 0 & c_2 a_2 \\ s_2 & c_2 & 0 & s_2 a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Example: Planar Robot Arm

- Specification: ${}^{0}\vec{v}_0 = \vec{0},\ {}^{0}\vec{\omega}_0 = \vec{0}$
- Derive by:

$$\boxed{\begin{aligned} {}^{i+1}\vec{\omega}_{i+1} &= {}^{i+1}_{i}R \cdot \left({}^{i}\vec{\omega}_i + \dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i} \right) \\ {}^{i+1}\vec{v}_{i+1} &= {}^{i+1}_{i}R\left({}^{i}\vec{v}_i + {}^{i}\vec{\omega}_{i+1} \times {}^{i}\vec{u}_{i+1} \right) \end{aligned}}$$

$$\begin{aligned} {}^{0}\vec{\omega}_1 &= {}^{0}\vec{\omega}_0 + \dot{\theta}_1\ {}^{0}\vec{e}_{z_0} \\ &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} {}^{1}\vec{\omega}_1 &= {}^{1}_{0}R \cdot {}^{0}\vec{\omega}_1 \\ &= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} {}^{1}\vec{v}_1 &= {}^{1}_{0}R\left({}^{0}\vec{v}_0 + {}^{0}\vec{\omega}_1 \times {}^{0}\vec{u}_1 \right) \\ &= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} c_1 a_1 \\ s_1 a_1 \\ 0 \end{pmatrix} \right] \\ &= \begin{bmatrix} 0 \\ a_1 \dot{\theta}_1 \\ 0 \end{bmatrix} \end{aligned}$$

Example: Planar Robot Arm

$$
{}^{1}\vec{\omega}_2 = {}^{1}\vec{\omega}_1 + \dot{\theta}_2\, {}^{1}\vec{e}_{z_1}
= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}
$$

$$
{}^{2}\vec{\omega}_2 = {}^{2}_{1}R \cdot {}^{1}\vec{\omega}_2
= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}
$$

$$
{}^{2}\vec{v}_2 = {}^{2}_{1}R\left({}^{1}\vec{v}_1 + {}^{1}\vec{\omega}_2 \times {}^{1}\vec{u}_2 \right)
= {}^{2}_{1}R \cdot \left[\begin{pmatrix} 0 \\ a_1\dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} c_2 a_2 \\ s_2 a_2 \\ 0 \end{pmatrix} \right]
$$

$$
= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_2 a_2(\dot{\theta}_1 + \dot{\theta}_2) \\ a_1\dot{\theta}_1 + c_2 a_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}
= \begin{bmatrix} s_2 a_2 \dot{\theta}_1 \\ a_1 c_2 \dot{\theta}_1 + a_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}
$$

Example: Planar Robot Arm

TCP linear velocity with respect to the base coordinate system

$$ {}^{0}_{2}R = {}^{0}_{1}R \cdot {}^{1}_{2}R = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}\vec{v}_2 = {}^{0}_{2}R \; {}^{2}\vec{v}_2 = \begin{bmatrix} -s_1 a_1 \dot{\theta}_1 - s_{12} a_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ c_1 a_1 \dot{\theta}_1 + c_{12} a_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} $$

$$ {}^{0}\vec{\omega}_2 = {}^{0}_{2}R \; {}^{2}\vec{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

Application of Jacobian Matrix

Let $\vec{y} = f(\vec{x})$ with $\vec{x} \in \mathbb{R}^m, \vec{y} \in \mathbb{R}^n$.

$$\begin{aligned} y_1 &= f_1(x_1, x_2, \dots, x_m) \\ y_2 &= f_2(x_1, x_2, \dots, x_m) \\ &\vdots \\ y_n &= f_n(x_1, x_2, \dots, x_m) \end{aligned}$$

$$\begin{aligned} dy_1 &= \frac{df_1}{dx_1}dx_1 + \frac{df_1}{dx_2}dx_2 + \cdots + \frac{df_1}{dx_m}dx_m \\ dy_2 &= \frac{df_2}{dx_1}dx_1 + \frac{df_2}{dx_2}dx_2 + \cdots + \frac{df_2}{dx_m}dx_m \\ &\vdots \\ dy_n &= \frac{df_n}{dx_1}dx_1 + \frac{df_n}{dx_2}dx_2 + \cdots + \frac{df_n}{dx_m}dx_m \end{aligned}$$

Application of Jacobian Matrix

- Vector notation

$$\begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix} = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \cdots & \frac{df_1}{dx_m} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \cdots & \frac{df_2}{dx_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{df_n}{dx_1} & \frac{df_n}{dx_2} & \cdots & \frac{df_n}{dx_m} \end{bmatrix} \cdot \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix} = J(\vec{x})d\vec{x}$$

- $d\vec{y} = df(\vec{x}) = \frac{df(\vec{x})}{d\vec{x}} d\vec{x} = J(\vec{x})d\vec{x}$ with Jacobian Matrix $J(\vec{x}) = \frac{df(\vec{x})}{d\vec{x}}$

Application of Jacobian Matrix

- Derivation of the function $f(x)$ w.r.t. time yields

$$\frac{d\vec{y}}{dt} = \frac{df(\vec{x})}{dt} = J(\vec{x})\frac{d\vec{x}}{dt} \quad \text{or} \quad \dot{\vec{y}} = J(\vec{x})\dot{\vec{x}}$$

- Jacobian matrix (robotics): Relationship between end effector velocity $\dot{\vec{y}}$ and joint velocities $\dot{\vec{\theta}}$
  - $\dot{\vec{y}} = J(\vec{\theta})\dot{\vec{\theta}}$ with vector notation $\dot{\vec{y}} = \left(\dot{x}, \dot{y}, \dot{z}, \dot{\alpha}, \dot{\beta}, \dot{\gamma}\right)^T$
- Number of columns m = movement/joint degrees of freedom
- Number of rows n = degree of freedom in Cartesian space

Application of Jacobian Matrix

- Transformation of a square 6×6 Jacobian matrix in another CS:

$$
{}^{0}J(\vec{\theta}) = \underbrace{\begin{bmatrix} {}^{0}_{1}R & 0 \\ 0 & {}^{0}_{1}R \end{bmatrix}}_{6\times 6} \cdot \ {}^{1}J(\vec{\theta})
$$

- Rest of the procedure
  - Determine ${}^{m}\vec{v}_m$ and ${}^{m}\vec{\omega}_m$ as shown
  - Transform with the above equation in ${}^{0}\vec{v}_m$ and ${}^{0}\vec{\omega}_m$

Application of Jacobian Matrix

Using ${}^{0}\vec{v}_2$ from the example above:

$$\dot{\vec{y}} = {}^{0}\vec{v}_2 = \begin{bmatrix} -s_1 a_1 \dot{\theta}_1 - s_{12} a_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ c_1 a_1 \dot{\theta}_1 + c_{12} a_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -s_1 a_1 - s_{12} a_2 & -s_{12} a_2 \\ c_1 a_1 + c_{12} a_2 & c_{12} a_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

with $J(\vec{\theta}) = \begin{bmatrix} -s_1 a_1 - s_{12} a_2 & -s_{12} a_2 \\ c_1 a_1 + c_{12} a_2 & c_{12} a_2 \\ 0 & 0 \end{bmatrix}$

Application of Jacobian Matrix

- Considering the angular velocity:

$$\dot{\vec{y}} = \begin{bmatrix} {}^{O}\vec{v}_2 \\ {}^{O}\vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} {}^{O}v_{2_x} \\ {}^{O}v_{2_y} \\ {}^{O}v_{2_z} \\ {}^{O}\omega_{2_x} \\ {}^{O}\omega_{2_y} \\ {}^{O}\omega_{2_z} \end{bmatrix} = \begin{bmatrix} -s_1 a_1 - s_{12} a_2 & -s_{12} a_2 \\ c_1 a_1 + c_{12} a_2 & c_{12} a_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

- Further possibility for the calculation of the Jacobian matrix: derivation of the forward kinematics

Inverse Jacobian Matrix

- Calculation of joint angular velocities from Cartesian velocities with inverse Jacobian matrix
  $$\dot{\vec{\theta}} = J(\vec{\theta})^{-1}\dot{\vec{y}}$$ Solution, if $\det(J) \neq 0$

- Not square → Cartesian degrees of freedom greater than joints degrees of freedom
  1. Elimination of linear dependent lines in J → Invertible matrix
  2. Least-square-method as an approximation

$$\dot{\vec{\theta}} = (J^T J)^{-1} J^T \dot{\vec{y}}$$

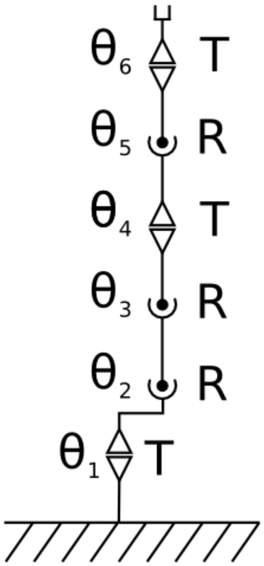
Inverse Jacobian Matrix

- Not square → Joint degrees of freedom greater than Cartesian degrees of freedom
  - There are a lot of solutions

1. Block degrees of freedom of movement so that J square
2. Introduce constraints (collision avoidance)

Singularities

- Robot configuration often with singular Jacobian matrices, thus losing Cartesian degrees of freedom
- Types of singularities
  - At the edge of the working space
  - Inside the workroom



e.g. shown typical industry robot, where $\theta_5 = 0$, θ_4 and θ_6 act in the same direction, i.e. one degree of freedom is lost

- Attention: In the vicinity of singularities very big joint velocities can result from small Cartesian velocities

Singularities, Example: Planar Robot

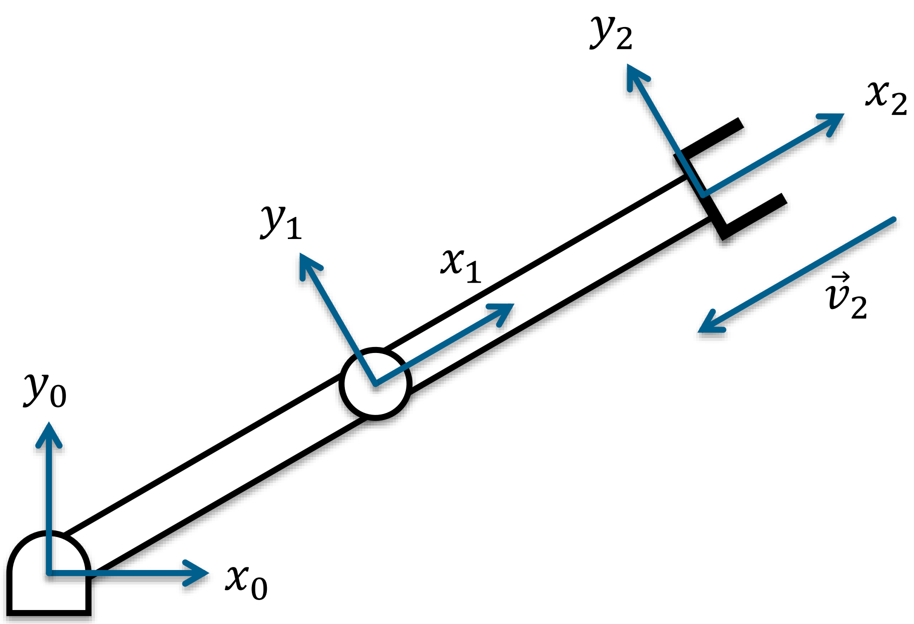
- Singular position of the planar robot
- Jacobian matrix: $J(\theta_1, \theta_2) = \begin{bmatrix} -s_1 a_1 - s_{12} a_2 & -s_{12} a_2 \\ c_1 a_1 + c_{12} a_2 & c_{12} a_2 \end{bmatrix}$
- Determinant: $\det(J) = a_1 a_2 \sin(\theta_2)$
- Singularity $(\det = 0)$: $a_1 a_2 \sin(\theta_2) = 0 \rightarrow \theta_2 = 0$ and $\theta_2 = \pi$
- Relevant for practice: $\theta_2 = 0$, i.e. robotic arm fully extended (singularity at the edge of the workspace)

Singularities, Example: Planar Robot

- Inverse Jacobian matrix

$$J^{-1}(\vec{\theta}) = \frac{1}{a_1 a_2 s_2} \begin{bmatrix} a_2 c_{12} & a_2 s_{12} \\ -a_1 c_1 - a_2 c_{12} & -a_1 s_1 - a_2 s_{12} \end{bmatrix}$$

- For $\theta_2 \rightarrow 0 \Rightarrow \sin\theta_2 \rightarrow 0 \Rightarrow \dot{\theta}_1$ and $\dot{\theta}_2 \rightarrow \infty$



Velocity in Wheel-driven Robotic Systems

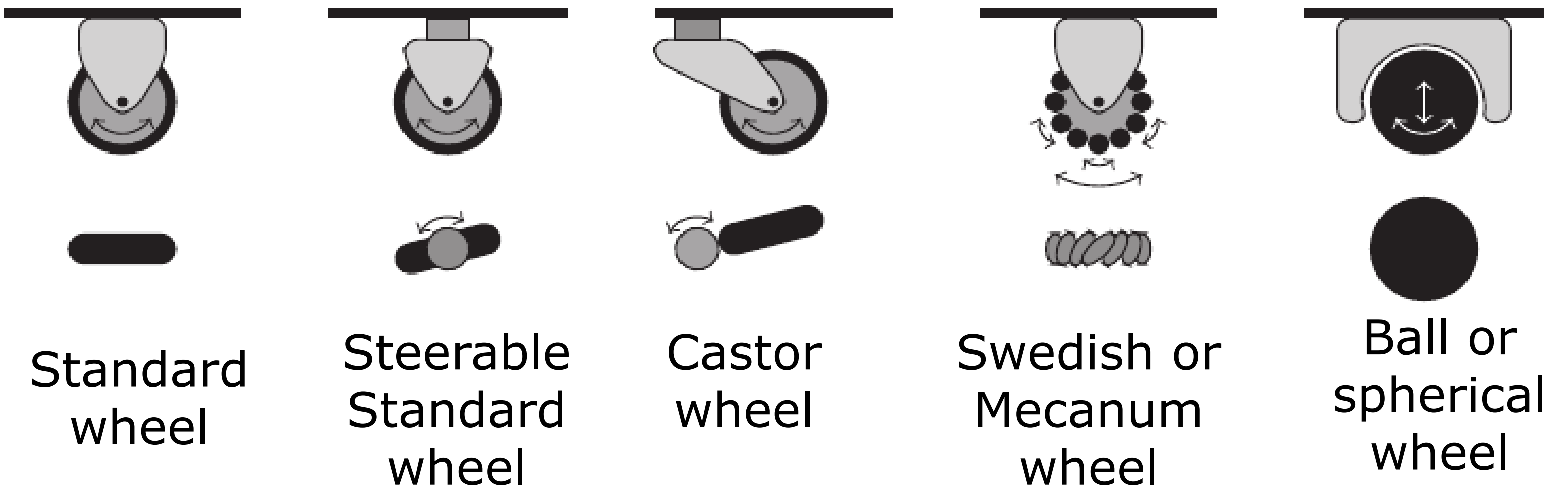
- The stepwise calculation of the linear and the rotational velocity applied to wheeled vehicles operating in a 2D environment.

- The question to be answered:
  - How the rotational velocity of each wheel can be determined, as the kinematic center is moved with a linear velocity $\dot{x}$ due to the x-axis, $\dot{y}$ due to the y-axis and $\dot{\theta}$ the rotational velocity around the z-axis.

- the inverse of this problem should be determined.

Velocity in Wheel-driven Robotic Systems

- The basic wheel types:



Velocity in Wheel-driven Robotic Systems

- for the standard and steerable standard wheel no sliding orthogonal to the wheel plane (velocity must be zero).
- The linear speed of each of these wheels in rolling direction $r\dot{\psi}$
  - r: the radius of the wheel
  - $\dot{\psi}$: the rotational speed of the wheel
- This formula could also be applied for the rolling speed of the castor wheel and the spherical wheel.
- If d is the offset between the wheel axis and the vertical axis of rotation, the linear velocity orthogonal to the wheel plane is the rotational velocity around the vertical axis times the length of the offset $(-d_c\dot{\beta})$.

Velocity in Wheel-driven Robotic Systems

- In Swedish or Mecanum wheel:
  - Passive rollers are mounted in an angle γ (normally $45^{\circ} or\ 90^{\circ}$)
  - $r\dot{\psi}\cos\gamma$:The linear velocity in rolling direction .

- *the Swedish or Mecanum wheel is able to move in an omnidirectional way*
- $\Rightarrow\ r\dot{\psi}\sin\gamma + r_{pr}\dot{\psi}_{pr}$: The speed orthogonal to the wheel plane
  - r_{pr}: the radius
  - $\dot{\psi}_{pr}$: the rotational speed of the roller

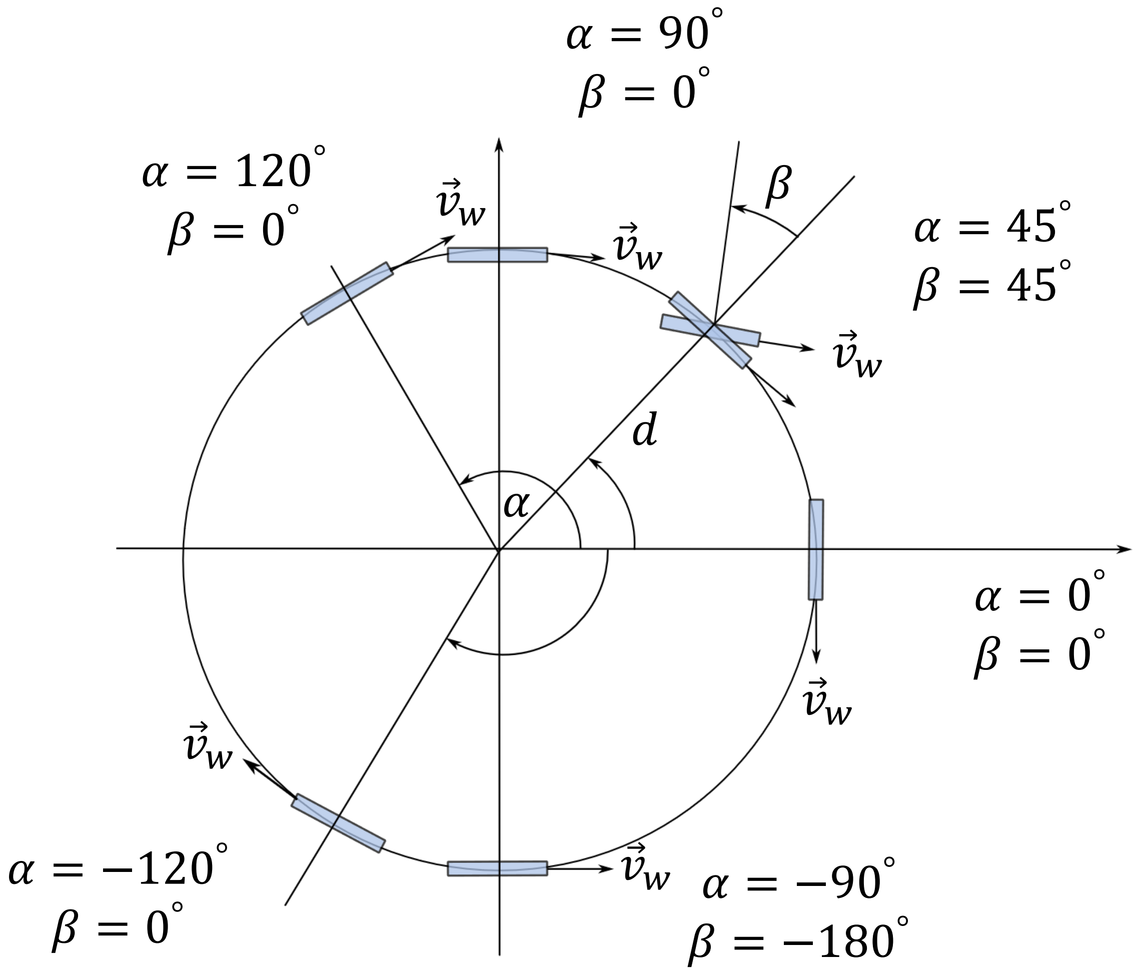
Velocity in Wheel-driven Robotic Systems

- To determine the velocity of a wheel due to the velocity vector $\vec{v} = (\dot{x}, \dot{y}, \dot{z})^T$ of the kinematic center, one must first define the robot coordinate frame, which has its origin in the kinematic center of the vehicle.
- $\alpha = 0$, if the normal vector of the wheel plane is located on the x-axis and has the same orientation.
- β: Angle between the straight line through the kinematic center and the fixing point of the wheel and the y-axis of the wheel frame.
- d: is the distance from the kinematic center to the fixing point of the wheel on the chassis.

Velocity in Wheel-driven Robotic Systems

- The wheel coordinate system has its x-axis in the rolling direction and the y-axis as the normal to the wheel plane.
- The linear speed of the wheel is in the direction of the x-axis of the wheel frame.
- This parameter definition can be used for all wheel types.
- In Swedish or Mecanum wheel:
  - γ: Angle between the x-axis of the wheel and rolling axis of the rollers.

Velocity in Wheel-driven Robotic Systems



Velocity in Wheel-driven Robotic Systems

- Supposing the velocity vector $\vec{\mathrm{v}} = (\dot{\mathrm{x}}, \dot{\mathrm{y}}, \dot{\mathrm{z}})^{\mathrm{T}}$ of the kinematic center is given, equation

$$ {}^{2}\vec{v}_{2} = \begin{bmatrix} c\alpha\dot{x} + s\alpha\dot{y} \\ -s\alpha\dot{x} + c\alpha\dot{y} \\ 0 \end{bmatrix} \qquad (8.5) $$

- Due to the translation d,

$$ {}^{3}\vec{v}_{3} = \begin{bmatrix} c\alpha\dot{x} + s\alpha\dot{y} \\ -s\alpha\dot{x} + c\alpha\dot{y} + d\dot{\theta} \\ 0 \end{bmatrix} \qquad (8.6) $$

Velocity in Wheel-driven Robotic Systems

- The last rotation around the z-axis with angle $\beta - 90^{\circ}$ transfers the x-axis of the last frame to the rolling direction of the wheel.
- For the calculation of ${}^{4}\vec{v}_{4}$ in the next equation:
  - $\sin(\beta - 90^{\circ}) = -\cos(\beta)$ and $\cos(\beta - 90^{\circ}) = \sin(\beta)$

$$
{}^{4}\vec{v}_{4} = \begin{bmatrix} c(\alpha+\beta)\dot{x} - c(\alpha+\beta)\dot{y} - c\beta d\dot{\theta} \\ c(\alpha+\beta)\dot{x} + s(\alpha+\beta)\dot{y} + s\beta d\dot{\theta} \\ 0 \end{bmatrix} \qquad (8.7)
$$

- Equalize the linear velocity vector of the standard wheel to that of equation (8.7):

$$
\begin{bmatrix} s(\alpha+\beta) & -c(\alpha+\beta) & -c\beta d \\ c(\alpha+\beta) & s(\alpha+\beta) & s\beta d \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} r\dot{\psi} \\ 0 \\ 0 \end{bmatrix} \qquad (8.8)
$$

Velocity in Wheel-driven Robotic Systems

- In steerable standard wheel,
  - equation (8.8) can be used in the same way, if the fixed angle β is replaced by a function $\beta(t)$.
- This equation could also be applied to the spherical wheel
  - because of the forces which affect the wheel and change $\beta(t)$, only a linear velocity in the rolling direction exists).
- In castor wheel,
  - the y-component of the velocity vector is depending on the angular velocity $\dot{\beta}$ and the length of the rod.

$$\begin{bmatrix} s(\alpha+\beta) & -c(\alpha+\beta) & -c\beta d \\ c(\alpha+\beta) & s(\alpha+\beta) & s\beta d \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} r\dot{\psi} \\ -d_c\dot{\beta} \\ 0 \end{bmatrix} \tag{8.9}$$

Velocity in Wheel-driven Robotic Systems

- The Swedish or Mecanum wheel is able to move in an omnidirectional way.

  ⇒ lateral movement of the wheel should be possible and can be calculated with equation:

$$\begin{bmatrix} s(\alpha+\beta+\gamma) & -c(\alpha+\beta+\gamma) & -c(\beta+\gamma)d \\ c(\alpha+\beta+\gamma) & s(\alpha+\beta+\gamma) & s(\beta+\gamma)d \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} r\dot{\psi}\cos\gamma \\ r\dot{\psi}\cos\gamma + r_{pr}\dot{\psi}_{pr} \\ 0 \end{bmatrix} \tag{8.10}$$

Kinematics of a Differential Vehicle

- First the wheel types used have to be determined.
- This type of robot has two fixed standard wheels which are mounted on one axis.
- The kinematic center is located in the middle of the axis;
- the distance between the kinematic center and each wheel should be d.
- The coordinate system to define the parameters must be specified.
- The origin of this frame lies on the kinematic center.
- One solution for modelling the wheel configuration is to place the wheels on the y-axis of the coordinate frame.
- The parameters are:

$$\alpha_l = 90^{\circ}, \beta_l = 0^{\circ}, \alpha_r = -90^{\circ}, \beta_r = 180^{\circ}$$

Kinematics of a Differential Vehicle

An example of a differential drive robot is Marvin, the mobile vehicle of the University of Kaiserslautern

Kinematics of a Differential Vehicle

- Based on equation (8.8). one obtains for the left and the right wheel:

$$
\begin{aligned}
s(\alpha_l + \beta_l)\dot{x} - c(\alpha_l + \beta_l)\dot{y} - c\beta_l d\dot{\theta} &= r_l\dot{\psi}_l \\
s(\alpha_l + \beta_l)\dot{x} + s(\alpha_l + \beta_l)\dot{y} + s\beta_l d\dot{\theta} &= 0 \\
s(\alpha_r + \beta_r)\dot{x} - c(\alpha_r + \beta_r)\dot{y} - c\beta_r d\dot{\theta} &= r_r\dot{\psi}_r \\
s(\alpha_r + \beta_r)\dot{x} + s(\alpha_r + \beta_r)\dot{y} + s\beta_r d\dot{\theta} &= 0
\end{aligned}
\qquad (8.11)
$$

- If the above mentioned parameters are inserted, the following equation will result:

$$
\begin{aligned}
\dot{x} - d\dot{\theta} &= r_l\dot{\psi}_l \\
\dot{y} &= 0 \\
\dot{x} + d\dot{\theta} &= r_r\dot{\psi}_r \\
\dot{y} &= 0
\end{aligned}
\qquad (8.12)
$$

Kinematics of a Differential Vehicle

- After solving the equation system on receives:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(r_l\dot{\psi}_l + r_r\dot{\psi}_r) \\ 0 \\ \frac{1}{2d}(-r_l\dot{\psi}_l + r_r\dot{\psi}_r) \end{bmatrix} \tag{8.13}$$

Kinematics of Omnidirectional Vehicles

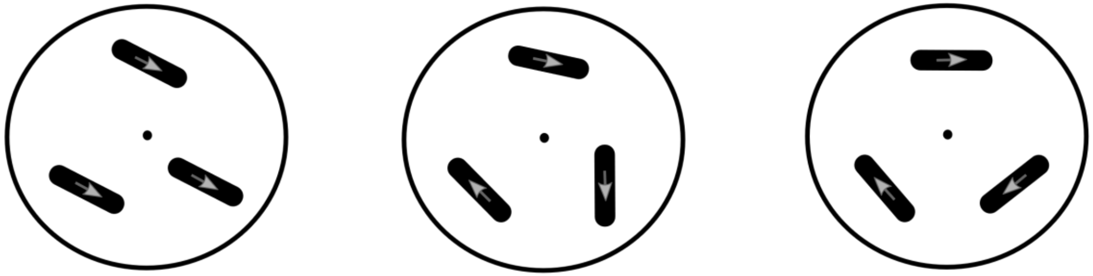
- To increase the mobility of a vehicle, an omnidirectional drive can be used.

The climbing robot Cromsci of the University of Kaiserslautern.

Wheel setting

Kinematics of Omnidirectional Vehicles

- For example, is equipped with such a drive, in which 3 steerable standard wheels are mounted with an angle displacement of 120° between them.



Kinematics of Omnidirectional Vehicles

- Model the wheel configuration as shown in figure (8.8) with the kinematic center in the middle of the robot.
- The front wheel is located on the x-axis, the two rear wheels have a displacement to the front wheel of $\pm 120^{\circ}$.
- As shown in figure (8.8),

$$\alpha_1 = 0^{\circ}, \alpha_2 = 120^{\circ}, \alpha_3 = -120^{\circ}$$

- $\beta_{1,2,3}$: the control parameters to determine the direction of the vehicle movements.
- d; distance between the wheel's contact point and the robot center.

Kinematics of Omnidirectional Vehicles

- For the navigation of Cromsci it is necessary to calculate based on the desired linear and rotational velocities of the kinematic center $(\dot{x}, \dot{y}, \dot{\theta})^T$
- The single wheel velocities and the orientations $(r_1\dot{\psi}_1, r_2\dot{\psi}_2, r_3\dot{\psi}_3, \beta_1, \beta_2, \beta_3)$
- Applying equation (8.8) for each wheel leads to the following equation systems:

$$
\begin{aligned}
s(\alpha_1+\beta_1)\dot{x} - c(\alpha_1+\beta_1)\dot{y} - d\cdot c(\beta_1)\dot{\theta} &= r_1\dot{\psi}_1 \\
s(\alpha_2+\beta_2)\dot{x} - c(\alpha_2+\beta_2)\dot{y} - d\cdot c(\beta_2)\dot{\theta} &= r_2\dot{\psi}_2 \\
s(\alpha_3+\beta_3)\dot{x} - c(\alpha_3+\beta_3)\dot{y} - d\cdot c(\beta_3)\dot{\theta} &= r_3\dot{\psi}_3 \qquad (8.14) \\
s(\alpha_1+\beta_1)\dot{x} + s(\alpha_1+\beta_1)\dot{y} + d\cdot c(\beta_1)\dot{\theta} &= 0 \\
s(\alpha_2+\beta_2)\dot{x} + s(\alpha_2+\beta_2)\dot{y} + d\cdot c(\beta_2)\dot{\theta} &= 0 \\
s(\alpha_3+\beta_3)\dot{x} + s(\alpha_3+\beta_3)\dot{y} + d\cdot c(\beta_3)\dot{\theta} &= 0
\end{aligned}
$$

Kinematics of Omnidirectional Vehicles

- Based on the last 3 equations of (8.14), the steering angles $\beta_i,\ i = 1{,}2{,}3$ are determined:

$$s(\alpha_i + \beta_i)\dot{x} + s(\alpha_i + \beta_i)\dot{y} + d \cdot c(\beta_i)\dot{\theta} = 0$$

$$\Longrightarrow c(\alpha_i) \cdot c(\beta_i).\dot{x} - s(\alpha_i) \cdot s(\beta_i).\dot{x}$$

$$+\ s(\alpha_i) \cdot c(\beta_i) \cdot \dot{y} + c(\alpha_i) \cdot s(\beta_i) \cdot \dot{y} + d \cdot s(\beta_i) \cdot \dot{\theta} = 0$$

$$\Longrightarrow c(\beta_i) \cdot (c(\alpha_i) \cdot \dot{x} + s(\alpha_i) \cdot \dot{y}) = s(\beta_i) \cdot \left(s(\alpha_i) \cdot \dot{x} - c(\alpha_i) \cdot \dot{y} - d\dot{\theta}\right)$$

(8.15)

$$\Rightarrow \quad \tan(\beta_i) = \frac{s(\beta_i)}{c(\beta_i)} = \frac{c(\alpha_i)\cdot\dot{x}+s(\alpha_i)\cdot\dot{y}}{s(\alpha_i)\cdot\dot{x}-c(\alpha_i)\cdot\dot{y}-d\cdot\dot{\theta}}$$

$$\Longrightarrow \beta_i = \arctan 2((c(\alpha_i) \cdot \dot{x} + s(\alpha_i) \cdot \dot{y}), (s(\alpha_i) \cdot \dot{x} - c(\alpha_i) \cdot \dot{y} - d \cdot \dot{\theta}))$$

From equation (8.14) the angular velocity of the wheel $\dot{\psi}_i$ can be calculated using β_i:

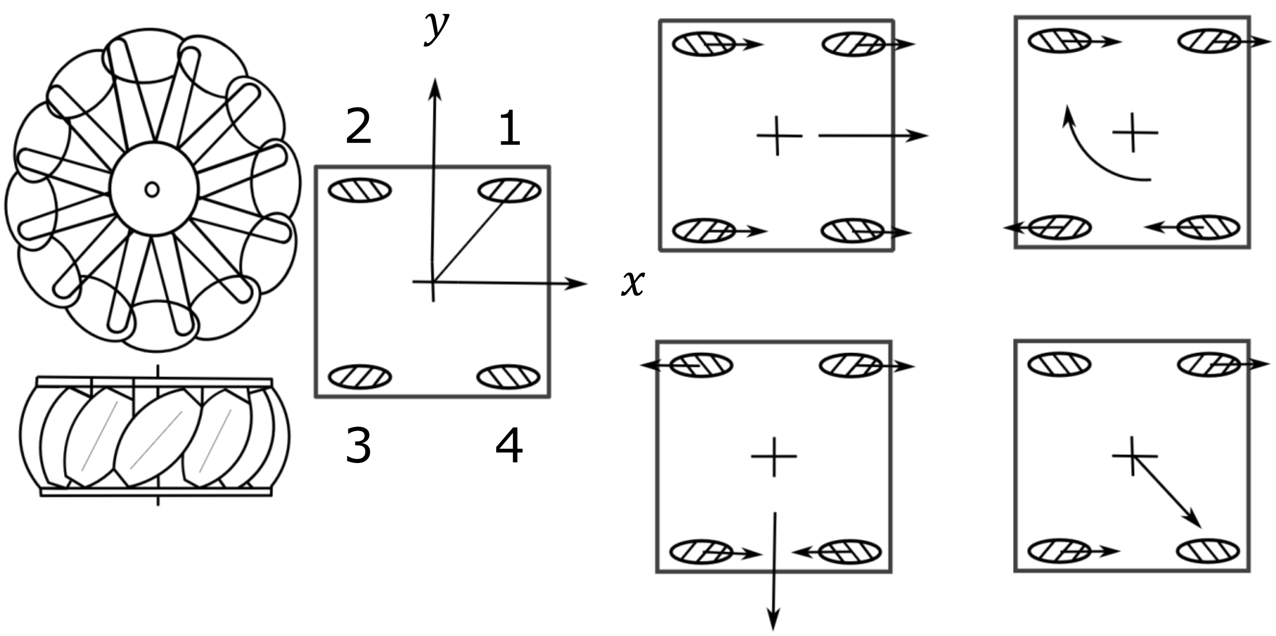
$$\dot{\psi}_i = \frac{1}{r_i}(s(\alpha_i + \beta_i)\dot{x} - c(\alpha_i + \beta_i)\dot{y} - d \cdot c(\beta_i)\dot{\theta}) \qquad (8.16)$$

Kinematics of a vehicle with Mecanum Wheels

- Convex cylinders arranged in a 45° angle relative to the wheel plane.
- Two pairs of independently driven Mecanum wheels are sufficient to enable omnidirectional movement.
- The orientation of the rollers of the wheels lying on a common diagonal axis is equal.
- The rollers of the other two wheels are oriented in the opposite direction.
- The rollers of the other two wheels are oriented in the opposite direction.

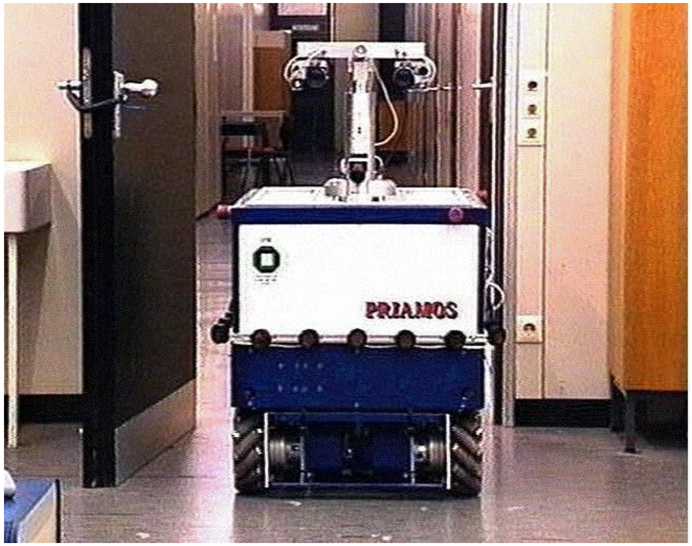
Kinematics of a vehicle with Mecanum Wheels

- In figure, 4 typical movements of the vehicle are shown.
- If all wheels move with the same velocity in the same direction, the robot drives straight ahead.
- The machine will turn if the right and left wheels move in opposite direction with the same velocity.



Kinematics of a vehicle with Mecanum Wheels

- A sideward motion is possible if the neighboring wheels move in opposite direction with the same velocity.
- A diagonal motion results if the two wheels on the diagonal move in the same direction with the same velocity.
- This type of drive was applied for the vehicles PRIAMOS of Prof. Dillmann's research group at the University of Karlsruhe.



Kinematics of a vehicle with Mecanum Wheels

- To set up the kinematic equation, the parameters (α, β, γ) for each wheel must be determined.
- The order of the wheels is shown in figure in slide 52.
- The parameters for four wheels are:

$$
\begin{array}{lll}
\alpha_1 = 45^\circ, & \beta_1 = 45^\circ, & \gamma_1 = -45^\circ \\
\alpha_2 = 135^\circ, & \beta_2 = -45^\circ, & \gamma_2 = 45^\circ \\
\alpha_3 = -135^\circ, & \beta_3 = 225^\circ, & \gamma_3 = -45^\circ \\
\alpha_4 = -45^\circ, & \beta_4 = 135^\circ, & \gamma_4 = 45^\circ
\end{array}
$$

Kinematics of a vehicle with Mecanum Wheels

- Using equation (8.10) and supposing all driven wheels have the same radius r, the same distance d from the kinematic center and the above mentioned parameters for α, β, γ are inserted, we receive:

$$s(45^\circ)\dot{x} - c(45^\circ)\dot{y} - d\dot{\theta} = r \cdot c(-45^\circ)\dot{\psi}_1 \qquad (8.17)$$

$$s(135^\circ)\dot{x} - c(135^\circ)\dot{y} - d\dot{\theta} = r \cdot c(45^\circ)\dot{\psi}_2 \qquad (8.18)$$

$$s(45^\circ)\dot{x} - c(45^\circ)\dot{y} - d\dot{\theta} = r \cdot c(-45^\circ)\dot{\psi}_3 \qquad (8.19)$$

$$s(135^\circ)\dot{x} - c(135^\circ)\dot{y} - d\dot{\theta} = r \cdot c(45^\circ)\dot{\psi}_1 \qquad (8.20)$$

Kinematics of a vehicle with Mecanum Wheels

- Based on this equation system, the velocities $\dot{x}, \dot{y}, \dot{\theta}$ of the kinematic center can be calculated:

$$\begin{aligned}\dot{x} &= \frac{r}{4}(\dot{\psi}_1 + \dot{\psi}_2 + \dot{\psi}_3 + \dot{\psi}_4)\\ \dot{y} &= \frac{r}{4}(-\dot{\psi}_1 + \dot{\psi}_2 - \dot{\psi}_3 + \dot{\psi}_4) \qquad (8.21)\\ \dot{\theta} &= \frac{r}{d\sqrt{2}}(\dot{\psi}_1 - \dot{\psi}_2 - \dot{\psi}_3 + \dot{\psi}_4)\end{aligned}$$

- The velocity vector of the kinematic center can be determined as:

- $$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \frac{r_{wheel}}{4} \cdot \begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 1 & -1 & 1\\ C & -C & -C & -C\end{bmatrix}\begin{bmatrix}\dot{\psi}_1\\ \dot{\psi}_2\\ \dot{\psi}_3\\ \dot{\psi}_4\end{bmatrix} \qquad (8.22)$$

With $C = \frac{2\sqrt{2}}{d}$, and $\dot{\psi}_4 = \dot{\psi}_1 + \dot{\psi}_2 - \dot{\psi}_3$

Poses Calculation Based on velocity

- Using equation (8.23) and time interval Δt, the incremental paths can be determined for 2D navigation as:

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} \cdot \Delta t \qquad (8.23)$$

- Assume the velocity $v = (v_x(t), v_y(t))_A$ given in a robot fixed coordinate frame (x_A, y_A), angular velocity $\omega(t)$ and the robot pose (x, y, θ) in the world coordinate system given:

$$\theta(t) = \int_0^t \omega(\tau) d\tau + \theta_0 \qquad (8.24)$$

$$x(t) = \int_0^t \dot{x}(\tau) d\tau + x_0 \qquad (8.25)$$

$$y(t) = \int_0^t \dot{y}(\tau) d\tau + y_0 \qquad (8.26)$$

- Vehicle velocities are $\dot{x}$ due to the x-axis, $\dot{y}$ due to the y-axis and $\omega = \dot{\theta}$ around z-axis.

Static Forces/Moments

- Calculation without consideration of movements
- Example: How high do torques have to be in order to keep an object of mass m in a certain position with TCP?
- Solution idea
  - Propagate powers and moments from link to link
  - Calculate a force/moment balance for each member
  - Start with the TCP
- $\vec{f}_i$: Force that attacks on link through link $i-1$
- $\vec{n}_i$: Torque (Moment) that attacks link through link $i-1$
- Forces/Moment equation
  (Influence of the next higher link)

$$ {}^{i}\vec{f}_i = {}^{i}\vec{f}_{i+1} \qquad {}^{i}\vec{n}_i = {}^{i}\vec{n}_{i+1} + {}^{i}\vec{u}_{i+1} \times {}^{i}\vec{f}_{i+1} $$

Static Forces/Moments: Propagation

- Static propagation of the forces /moments from link to link
- Forces at link i

$$^{i}\vec{f}_i = {}^{\;\;i}_{i+1}R \cdot {}^{i+1}\vec{f}_{i+1}$$

- Moment at link i

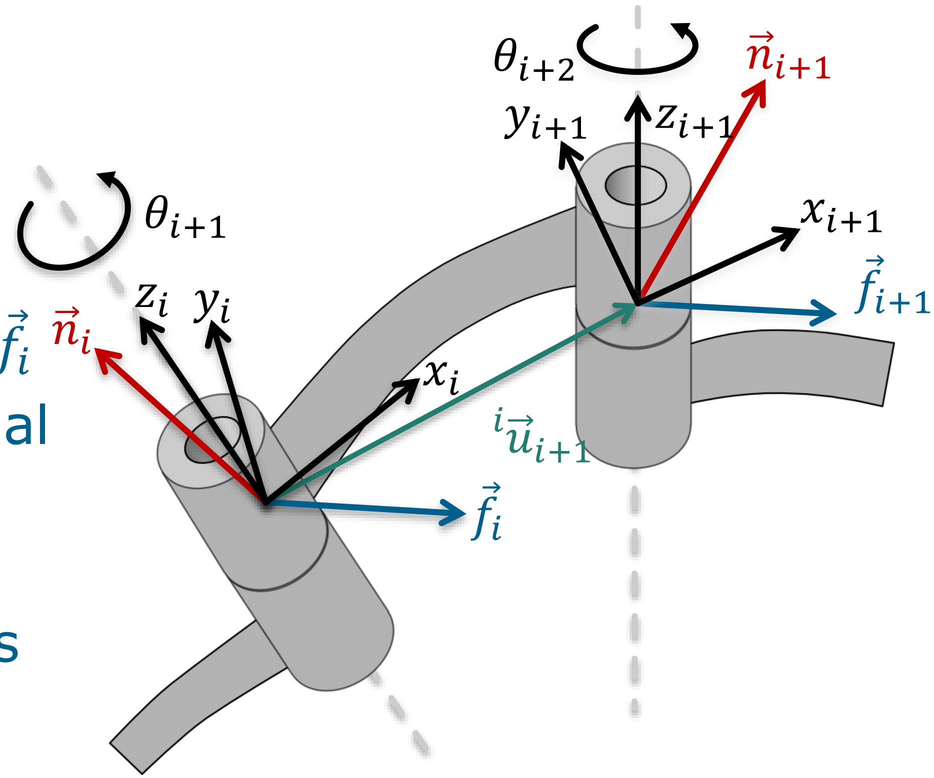
$$^{i}\vec{n}_i = {}^{\;\;i}_{i+1}R \cdot {}^{i+1}\vec{n}_{i+1} + {}^{i}\vec{u}_{i+1} \times {}^{i}\vec{f}_i$$

- Required moment in rotational joints

$$\tau_{i+1} = {}^{i}\vec{n}_i^{T} \cdot {}^{i}\vec{e}_{z_i}$$

- Required force in linear joints

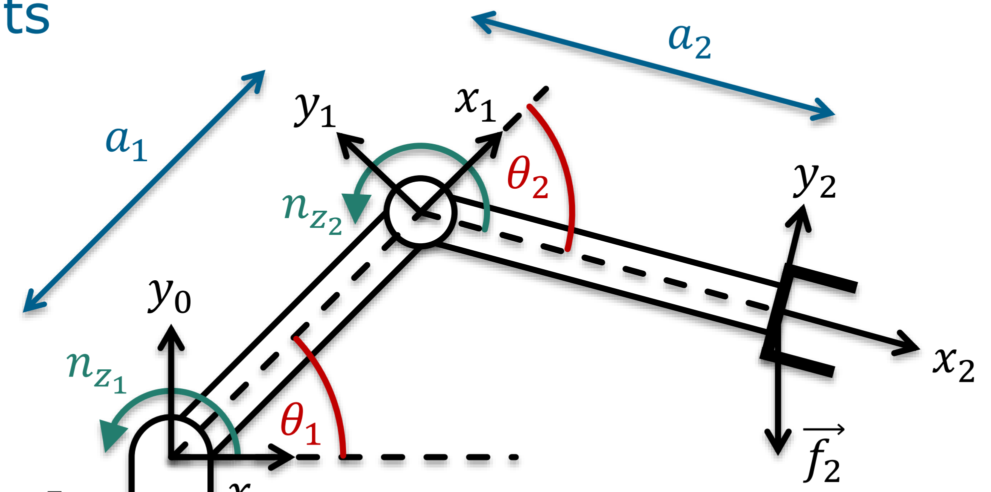
$$\tau_{i+1} = {}^{i}\vec{f}_i^{T} \cdot {}^{i}\vec{e}_{z_i}$$



Static Forces/Moments: Example

- Given: Forces f, applied at the TCP
- Desired: Torques in the joints

$$ {}^2\vec{f}_2 = \begin{bmatrix} {}^2f_{2_x} \\ {}^2f_{2_y} \\ 0 \end{bmatrix} \qquad {}^2\vec{n}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$



$$ \begin{aligned} {}^1\vec{n}_1 &= {}^1\vec{n}_2 + {}^1\vec{u}_2 \times {}^1\vec{f}_1 \\ &= {}^1\vec{u}_2 \times \left({}^1_2R \cdot {}^2\vec{f}_2\right) \\ &= \begin{bmatrix} a_2c_2 \\ a_2s_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_2 \cdot {}^2f_{2_x} - s_2 \cdot {}^2f_{2_y} \\ s_2 \cdot {}^2f_{2_x} - c_2 \cdot {}^2f_{2_y} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ a_2 \cdot {}^2f_{2_y} \end{bmatrix} \end{aligned} $$

Static Forces/Moments: Example

$$
{}^{0}\vec{n}_0 = {}^{0}_{1}R \cdot {}^{1}\vec{n} + {}^{0}\vec{u}_1 \times {}^{0}\vec{f}_0
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ a_2 \cdot {}^{2}f_{2_y} \end{bmatrix} + \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_{12} \cdot {}^{2}f_{2_x} - s_{12} \cdot {}^{2}f_{2_y} \\ s_{12} \cdot {}^{2}f_{2_x} - c_{12} \cdot {}^{2}f_{2_y} \\ 0 \end{bmatrix}
$$

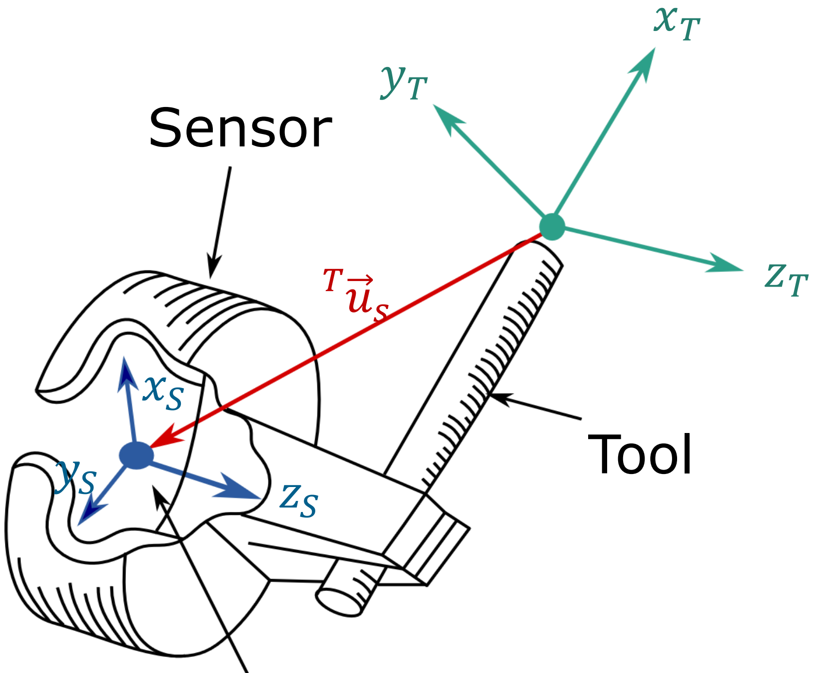
$$
= \begin{bmatrix} 0 \\ 0 \\ a_2 \cdot {}^{2}f_{2_y} + s_2 a_1 \cdot {}^{2}f_{2_x} + c_2 a_1 \cdot {}^{2}f_{2_y} \end{bmatrix}
$$

$$
\tau_1 = a_2 \cdot {}^{2}f_{2_y} + s_2 a_1 \cdot {}^{2}f_{2_x} + c_2 a_1 \cdot {}^{2}f_{2_y}
$$

$$
\tau_2 = a_2 \cdot {}^{2}f_{2_y}
$$

Transformation of Forces: Application Example

- TCP grips tool→ Load cell measures forces and moments in the wrist
- Desired: Forces and torques at the end of the tool



Force/Moment Calculation with Jacobian Matrix

- Contemplation of the virtual work in the Cartesian space - and in the configuration space
- Work, which is caused by the forces and moments $\vec{\eta}$ acting on the TCP, must be equal to the work that is applied in the joints by adjusting forces and setting moments $\vec{\tau}$
- $$\vec{\eta}^T \cdot \dot{\vec{y}} = \vec{\tau}^T \cdot \dot{\vec{\theta}} \qquad (8.27)$$
  - $\vec{\eta} = \begin{bmatrix} \vec{f}_{TCP} \\ \vec{n}_{TCP} \end{bmatrix}$: 6×1, Cartesian force-/moment vector at TCP
  - $\dot{\vec{y}}$: 6×1, infinitesimal offset vector of TCP
  - $\vec{\tau}$: 6×1, force-/moment vector in joints
  - $\dot{\vec{\theta}}$: 6×1, change of joint positions

Force/Moment Calculation with Jacobian Matrix

- By inserting the relationship $\dot{\vec{y}} = J(\vec{\theta}) \cdot \dot{\vec{\theta}}$ $(8 \cdot 27)$ can be transformed into $\vec{\eta}^T \cdot J(\vec{\theta}) \cdot \dot{\vec{\theta}} = \vec{\tau}^T \cdot \dot{\vec{\theta}}$
- Thus, $\vec{\eta}^T \cdot J(\vec{\theta}) = \vec{\tau}^T$ and $\vec{\tau} = J^T(\vec{\theta}) \cdot \vec{\eta}$

Coming up next…

Dynamics Modeling