UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 10 — FINAL

Due: Friday, May 8, 2020, by 5PM

Material: The final covers all the material so far.

Due date: Friday, May 8, 2020 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

Logistics: The exam consists of this page plus 2 pages of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than 5 hours cumulative time spent on these problems. You may take as many breaks as you like, not counted against the 5 hours. **You should not be consulting references, working on the problems, or discussing with others during the breaks**.

Resources: The final is **not collaborative**. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Griffiths, and solution sets on the course website. **You may not consult any other material**, including other textbooks, the web (except for the current Phys. 402 website), material from previous years' Phys. 402 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. **Gradually changing waveguide**. Suppose we have a hollow waveguide running in the z direction for several kilometers. At every z the cross-section is a rectangle with sides a, b where $a \geq b$. Suppose we send in a wave that excites the TE_{mn} mode with some frequency ω and wavenumber k that satisfy the dispersion relation [Griffiths Eq. (9.187)]

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]} = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}\,, \tag{1}$$

and the amplitude is given by some B_0 in [Griffiths Eq. (9.186)]

$$B_z = B_0 \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})\,. \tag{2}$$

   (a) What is the average energy flux $\langle \boldsymbol{S} \rangle$ (averaged over one period of the wave)? What is the averaged energy flux through the whole cross-section, $\int \langle \boldsymbol{S} \rangle \cdot d\boldsymbol{a}$? (The result of Griffiths' problem 9.11 [time-averaging a product using complex exponentials] might be helpful).

   Now suppose that this waveguide's cross-section changes very slowly in z, so that $a = a(z)$ and $b = b(z)$. Very slowly here means that $\frac{1}{a}\frac{da}{dz} \ll k$ and similarly for b. For simplicity we will assume that the aspect ratio a/b remains constant.

   (b) Will ω change with z? What about k?

   (c) Now the amplitude $B_0(z)$ will have to slowly vary with z. Find a differential equation that would allow you to solve for $B_0(z)$ if somebody gave you $a(z)$ (and thus they are also giving you $b(z)$ since their ratio is constant).

   (d) Find a simple combination of a, B_0, and k that is constant along z.

   (e) What will happen if a gradually shrinks too small?

2. **Integral identities**. For the following problems, you can assume that as you go to very large distances, the electric field decays as $1/r^2$, and the magnetic field decays as $1/r^3$.

   (a) How quickly can the vector potential $\boldsymbol{A}$ decay at large r?

   (b) Prove the following integral identity, for any two vector fields $\boldsymbol{V}, \boldsymbol{W}$ integrated over volume $\mathcal{V}$:

$$\int_{\mathcal{V}} \boldsymbol{W} \cdot (\nabla \times \boldsymbol{V})\, d^3r = \int_{\partial\mathcal{V}} (\boldsymbol{V} \times \boldsymbol{W}) \cdot d\boldsymbol{a} + \int_{\mathcal{V}} \boldsymbol{V} \cdot (\nabla \times \boldsymbol{W})\, d^3r\,. \tag{3}$$

   (c) Now combine everything to show that the following integral over all space and time vanishes:

$$\int_{-\infty}^{+\infty} \int_{\text{All space}} (\boldsymbol{E} \cdot \boldsymbol{B})\, d^3r\, dt = 0\,. \tag{4}$$

   [Hint: use the potential formulation, and assume that the fields also vanish as $t \to \pm\infty$.]

3. **Radiation reaction from a rotor with two charges**. One problem set 8, we considered a rotor of length $2b$ laying in the $x - y$ plane, with a charge $+q$ at one end and $-q$ at the other end, spinning about the z axis at angular frequency ω. In that problem, we found the dipole radiation carried an energy flux (integrated over all angles):

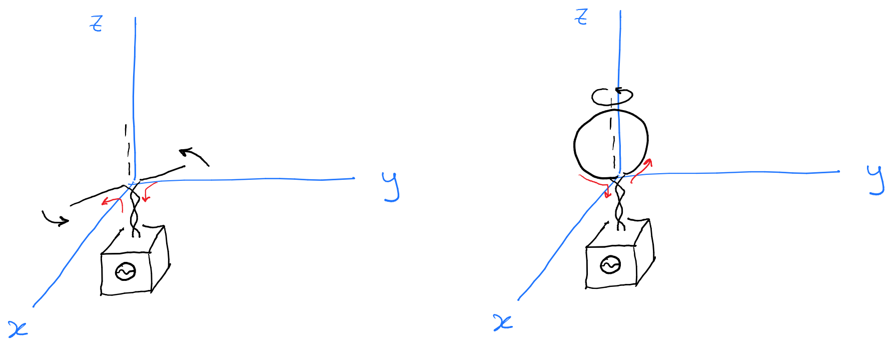
$$\langle \frac{dE}{dt} \rangle = \frac{2\mu_0 q^2 b^2 \omega^4}{3\pi c}\,. \tag{5}$$

   Let us now give each of these charges a mass m, and let the rotor not contribute to the moment of inertia.

   (a) Since energy is leaving, the spin rate $\omega(t)$ will slowly decrease. Find a differential equation for $d\omega/dt$ and solve for $\omega(t)$.

(b) Now compute the radiation-reaction force on each particle. Use this to compute the torque on the system and so find another expression for $d\omega/dt$. Do they give the same result? Why or why not?

4. **Two rotating antennas**. You might have seen a rotating antenna at an airport. Let's try to make two simple models. Suppose we take the simple split dipole antenna, at left, and the loop ("magnetic dipole") antenna, at right:



Each half of the split dipole has length b, and it is lying along direction $\hat{\boldsymbol{n}}$ somewhere in the $x - y$ plane. Meanwhile the loop antenna has radius b, and its normal vector $\hat{\boldsymbol{n}}$ lies somewhere in the $x - y$ plane.

In each scenario, the antenna is hooked up to a source of alternating current $I(t) = I_0 \cos(\Omega_s t)$, denoted with red arrows for one part of the oscillation. Each antenna is *also* hooked up to a motor that makes it rotate around the $\hat{\boldsymbol{z}}$ axis with a different angular frequency ω_r, denoted with the black arrows.

We're interested in describing the radiation at some point $\boldsymbol{r}$ at a large distance away from an antenna.

(a) In order to control the multipole expansion, we need to satisfy certain relationships between things like b, r, ω_r, and Ω_s. Explain which quantities must be very small or large, and why.

For the remainder of the problem, you can assume that $\omega_r \ll \Omega_r$, if you didn't already assume that above.

(b) For the split dipole, what is the electric dipole $\boldsymbol{p}(t)$ as a function of time? For the loop antenna, what is the magnetic dipole $\boldsymbol{m}(t)$ as a function of time?

(c) For each of the two models (split dipole, loop antenna) give the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$ at the very distant point $\boldsymbol{r}$. We only need the $\mathcal{O}(1/r)$ part, and you can use all the approximations above. [Hint: It will probably be simplest to first work out a coordinate-independent result, and the use it to find $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ components of the fields.]

5. Griffiths problem 12.47 (transform a plane electromagnetic wave to a new frame).