UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

Problem Set 6

Due: Thurs., Apr. 16, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Abstract 3+1 basics**. Let $(\mathcal{M}, g_{ab})$ be a 4-manifold with Lorentzian metric g (with Levi-Civita connection ∇), and suppose we have a foliation by hypersurfaces of a global "time" function $t : \mathcal{M} \to \mathbb{R}$, each leaf being a spacelike hypersurface Σ_t. Let the future-pointing, timelike, unit normal be

$$n^a \equiv -\alpha \nabla^a t = \frac{-\nabla^a t}{\sqrt{-g^{ab}\nabla_a t \nabla_b t}}\,, \tag{1}$$

where the lapse α is defined through $g^{ab}\nabla_a t \nabla_b t = -1/\alpha^2$.

(a) Show that the one-form field n_a is *irrotational*, which is to say that it satisfies $n_{[a}\nabla_b n_{c]} = 0$.

Now recall that we decompose each tangent space into the subspace $T\Sigma_t$ and the orthogonal complement in the span of n^a, by writing the induced metric γ_{ab} on $T\Sigma_t$ (sometimes called the "first fundamental form"),

$$\gamma_{ab} = g_{ab} + n_a n_b\,. \tag{2}$$

Then the (1,1) version $\gamma^a{}_b$ is an (idempotent) projection operator. The induced metric γ_{ab} has Levi-Civita connection D_a. This connection only knows how to act on purely spatial tensors – recall that a tensor is purely spatial iff it vanishes when n^a is contracted into any index slot. The simple way to define D_a is

$$D_c T^{a_1 a_2 \cdots}{}_{b_1 \cdots} = \gamma_c^{c'} \gamma_{a'_1}^{a_1} \cdots \gamma_{b_1}^{b'_1} \cdots \nabla_{c'} T^{a'_1 a'_2 \cdots}{}_{b'_1 \cdots}\,. \tag{3}$$

(b) Show that D_a is indeed metric-compatible with γ_{bc}.

(c) Show that the Leibniz rule $D_a(v^b w_b) = v^b(D_a w_b) + (D_a v^b) w_b$ holds only if v^b and w_b are purely spatial.

If an observer was moving along a world-line with tangent n^a, then her proper acceleration 4-vector would be $a^a \equiv n^b \nabla_b n^a$.

(d) Show that the acceleration vector is a purely spatial vector.

(e) Show that the acceleration can be written in terms of the spatial gradient of the lapse function,

$$a_a = D_a \ln \alpha\,. \tag{4}$$

Now we're interested in the extrinsic curvature (sometimes called the "second fundamental form"), found by studying how n^a varies from point to point. Our convention is

$$K_{ab} \equiv -\gamma_a^c \nabla_c n_b\,, \tag{5}$$

which is a purely spatial tensor.

(f) To ensure this is purely spatial we did not need a γ projector on the b index – show why.

(g) Show the equality $K_{ab} = -\nabla_a n_b - n_a a_b$

(h) Show that K_{ab} is a symmetric tensor.

(i) Show the equality $K_{ab} = -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{ab}$.

(j) Show that $\mathcal{L}_{\boldsymbol{n}} K_{ab}$ is purely spatial.

2. **A coordinate example**. Take the Schwarzschild spacetime in standard Schwarzschild coordinates. Define a foliation by level sets of the function

$$T = t + 4M\left[\sqrt{\frac{r}{2M}} + \frac{1}{2}\ln\left(\frac{\sqrt{r/2M}-1}{\sqrt{r/2M}+1}\right)\right]. \tag{6}$$

   (a) Compute the unit normal n_a and the induced metric γ_{ab}.

   (b) Calculate the extrinsic curvature K_{ab}.

   (c) Show that the Schwarzschild metric can be rewritten using T as a time coordinate instead of t, resulting in:

$$ds^2 = -dT^2 + (dr + \sqrt{2M/r}\; dT)^2 + r^2 d\Omega^2 . \tag{7}$$

   From this calculation you can see that the $T = \text{const.}$ surfaces are intrinsically flat.

3. **A 4+1 example**. Let's start from 5-dimensional Minkowski space with coordinates z^A,

$$ds^2 = \eta_{AB} dz^A dz^B = -(dz^0)^2 + (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 . \tag{8}$$

   We construct a map from a 4-dimensional manifold with coordinates $x^a = (t, \chi, \theta, \phi)$ into this 5-d manifold, thus defining a 4-d hypersurface in 5-d. The coordinate maps for embedding are $z^A(x^a)$:

$$z^0 = a\sinh(t/a)\,, \qquad z^1 = a\cosh(t/a)\cos\chi\,, \qquad z^2 = a\cosh(t/a)\sin\chi\cos\theta\,, \tag{9}$$

$$z^3 = a\cosh(t/a)\sin\chi\sin\theta\cos\phi\,, \qquad z^4 = a\cosh(t/a)\sin\chi\sin\theta\sin\phi\,. \tag{10}$$

   (a) Find a single coordinate function $\Phi(z^A)$ such that $\Phi = 0$ defines the same submanifold.

   (b) Compute the unit normal n^A and the tangent vectors $e^A_{(a)} = \partial z^A/\partial x^a$.

   (c) Compute the induced 4-metric $\gamma_{ab} = \eta_{AB} e^A_a e^B_b$, and comment on the physical meaning of this metric.

   (d) Compute the extrinsic curvature K_{ab}, then use the Gauss-Codazzi equations to show that the 4-metric is a metric of constant curvature,

$${}^{(4)}R_{abcd} = \frac{1}{a^2}(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc})\,. \tag{11}$$