UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2019

Problem Set 5 — MIDTERM

Due: Monday, Mar. 25, 2019, by 5PM

Material: The midterm covers the material so far except for last week and this week (linear oscillators).

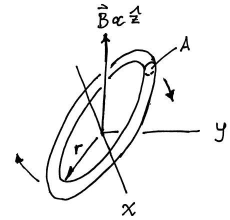
Due date: Monday, Mar. 25, 2019 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

Logistics: The exam consists of this page plus two pages of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than 5 hours cumulative time spent on these problems. You may take as many breaks as you like, not counted against the 5 hours. **You should not be consulting references, working on the problems, or discussing with others during the breaks**.

Resources: The midterm and final are **not collaborative**. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Griffiths, and solution sets on the course website. **You may not consult any other material**, including other textbooks, the web (except for the current Phys. 402 website), material from previous years' Phys. 402 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. **Spinning ring in magnetic field**. Suppose we are freely floating in space, and there is a magnetic field of strength B oriented along the $\hat{z}$ axis. We have a conducting circular ring of radius r and conductivity σ. The ring is made of wire whose cross-sectional area is A.



   (a) In terms of the given quantities, what is the ring's total resistance R?

   We set this ring spinning about the $\hat{x}$ axis as shown in the figure. Let the spin rate be ω (radians per second) which is approximately constant.

   (b) What is the magnetic flux through the ring, as a function of time? What is the induced EMF around the ring, again as a function of time?

   Suppose the ring also has some self-inductance L. Now you can model the ring as a circuit with a source of EMF, a resistor, and an inductor.

   (c) Draw the circuit diagram.

   (d) Write down a differential equation for the current as a function of time. Solve for $I(t)$.

   (e) What is the instantaneous power that is dissipated in the circuit? What is the average power $\langle P \rangle$ over one period of the rotation?

   The energy that is being dissipated is coming, of course, from the rotational kinetic energy of the ring. Thus ω will not be constant and instead will slowly decrease. Let the moment of inertia of the ring about its spin axis be $\mathcal{I}$ (not to be confused with I).

   (f) Write a differential equation for the spin rate $\omega(t)$. You may use the averaged power $\langle P \rangle$ in place of the instantaneous power.

   From the form of this differential equation, you should be able to identify a characteristic frequency ω_c such that there is qualitatively different behavior when $\omega \ll \omega_c$ or $\omega \gg \omega_c$.

   (g) What is ω_c? (Factors of order unity don't matter here, only the functional dependence on physical quantities)

   (h) Suppose $\omega \ll \omega_c$. Solve for the (approximate) time dependence $\omega(t)$.

2. **Magnetic field between two coils**. We have two circular current loops in our lab, each of radius R. We place them both on the z axis with their symmetry axes also along z. One of them is at height $z = +a$, and the other is at height $z = -a$. We run the same current I through both coils.

   (a) Write down an integral for the mutual inductance between the two loops. (This should be an ordinary one-dimensional integral; make as much progress as possible but you may not be able to completely evaluate it)

   (b) What is the magnetic field along the z axis, $\boldsymbol{B}(\rho = 0, z)$, in the region between the two loops? Show that $\left.\frac{\partial B_z}{\partial z}\right|_{z=0} = 0$.

   (c) Find the value of a that will make the z component of the magnetic field more uniform at the center, so as to satisfy

$$\left.\frac{\partial^2 B_z}{\partial z^2}\right|_{z=0} = 0\,.$$

   (d) Close to the z axis, the magnetic field can be treated as a Taylor series in ρ (there can not be any ϕ dependence because of symmetry). Using one of the Maxwell equations, determine the first two nonvanishing terms in the series for $B_\rho(\rho, z)$ [hint: they will depend on derivatives of $B_z(0, z)$]. What happens at $z = 0$ with the above choice of a?

3. **Taking lots of derivatives**. Suppose we create an electromagnetic field given by

$$\boldsymbol{E} = (\beta \sin y \,\hat{\boldsymbol{x}} + \beta \cos x \,\hat{\boldsymbol{y}} + \alpha \sin \,\hat{\boldsymbol{z}})e^{-t} \,,$$
$$\boldsymbol{B} = -\beta(\cos y + \sin x)e^{-t} \,\hat{\boldsymbol{z}} \,.$$

   (a) Show that $\nabla \cdot \boldsymbol{B} = 0$ is satisfied.

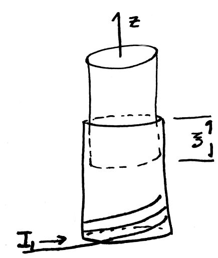
   (b) Show that $\nabla \times \boldsymbol{E} = -\frac{\partial B}{\partial t}$ is satisfied.

   (c) What charge density ρ and current density $\boldsymbol{J}$ would produce this electromagnetic field?

   (d) Show that $\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$ is satisfied.

   (e) Compute the Poynting vector $\boldsymbol{S}$ of this field configuration.

4. **Partially overlapping solenoids**. Suppose we have two almost-identical solenoids: they have length ℓ, with N total turns, giving $n \equiv N/\ell$ turns per unit length, and the wire in each solenoid has a total resistance R. One of them has radius $r + \epsilon$ and the other has radius r, so they can just barely nest within each other (but for calculation purposes we can consider both of them to have radius r).
We place both of these solenoids along the z axis, with a length ξ of overlap (see figure). However this overlap is not fixed – the two solenoid can move in response to forces, and thus $\xi(t)$ may depend on time.



   (a) Compute the mutual inductance $M(\xi)$ of the two solenoids, which will depend on ξ.

   (b) Suppose a constant current I_1 is run through the first solenoid (counter-clockwise as viewed from above). What is the mutually-induced current I_2 in the second solenoid? Is it clockwise or counter-clockwise as viewed from above? (You may ignore self-inductance).

   (c) What is the magnetic field $\boldsymbol{B}$ inside the three regions (only solenoid 1, overlap region, and only solenoid 2)?

   (d) Compute the energy stored in the magnetic field in all three regions added together.

   (e) Argue whether the amount of overlap ξ will increase or decrease.

5. **Energy/momentum transfer**. Suppose a plane electromagnetic wave is traveling through vacuum (constants ϵ_0 and μ_0) in the $\hat{z}$ direction, with frequency ω, with the electric field linearly polarized in the $\hat{x}$ direction with amplitude E_{0I}.

   (a) This wave is incident on a perfectly absorbing sheet lying in the $x - y$ plane. How much energy does the sheet absorb per unit time, per unit area? How much momentum?

   (b) Now suppose we replace the perfect absorber with a perfectly reflecting mirror. How much energy does the mirror absorb per unit time, per unit area? How much momentum?

   Next suppose we replace the perfect reflector with a partially-transmitting sheet of linear medium with electric permittivity ϵ_2 and magnetic permeability μ_0. This sheet has some finite thickness but let us focus only on the first interface, between vacuum and the material, and ignore everything that happens downstream.

   (c) For normal incidence, what are the reflected and transmitted electric and magnetic fields in terms of the incident field?

   (d) What is the momentum density $\wp$ in the transmitted field? (Hint: how do the permittivity and permeability enter into $\wp$?)

   (e) What is the momentum density, separately, in (i) the incident field, and (ii) the reflected field?

   (f) What is the momentum density in the *sum* of the incident and reflected fields?

   (g) Which of these results do you think is the correct way to compute the momentum transferred to the partially-transmitting sheet? Justify your claim.