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Random Walks in the Quarter-Plane: Advances in Explicit Criterions for the Finiteness of the Associated Group in the Genus 1 Case

Guy Fayolle[*]   *Roudolf Iasnogorodski*[†]

Abstract

In the book [3], original methods were proposed to determine the invariant measure of random walks in the quarter plane with small jumps, the general solution being obtained via reduction to boundary value problems. Among other things, an important quantity, the so-called *group of the walk*, allows to deduce theoretical features about the nature of the solutions. In particular, when the *order* of the group is finite, necessary and sufficient conditions have been given in [3] for the solution to be rational or algebraic. In this paper, when the underlying algebraic curve is of genus 1, we propose a concrete criterion ensuring the finiteness of the group. It turns out that this criterion is always tantamount to the cancellation of a single constant, which can be expressed as the determinant of a matrix of order 3 or 4, and depends in a polynomial way on the coefficients of the walk.

1   Introduction

In a probabilistic framework, we consider a piecewise homogeneous random walk with sample paths in $\mathbb{Z}_+^2$, the lattice in the positive quarter plane. In the strict interior of $\mathbb{Z}_+^2$, the size of the jumps is 1, and $\{p_{ij}, |i|, |j| \leqslant 1\}$ will denote the generator of the process for this region. Thus a transition $(m, n) \to (m + i, n + j), m, n > 0$, can take place with probability p_{ij}, and

$$\sum_{|i|,|j| \leqslant 1} p_{ij} = 1.$$

On the other hand, no strong assumption is made about the boundedness of the upward jumps on the axes, neither at $(0, 0)$. In addition, the downward jumps on the x [resp. y] axis are bounded by L [resp. M], where L and M are arbitrary finite integers.

Needless to recall that a huge amount of work has been devoted to the analysis of this process since the early 1970s, both from analytic and probabilistic points of view. Today, it can be asserted that the main issues at stake (e.g. ergodicity conditions, computation of the invariant measure, etc) have been settled.

In the book [3], an important quantity was studied, the so-called *group of the walk*, originally introduced in [8, 9]. When the *order* of this group is finite, the nature of the solutions can be fully characterized. In particular, necessary and sufficient conditions have been obtained in [3] for these solutions to be rational or algebraic. In brief, as explained hereafter in Sections 1.1 and 1.2, this

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group exchanges the two roots of the algebraic curve defined by $Q(x, y) = 0$ (see equation (1.1)), the *genus* of which is either 0 or 1.

In a combinatorial context (enumeration of lattice walks), it was possible, by following the approach of [3], to analyze the nature of the bivariate counting generating functions and to discriminate between algebraicity and holonomy.

On the other hand, when the genus of the random walk is equal to 0, an effective criterion giving the order of the group has been provided in [5]: then the group is infinite, except precisely when the drift vector of the walk in the interior of $\mathbb{Z}_+^2$ is equal to 0, where finiteness is quite possible.

The genus 1 case is more difficult and solved in this paper, in the sense that we propose explicit necessary and sufficient criteria for the finiteness of the group.

1.1   The basic functional equations (see [3], chapters 2 and 5)

The invariant measure $\{\pi_{i,j}, i, j \geq 0\}$ does satisfy the fundamental functional equation

$$Q(x, y)\pi(x, y) = q(x, y)\pi(x) + \widetilde{q}(x, y)\widetilde{\pi}(y) + \pi_0(x, y), \tag{1.1}$$

with

$$
\begin{cases}
\pi(x, y) = \displaystyle\sum_{i,j \geqslant 1} \pi_{ij} x^{i-1} y^{j-1}, \\[2mm]
\pi(x) = \displaystyle\sum_{i \geqslant L} \pi_{i0} x^{i-L}, \quad \widetilde{\pi}(y) = \sum_{j \geqslant M} \pi_{0j} y^{j-M}, \\[2mm]
Q(x, y) = xy\left[1 - \displaystyle\sum_{i,j \in \mathcal{S}} p_{ij} x^i y^j\right], \quad \sum_{i,j \in \mathcal{S}} p_{ij} = 1, \\[2mm]
q(x, y) = x^L\left[\displaystyle\sum_{i \geqslant -L, j \geqslant 0} p'_{ij} x^i y^j - 1\right] \equiv x^L(P_{L0}(x, y) - 1), \\[2mm]
\widetilde{q}(x, y) = y^M\left[\displaystyle\sum_{i \geqslant 0, j \geqslant -M} p''_{ij} x^i y^j - 1\right] \equiv y^M(P_{0M}(x, y) - 1), \\[2mm]
\pi_0(x, y) = \displaystyle\sum_{i=1}^{L-1} \pi_{i0} x^i \left[P_{i0}(x, y) - 1\right] + \sum_{j=1}^{M-1} \pi_{0j} y^j \left[P_{0j}(x, y) - 1\right] + \pi_{00}(P_{00}(xy) - 1).
\end{cases}
$$

In equation (1.1), $\mathcal{S}$ is the set of allowed jumps, the unknown functions $\pi(x, y), \pi(x), \widetilde{\pi}(y)$ are sought to be analytic in the region $\{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$, and continuous on their respective boundaries. In addition, $q, \widetilde{q}, q_0, P_{i0}, P_{0j}$, are given probability generating functions supposed to have suitable analytic continuations (as a rule, they are polynomials when the jumps are bounded).

The function $Q(x, y)$, often referred to as the *kernel* of (1.1), can be rewritten in the two following equivalent forms

$$Q(x, y) = a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y), \tag{1.2}$$

where

$$
\begin{aligned}
a(x) &= p_{1,1}x^2 + p_{0,1}x + p_{-1,1} & \widetilde{a}(y) &= p_{1,1}y^2 + p_{1,0}y + p_{1,-1}, \\
b(x) &= p_{1,0}x^2 + (p_{0,0} - 1)x + p_{-1,0} & \widetilde{b}(y) &= p_{0,1}y^2 + (p_{0,0} - 1)y + p_{0,-1}, \\
c(x) &= p_{1,-1}x^2 + p_{0,-1}x + p_{-1,-1} & \widetilde{c}(y) &= p_{-1,1}y^2 + p_{-1,0}y + p_{-1,-1}.
\end{aligned}
$$

We shall also need the discriminants

$$D(x) \stackrel{\text{def}}{=} b^2(x) - 4a(x)c(x), \qquad \widetilde{D}(y) \stackrel{\text{def}}{=} \widetilde{b}^2(y) - 4\widetilde{a}(y)\widetilde{c}(y). \tag{1.3}$$

The polynomials D and $\widetilde{D}$ are of degree 4, respectively in x and y, with dominant coefficients

$$d_4 = p_{1,0}^2 - 4p_{1,1}p_{1,-1}, \qquad \widetilde{d}_4 = p_{0,1}^2 - 4p_{1,1}p_{-1,1}. \tag{1.4}$$

More information is given in Appendix A

1.2   The group and the genus

Let $\mathbb{C}(x), \mathbb{C}(y)$ and $\mathbb{C}(x,y)$ denote the respective fields of rational functions of x, y and (x,y) over $\mathbb{C}$. Since in general Q is assumed to be irreducible, the quotient field $\mathbb{C}(x,y)$ with respect to Q is also a field and will be denoted by $\mathbb{C}_Q(x,y)$.

Definition 1.1. The *group of the random walk* is the Galois group $\mathcal{H} = \langle \xi, \eta \rangle$ of automorphisms of $\mathbb{C}_Q(x,y)$ generated by ξ and η given by

$$\xi(x,y) = \left(x, \frac{c(x)}{y\,a(x)} \right), \qquad \eta(x,y) = \left(\frac{\widetilde{c}(y)}{x\,\widetilde{a}(y)}, y \right).$$

Here ξ and η are involutions satisfying $\xi^2 = \eta^2 = I$.

Let

$$\delta = \xi\eta,$$

the product, which is non-commutative except for $\delta^2 = I$, in which case the group is of order 4 (see [3] and the sequel of this paper). Then $\mathcal{H}$ has a normal cyclic subgroup $\mathcal{H}_0 = \{\delta^i, i \in \mathbb{Z}\}$, which is finite or infinite, and $\mathcal{H}/\mathcal{H}_0$ is a group of order 2.

Then the group $\mathcal{H}$ is finite of order $2n$ if, and only if,

$$\delta^n = I. \tag{1.5}$$

We shall write $f_\alpha = \alpha(f)$ for any automorphism $\alpha \in \mathcal{H}$ and any function $f \in \mathbb{C}_Q(x,y)$. As briefly depicted in Section A of the Appendix, the fundamental equation (1.1), together with ξ, η, δ, can be *lifted onto the universal covering* $\mathbb{C}$ (the finite complex plane).

In [3], the group $\mathcal{H}$ of the random walk was shown to be of even order $2n, n = 2, \ldots, \infty$, and we were able to characterize completely the solutions of the basic functional equation. Moreover, when n is supposed to be finite and the functions $q, \widetilde{q}, q_0$ are polynomials, we also gave necessary and sufficient conditions for these solutions to be rational or algebraic.

2   Topics on the conditions for $\mathcal{H}$ to be finite

Finding an exact explicit form for n to be finite is a deep question, which has many connections with some classical problems in algebraic geometry. As quoted in equation (A.2) of AppendixA, a necessary and sufficient condition was provided in [3], which, albeit theoretically nice (!), is not really easy to check by calculus. The main goal of the present paper is precisely to transform this condition into closed-form expressions, algebraically tractable. As we shall see, there are some structural differences due to the parity of the number n defined in Section 1.2, remembering that the order of the group is $2n$. Therefore, as a kind of introductory purpose, we will consider first the groups of order $4, 6, 8$.

Recalling that $\mathcal{H}$ is generated by the elements ξ and η, we can define the homomorphism

$$h(R(x,y)) \overset{\text{def}}{=} R(h(x), h(y)), \quad \forall h \in \mathcal{H}, \ \forall R \in \mathbb{C}_Q(x,y).$$

Clearly, 2 elements h_1, h_2 of $\mathcal{H}$ are identical if, and only if,

$$h_1(x) = h_2(x), \quad h_1(y) = h_2(y).$$

In addition, for any $R \in \mathbb{C}_Q(x,y)$, the following important equivalences hold:

$$\begin{cases} \xi(R) = R \iff R \in \mathbb{C}(x), \\ \eta(R) = R \iff R \in \mathbb{C}(y), \end{cases} \tag{2.1}$$

so that $\mathbb{C}(x)$ (resp. $\mathbb{C}(y)$ are the elements of $\mathbb{C}_Q(x,y)$ invariant with respect to ξ (resp. η). Indeed, let

$$R(x,y) = \frac{P_0(x,y)}{P_1(x,y)}, \tag{2.2}$$

where $P_0(x, y), P_0(x, y)$ are polynomials in x, y. Since $Q(x, y)$ is a polynomial of second degree in y, (2.2) yields at once

$$R(x, y) = \frac{A_1(x)y + A_0(x)}{B_1(x)y + B_0(x)} \qquad (\text{mod } Q(x, y)).$$

When R is supposed to be invariant with respect to ξ, it follows that

$$\frac{A_1(x)y + A_0(x)}{B_1(x)y + B_0(x)} = \frac{A_1(x)y_\xi + A_0(x)}{B_1(x)y_\xi + B_0(x)} \qquad \text{mod } Q(x, y). \tag{2.3}$$

As $y \neq y_\xi$, we obtain from (2.3)

$$A_1(x)B_0(x) = A_0(x)B_1(x),$$

so that

$$R(x, y) = \begin{cases} \dfrac{A_0(x)}{B_0(x)}, & \text{if } B_0 \neq 0, \\[2mm] \dfrac{A_1(x)}{B_1(x)} & \text{otherwise,} \end{cases} \tag{2.4}$$

which proves (2.1). ∎

Definition 2.1. Introduce the matrix

$$\mathbb{P} = \begin{pmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{pmatrix}, \tag{2.5}$$

and let $\vec{C}_1, \vec{C}_2, \vec{C}_3$ (resp. $\vec{D}_1, \vec{D}_2, \vec{D}_3$) denote the column vectors of $\mathbb{P}$ (resp. of $\mathbb{P}^T$, the tranpose matrix of $\mathbb{P}$). ∎

The simple following property will be very useful.

Proposition 2.2. *Assume there exists a positive integer s, such that*

$$\delta^s(x) = x. \tag{2.6}$$

Then $\delta^s = I$ and the group is of order $2s$, where s stands for the smallest integer with property (2.6).

Proof. Each of the three following permutations

$$x \Longleftrightarrow y, \quad \delta \Longleftrightarrow \delta^{-1}, \quad \mathbb{P} \Longleftrightarrow \mathbb{P}^T,$$

implies the two other ones. Hence, the quantity $\rho(x, y, k) \overset{\text{def}}{=} \delta^k(x).\delta^{-k}(y)$, for any integer $k \geq 1$, remains invariant by permuting $\mathbb{P}$ with $\mathbb{P}^T$.

Assume first $s = 2m$. Then (2.6) becomes $\delta^m(x) = \delta^{-m}(x)$, and

$$\rho(x, y, m) = \delta^{-m}(x).\delta^{-m}(y) = \delta^m(x).\delta^m(y),$$

where the second equality is obtained by replacing $\mathbb{P}$ by $\mathbb{P}^T$. Then, comparing with the definition of $\rho(x, y, m)$, we get $\delta^m(y) = \delta^{-m}(y)$, which yields in turn $\delta^s(y) = y$, whence $\delta^s = I$.

The argument works exactly the same way if s is odd, say $s = 2m + 1$. Indeed, in this case we have

$$\rho(x, y, m) = \delta^{-(m+1)}(x).\delta^{-m}(y) = \delta^m(x).\delta^{m+1}(y),$$

(by exchanging again $\mathbb{P}$ with $\mathbb{P}^T$), which implies $\delta^{m+1}(y) = \delta^{-m}(y)$, that is $\delta^s(y) = y$, concluding the proof of the proposition. ∎

Corollary 2.3.

(i) *If there exists s such that $\delta^s(x) = r(x)$, where $r(x)$ represents a rational fraction of x, then $\delta^{2s}(x) = x$ and the group is of order $4s$.*

(ii) If there exists s such that $\delta^s(x) = t(y)$, where $t(y)$ represents a rational fraction of y, then $\delta^{2s+1}(x) = x$ and the group is of order $4s + 2$.

In both cases, s stands for the smallest integer with the corresponding property.

Proof. Remark first the identities $\xi\delta^s\xi = \delta^{-s}$ and $\eta\delta^s\xi = \delta^{-s-1}$.
So, we have the following chain of equalities.

$$\delta^s(x) = r(x) \implies \xi\delta^s\xi(x) = \delta^s(x) \iff \delta^{-s}(x) = \delta^s(x) \iff \delta^{2s}(x) = x.$$

Similarly

$$\delta^s(x) = t(y) \implies \eta\delta^s\xi(x) = \delta^s(x) \iff \delta^{-s-1}(x) = \delta^s(x) \iff \delta^{2s+1}(x) = x.$$

In both cases, the conclusion follows from Proposition 2.2, in which s is replaced respectively by $2s$ and $2s + 1$. ∎

Lemma 2.4. *On the algebraic curve $\{Q(x, y) = 0\}$, the following general relations hold.*

$$\begin{cases} \eta(x) &= \dfrac{xv(y) - u(y)}{xw(y) - v(y)}, \\[2mm] \xi(y) &= \dfrac{y\widetilde{v}(x) - \widetilde{u}(x)}{y\widetilde{w}(x) - \widetilde{v}(x)}, \end{cases} \tag{2.7}$$

where u, v, w, h (resp. $\widetilde{v}, \widetilde{v}, \widetilde{w}, \widetilde{h}$) are polynomials of degree ≤ 2.
In particular, there exist affine solutions

$$(u(y), v(y), w(y))^T = \vec{A}y + \vec{B}, \ (\widetilde{u}(x), \widetilde{v}(x), \widetilde{w}(x))^T = \vec{E}x + \vec{F}, \tag{2.8}$$

with column vectors

$$\vec{A} = (u_0, v_0, w_0)^T, \ \vec{B} = (u_1, v_1, w_1)^T, \ \vec{E} = (\widetilde{u}_0, \widetilde{v}_0, \widetilde{w}_0)^T, \ \vec{F} = (\widetilde{u}_1, \widetilde{v}_1, \widetilde{w}_1)^T,$$

given by

$$\begin{cases} \vec{A} &= (\alpha\vec{C}_2 + \beta\vec{C}_1) \times \vec{C}_3, \\ \vec{B} &= \vec{C}_1 \times (\alpha\vec{C}_3 + \beta\vec{C}_2), \\ \vec{E} &= (\widetilde{\alpha}\vec{D}_2 + \widetilde{\beta}\vec{D}_1) \times \vec{D}_3, \\ \vec{F} &= \vec{D}_1 \times (\widetilde{\alpha}\vec{D}_3 + \widetilde{\beta}\vec{D}_2), \end{cases} \tag{2.9}$$

where $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}$ are arbitrary complex constants, and the operator "$\times$" stands for the cross vector product. In addition, when $\mathbb{P}$ is of rank 3, the vectors $\vec{A}, \vec{B}, \vec{E}, \vec{F}$ do not vanish. Choosing in (2.9) $\alpha = \widetilde{\alpha} = 0, \beta = \widetilde{\beta} = 1$, gives

$$\begin{cases} u(y) = y\Delta_{13} - \Delta_{12}, & \widetilde{u}(x) = x\Delta_{31} - \Delta_{21}, \\ v(y) = y\Delta_{23} - \Delta_{22}, & \widetilde{v}(x) = x\Delta_{32} - \Delta_{22}, \\ w(y) = y\Delta_{33} - \Delta_{32}, & \widetilde{w}(x) = x\Delta_{33} - \Delta_{23}, \end{cases} \tag{2.10}$$

where Δ_{ij} denotes the cofactor of the $(i, j)^{th}$ entry of the matrix $\mathbb{P}$ given in (2.5).

Proof. We proceed by construction, assuming $\eta(x)$ is given by the following expression

$$\eta(x) = \frac{xv(y) - u(y)}{xw(y) - h(y)}, \tag{2.11}$$

where u, v, w, h are affine functions of y. Then, by using the basic relation

$$\eta(x) = \frac{\widetilde{c}(y)}{x\widetilde{a}(y)}$$

in (2.11), we obtain

$$\widetilde{a}(y)v(y)x^2 - [\widetilde{a}(y)u(y) + \widetilde{c}(y)w(y)]x + \widetilde{c}(y)h(y) = 0 \pmod{Q(x, y)},$$

which must be proportional to (1.2) written in the form

$$\widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y) = 0.$$

This implies the two identities, $\forall y \in \mathbb{C}$,

$$\begin{cases} \widetilde{a}(y)u(y) + \widetilde{b}(y)v(y) + \widetilde{c}(y)w(y) = 0, \\ h(y) = v(y). \end{cases} \tag{2.12}$$

Hereafter, we assume the matrix $\mathbb{P}$ is of rank 3. Indeed, it will be shown in the next section that *this is always the case, except when the group is of order* 4.

So, letting $\mathcal{V}$ denote the vector space of polynomials of degree ≤ 2, the polynoms $\widetilde{a}, \widetilde{b}, \widetilde{c}$ form a base of $\mathcal{V}$, and we can write

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathcal{M} \begin{pmatrix} \widetilde{a} \\ \widetilde{b} \\ \widetilde{c} \end{pmatrix},$$

where $\mathcal{M}$ is an unspecified constant matrix. Then the first equation of (2.12) says that one must look for elements $(u, v, w) \in \mathcal{V}^3$, such that the two vectors (u, v, w) and $(\widetilde{a}, \widetilde{b}, \widetilde{c})$ are *orthogonal*. The bilinear mapping $(u, v, w).(\widetilde{a}, \widetilde{b}, \widetilde{c})^T$ gives rise to an associated quadratic form (in the variables $\widetilde{a}, \widetilde{b}, \widetilde{c}$)

$$B(\widetilde{a}, \widetilde{b}, \widetilde{c}) = (\widetilde{a}, \widetilde{b}, \widetilde{c}) \, \mathcal{M}^* (\widetilde{a}, \widetilde{b}, \widetilde{c})^T,$$

where $\mathcal{M}^*$ is a symmetric matrix. Consequently, since $\widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y)$ build a base of $\mathcal{V}^3$, $B(.)$ will be identically zero, $\forall y \in \mathbb{C}$, if, and only if, $\mathcal{M}^* = 0$, which implies that $\mathcal{M}$ is a skew-symmetric matrix

$$\mathcal{M} = \begin{pmatrix} 0 & -\varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{pmatrix}.$$

Next, we can characterize the 3 basic families $u(y), v(y), w(y)$, which among other things generate (2.8) by linear combination.

1. u, v, w of degree 1. It suffices to choose

$$\varepsilon_3 = p_{-1,1}, \ \varepsilon_2 = p_{0,1}, \ \varepsilon_1 = p_{1,1}.$$

2. u, v, w without constant terms, so that $u/y, v/y, w/y$ are admissible and still of degree 1. It suffices to choose

$$\varepsilon_3 = p_{-1,-1}, \ \varepsilon_2 = p_{0,-1}, \ \varepsilon_1 = p_{1,-1}.$$

3. u, v, w of degree 2. It suffices to choose

$$\varepsilon_3 = p_{-1,0}, \ \varepsilon_2 = p_{0,0} - 1, \ \varepsilon_1 = p_{1,0}.$$

The proof of the lemma is terminated.   ∎

Lemma 2.5. *Let γ an endomorphism defined on the algebraic surface $\{Q(x,y) = 0\}$, which is supposed to be invariant on the field $\mathbb{C}(x)$ of rational functions of x, and such that*

$$\gamma(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)}, \tag{2.13}$$

where e, f, g, h are polynomials of degree 1 in x. (Note that this is always possible, as shown in Lemma 2.4). Then, for γ to be an involution, the condition $f(x) + h(x) \equiv 0$ is necessary and sufficient.

Proof. Applying γ to both terms of (2.13) yields the equality

$$\gamma^2(y) = \frac{y[f^2(x) - e(x)g(x)] - e(x)[f(x) + h(x)]}{yg(x)[f(x) + h(x)] - e(x)g(x) + h^2(x)}.$$

If $f(x) + h(x) = 0$, then we have immediately $\gamma^2(y) = y$, and consequently $\gamma^2 = I$, showing at once that $f(x) + h(x) = 0$ is a sufficient condition for γ to be an involution.

On the other hand, we have

$$g(x)(f(x) + h(x))y^2 + (h^2(x) - f^2(x))y + e(x)(f(x) + h(x)) = 0.$$

Comparing the last equation with (1.2), we obtain (omitting the variable x)

$$\frac{g(f + h)}{a} = \frac{h^2 - f^2}{b} = \frac{e(f + h)}{c} \quad \mod Q(x, y). \tag{2.14}$$

Assume for a while $f + h \not\equiv 0$. Then, by (2.14),

$$\begin{cases} bg = a(h - f), \\ cg = ae. \end{cases} \tag{2.15}$$

The second degree polynomials $a(x)$ and $b(x)$ are relatively prime. Indeed, the roots of $a(x)$ are either both negative, or complex conjugate. But $b(x)$ does not admit roots with a negative real part, since $p_{00} - 1 < 0$. So, $a(x)$ should divide $g(x)$, which is impossible.

So, in equation (2.13), we must have $f(x) + h(x) = 0, \forall x \in \mathbb{C}$, except when a, b, c are all of degree 1, which corresponds to the *singular random walk* $p_{11} = p_{10} = p_{1,-1} = 0$ introduced in [3]. But we note in this latter case, that $Q(x, y)$ is of degree 1 in x, and the genus of the algebraic curve is zero. The lemma is proved. ∎

Remark 2.6. The result of Lemma 2.5 does not hold if polynomials e, f, h are not of degree 1. For instance, one can check directly from (2.15) that, if g or e are taken to be of degree 2, then any involution γ has necessarily the form

$$\gamma(y) = \frac{(h - b)y - c}{ay + h},$$

2.1   Criterion for groups of Order 4

For the sake of completeness, we give the following proposition togethether with its proof, which were obtained in [3].

Proposition 2.7. *The group $\mathcal{H}$ is of order 4 if, and only if,*

$$\begin{vmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{vmatrix} = 0, \tag{2.16}$$

and this is the only case where the matrix $\mathbb{P}$ has rank 2.

Proof. Recalling that $\delta \overset{\text{def}}{=} \xi\eta$, the equality $\delta^2 = I$ can be rewritten as

$$\xi\eta = \eta\xi,$$

which by Proposition 2.2 is for instance equivalent to

$$\xi\eta(x) = \eta(x),$$

where we have used $\xi(x) = x$. So, $\eta(x)$ is left invariant by ξ, which implies

$$\eta(x) \in \mathbb{C}(x).$$

Finally, η is an involution ($\eta^2 = I$) and a conformal automorphism on $\mathbb{C}(x)$. Consequently, η is indeed a fractional linear transform of the type

$$\eta(x) = \frac{rx + s}{tx - r},$$

where all coefficients belong to $\mathbb{C}$. The following chain of equivalences hold.

$$\eta(x) = \frac{rx+s}{tx-r} \quad \Leftrightarrow \quad tx\eta(x) = r(x+\eta(x)) + s$$

$$\Leftrightarrow \quad 1,\ x+\eta(x),\ x\eta(x) \text{ are linearly dependent on } \mathbb{C}$$

$$\Leftrightarrow \quad 1,\ -\frac{\widetilde{b}(y)}{\widetilde{a}(y)},\ \frac{\widetilde{c}(y)}{\widetilde{a}(y)} \text{ are linearly dependent on } \mathbb{C}$$

$$\Leftrightarrow \quad \widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y) \text{ are also linearly dependent on } \mathbb{C},$$

where equation (1.2) has been used in the form

$$Q(x,y) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y).$$

It is worth remarking that, starting from $\xi(y)$, the same argument would involve the transpose matrix $\mathbb{P}^T$, leading thus (as expected!) to the same criterion (2.16). The proof of the lemma is concluded.   ∎

2.2 Criterion for groups of Order 6

Proposition 2.8.   *$\mathcal{H}$ is of order 6 if, and only if,*

$$\begin{vmatrix} \Delta_{11} & \Delta_{21} & \Delta_{12} & \Delta_{22} \\ \Delta_{12} & \Delta_{22} & \Delta_{13} & \Delta_{23} \\ \Delta_{21} & \Delta_{31} & \Delta_{22} & \Delta_{32} \\ \Delta_{22} & \Delta_{32} & \Delta_{23} & \Delta_{33} \end{vmatrix} = 0. \tag{2.17}$$

where the Δ_{ij}'s have be given in Lemma 2.4.

This proposition, up to some typos, was stated without explicit proof in [3, equation (4.1.6)]
Proof.   In this case $(\xi\eta)^3 = I$, which is equivalent to

$$\eta\xi\eta = \xi\eta\xi. \tag{2.18}$$

Applying (2.18) for instance to x, we get

$$\xi\eta(x) = \eta\xi\eta(x),$$

which shows that $\xi\eta(x)$ is invariant with respect to η and consequently is a rational function of y, remembering one is dealing with the field of rational functions. Similarly, $\eta\xi(y)$, invariant with respect to ξ, is a rational function of x. Hence (2.18) is plainly equivalent to

$$\begin{cases} \xi\eta(x) = P(y), \\ \eta\xi(y) = R(x), \end{cases}$$

where P and R are rational. Then

$$y = R(\xi\eta(x)) = R \circ P(y),$$

or, equivalently,

$$R \circ P = I, \tag{2.19}$$

so that P and R are fractional linear transforms. Thus (2.19) yields the relation

$$\xi(y) = \frac{p\eta(x) + q}{r\eta(x) + s}, \tag{2.20}$$

which imposes a linear dependence on $\mathbb{C}$ between the four elements $1, \xi(y), \eta(x), \xi(y)\eta(x)$, with 4 unknown constants (in fact 3 by homogeneity). Our goal is to avoid the pitfall of entering tedious (and harmful!) computations.

Starting from equation (2.7), we choose $\eta(x)$ by means of (2.10), that is

$$\eta(x) = \frac{y(x\Delta_{23} - \Delta_{13}) - x\Delta_{22} + \Delta_{12}}{y(x\Delta_{33} - \Delta_{23}) - x\Delta_{32} + \Delta_{22}}. \tag{2.21}$$

Instantiating now (2.21) in (2.20), we obtain

$$\xi(y) = \frac{y[p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23})] + p(\Delta_{12} - x\Delta_{22}) + q(\Delta_{22} - x\Delta_{32})}{y[r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23})] + r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32})} \tag{2.22}$$

Then, rewriting (2.22) in the form proposed in (2.13), namely

$$\xi(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)},$$

where

$$\begin{cases} e(x) &= p(x\Delta_{22} - \Delta_{12}) + q(x\Delta_{32} - \Delta_{22}), \\ f(x) &= p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23}), \\ g(x) &= r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23}), \\ h(x) &= r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32}), \end{cases} \tag{2.23}$$

we are in a position to compare system (2.23) with the solution presented in equation (2.9) of Lemma 2.4.

Indeed, letting $\vec{\varphi}(x) = (e(x), f(x), g(x))^T \overset{\text{def}}{=} \vec{\varphi}_0 + x\vec{\varphi}_1$, we get from (2.9) the vector equation

$$\vec{\varphi}_0 + x\vec{\varphi}_1 = \vec{E} + x\vec{F},$$

which, after renaming the constants $\widetilde{\alpha}$ and $\widetilde{\beta}$ as c and $-d$ respectively, yields in turn 6 linear equations, namely

$$\begin{cases} c\Delta_{11} + d\Delta_{21} + p\Delta_{12} + q\Delta_{22} = 0, \\ c\Delta_{12} + d\Delta_{22} + p\Delta_{13} + q\Delta_{23} = 0, \\ c\Delta_{13} + d\Delta_{23} + r\Delta_{13} + s\Delta_{23} = 0, \\ c\Delta_{21} + d\Delta_{31} + p\Delta_{22} + q\Delta_{32} = 0, \\ c\Delta_{22} + d\Delta_{32} + p\Delta_{23} + q\Delta_{33} = 0, \\ c\Delta_{23} + d\Delta_{33} + r\Delta_{23} + s\Delta_{33} = 0. \end{cases} \tag{2.24}$$

In addition, we have to take into account the constraint $f(x) + h(x) = 0$, which is tantamount to

$$\begin{cases} p\Delta_{23} + q\Delta_{33} = r\Delta_{22} + s\Delta_{32}, \\ p\Delta_{13} + q\Delta_{23} = r\Delta_{12} + s\Delta_{22}. \end{cases} \tag{2.25}$$

So, the final step is to analyze the feasability of the global linear system formed by the intersection of (2.24) and (2.25). Altogether, we are left with 8 equations with respect to 6 unknown variables c, d, p, q, r, s.

Equations of system (2.24), referred to as $1, 2 \ldots 6$, can be split into two sets:

(a) The set $(1, 2, 4, 5)$, forming an homogeneous linear system of 4 equations with 4 unknowns

$$\begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{12} & \Delta_{22} \\ \Delta_{12} & \Delta_{22} & \Delta_{13} & \Delta_{23} \\ \Delta_{21} & \Delta_{31} & \Delta_{22} & \Delta_{32} \\ \Delta_{22} & \Delta_{32} & \Delta_{23} & \Delta_{33} \end{pmatrix} \begin{pmatrix} c \\ d \\ p \\ q \end{pmatrix} = 0. \tag{2.26}$$

(b) The set $(3, 6)$, which can easily be rewritten as

$$\begin{cases} (c + r)\Delta_{13} + (d + s)\Delta_{23} = 0, \\ (c + r)\Delta_{23} + (d + s)\Delta_{33} = 0. \end{cases} \tag{2.27}$$

Clearly, system (2.26) has a non trivial solution, if and only if condition (2.17) holds, which hence is necessary for the group to be of order 6. To prove its sufficiency, we have to consider also systems (2.27), (2.25).

The determinant of system (2.27) is equal to $\Delta_{13}\Delta_{33} - \Delta_{23}^2$. But the matrix $\mathbb{P}$, introduced in (2.5), has all its entries (i, j) positive, except $(2, 2) = p_{00} - 1 < 0$, so that

$$\Delta_{13}\Delta_{33} - \Delta_{23}^2 \leqslant 0.$$

Moreover, the equality $\Delta_{13}\Delta_{33} - \Delta_{23}^2 = 0$ takes place only for special values of the jump probabilities p_{ij}'s, corresponding to simple singular random walks (see [3]), which we do not consider here. Consequently (2.27) has only the trivial solutions, so that

$$c + r = d + s = 0. \tag{2.28}$$

Now, by (2.28), one replaces r and s respectively by $-c$ and $-d$ in (2.25), and we get 2 equations coinciding in fact with the equations $(2, 5)$ of system (2.24). It is worth remarking that we found the two equations of (2.25) are implicitly satisfied, but it was useful to check this fact, just for the sake of completeness! The proof of the lemma is concluded.   ■

Now, we shall attack the general situation by slitting into the two possible situations $n = 2m$ and $n = 2m + 1$.

2.3   Criterion for groups of order $4m$

In the rest of this section, we refer without further comment to the notation and formulae of Sections A and B of the Appendix. Although the next proposition might appear almost immediate from (A.2), proved in [3], we give a self-contained new proof, quite in the scope of the present study.

Proposition 2.9. *The group $\mathcal{H}$ is of order $4m$ if and only if the Weierstrass function $\wp$ with periods (ω_1, ω_2) satisfies the equation*

$$\wp(m\omega_3) = \wp(\omega_2/2). \tag{2.29}$$

Proof.   Recalling that $\delta = \xi\eta$, with $\xi(x) = x$, $\eta(y) = y$, $\xi^2 = \eta^2 = I$, we have here $\delta^{2m} = I$, that is

$$(\xi\eta)^m = (\eta\xi)^m. \tag{2.30}$$

By applying equation (2.30) at x (or even at an arbitrary element of $\mathbb{C}(x)$), and replacing $\xi(x)$ by x in the right-hand side member, we obtain

$$\xi\delta^m(x) = \delta^m(x),$$

which shows that the involution $\delta^m(x)$ is invariant with respect to ξ, and hence is an element of $\mathbb{C}(x)$. This can be summarized by

$$\delta^m(x) = F(x) = \frac{xf - e}{xg - f}, \tag{2.31}$$

where $F(x)$ is a simple *fractional linear transform*, with constants e, f, g, to be determined.

From Proposition 2.2 and Corollary 2.3, it is worth recalling that (2.31) contains exactly the same information as condition (2.30).

Thus, equation (2.31) implies the existence of a linear dependence between the functions

$$x.\delta^m(x), \ x + \delta^m(x), \ \mathbf{1}, \tag{2.32}$$

where $\mathbf{1}$ denotes an arbitrary constant function and the symbol "." dwells on the ordinary scalar product. Then, after a lifting onto the universal covering and a translation of $-m\omega_3/2$, condition (2.32) gives rise to the following lemma.

Lemma 2.10. *For the group to be of order $4m$, a necessary and sufficient condition is that the three functions*

$$x(\omega - m\omega_3/2).x(\omega + m\omega_3/2), \ x(\omega - m\omega_3/2) + x(\omega + m\omega_3/2), \ \mathbf{1}, \tag{2.33}$$

be linearly dependent, $\forall \omega \in \mathbb{C}$.   ■

Proof.   In agreement with (B.7), let

$$S = x(\omega + m\omega_3/2) + x(\omega - m\omega_3/2), \quad P = x(\omega + m\omega_3/2).x(\omega - m\omega_3/2),$$
$$X = \wp(\omega), \quad Y = \wp(m\omega_3/2).$$

By (A.1), $x(\omega)$ is homographic in X, and hence Lemma 2.10 amounts to saying that $S, P, \mathbf{1}$, considered as functions of X, are linearly dependent.

By system (B.9) together with with definition (B.10), one verifies immediately that the existence of constants e, f, g satisfying

$$eS + fP + g = 0, \quad \forall X \in \mathbb{C}, \tag{2.34}$$

is synonymous with the linear dependence

$$uA_1 + vB_1 + w(X - Y)^2 = 0, \quad \forall X \in \mathbb{C}, \tag{2.35}$$

where the constants u, v, w fulfil the linear sytem

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2p & p^2 & 1 \\ q - 2rp & p(q - rp) & -r \\ 4r(pr - q) & 2(q - rp)^2 & 2r^2 \end{pmatrix} \begin{pmatrix} e \\ f \\ g \end{pmatrix}. \tag{2.36}$$

But the determinant of system (2.36) is exactly equal to $-2q^3$ (independent of p and r, as expected) and does never vanish (see (B.8)). Consequently, equations (2.34) and (2.35) are in fact equivalent.

Ultimately, we have to extract the vector coefficients of A_1, B_1, D, which from Lemma B.1 are *polynomials of second degree* in X, so that condition (2.33) amounts to

$$\Delta(Y) \stackrel{\text{def}}{=} \begin{vmatrix} 4Y & 4Y^2 - g_2 & -(g_2 Y + 2g_3) \\ 2Y^2 & g_2 Y + 2g_3 & 2g_3 Y + g_2^2/8 \\ 1 & -2Y & Y^2 \end{vmatrix} = 0. \tag{2.37}$$

The determinant (2.37) is equal to

$$8\,Y^6 - 10\,g_2 Y^4 - 40\,g_3 Y^3 - 5/2\,g_2^2\,Y^2 - 2\,g_2\,g_3 Y - 4g_3^2 + 1/8\,g_2^3,$$

but it has the pleasant property of being the product of three explicit second degree polynomials, namely

$$\Delta(Y) = 8[Y^2 - 2e_1 Y - (e_1^2 + e_2 e_3)][Y^2 - 2e_2 Y - (e_2^2 + e_3 e_1)][Y^2 - 2e_3 Y - (e_3^2 + e_1 e_2)], \tag{2.38}$$

where e_1, e_2, e_3 are defined in (B.2) and (B.3). Let P_1, P_2, P_3 denote the 3 polynomials coming in (2.38), respectively from left to right, with their corresponding reduced discriminants

$$\delta_1 = (e_1 - e_2)(e_1 - e_3) > 0,$$
$$\delta_2 = (e_2 - e_3)(e_2 - e_1) < 0,$$
$$\delta_3 = (e_3 - e_1)(e_3 - e_2) > 0.$$

On the real interval $[0, \omega_2]$, $\wp(\omega)$ is positive and reaches its minimum at $\omega = \omega_2/2$, with $\wp(\omega_2/2) = e_1$. So, we can immediately eliminate P_2, which has two complex roots. As for P_3, which has two real roots, one of them (at least) being negative, we have

$$P_3(e_1) = e_1^2 - 2e_1 e_3 - e_3^2 - e_1 e_2 = (e_1 - e_2)(e_1 - e_3) > 0.$$

But e_1 is larger than the two roots of P_3, which therefore never cancel $\Delta(Y)$. Thus we are left with the real roots in Y of $P_1(Y)$, of which the sole one bigger than e_1 is admissible, namely

$$e_1 + \sqrt{(e_1 - e_2)(e_1 - e_3)} \equiv \wp\left(\frac{\omega_2}{4}\right).$$

Finally, we have shown that (2.37) holds if and only if

$$\wp(m\omega_3/2) = \wp(\omega_2/4).$$

To conclude the proof of Proposition 2.9, it suffices to remark that, as we work inside the fundamental parallelogram (ω_1, ω_2), the relations $\frac{m\omega_3}{2} = \pm\frac{\omega_2}{4}$ and $m\omega_3 = \frac{\omega_2}{2}$ are equivalent mod (ω_2). ∎

The calculation of $\wp(m\omega_3/2)$ could be carried out from (B.4), via the recursive relationship

$$\wp((l+1)\omega_3/2) + \wp((l-1)\omega_3/2) = \frac{(\wp(l\omega_3/2) + \wp(\omega_3/2))(4\wp(l\omega_3/2)\wp(\omega_3/2) - g_2) - 2g_3}{2(\wp(l\omega_3/2) - \wp(\omega_3/2))^2},$$

siince the value of $\wp(\omega_3/2)$ is directly obtained from (A.1) and [3, Section 3.3]. Yet, one has to admit that the partial fraction giving $\wp(m\omega_3/2)$ in terms of $\wp(\omega_3/2)$ is hardly directly expoitable in Proposition 2.9.

Therefore, in the next sections, we shall pursue with the analysis of the crucial condition (2.32), having in mind the obtention of concrete formulas by means of *determinants* expressed in terms of the entries of the matrix $\mathbb{P}$ defined in (2.5).

2.3.1   The case $m = 2k$

When $m = 2k$, applying the operator δ^{-k} in (2.32) is equivalent to say, after a direct manipulation, that

$$\delta^k(x).\xi\delta^k(x), \ \delta^k(x) + \xi\delta^k(x), \ \mathbf{1}, \tag{2.39}$$

are linearly dependent. But we know that $\delta^k(x).\xi\delta^k(x)$ and $\delta^k(x) + \xi\delta^k(x)$ are elements of $\mathbb{C}(x)$, and by (B.8), (B.4), (B.5), they are in fact ratios of polynomials of degree 2 in x with the same denominator.

In addition, letting $\zeta_j(x) \overset{\text{def}}{=} \delta^j(x) + \xi\delta^j(x)$, the following recursive scheme holds.

$$\begin{cases} \zeta_0(x) &= 2x, \ \zeta_1(x) = \eta(x) + \delta(x), \\ \zeta_j(x) &= \zeta_{j-1}(\zeta_1(x)) - \zeta_{j-2}(x), \ \forall j \geqslant 2. \end{cases} \tag{2.40}$$

2.3.2   The case $m = 2k + 1$

Upon applying here the operator $\delta^{-(k+1)}$ in (2.32) and using the identity $\delta^{-(k+1)}(x) = \eta\delta^k(x)$, we obtain that

$$\delta^k(x).\eta\delta^k(x), \ \delta^k(x) + \eta\delta^k(x), \ \mathbf{1} \tag{2.41}$$

are linearly dependent. One checks immediately that $\delta^k(x).\eta\delta^k(x)$ and $\delta^k(x) + \eta\delta^k(x)$ are invariant with respect to η, and thus are elements of $\mathbb{C}(y)$. Then, exchanging the role of the variables x and y, a uniformization similar to (A.1) would show that $\delta^k(x).\eta\delta^k(x)$ and $\delta^k(x) + \eta\delta^k(x)$ are ratios of polynomials of degree 2 in y with the same denominator.

2.3.3   Explicit criterion for the groupe of order 8

This corresponds to $m = 2$ in the preceding section. In this case explicit conditions can be carried out, both manually and also with any good computer algebra system. Hereafter, we only list the main results.

In agreement with Section 2.3.1, and using again $\xi^2 = I$, the functions

$$\eta(x).\delta(x), \ \eta(x) + \delta(x), \ \mathbf{1},$$

are in $\mathbb{C}(x)$ and are sought to be linearly dependent on $\mathbb{C}$. Setting

$$\eta(x) + \delta(x) = \frac{P_1(x)}{Q_1(x)}, \quad \eta(x).\delta(x) = \frac{R_1(x)}{Q_1(x)},$$

the polynomials P_1, Q_1, R_1 satisfy the relations

$$\begin{cases} P_1 = -2\,S_2\,T_2 + S_3\,T_1 + S_1\,T_3, \\ Q_1 = S_2^2 - S_3\,S_1, \\ R_1 = T_2^2 - T_3\,T_1, \end{cases} \tag{2.42}$$

with

$$S_1(x) = p_{10}\, c(x) - p_{1,-1}\, b(x) \qquad = \Delta_{31}\, x - \Delta_{21},$$

$$T_1(x) = \frac{p_{-1,-1}\, b(x) - p_{-1,0}\, c(x)}{x} = -\Delta_{21}\, x + \Delta_{11},$$

$$S_2(x) = p_{11}\, c(x) - p_{1,-1}\, a(x) \qquad = -\Delta_{32}\, x + \Delta_{22},$$

$$T_2(x) = \frac{p_{11}\, a(x) - p_{-1,1}\, c(x)}{x} = \Delta_{22}\, x - \Delta_{12},$$

$$S_3(x) = p_{11}\, b(x) - p_{10}\, a(x) \qquad = \Delta_{33}\, x - \Delta_{23},$$

$$T_3(x) = \frac{p_{-1,0}\, a(x) - p_{-1,1}\, b(x)}{x} = -\Delta_{23}\, x + \Delta_{13},$$

the Δ_{ij}'s being the cofactors of the matrix $\mathbb{P}$ introduced in Lemma 2.4. The derivation of (2.42) has been obtained by a direct algebra. Indeed, the quantities $\eta(x) + \delta(x)$ and $\eta(x).\delta(x)$ are a priori ratios of polynomials of *4th degree*, which miraculously simplify to ratios of second degree polynomials. . .

So, the condition ensuring the linear dependence between P_1, Q_1, R_1 given by (2.42) leads to the following

Proposition 2.11.   *The group $\mathcal{H}$ is of order 8 if, and only if, the third order determinant*

$$\begin{vmatrix} \begin{matrix} 2\,\Delta_{22}\Delta_{32} \\ -(\Delta_{21}\Delta_{33} + \Delta_{31}\Delta_{23}) \end{matrix} & \begin{matrix} -2\,(\Delta_{22}^2 + \Delta_{12}\Delta_{32} - \Delta_{21}\Delta_{23}) \\ + \Delta_{11}\Delta_{33} + \Delta_{31}\Delta_{13} \end{matrix} & \begin{matrix} 2\,\Delta_{12}\Delta_{22} \\ -(\Delta_{11}\Delta_{23} + \Delta_{21}\Delta_{13}) \end{matrix} \\ \Delta_{32}^2 - \Delta_{31}\Delta_{33} & -2\,\Delta_{32}\Delta_{22} + \Delta_{31}\Delta_{23} + \Delta_{21}\Delta_{33} & \Delta_{22}^2 - \Delta_{21}\Delta_{23} \\ \Delta_{22}^2 - \Delta_{21}\Delta_{23} & -2\,\Delta_{22}\Delta_{12} + \Delta_{11}\Delta_{23} + \Delta_{13}\Delta_{21} & \Delta_{12}^2 - \Delta_{11}\Delta_{13} \end{vmatrix} \qquad (2.43)$$

is equal to zero. (Due to the size of the printing output, each element of the first line of the matrix in (2.43) *has been split vertically as the sum of two terms)* ∎

Remark 2.12. It is interesting to note that the polynomials $S_i, T_i, i = 1, 2, 3$, coincide with the ones appearing in (2.10).

Remark 2.13. When $m = 2$, i.e. a group of order 8, the line of argument developed in Section 2.2 could be applied, but at the expense of intricate computations. Indeed by Lemma 2.4,

$$\eta(x) = \frac{xv(y) - u(y)}{xw(y) - v(y)}, \qquad \xi\eta(x) = \frac{xv(\xi(y)) - u(\xi(y))}{xw(\xi(y)) - v(\xi(y))},$$

whence

$$\xi\eta(x) + \eta(x) = \frac{(xv(\xi(y)) - u(\xi(y)))(xw(y) - v(y)) + (xv(y) - u(y))(xw(\xi(y)) - v(\xi(y)))}{(xw(\xi(y)) - v(\xi(y)))(xw(y) - v(y))}, \quad (2.44)$$

$$\xi\eta(x).\eta(x) = \frac{(xv(\xi(y)) - u(\xi(y)))(xv(y) - u(y))}{(xw(\xi(y)) - v(\xi(y)))(xw(y) - v(y))}. \qquad (2.45)$$

Setting for a while in (2.44) and (2.45)

$$\xi\eta(x).\eta(x) = \frac{K_1}{L}, \quad \xi\eta(x) + \eta(x) = \frac{K_2}{L},$$

we must test a possible linear dependence between the functions

$$K_1, \ K_2, \ L,$$

which are all elements of $\mathbb{C}(x)$. Sketching the global calculus, we have for instance

$$L = (xw(y) - v(y))(xw(\xi(y)) - v(\xi(y)))$$
$$= w(y)w(\xi(y))x^2 - x[v(y)w(\xi(y)) + w(y)v(\xi(y))] + v(y)v(\xi(y)).$$

A typical term coming in L is $w(y)w(\xi(y))$, which from Lemma 2.4 equals

$$w(y)w(\xi(y)) = w_0^2 + w_0 w_1(y + \xi(y)) + w_1^2 y\xi(y)$$
$$= w_0^2 - w_0 w_1 \frac{b(x)}{a(x)} + w_1^2 \frac{c(x)}{a(x)}.$$

Analogously, the other parts of the puzzle give

$$a(x)L = a(x)(xw_0 - v_0)^2 - b(x)(xw_0 - v_0)(xw_1 - v_1) + c(x)(xw_1 - v_1)^2,$$
$$a(x)K_1 = a(x)(xv_0 - u_0)^2 - b(x)(xv_0 - u_0)(xv_1 - u_1) + c(x)(xv_1 - u_1)^2,$$
$$a(x)K_2 = a(x)h_a(x) + b(x)h_b(x) + c(x)h_c(x),$$

where

$$h_a(x) = 2[v_0 w_0 x^2 - v_0^2 x + v_0 u_0],$$
$$h_b(x) = -(v_1 w_0 - v_0 w_1)x^2 + (2v_0 v_1 + u_0 w_1 + u_1 w_0)x - (u_0 v_1 + u_1 v_0),$$
$$h_c(x) = 2[v_1 w_1 x^2 - (v_1^2 + u_1 w_1)x + v_1 u_1].$$

At the very end, this machinery would require to deal with rational fractions of *4th degree*, implying tedious computations, which are partly avoided by means of the procedure proposed above. ∎

2.4   Criterion for groups of order $4m + 2$

Condition (1.5) reads

$$\delta^{2m+1} = I, \tag{2.46}$$

which by Proposition 2.2 and Corollary 2.3 is equivalent to

$$\eta(\delta^m(x)) = \delta^m(x),$$

or, by a simple algebra using $\xi(x) = x$, to

$$\delta^m(x) = G(y) \in \mathbb{C}(y). \tag{2.47}$$

Similarly, upon applying (2.46) to y, we get

$$\delta^{-m}(y) = \delta^{m+1}(y) = \xi(\delta^{-m}(y)),$$

so that

$$\delta^{-m}(y) = F(x) \in \mathbb{C}(x).$$

Then, applying δ^{-m} to both members of (2.47), we get

$$x = \delta^{-m}(G(y)) = G(\delta^{-m}(y)) = G \circ F(x),$$

which shows that $G \circ F = I$, and hence G and F are simple fractional linear transforms. Setting for instance

$$G(y) = -\frac{py + q}{ry + s},$$

where p, q, r, s are arbitrary complex constants, the problem is to achieve the linear relation

$$r\, y\delta^m(x) + s\, \delta^m(x) + py + q = 0 \quad \mod Q(x, y), \tag{2.48}$$

which is necessary and sufficient for the group to be of order $4m + 2$.

3   Some general results

We state below three theorems allowing to conclude that, for the group to be finite, there is a *unique condition* tantamout to the cancellation of a *determinant*, the elements of which are intricate functions of the coefficients of the transition matrix $\mathbb{P}$, but nonetheless recursively computable.

- The determinant is of order 3, for groups of order $4m, m \geq 1$.
- The determinant is of order 4, for groups of order $4m + 2, m \geq 1$.

3.1 A theorem about δ^s

The following important fact holds.

Theorem 3.1. *For any integer $s \geq 1$, we have*

$$\delta^s(x) = \frac{y\, U_s(x) + V_s(x)}{W_s(x)} \quad \mod Q(x,y), \tag{3.1}$$

where U_s, V_s, W_s are second degree polynomials.

Proof.   For $s = 1$, a direct (however slightly tedious) computation with the (x,y) variables can be worked out. But, already for $s = 2$, one seems to reach the limits of human computational abilities, which are definitely exceeded for $s \geq 3$! A formal verification through the *Maple 18* Computer Algebra System has been carried out for $s = 2, 3$, but seems hardly exploitable as soon as $s > 3$. Hereafter, we propose a simple proof by using again the uniformisation (A.1).

Let

$$\delta^s(x) - \xi \delta^s(x) \stackrel{\text{def}}{=} 2H, \ X = \wp(\omega), \ Y = \wp(s\omega_3).$$

Then, with the notation of Appendix B,

$$\delta^s(x) = \frac{S(\omega, s\omega_3)}{2} + H,$$

where $S(\omega, s\omega_3)$ is given by (B.9).

Now, by (B.4), (B.5), (B.8) and the addition formula for the $\wp$ function, we can write

$$H = \frac{q[\wp(s\omega_3 - \omega) - \wp(s\omega_3 + \omega)]}{2[\wp(s\omega_3 + \omega) - r][\wp(s\omega_3 - \omega) - r]}$$

$$= \frac{q\, \wp'(\omega)\wp'(s\omega_3)}{2(X - Y)^2[B(s\omega_3, \omega) - rA(s\omega_3, \omega) + r^2]} = \frac{q\, \wp'(\omega)\wp'(s\omega_3)}{D(X,Y)},$$

where $D(X,Y)$, introduced in (B.10), is a polynomial of second degree in X and Y. On the other hand, by (A.1) and (B.8), we have

$$\wp'(\omega) = \frac{2q\,[2a(x)y + b(x)]}{(x - p)^2},$$

and hence

$$H = \frac{2q^2 \wp'(s\omega_3)[2a(x)y + b(x)]}{(x - p)^2 D(X,Y)}.$$

Since x is a homographic function of X, it follows that the denominator of H is a second degree polynomial in the variable $(x - p)$, where

$$U_s(x) = 4q^2 \wp'(s\omega_3)a(x),$$
$$V_s(x) = 2q^2 \wp'(s\omega_3)b(x) + (x - p)^2 \left[2pB_1(X,Y) + (q - 2pr)A_1(X,Y) + 4r(pr - q)(X - Y)^2\right],$$
$$W_s(x) = (x - p)^2 D(X,Y).$$

Moreover, the coefficients involve the Weierstrass $\wp$ function at points of the type $k\omega_3$, k integer, which can be computed via standard recursive schemes. The proof of (3.1) is concluded.   ∎

3.2 Form of the general criterion

The main results are formulated in the next two theorems.

Theorem 3.2. *The finiteness of the group is always equivalent to the cancellation of a single constant, which depends on the entries of $\mathbb{P}$ in a polynomial way. In other words, the group is finite if and only if the non-negative (p_{ij})'s belong to the intersection of some algebraic hypersurface with the hyperplane $\sum p_{ij} = 1$.*

Proof.   By construction, we can a priori write

$$\delta^s(x) = M_s(x)y + N_s(x) \mod Q(x,y), \tag{3.2}$$

where M_s and N_s are rational fractions, whose numerators and denominators are polynomials of unknown degrees, but with coefficients given in terms of polynomials of the entries of $\mathbb{P}$. Moreover the decomposition (3.2) is unique, so that, comparing with (3.1), we have

$$\begin{cases} M_s(x) = \dfrac{4q^2 \wp'(s\omega_3)a(x)}{W_s(x)}, \\ N_s(x) = \dfrac{V_s(x)}{W_s(x)}, \end{cases}$$

where V_s, W_s are the second degree polynomials given by Theorem 3.1. Moreover, by homogeneity, we can always rewrite

$$M_s(x) = \frac{A_s a(x)}{F_s(x)},$$

where A_s, K_s are real constants with

$$A_s = K_s[4q^2 \wp'(s\omega_3)], \quad F_s(x) = K_s W_s(x).$$

In particular, by Corollary 2.3, the group is of order $4s$ if and only if $M_s \equiv 0$, or equivalently $A_s = 0$, where now A_s *depends only on the entries of* $\mathbb{P}$ in a complicated polynomial form.

When the group is of order $4s+2$, we can exchange the role of x and y in the uniformisation (A.1), by uniformizing indeed $y(\omega)$. Then, *mutatis mutandis*, this yields

$$\delta^s(x) = \frac{\widetilde{A}_s \widetilde{a}(y)x + \widetilde{V}_s(y)}{\widetilde{F}_s(y)} \mod Q(x,y),$$

where $\widetilde{F}_s, \widetilde{V}_s$ are second degree polynomials. Referring again to Corollary 2.3, we conclude that the group is of order $4s+2$ if and only if $\widetilde{A}_s = 0$, with $\widetilde{A}_s$ depending on the coefficients of $\mathbb{P}$ in a polynomial way. The proof of the theorem is terminated. ∎

Theorem 3.3. *For the group $\mathcal{H}$ to be finite, the necessary and sufficient condition is* $\det(\Omega) = 0$, *where Ω is a matrix of order* 3 *(resp. 4) when the group is of order $4m$ (resp. $4m+2$).*

Proof.   We only present a sketch of the main line of argument.

- When the group $\mathcal{H}$ is of order $4m$, Theorem 3.1 and conditions (2.39) or (2.41) show that we are indeed left with a system of 3 homogeneous linear equations with 3 unknowns.

- The situation for $\mathcal{H}$ to be of order $4m + 2$ is slightly more complicated. Indeed, according to Theorem 3.1, in order to satisfy condition (2.48), we end up with 6 linear homogeneous equations and only 4 unknowns. By Theorem 3.2, the condition for the the group to be finite is always equivalent to the cancellation of the constant A_s. This shows that among these 6 relations exactly 4 of them are independent, and so the condition reduces to the cancellation of a determinant of order 4, saying that a linear system of 4 equations with 4 unknowns must have a non trivial solution.

- In both cases, the algebraic hypersurfaces defined by $A_s = 0$ and $\det(\Omega) = 0$ are in fact identical.

The proof of the theorem is concluded. ∎

3.2.1   Miscellaneous questions

We list below some technical questions, which very likely could be solved up to some effort..!

- The degree of the hypersurface coming in Theorem 3.2 can be determined, by using the arguments appearing in the proof, together with the recursive relationships involving the Weierstrass functions.

- By continuity with respect to the parameters, we can think that it is possible to find random walks having a group of any finite prescribded order. Also, the case of genus 0 could be adressed by the same of type of argument (see also [5] in this respect).

4 Some examples

4.1 $\mathcal{H}$ of order 4

(i) The product of 2 independent random walks inside the quarter plane, so that

$$\sum_{i,j} p_{ij} x^i y^j = p(x)\widetilde{p}(y).$$

(ii) The *simple* random walk where $\sum p_{ij} x^i y^j = p(x) + \widetilde{p}(y)$. Thus, in the interior of the quarter plane, $p_{ij} \neq 0$ if, and only if, i or j is zero.

(iii) The case (a) in figure 4.1, which can be viewed as a simple queueing network with parallel arrivals and internal transfers.

4.2 $\mathcal{H}$ of order 6

Among these are the cases studied in [2], [1] and [6], which are represented in figure 4.1 (b), (c) respectively, where, as before, only jumps inside the quarter plane have been drawn.
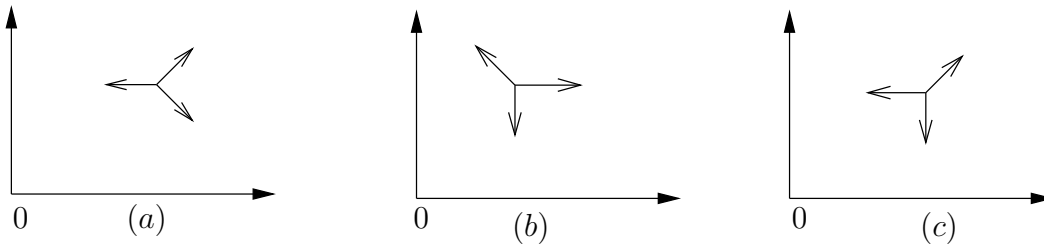

Fig. 4.1: Some groups of order 6

A About the uniformization of $Q(x, y) = 0$ (see [3], chapter 3)

Let $X(y)$ [resp. $Y(x)$] be the algebraic function defined by $Q(X(y), y) = 0$ [resp. $Q(x, Y(x)) = 0$]. This function has two *branches*, say X_0 and X_1 [resp. Y_0 and Y_1].

We shall denote by $\{x_\ell\}_{1 \leqslant \ell \leqslant 4}$ the four roots of $D(x)$, which are the branch points of the Riemann surface

$$\mathscr{K} = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}.$$

They are enumerated in such a way that $|x_1| \leqslant |x_2| \leqslant |x_3| \leqslant |x_4|$.

Moreover $x_1 \leqslant x_2$, $[x_1 x_2] \subset [-1, +1]$ and $0 \leqslant x_2 \leqslant x_3$.

When the associated Riemann surface is of genus 1, the algebraic curve $Q(x, y) = 0$ admits a *uniformization* given in terms of the Weierstrass $\wp$ function with periods ω_1, ω_2 and its derivatives. Indeed, letting

$$
\begin{aligned}
D(x) &= b^2(x) - 4a(x)c(x) \stackrel{\text{def}}{=} d_4 x^4 + d_3 x^3 + d_2 x^2 + d_1 x + d_0, \\
z &\stackrel{\text{def}}{=} 2a(x)y + b(x),
\end{aligned}
$$

Then the following formulae hold (see [3], Section 3.3).

(i) If $d_4 \neq 0$ (4 finite branch points $x_1, \ldots x_4$) then $D'(x_4) > 0$ and

$$
\begin{cases}
x(\omega) = x_4 + \dfrac{D'(x_4)}{\wp(\omega) - \frac{1}{6} D''(x_4)}, \\[3mm]
z(\omega) = \dfrac{D'(x_4)\wp'(\omega)}{2\left(\wp(\omega) - \frac{1}{6} D''(x_4)\right)^2}.
\end{cases}
\tag{A.1}
$$

(ii) If $d_4 = 0$ (3 finite branch points x_1, x_2, x_3 and $x_4 = \infty$) then

$$\begin{cases} x(\omega) = \dfrac{\wp(\omega) - \dfrac{d_2}{3}}{d_3}, \\[4mm] z(\omega) = -\dfrac{\wp'(\omega)}{2d_3}. \end{cases}$$

We also have

$$\omega_1 = 2i \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-D(x)}}, \qquad \omega_2 = 2 \int_{x_2}^{x_3} \frac{\mathrm{d}x}{\sqrt{D(x)}}, \qquad \omega_3 = 2 \int_{X(y_1)}^{x_1} \frac{\mathrm{d}x}{\sqrt{D(x)}},$$

noting that ω_1 is purely imaginary, while $0 < \omega_3 < \omega_2$. It was proved that the group $\mathcal{H}$ is finite of order $2n$ if, and only if,

$$n\omega_3 = 0 \quad \mathrm{mod}\ (\omega_1, \omega_2),$$

or, since ω_3 is real,

$$n\omega_3 = 0 \quad \mathrm{mod}\ (\omega_2), \tag{A.2}$$

where n stands for the minimal positive integer with this property. On the *universal covering* $\mathbb{C}$ (the finite complex plane), the automorphisms introduced above in Section 1.2 become (see [3], Section 3.3)

$$\xi^*(\omega) = -\omega + \omega_2, \qquad \eta^*(\omega) = -\omega + \omega_2 + \omega_3, \qquad \delta^*(\omega) = \eta^*\xi^* = \omega + \omega_3, \tag{A.3}$$

and here $\delta = \xi\eta$ corresponds to $\delta^* = \eta^*\xi^*$ (note the permutation of the letters!). For any $f(x, y) \in C_Q(x, y)$,

$$\delta(f(x, y)) = f(\delta(x), \delta(y)) = f(x(\delta^*(\omega)), y(\delta^*(\omega))), \ \omega \in \mathbb{C}.$$

In particular,

$$\begin{cases} \xi\eta(x) = x(\eta^*\xi^*(\omega)) = x(\omega + \omega_3), \\ \eta(x) = x(\eta^*(\omega)) = x(-\omega + \omega_2 + \omega_3) = x(\omega - \omega_3). \end{cases} \tag{A.4}$$

B   Some symmetric quantities of the $\wp$ function

From the form of $x(\omega)$ in (A.1) and the expressions (A.3) of the automorphisms ξ^*, η^*, δ^*, it appears we need to calculate the quantities

$$A(u, v) \overset{\mathrm{def}}{=} \wp(u + v) + \wp(u - v), \quad B(u, v) \overset{\mathrm{def}}{=} \wp(u + v)\wp(u - v)$$

in terms of rational functions of $\wp(u)$ and $\wp(v)$.

Letting $\wp'(u)$ denote the derivative of $\wp(u)$ with respect to u, it is well known (see e.g. [7]) that $\wp$ is even, $\wp'(z)$ is odd, and they satisfy the algebraic differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \tag{B.1}$$

where the quantities g_2, g_3 are constants often called *the invariants*. Equation (B.1) admits also of the classical form

$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3), \tag{B.2}$$

where

$$e_1 + e_2 + e_3 = 0, \ g_2 = -4(e_1 e_2 + e_2 e_3 + e_3 e_1), \ g_3 = 4e_1 e_2 e_3,$$

$$e_1 = \wp\left(\frac{\omega_2}{2}\right), \quad e_2 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right), \quad e_3 = \wp\left(\frac{\omega_1}{2}\right).$$

In the present situation, all the e_i's are real and satisfy

$$e_1 > 0, \quad e_3 < 0, \quad e_1 > e_2 > e_3. \tag{B.3}$$

We shall also need the *addition theorem* (see [7]), which reads, for all $u, v, u \pm v \neq 0$,

$$\wp(u+v) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2 - \wp(u) - \wp(v).$$

It is important to note that $A(u,v)$ and $B(u,v)$ are symmetric and even w.r.t. (u,v).

Let us set for a while $X \overset{\text{def}}{=} \wp(u)$, $Y \overset{\text{def}}{=} \wp(v)$.

Lemma B.1.

$$A(u,v) = \frac{(X+Y)(4XY - g_2) - 2g_3}{2(X-Y)^2}, \tag{B.4}$$

$$B(u,v) = \frac{(XY)^2 + \frac{g_2}{2}XY + g_3(X+Y) + \frac{g_2^2}{16}}{(X-Y)^2}. \tag{B.5}$$

Proof. The parity properties allow to write

$$A(u,v) = \frac{\wp'^2(u) + \wp'^2(v) - 4(\wp^2(u) - \wp^2(v))(\wp(u) - \wp(v))}{2(\wp(u) - \wp(v))^2},$$

which by (B.1) yields directly (B.4). As for $B(u,v)$, we have

$$B(u,v) = \frac{[(X+Y)(4XY - g_2) - 2g_3]^2 - 4(4X^3 - g_2 X - g_3)(4Y^3 - g_2 Y - g_3)}{16(X-Y)^4}. \tag{B.6}$$

Then it is quite reasonable to guess the last expression reduces in fact to a rational fraction of second degree. This is clear in our particular context, where we shall take for example $u = \omega$ and $v = m\omega_3/2$, but it turns out to be true for arbitrary u, v. Indeed, the numerator of (B.6) can be factorized by $(X-Y)^2$, which leads to (B.5) (this last property has been enlighted with the help of the *Maple 18* Computer Algebra System).

The proof of the lemma is concluded. ∎

Now, by Lemma B.1 and equation (A.1), we are in a position to compute

$$S(u,v) \overset{\text{def}}{=} x(u+v) + x(u-v), \quad P(u,v) \overset{\text{def}}{=} x(u+v)x(u-v). \tag{B.7}$$

For the sake of brevity, It will be convenient to write

$$x(\omega) = p + \frac{q}{\wp(\omega) - r}, \tag{B.8}$$

where p, q, r are known constants coming in equation (A.1). Then the following functional algebraic relations, giving $S(u,v)$ and $P(u,v)$ in terms of $A(u,v)$ and $B(u,v)$, are straightforward.

$$\begin{cases} S = \dfrac{2pB + (q - 2pr)A + 2r(pr - q)}{B - rA + r^2}, \\ P = \dfrac{p^2 B + p(q - pr)A + (pr - q)^2}{B - rA + r^2}. \end{cases} \tag{B.9}$$

Ad libitum, we shall specify the variables only whenever needed: for instance, instead of S, we shall write $S(X,Y)$, or $S(\omega, s\omega_3)$, etc. So, by using (B.4), (B.5), (B.9), we obtain the final expressions of S and P, which, as expected, are rational functions of *second degree* with respect to X and Y separately, and thus also with respect to $x(u)$ and $x(v)$.

Moreover, setting

$$A_1 = 2(X-Y)^2 A, \quad B_1 = 2(X-Y)^2 B, \quad D = 2(X-Y)^2(B - rA + r^2), \tag{B.10}$$

one sees immediately that S and P can be expressed as ratios of polynomials of second degree in x through the homography (B.8).

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