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A Kernel Method for Exact Tail Asymptotics — Random Walks in the Quarter Plane

(In memory of Dr. Philippe Flajolet)

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Abstract

In this paper, we propose a kernel method for exact tail asymptotics of a random walk to neighborhoods in the quarter plane. This is a two-dimensional method, which does not require a determination of the unknown generating function(s). Instead, in terms of the asymptotic analysis and a Tauberian-like theorem, we show that the information about the location of the dominant singularity or singularities and the detailed asymptotic property at a dominant singularity is sufficient for the exact tail asymptotic behaviour for the marginal distributions and also for joint probabilities along a coordinate direction. We provide all details, not only for a "typical" case, the case with a single dominant singularity for an unknown generating function, but also for all non-typical cases which have not been studied before. A total of four types of exact tail asymptotics are found for the typical case, which have been reported in the literature. We also show that on the circle of convergence, an unknown generating function could have two dominant singularities instead of one, which can lead to a new periodic phenomena. Examples are illustrated by using this kernel method. This paper can be considered as a systematic summary and extension of existing ideas, which also contains new and interesting research results.

Keywords: random walks in the quarter plane; stationary distribution; generating function; kernel methods; singularity analysis; exact tail asymptotics; light tail

1 Introduction

Two-dimensional discrete random walks in the quarter plane are classical models, that could be either probabilistic or combinatorial. Studying these models is important and often fundamental for both theoretical and applied purposes. For a stable probabilistic model, it is of significant interest to study its stationary probabilities. However, only for very limited special cases, a closed-form solution is available for the stationary probability distribution. This adds value to studying tail asymptotic properties in stationary probabilities, since performance bounds and approximations can often be developed from the tail asymptotic property. The focus of this paper is to characterize exact tail asymptotics. Specifically, we propose a kernel method to systematically study the exact tail behaviour for the stationary probability distribution of the random walk in the quarter plane.

The kernel method proposed here is an extension of the classical kernel method, first introduced by Knuth [28] and later developed as the kernel method by Banderier *et al.* [3]. The standard kernel method

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deals with the case of a functional equation of the fundamental form $K(x,y)F(x,y) = A(x,y)G(x) + B(x,y)$, where $F(x,y)$ and $G(x)$ are unknown functions. The key idea in the kernel method is to find a branch $y = y_0(x)$, such that, at $(x, y_0(x))$, the kernel function is zero, or $K(x, y_0(x)) = 0$. When analytically substituting this branch into the right hand side of the fundamental form, we then have $G(x) = -B(x, y_0(x))/A(x, y_0(x))$, and hence,

$$F(x,y) = \frac{-A(x,y)B(x,y_0(x))/A(x,y_0(x)) + B(x,y)}{K(x,y)}.$$

However, applying the above idea to the fundamental form of a two-dimensional random walk does not immediately lead to a determination of the generating function $P(x,y)$. Instead, it provides a relationship between two unknown generating functions $\pi_1(x)$ and $\pi_2(y)$, referred to as the generating functions for the boundary probabilities. This is the key challenge in the analysis of using a kernel method. Therefore, a good understanding on the interlace of these two functions is crucial.

Following the early research by Malyshev [39, 40], the algebraic method targeting on expressing the unknwon generating functions was further systematically updated in Fayolle, Iasnogorodski and Malyshev [11] based on the study of the kernel equation. The authors indicated in their book that: "Even if asymptotic problems were not mentioned in this book, they have many applications and are mostly interesting for higher dimensions." The proposed kernel method in this paper is a continuation of the study in [11]. Research on tail asymptotics for various models following the method (determination of the unknown generating function(s) first) of [11] or other closely related methods can be found in Flatto and McKean [14], Fayolle and Iasnogorodski [9], Fayolle, King and Mitrani [10], Cohen and Boxma [7], Flatto and Hahn [15], Flatto [16], Wright [55], Kurkova and Suhov [30], Bousquet-Melou [6], Morrison [50], Li and Zhao [34, 35], Guillemin and Leeuwaarden [20], and Li, Tavakoli and Zhao [32].

Different from the work mentioned above, which requires characterizing or expressing the unknown generating function, such as a closed-form solution or an integral expression through boundary value problems, the proposed kernel method only requires the information about the dominant singularities of the unknown function, including the location and detailed asymptotic property at the dominant singularities. Because of this, the method makes it possible to systematically deal with all random walks instead of a model based treatment. In a recent research, Li and Zhao [36] applied this method to a specific model, and Li, Tavakoli and Zhao [32] to the singular random walks. For exact tail asymptotics without a determination of the unknown generating function(s) or Laplace transformation function(s), different methods were used in the following studies: Abate and Whitt [1], Lieshout and Mandjes [37], Miyazawa and Rolski [48], Dai and Miyazawa [8].

Other methods for studying two-dimensional problems, including exact tail asymptotics, also exist, for example, based on large deviations, on properties of the Markov additive process (including matrix-analytic methods), or on asymptotic properties of the Green functions. References include Borovkov and Mogul'skii [5], McDonald [41], Foley and McDonald [17, 18, 19], Khanchi [24, 25], Adan, Foley and McDonald [2], Raschel [52], Miyazawa [44, 45, 46], Kobayashi and Miyazawa [26], Takahashi, Fujimoto and Makimoto [53], Haque [21], Miyazawa [43], Miyazawa and Zhao [49], Kroese, Scheinhardt and Taylor [29], Haque, Liu and Zhao [22], Li and Zhao [33], Motyer and Taylor [51], Li, Miyazawa and Zhao [31], He, Li and Zhao [23], Liu, Miyazawa and Zhao [38], Tang and Zhao [54], Kobayashi, Miyazawa and Zhao [27], among others. For more references, people may refer to a recent survey on tail asymptotics of multi-dimensional reflecting processes for queueing networks by Miyazawa [47].

The main focus of this paper is to propose a kernel method for exact tail asymptotics of random walks in the quarter plane following the ideal in [11], based on which a complete description of the exact tail asymptotics for stationary probabilities of a non-singular genus 1 random walk is obtained.

We claim that the unknown generating function $\pi_1(x)$, or equivalently, $\pi_2(y)$, has either one or two dominant singularities. For the case of either one dominant singularity, or two dominant singularities with different asymptotic properties, a total of four types of exact tail asymptotics exists: (1) exact geometric decay; (2) a geometric decay multiplied by a factor of $n^{-1/2}$; (3) a geometric decay multiplied by a factor of $n^{-3/2}$; and (4) a geometric decay multiplied by a factor of n. These results are essentially not new (for examples see references [5, 17, 19, 45, 25]) except that the fourth type is missing from previous studies for the discrete random walk, but was reported for the continuous random walk in [8]. For the case of two dominant singularities with the same asymptotic property, a new periodic phenomena in the tail asymptotic property is discovered, which has not been reported in previous literature. For the tail asymptotic behaviour of the non-boundary joint probabilities along a coordinate direction, a new method based on recursive relationships of probability generating functions will be applied, which is an extension of the idea used in [36].

For an unknown generating function of probabilities, a Tauberian-like theorem is used as a bridge to link the asymptotic property of the function at its dominant singularities to the tail asymptotic property of its coefficient, or in our case, stationary probabilities. This theorem does not require the monitonicity in the probabilities, which is required by a standard Tauberian theorem and cannot be verified in general, or Heaviside operational calculus, which is usually very difficult to be rigorous. However, the price paid for applying the Tauberian-like theorem requires more in analyticity of the function and detailed information about all dominant singularities, or singularities on the circle of convergence. Therefore we need to provide information about how many singularities exist on the circle of convergence and their detailed properties, such as the nature of the singularity and the multiplicity in the case of the pole, for the random walk. It is not always true that only one singularity exists on the circle of convergence. Technical details are needed to address these issues.

The kernel method immediately leads to exact tail asymptotics in the boundary probabilities, in both directions, based on which exact tail asymptotics in a marginal distribution will become clear. However, it does not directly lead to exact tail asymptotic properties for the joint probabilities along a coordinate direction, except for the boundary probabilities as mentioned above. Therefore, further efforts are required. In this paper, we propose a method, based on difference equations of the unknown generating functions, to do the asymptotic analysis, which successfully overcomes the hurdle for exact tail asymptotics for joint probabilities.

The rest of the paper is organized into eight sections. In section 2, after the model description, the so-called fundamental form for the random walk in the quarter plane is provided, together with a stability condition. Section 3 contains necessary properties for the two branches (or an algebraic function) defined by the kernel equation and for the branch points of the branches. These properties are either directly from [11] or further refinements. Section 4 consists of six subsections for the purpose of characterizing the asymptotic properties of the unknown generating functions $\pi_1(x)$ and $\pi_2(y)$ at their dominant singularities. Specifically, two Tauberian-like theorems are introduced in subsection 1; the interlace between the two unknown generating functions is discussed in subsection 2, which plays a key role in the proposed kernel method; detailed properties for singularities of the unknown generating functions are obtained in subsections 3–5, which finally lead to the main theorem (Theorem 4.8) in this section provided in the last subsection. In Section 5, asymptotic analysis for the boundary generating functions is carried out, which directly leads to the tail asymptotics for the boundary probabilities in terms of the Tauberian-like theorem. In Section 6, based on the asymptotic results obtained for the generating function of boundary probabilities in the previous section, and the fundamental form, exact tail asymptotic properties for the two marginal distributions are provided. Exact tail asymptotic properties for joint probabilities along a coordinate direction is addressed in Section 7, which is not a direct result from the kernel method. Instead, we propose a difference equation method to carry out an asymptotic analysis of a sequence of unknown generating functions. The last section contributes to some concluding remarks and two examples by applying the kernel method.

2 Description of the Random Walk

The random walk in the quarter plane used in this paper to demonstrate the kernel method is a reflected random walk or a Markov chain with the state space $\mathbb{Z}_+^2 = \{(m,n); m,n$ are non-negative integers $\}$. To describe this process, we divide the whole quadrant $\mathbb{Z}_+^2$ into four regions: the interior $S_+ = \{(m,n); m,n = 1,2,\ldots\}$, horizontal boundary $S_1 = \{(m,0); m = 1,2,\ldots\}$, vertical boundary $S_2 = \{(0,n); n = 1,2,\ldots\}$, and the origin $S_0 = \{(0,0)\}$, or $\mathbb{Z}_+^2 = S_+ \cup S_1 \cup S_2 \cup S_0$. In each of these regions, the transition is homogeneous. Specifically, let X_+, X_1, X_2 and X_0 be random variables having the distributions, respectively, $p_{i,j}$ with $i,j = 0,\pm 1$; $p_{i,j}^{(1)}$ with $i = 0,\pm 1$ and $j = 0,1$; $p_{i,j}^{(2)}$ with $i = 0,1$ and $j = 0,\pm 1$; and $p_{i,j}^{(0)}$ with $i,j = 0,1$. Then, the transition probabilities of the random walk (Markov chain) $L_t = (L_1(t), L_2(t))$ are given by

$$P(L_{t+1} = (m_2,n_2) | L_t = (m_1,n_1)\} = \\ \begin{cases} P(X_+ = (m_2 - m_1, n_2 - n_1)), & \text{if } (m_2,n_2) \in S, (m_1,n_1) \in S_+, \\ P(X_k = (m_2 - m_1, n_2 - n_1)), & \text{if } (m_2,n_2) \in S, (m_1,n_1) \in S_k \text{ with } k = 0,1,2. \end{cases}$$

2.1 Ergodicity conditions

A stability (ergodic) condition can be found in Theorem 3.3.1 of Fayolle, Iasnogorodski and Malyshev [11], which has been amended by Kobayashi and Miyazawa as Lemma 2.1 in [26]. This condition is stated in terms of the drift vectors defined by

$$\begin{aligned} M &= (M_x, M_y) = \Big(\sum_i i\Big(\sum_j p_{i,j}\Big), \sum_j j\Big(\sum_i p_{i,j}\Big)\Big), \\ M^{(1)} &= (M_{x,}^{(1)} M_y^{(1)}) = \Big(\sum_i i\Big(\sum_j p_{i,j}^{(1)}\Big), \sum_j j\Big(\sum_i p_{i,j}^{(1)}\Big)\Big), \\ M^{(2)} &= (M_{x,}^{(2)} M_y^{(2)}) = \Big(\sum_i i\Big(\sum_j p_{i,j}^{(2)}\Big), \sum_j j\Big(\sum_i p_{i,j}^{(2)}\Big)\Big). \end{aligned}$$

Theorem 2.1 (Theorem 3.3.1 in [11] and Lemma 2.1 in [26]) *When $M \neq 0$, the random walk is ergodic if and only if one of the following three conditions holds:*

1. $M_x < 0$, $M_y < 0$, $M_x M_y^{(1)} - M_y M_x^{(1)} < 0$ *and* $M_y M_x^{(2)} - M_x M_y^{(2)} < 0$;
2. $M_x < 0$, $M_y \geq 0$, $M_y M_x^{(2)} - M_x M_y^{(2)} < 0$ *and* $M_x^{(1)} < 0$ *if* $M_y^{(1)} = 0$;
3. $M_x \geq 0$, $M_y < 0$, $M_x M_y^{(1)} - M_y M_x^{(1)} < 0$ *and* $M_y^{(2)} < 0$ *if* $M_x^{(2)} = 0$.

Throughout the paper, we make the following assumption, unless otherwise specified:

Assumption 1 *The random walk L_t is irreducible, positive recurrent and aperiodic.*

Under Assumption 1, let $\pi_{m,n}$ be the unique stationary probability distribution of the random walk.

Remark 2.1 *It should be noted that for a stable random walk, the condition $M \neq 0$ is equivalent to that both sequences $\{\pi_{m,0}\}$ and $\{\pi_{0,n}\}$ are light-tailed (for example, see Lemma 3.3 of [26]), which is not our focus of this paper. Therefore, Theorem 2.1 provides a necessary and sufficient stability condition for the light-tailed case.*

2.2 Fundamental Form

Define the following generating functions of the probability sequences for the interior states, horizontal boundary states and vertical boundary states, respectively,

$$\begin{aligned} \pi(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \pi_{m,n} x^{m-1} y^{n-1}, \\ \pi_1(x) &= \sum_{m=1}^{\infty} \pi_{m,0} x^{m-1}, \\ \pi_2(y) &= \sum_{n=1}^{\infty} \pi_{0,n} y^{n-1}. \end{aligned}$$

The so-called fundamental form of the random walk provides a functional equation relating the three unknown generating functions $\pi(x,y)$, $\pi_1(x)$ and $\pi_2(y)$. To state the fundamental form, we define

$$\begin{aligned} h(x,y) &= xy\left(\sum_{i=-1}^{1}\sum_{j=-1}^{1} p_{i,j} x^i y^j - 1\right) \\ &= a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y), \\ h_1(x,y) &= x\left(\sum_{i=-1}^{1}\sum_{j=0}^{1} p_{i,j}^{(1)} x^i y^j - 1\right) \\ &= a_1(x)y + b_1(x) = \widetilde{a}_1(y)x^2 + \widetilde{b}_1(y)x + \widetilde{c}_1(y), \\ h_2(x,y) &= y\left(\sum_{i=0}^{1}\sum_{j=-1}^{1} p_{i,j}^{(2)} x^i y^j - 1\right) \\ &= \widetilde{a}_2(y)x + \widetilde{b}_2(y) = a_2(x)y^2 + b_2(x)y + c_2(x), \\ h_0(x,y) &= \left(\sum_{i=0}^{1}\sum_{j=0}^{1} p_{i,j}^{(0)} x^i y^j - 1\right) \\ &= a_0(x)y + b_0(x) = \widetilde{a}_0(y)x + \widetilde{b}_0(y), \end{aligned}$$

where

$$\begin{aligned}
a(x) &= p_{-1,1} + p_{0,1}x + p_{1,1}x^2, \\
b(x) &= p_{-1,0} - (1 - p_{0,0})x + p_{1,0}x^2, \\
c(x) &= p_{-1,-1} + p_{0,-1}x + p_{1,-1}x^2, \\
\tilde{a}(y) &= p_{1,-1} + p_{1,0}y + p_{1,1}y^2, \\
\tilde{b}(y) &= p_{0,-1} - (1 - p_{0,0})y + p_{0,1}y^2, \\
\tilde{c}(y) &= p_{-1,-1} + p_{-1,0}y + p_{-1,1}y^2,
\end{aligned}$$

$$\begin{aligned}
a_1(x) &= p^{(1)}_{-1,1} + p^{(1)}_{0,1}x + p^{(1)}_{1,1}x^2, b_1(x) = p^{(1)}_{-1,0} - \left(1 - p^{(1)}_{0,0}\right) x + p^{(1)}_{1,0}x^2, \\
\widetilde{a}_1(y) &= p^{(1)}_{1,0} + p^{(1)}_{1,1}y, \widetilde{b}_1(y) = p^{(1)}_{0,0} - 1 + p^{(1)}_{0,1}y, \widetilde{c}_1(y) = p^{(1)}_{-1,0} + p^{(1)}_{-1,1}y \\
a_2(x) &= p^{(2)}_{0,1} + p^{(2)}_{1,1}x, b_2(x) = p^{(2)}_{0,0} - 1 + p^{(2)}_{1,0}x, c_2(x) = p^{(2)}_{0,-1} + p^{(2)}_{1,-1}x \\
\widetilde{a}_2(y) &= p^{(2)}_{1,-1} + p^{(2)}_{1,0}y + p^{(2)}_{1,1}y^2, \widetilde{b}_2(y) = p^{(2)}_{0,-1} - \left(1 - p^{(2)}_{0,0}\right) y + p^{(2)}_{0,1}y^2, \\
a_0(x) &= p^{(0)}_{0,1} + p^{(0)}_{1,1}x, b_0(x) = p^{(0)}_{1,0}x - \left(1 - p^{(0)}_{0,0}\right), \\
\widetilde{a}_0(y) &= p^{(0)}_{1,0} + p^{(0)}_{1,1}y, \widetilde{b}_0(y) = p^{(0)}_{0,1}y - \left(1 - p^{(0)}_{0,0}\right).
\end{aligned}$$

The basic equation of the generating function of the joint distribution, or the fundamental form of the random walk, is given by

$$-h(x,y)\pi(x,y) = h_1(x,y)\pi_1(x) + h_2(x,y)\pi_2(y) + h_0(x,y)\pi_{0,0}. \tag{2.1}$$

The reason for the above functional equation to be called fundamental is largely due to the fact that through analysis of this equation, the unknown generating functions can be determined or expressed, for example, through algebraic methods and boundary value problems as illustrated in Fayolle, Iasnogorodski and Malyshev [11]. The kernel method presented here also starts with the fundamental form, but without expressing generating functions first.

Remark 2.2 *The generating function $\pi(x,y)$ is defined for $m,n > 0$, excluding the boundary probabilities. (2.1) was proved in (1.3.6) in [11]. Based on (2.1), one can also obtain a similar fundamental form using generating functions including boundary probabilities:* $\Pi(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \pi_{m,n}x^m y^n$, $\Pi_1(x) = \sum_{m=0}^{\infty} \pi_{m,0}x^m$ *and* $\Pi_2(y) = \sum_{n=0}^{\infty} \pi_{0,n}y^n$.

For the conclusion of this section, we can easily check the following expressions, some of which will be needed in later sections:

$$M_y = a(1) - c(1) = \widetilde{a}'(1) + \widetilde{b}'(1) + \widetilde{c}'(1), \qquad M_x = \widetilde{a}(1) - \widetilde{c}(1) = a'(1) + b'(1) + c'(1), \tag{2.2}$$

$$M^{(1)}_y = a_1(1) = \widetilde{a}'_1(1) + \widetilde{b}'_1(1) + \widetilde{c}'_1(1), \qquad M^{(1)}_x = \widetilde{a}_1(1) - \widetilde{c}_1(1) = a'_1(1) + b'_1(1), \tag{2.3}$$

$$M^{(2)}_y = a_2(1) - c_2(1) = \widetilde{a}'_2(1) + \widetilde{b}'_2(1), \qquad M^{(2)}_x = \widetilde{a}_2(1) = a'_2(1) + b'_2(1) + c'_2(1). \tag{2.4}$$

3 Branch Points And Functions Defined by the Kernel Equation

The property of the random walk relies on the property of the kernel function h and functions h_1 and h_2. The kernel function plays a key role in the kernel method.

Definition 3.1 *A random variable is called non-singular if the kernel function $h(x, y)$, as a polynomial in the two variables x and y, is irreducible (equivalently, if $h = fg$ then either f or g is a constant) and quadratic in both variables.*

Throughout the paper unless otherwise specified, we make the second assumption below.

Assumption 2 *The random considered is non-singular.*

The non-singular condition for a random walk is closely related to the irreducibility of the marginal processes $L_1(t)$ and $L_2(t)$, but they are not the same concept. A necessary and sufficient condition for a random walk to be singular is given, in terms of $p_{i,j}$, in Lemma 2.3.2 in [11]. Study on tail asymptotics for a singular random walk is either easier or similar to the non-singular case, which can be found in Li, Tavakoli and Zhao [32].

The starting point of our analysis is the set of all pairs (x, y) satisfy the kernel equation, or

$$B = \{(x, y) \in \mathbb{C}^2 : h(x, y) = 0\},$$

where $\mathbb{C}$ is the set of all complex numbers. The kernel function can be considered as a quadratic form in either x or y with the coefficients being functions of y or x, respectively. Therefore, the kernel equation can be written as

$$a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y) = 0. \tag{3.1}$$

For a fixed x, the two solutions to the kernel equation as a quadratic form in y are given by

$$Y_\pm(x) = \frac{-b(x) \pm \sqrt{D_1(x)}}{2a(x)}$$

if $a(x) \neq 0$, where $D_1(x) = b^2(x) - 4a(x)c(x)$. Notice that non-singularity implies that $a(x) \not\equiv 0$ and, therefore, only up to two values of x could lead to $a(x) = 0$ since $a(x)$ is a polynomial of degree up to 2.

Similarly, for a fixed y, the two solutions to the kernel equation as a quadratic form in x are given by

$$X_\pm(y) = \frac{-\widetilde{b}(y) \pm \sqrt{D_2(y)}}{2\widetilde{a}(y)},$$

where $D_2(y) = \widetilde{b}^2(y) - 4\widetilde{a}(y)\widetilde{c}(y)$.

It is important to study the set B, or equivalently $Y_\pm(x)$ or $X_\pm(y)$, since for all $(x, y) \in B$ with $|\pi(x, y)| < \infty$, the right hand side of the fundamental form is also zero, which provides a relationship between the two unknown generating functions π_1 and π_2. In the above, $\sqrt{D_1(x)}$ is well-defined if $D_1(x) \geq 0$ and similarly $\sqrt{D_2(y)}$ is well-defined if $D_2(y) \geq 0$. As a function of a complex variable, the square root is a two-valued function. To specify a branch, when z is complex, $\sqrt{z}$ is defined such that $\sqrt{1} = 1$.

Let $z = D_1(x)$. Then, both $Y_-(x)$ and $Y_+(x)$ are analytic as long as $z \notin (-\infty, 0]$ and $a(x) \neq 0$. For these two functions, we start from a region, in which they are analytic, and consider an analytic continuation of these two functions. In this consideration, the key is the continuation of $\sqrt{D_1(x)}$.

Definition 3.2 *A branch point of $Y_\pm(x)$ ($X_\pm(y)$) is a value of x (y) such that $D_1(x) = 0$ ($D_2(y) = 0$).*

To discuss the branch points, notice that the discriminant D_1 (D_2) is a polynomial of degree up to four. Since the two cases are symmetric, we discuss $D_1(x)$ in detail only. Rewrite $D_1(x)$ as

$$D_1(x) = d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0,$$

where

$$\begin{aligned}
d_0 &= p_{-1,0}^2 - 4p_{-1,1}p_{-1,-1},\\
d_1 &= 2p_{-1,0}(p_{0,0} - 1) - 4(p_{-1,1}p_{0,-1} + p_{0,1}p_{-1,-1}),\\
d_2 &= (p_{0,0} - 1)^2 + 2p_{1,0}p_{-1,0} - 4(p_{1,1}p_{-1,-1} + p_{1,-1}p_{-1,1} + p_{0,1}p_{0,-1}),\\
d_3 &= 2p_{1,0}(p_{0,0} - 1) - 4(p_{1,1}p_{0,-1} + p_{0,1}p_{1,-1}),\\
d_4 &= p_{1,0}^2 - 4p_{1,1}p_{1,-1}.
\end{aligned}$$

It can be easily checked that $d_1 \leq 0$ and $d_3 \leq 0$.

When D_1 is a polynomial of degree 4 (or $d_4 \neq 0$), there are four branch points, denoted by x_i (y_i), $i = 1, 2, 3, 4$. Without loss of generality, we assume that $|x_1| \leq |x_2| \leq |x_3| \leq |x_4|$. When the degree of $D_1(x)$ is $d < 4$, for convenience, we let $x_{d+k} = \infty$ for integer $k > 0$ such that $d + k \leq 4$. For example, if $d = 3$, then $x_4 = \infty$. This can be justified by the following: consider the polynomial $\widetilde{D}_1(\widetilde{x}) = D_1(x)/x^4$ in $\widetilde{x}$, where $\widetilde{x} = 1/x$. Then, $\widetilde{x} = 0$ is a d-tuple zero of $\widetilde{D}_1(\tilde{x})$, and therefore $x = \infty$ can be viewed as a d-tuple zero of $D_1(x)$.

The following lemma characterizes the branch points of $Y_\pm(x)$ for all non-singular random walks, including the heave-tailed case, or the case of $M = 0$.

Lemma 3.1 ***1.*** *For a non-singular random walk with $M_y \neq 0$, $Y_\pm(x)$ has two branch points x_1 and x_2 inside the unit circle and another two branch points x_3 and x_4 outside the unit circle. All these branch points lie on the real line. More specifically,*

(1) *if $p_{1,0} > 2\sqrt{p_{1,1}p_{1,-1}}$, then $1 < x_3 < x_4 < \infty$;*

(2) *if $p_{1,0} = 2\sqrt{p_{1,1}p_{1,-1}}$, then $1 < x_3 < x_4 = \infty$;*

(3) *if $p_{1,0} < 2\sqrt{p_{1,1}p_{1,-1}}$, then $1 < x_3 \leq -x_4 < \infty$, where the equality holds if and only if $d_1 = d_3 = 0$.*

Similarly,

(4) *if $p_{-1,0} > 2\sqrt{p_{-1,1}p_{-1,-1}}$, then $0 < x_1 < x_2 < 1$;*

(5) *if $p_{-1,0} = 2\sqrt{p_{-1,1}p_{-1,-1}}$, then $x_1 = 0$ and $0 < x_2 < 1$;*

(6) *if $p_{-1,0} < 2\sqrt{p_{-1,1}p_{-1,-1}}$, then $0 < -x_1 \leq x_2 < 1$, where the equality holds if and only if $d_1 = d_3 = 0$.*

2. *For a non-singular random walk with $M_y = 0$ (in this case $M_x \neq 0$ since we are only considering the genus 1 case in this paper), either $x_2 = 1$ if $M_x < 0$; or $x_3 = 1$ if $M_x > 0$. In the latter case, the system is unstable.*

Proof. We only need to prove **3.** and **6.** since all other proofs can be found in Fayolle, Iasnogorodski and Malyshev [11] (Lemma 2.3.8 and Lemma 2.3.9). We provide details for **3.** since **6.** can be proved similarly. Suppose otherwise $x_3 > -x_4$. From $d_1 \le 0$ and $d_3 \le 0$, we obtain $D_1(-x_3) = -d_3 x_3^3 - d_1 x_3 > 0$. On the other hand, $D_1(-\infty) = -\infty$ since $d_4 < 0$, which implies that $D_1(x) = 0$ has a fifth root in $(-\infty, x_3)$, but this is impossible. The contradiction shows that $x_3 \le -x_4$. It is clear that the equality holds if and only if $d_1 = d_3 = 0$. □

Remark 3.1 *Similar results hold for the branch points y_i, $i = 1, 2, 3, 4$, of $X_\pm(y)$.*

Definition 3.3 *$p_{i,j}$ ($p_{i,j}^{(k)}$) is called X-shaped if $p_{i,j} = 0$ ($p_{i,j}^{(k)} = 0$) for all i and j such that $|i + j| = 1$. A random walk is called X-shaped if $p_{i,j}$ and also $p_{i,j}^{(k)}$ for $k = 1, 2$ are all X-shaped.*

Based on Lemma 3.1, we can prove the following result.

Corollary 3.1 *$x_3 = -x_4$ if and only if $p_{i,j}$ is X-shaped.*

Throughout the rest of the paper, we define $[x_3, x_4] = [-\infty, x_4] \cup [x_3, \infty]$ when $x_4 < -1$. Similarly, $[y_3, y_4] = [-\infty, y_4] \cup [y_3, \infty]]$ when $y_4 < -1$. We define the following cut planes:

$$\begin{aligned}
\widetilde{\mathbb{C}}_x &= \mathbb{C}_x \setminus [x_3, x_4], \\
\widetilde{\mathbb{C}}_y &= \mathbb{C}_y \setminus [y_3, y_4], \\
\widetilde{\widetilde{\mathbb{C}}}_x &= \mathbb{C}_x \setminus [x_3, x_4] \cup [x_1, x_2], \\
\widetilde{\widetilde{\mathbb{C}}}_y &= \mathbb{C}_y \setminus [y_3, y_4] \cup [y_1, y_2],
\end{aligned}$$

where $\mathbb{C}_x$ and $\mathbb{C}_y$ are the complex planes for x and y, respectively.

We now define two complex functions on the cut plane $\widetilde{\widetilde{\mathbb{C}}}_x$ based on $Y_\pm(x)$:

$$Y_0(x) = \begin{cases} Y_-(x), & \text{if } |Y_-(x)| \le |Y_+(x)|, \\ Y_+(x), & \text{if } |Y_-(x)| > |Y_+(x)|; \end{cases} \tag{3.2}$$

and

$$Y_1(x) = \begin{cases} Y_+(x), & \text{if } |Y_-(x)| \le |Y_+(x)|, \\ Y_-(x), & \text{if } |Y_-(x)| > |Y_+(x)|. \end{cases} \tag{3.3}$$

Obviously, Y_0 is the function of Y_- and Y_+ with the smaller modulus and Y_+ is the function with the larger modulus.

Functions $X_0(y)$ and $X_1(y)$ are defined on the cut plane $\widetilde{\mathbb{C}}_y$ in the same manner.

Remark 3.2 *A branch point of $Y_\pm(x)$ ($X_\pm(y)$) is also referred to as a branch point of $Y_0(x)$ and $Y_1(x)$ ($X_0(y)$ and $X_1(y)$).*

Remark 3.3 *It is not always the case that Y_0 is a continuation of Y_- and Y_1 a continuation of Y_+. However, for $x \in \widetilde{\widetilde{\mathbb{C}}}_x$ with $a(x) \ne 0$, $Y_0(x)$ and $Y_1(x)$ are still the two zeros of the kernel function $h(x, y)$. Parallel comments can be made on X_0 and X_1.*

A list of basic properties of Y_0 and Y_1 (X_0 and X_1) is provided in the following lemma.

Lemma 3.2 *1. For $|x|=1$, $|Y_0(x)|\le 1$ and $|Y_1(x)|\ge 1$, with equality only possibly for $x=\pm 1$. For $x=1$, we have*

$$Y_0(1) = \min\left(1, \frac{\sum p_{i,-1}}{\sum p_{i,1}} = \frac{c(1)}{a(1)}\right),$$
$$Y_1(1) = \max\left(1, \frac{\sum p_{i,-1}}{\sum p_{i,1}} = \frac{c(1)}{a(1)}\right);$$

for $x=-1$, the equality holds only if $p_{i,j}$ is X-shaped, for which we have

$$Y_0(-1) = -\min\left(1, \frac{\sum p_{i,-1}}{\sum p_{i,1}} = \frac{c(1)}{a(1)}\right),$$
$$Y_1(-1) = -\max\left(1, \frac{\sum p_{i,-1}}{\sum p_{i,1}} = \frac{c(1)}{a(1)}\right).$$

2. The functions $Y_i(x)$, $i=0,1$, are meromorphic in the cut plane $\tilde{\tilde{\mathbb{C}}}_x$. In addition,

(a) *$Y_0(x)$ has two zeros and no poles. Hence $Y_0(x)$ is analytic in $\tilde{\tilde{\mathbb{C}}}_x$;*

(b) *$Y_1(x)$ has two poles and no zeros.*

(c) *$|Y_0(x)|\le|Y_1(x)|$, in the whole cut complex plane $\tilde{\tilde{\mathbb{C}}}_x$, and equality takes place only on the cuts.*

3. The function $Y_0(x)$ can become infinite at a point x if and only if,

(a) *$p_{11}=p_{10}=0$, in this case, $x=x_4=\infty$; or*

(b) *$p_{-11}=p_{-10}=0$, in this case, $x=x_1=0$.*

Parallel conclusions can be made for functions $X_0(y)$ and $X_1(y)$.

Proof. This lemma contains results in Lemma 2.3.4 and Theorem 5.3.3 in [11]. First, according to (ii) of Theorem 5.3.3 in [11], the functions Y_0 and Y_1 defined in this paper coincide the functions Y_0 and Y_1 in [11] due to the uniqueness of the continuity. Then, all results in 1 come from Lemma 2.3.4 and Lemma 5.3.1 in [11] except for the expressions for $Y_0(-1)$ and $Y_1(-1)$, which can be obtained in the same fashion as for $Y_0(1)$ and $Y_1(1)$; results in 2 are given in (ii) of Theorem 5.3.3 in [11]; and the conclusion in 3 is the same as in (iii) Theorem 5.3.3 in [11]. $\square$

Remark 3.4 *All the above properties can be directly obtained through elementary analysis of the square root function.*

Throughout the rest of the paper, unless otherwise specified, we make the following assumption:

Assumption 3 *All branch points x_i and y_i, $i=1,2,3,4$, are distinct.*

A random walk satisfying Assumption 3 is called a genus 1 random walk.

Remark 3.5 *This assumption is equivalent to the assumption that the Riemann surface defined by the kernel equation has genus 1. The Riemann surface for the random walk is either genus 1 or genus 0. A necessary and sufficient condition for the random walk in the quarter plane to be genus 1 is given in Lemma 2.3.10 in [11]. Most of queueing application models are the case of genus 1. The genus 0 case can be analyzed similarly except for the heavy-tailed case, the case where $M = 0$. In general, analysis of the genus 0 case (except for the case of $M = 0$) could be less challenging since expressions for the unknown generating functions $\pi_1(x)$ and $\pi_2(y)$ are either explicit or less complex than for the genus 1 case, which can immediately lead to an analytic continuation of these unknown generating functions. Chapter 6 of [11] is devoted to the genus 0 case.*

Corollary 3.2 *For a non-singular genus 1 random walk, if $p_{i,j}$ is X-shaped, then all $p_{1,1}$, $p_{1,-1}$, $p_{-1,1}$ and $p_{-1,-1}$ are positive.*

Proof. If only one of $p_{1,1}$, $p_{1,-1}$, $p_{-1,1}$ and $p_{-1,-1}$ is zero, then the random walk is non-singular having genus 0 (Lemma 2.3.10 in [11]) and if at least two of them are zero, then the random walk is singular (Lemma 2.3.2 in [11]). □

Corollary 3.3 *For a stable random walk with $M \neq 0$,*

1. *If $p^{(1)}_{i,j}$ is X-shaped, then $p^{(1)}_{1,1}$ and $p^{(1)}_{-1,1}$ cannot be both zero; and*
2. *If $p^{(2)}_{i,j}$ is X-shaped, then $p^{(2)}_{1,1}$ and $p^{(2)}_{-1,-1}$ cannot be both zero.*

Proof. Otherwise, $p^{(k)}_{0,0} = 1$, $k = 1$ or 2, with which the random walk cannot be stable. □

For our purpose, more results about functions Y_0 and Y_1 (X_0 and X_1) are needed. Once again, we consider Y_0 and Y_1. X_0 and X_1 can be considered in the same way. Recall that Y_k ($k = 1, 2$) are defined on the cut plane $\mathbb{C}_x \setminus [x_3, x_4] \cup [x_1, x_2]$, where two slits $[x_1, x_2]$ and $[x_3, x_4]$ are removed from the complex plane such that the functions Y_k can stay always in one branch. Take the slit $[x_1, x_2]$ as an example. For $x' \in [x_1, x_2]$, the limit of $Y_k(x)$ when x approaches to x' from above the real axis is different from the limit as x approaches to x' from below the real axis. Let $x'' \in [x_1, x_2]$ be another point satisfying $x'' > x'$. By $Y_0[\overrightarrow{\underleftarrow{x'x''}}]$, we denote the image contour, which is the limit of $Y_0(x)$ from above the real axis when x traverses from x' to x'' and from below the real axis when x continues to traverse back from x'' to x'. For convenience, we say that $Y_0[\overrightarrow{\underleftarrow{x'x''}}]$ is the image of the contour $\overrightarrow{\underleftarrow{x'x''}}$, traversed from x' to x'' along the upper edge of the slit $[x', x'']$ and then back to x' along the lower edge of the slit. In this way, we can define the following image contours:

$$\mathscr{L} = Y_0[\overrightarrow{\underleftarrow{x_1x_2}}], \quad \mathscr{L}_{ext} = Y_0[\overrightarrow{\underleftarrow{x_3x_4}}]; \tag{3.4}$$

$$\mathscr{M} = X_0[\overrightarrow{\underleftarrow{y_1y_2}}], \quad \mathscr{M}_{ext} = X_0[\overrightarrow{\underleftarrow{y_3y_4}}], \tag{3.5}$$

respectively. Furthermore, for an arbitrary simple closed curve $\mathscr{U}$, by $G_{\mathscr{U}}$ we denote the interior domain bounded by $\mathscr{U}$ and by $G^c_{\mathscr{U}}$ the exterior domain.

The properties of the above image contours provided in the following lemma are important for the interlace between the two unknown functions $\pi_{(}x)$ and $\pi_2(y)$ discussed in the next section. To state the lemma, define the following determinant:

$$\Delta = \begin{vmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{vmatrix}.$$

Lemma 3.3 *For non-singular genus 1 random walk without branch points on the unit circle, we have the following properties:*

1. *The curve $\mathscr{M}$ and $\mathscr{M}_{ext}$ are simple, closed and symmetrical about the real axis in $\mathbb{C}_x$ plane. Moreover,*

(a) *If $\Delta > 0$, then*

$$[x_1, x_2] \subset G_{\mathscr{M}} \subset G_{\mathscr{M}_{ext}} \text{ and } [x_3, x_4] \subset G^c_{\mathscr{M}_{ext}};$$

(b) *If $\Delta < 0$, then*

$$[x_1, x_2] \subset G_{\mathscr{M}_{ext}} \subset G_{\mathscr{M}} \text{ and } [x_3, x_4] \subset G^c_{\mathscr{M}};$$

(c) *If $\Delta = 0$, then*

$$[x_1, x_2] \subset G_{\mathscr{M}_{ext}} = G_{\mathscr{M}} \text{ and } [x_3, x_4] \subset G^c_{\mathscr{M}}.$$

Entirely symmetric results hold for $\mathscr{L}$ and $\mathscr{L}_{ext}$.

2. *The branches X_i and Y_i have the following properties:*

(a) *Both $X_0(y)$ and $Y_0(x)$ are conformal mappings: $G_{\mathscr{M}} - [x_1, x_2] \underset{X_0(y)}{\overset{Y_0(x)}{\rightleftarrows}} G_{\mathscr{L}} - [y_1, y_2]$;*

(b) *$X_0(y) \in G_{\mathscr{M}} \cup G_{\mathscr{M}_{ext}}$ and $X_1(y) \in G^c_{\mathscr{M}} \cup G^c_{\mathscr{M}_{ext}}$. Symmetrically, $Y_0(x) \in G_{\mathscr{L}} \cup G_{\mathscr{L}_{ext}}$ and $Y_1(x) \in G^c_{\mathscr{L}} \cup G^c_{\mathscr{L}_{ext}}$;*

(c) *If $G_{\mathscr{M}} \subset G_{\mathscr{M}_{ext}}$, then*

$$\begin{aligned} X_0 \circ Y_0(t) &= t, \text{ if } t \in G_{\mathscr{M}}, \\ X_0 \circ Y_0(t) &\neq t, \text{ if } t \in G^c_{\mathscr{M}} \text{ and } X_0 \circ Y_0(G^c_{\mathscr{M}}) = G_{\mathscr{M}}. \end{aligned}$$

Symmetrically, if $G_{\mathscr{L}} \subset G_{\mathscr{L}_{ext}}$, then

$$\begin{aligned} Y_0 \circ X_0(t) &= t \text{ if } t \in G_{\mathscr{L}}, \\ Y_0 \circ X_0(t) &\neq t \text{ if } t \in G^c_{\mathscr{L}} \text{ and } Y_0 \circ X_0(G^c_{\mathscr{L}}) = G_{\mathscr{L}}. \end{aligned}$$

PROOF. A proof of the lemma can be found in Theorem 5.3.3 (i) and Corollary 5.3.5 in [11]. Parallel results when 1 is a branch point (or both 1 and -1 are branch points) can be found in Lemma 2.3.6, Lemma 2.3.9 and Lemma 2.3.10 of [11]. □

Remark 3.6 *Results in this lemma can also be directly proved through elementary analysis without using advanced mathematical concepts used in [11].*

4 Asymptotic Analysis of the Two Unknown Functions $\pi_1(x)$ And $\pi_2(y)$

The key idea of the kernel method is to consider all $(x, y) \in B$ such that the right hand side of the fundamental form is also zero, which provides a relationship between the two unknown functions $\pi_1(x)$ and $\pi_2(y)$. Then, the interlace between the unknown functions $\pi_1(x)$ and $\pi_2(y)$ plays the key role in the asymptotic analysis of these two functions, from which exact tail asymptotics of the stationary distribution can be determined according to asymptotic analysis of the unknown function at its singularities and the Tauberien-like theorem.

4.1 Tauberian-like theorems

Various approaches, say probabilistic or non-probabilistic, including analytic or algebraic, are available for exact geometric decay. However, asymptotic analysis seems unavoidable for exact non-geometric decay. A Tauberian, or Tauberian-like, theorem provides a tool of connecting the asymptotic property at dominant singularities of an analytic function at zero and the tail property of the sequence of coefficients in the Taylor series of the function. In our case, an unknown generating function of a probability sequence is analytic at zero. Since these probabilities are unknown, in general, it cannot be verified that the probability sequence is (eventual) monotone, which is a required condition for applying a standard Tauberian theorem. The tool used in this paper is a Tauberian-like theorem, which does not require this monitonicity. Instead, it imposes some extra condition on analyticity of the unknown generating function.

Let $A(z)$ be analytic in $|z| < R$, where R is the radius of convergence of the function $A(z)$. We first consider a special case in which R is the only singularity on the circle of convergence.

Remark 4.1 *It should be noticed that for an analytic function at 0, if the coefficients of the Taylor expansion are all non-negative, then the radius $R > 0$ of convergence is a singularity of the function according to the well-known Pringsheim's Theorem.*

Definition 4.1 (Definition VI.1 in Flajolet and Sedgewick [13]) *For given numbers $\varepsilon > 0$ and ϕ with $0 < \phi < \pi/2$, the open domain $\Delta(\phi, \varepsilon)$ is defined by*

$$\Delta(\phi, \varepsilon) = \{z \in \mathbb{C} : |z| < 1 + \varepsilon, z \neq 1, |z - 1| > \phi\}. \tag{4.1}$$

A domain is a Δ-domain at 1 if it is a $\Delta(\phi, \varepsilon)$ for some $\varepsilon > 0$ and $0 < \phi < \pi/2$. For a complex number $\zeta \neq 0$, a Δ-domain at ζ is defined as the image $\zeta \cdot \Delta(\phi, \varepsilon)$ of a Δ-domain $\Delta(\phi, \varepsilon)$ at 1 under the mapping $z \mapsto \zeta z$. A function is called Δ-analytic if it is analytic in some Δ-domain.

Remark 4.2 *The region $\Delta(\phi, \varepsilon)$ is an intended disk with the radius of $1 + \varepsilon$. Readers may refer to Figure VI.6 in [13] for a picture of the region. Throughout the paper, without otherwise stated, the limit of a Δ-analytic function is always taken in the Δ-domain.*

Theorem 4.1 (Tauberian-like theorem for single singularity) *Let $A(z) = \sum_{n\geq 0} a_n z^n$ be analytic at 0 with R the radius of convergence. Suppose that R is a singularity of $A(z)$ on the circle of convergence such that $A(z)$ can be continued to a Δ-domain at R. If for a real number $\alpha \notin \{0, -1, -2, \ldots\}$,*

$$\lim_{z \to R} (1 - z/R)^{\alpha} A(z) = g,$$

where g is a non-zero constant, then,

$$a_n \sim \frac{g}{\Gamma(\alpha)} n^{\alpha - 1} R^{-n},$$

where $\Gamma(\alpha)$ is the value of the gamma function at α.

Proof. This is a immediate consequence of Corollary VI.1 in [13] after the transform $z \mapsto Rz$. □

For the random walks studied in this paper, we will prove that the unknown generating function $\pi_1(x)$ ($\pi_2(y)$) has only one singularity on the circle of its convergence, except the X-shaped random walk for which the convergent radius R and $-R$ are the only singularities. To deal with the later case, we introduce the following Tauberian-like theorem for the case of multiple singularities.

Theorem 4.2 (Tauberian-like theorem for multiple singularities) *Let $A(z) = \sum_{n\geq 0} a_n z^n$ be analytic when $|z| < R$ and have a finite number of singularities ζ_k, $k = 1, 2, \ldots, m$ on the circle $|z| = R$ of convergence. Assume that there exists a Δ-domain Δ_0 at 1 such that A can be continued to intersection of the Δ-domains ζ_k at ζ_k, $k = 1, 2, \ldots, m$:*

$$D = \cap_{k=1}^{m} (\zeta_k \cdot \Delta_0).$$

If for each k, there exists a real number $\alpha_k \notin \{0, -1, -2, \ldots\}$ such that

$$\lim_{z \to \zeta_k} (1 - z/\zeta_k)^{\alpha_k} A(z) = g_k,$$

where g_k is a non-zero constant, then,

$$a_n \sim \sum_{k=1}^{m} \frac{g_k}{\Gamma(\alpha_k)} n^{\alpha_k - 1} \zeta_k^{-n}.$$

Proof. This is an immediate corollary of Theorem VI.5 in [13] for the case where α_k is real, $\beta_k = 0$, $\sigma_k(z) = \tau_k(z) = (1 - z)^{-\alpha_k}$ and $\sigma_{k,n} = \frac{g_k}{\Gamma(\alpha)} n^{\alpha_k - 1}$. $\square$

4.2 Interlace of the two unknown functions $\pi_1(x)$ and $\pi_2(y)$

The interlace of the unknown functions $\pi_1(x)$ and $\pi_2(y)$ is a key for asymptotic analysis of these functions. Let

$$\begin{aligned} \Gamma_a &= \{x \in \mathbb{C} : |x| = a\}, \\ D_a &= \{x : |x| < a\}, \\ \overline{D}_a &= \{x : |x| \leq a\}. \end{aligned}$$

When $a = 1$, we write $\Gamma = \Gamma_1$, $D = D_1$ and $\overline{D} = \overline{D}_1$.

We fist state two literature results on the continuation of the functions $\pi_1(x)$ and $\pi_2(y)$.

Lemma 4.1 (Theorem 3.2.3 in [11]) *For a stable non-singular random walk having genus 1, $\pi_1(x)$ is a meromorphic function in the complex cut plane $\widetilde{\mathbb{C}}_x$. Similarly, $\pi_2(y)$ is a meromorphic function in the complex cut plane $\widetilde{\mathbb{C}}_y$.*

This continuation result is crucial for tail asymptotic analysis. The following intuition might be helpful to see why such a continuation exist. When the right hand side of the fundamental form is zero, the x and y are related, say through the function $Y_0(x)$. Therefore, x_3 is the dominant singularity if there are no other singularities exist between $(1, x_3)$. Based on the expression for $\pi_1(x)$ obtained from the fundamental form, all other singularities come from the zeros of $h_1(x, Y_0(x))$, which are poles of $\pi_1(x)$, or the singularities of $\pi_2(Y_0(x))$. A similar intuition holds for the function $\pi_2(y)$. Based on the above intuition, it is reasonable to expect Lemma 4.1.

Remark 4.3 *An analytic continuation can be achieved through various methods. In [11] and [15], it was proved in terms of properties of Riemann surfaces. In [26] and [20], direct methods were used for a convergent region. For some cases, a simple proof exists by using the property of the conformal mapping*

Y_0 or X_0. For example, for the case of $M_y > 0$ and $M_x < 0$, we know, from Lemma 3.2-1, Lemma 3.1 and Lemma 3.2-2 respectively, that $|Y_0(x)| < 1$ for $|x| = 1$, $x_3 > 1$ and Y_0 is analytic in the cut plan. Therefore, it is not difficult to see that we can find an $\varepsilon > 0$ such that for $|x| < 1 + \varepsilon$, the function $\pi_2(Y_0(x))$ in (4.3) is analytic, which leads to the continuation of $\pi_1(x)$.

Lemma 4.2 (Lemma 2.2.1 in [11]) *Assume that the random walk is ergodic with $M \neq 0$ and the polynomial $h(x, y)$ is irreducible. Then, exists an $\varepsilon > 0$ such that the functions $\pi_1(x)$ and $\pi_2(y)$ can be analytically continued up to the circle $\Gamma_{1+\varepsilon}$ in their respective complex plane. Moreover, they satisfy the following equation in $D^2_{1+\varepsilon} \cap B$:*

$$h_1(x, y)\pi_1(x) + h_2(x, y)\pi_2(y) + h_0(x, y)\pi_{0,0} = 0.$$

PROOF. The analytic continuation is a direct consequence of Lemma 4.1 and the equation is directly from the fundamental form. $\square$

Theorem 4.3 ***1.*** *Function $\pi_2(Y_0(x))$ is meromorphic in the cut complex plane $\tilde{\tilde{\mathbb{C}}}_x$. Moreover, if $Y_0(x_3)$ is not a pole of $\pi_2(y)$, then x_3 is x_{dom} of $\pi_2(Y_0(x))$ and there exist $\varepsilon > 0$ and $0 < \phi < \pi/2$ such that*

$$\lim_{x \to x_3} \pi_2(Y_0(x)) = \pi_2(Y_0(x_3)) \quad \text{and} \quad \lim_{x \to x_3} \pi_2'(Y_0(x)) = \pi_2'(Y_0(x_3)).$$

Similarly, $\pi_1(X_0(y))$ is meromorphic in the cut complex plane $\tilde{\tilde{\mathbb{C}}}_y$. Moreover, if $X_0(y_3)$ is not a pole of $\pi_1(x)$, then y_3 is y_{dom} of $\pi_1(X_0(y))$ and there exist $\varepsilon > 0$ and $0 < \phi < \pi/2$ such that

$$\lim_{y \to y_3} \pi_1(X_0(y)) = \pi_1(X_0(y_3)) \quad \text{and} \quad \lim_{y \to y_3} \pi_1'(X_0(y)) = \pi_1'(X_0(y_3)).$$

2. *In cut plane $\tilde{\tilde{\mathbb{C}}}_x$, equation*

$$h_1(x, Y_0(x))\pi_1(x) + h_2(x, Y_0(x))\pi_2(Y_0(x)) + h_0(x, Y_0(x))\pi_{0,0} = 0 \tag{4.2}$$

holds except at a pole (if there is any) of $\pi_1(x)$ or $\pi_2(Y_0(x))$. Therefore,

$$\pi_1(x) = \frac{-h_2(x, Y_0(x))\pi_2(Y_0(x)) - h_0(x, Y_0(x))\pi_{0,0}}{h_1(x, Y_0(x))}, \tag{4.3}$$

except at zero of $h_1(x, Y_0(x))$, or at a pole (if there is any) of $\pi_1(x)$ or $\pi_2(Y_0(x))$.

Similarly, in the cut plane $\tilde{\tilde{\mathbb{C}}}_y$, equation

$$h_1(X_0(y), y)\pi_1(X_0(y) + h_2(X_0(y), y)\pi_2(y) + h_0(X_0(y), y)\pi_{0,0} = 0 \tag{4.4}$$

holds except at a pole (if there is any) of $\pi_2(y)$ or $\pi_1(X_0(y))$. Therefore,

$$\pi_2(y) = \frac{-h_1(X_0(y), y)\pi_1(X_0(y)) - h_0(X_0(y), y)\pi_{0,0}}{h_2(X_0(y), y)}, \tag{4.5}$$

except at a zero of $h_2(X_0(y), y)$, or at a pole (if there is any) of $\pi_2(y)$ or $\pi_1(X_0(y))$.

Proof. We only prove the result for functions of x and the result for functions of y can be proved in the same fashion.

1. From Lemma 3.1 and Lemma 4.1, $Y_0(x)$ is analytic in the cut complex plane $\tilde{\tilde{\mathbb{C}}}_x$ and $\pi_2(y)$ is meromorphic in the cut complex plane $\widetilde{\mathbb{C}}_y$, which implies $\pi_2(Y_0(x))$ is meromorphic in $\tilde{\tilde{\mathbb{C}}}_x$ if $Y_0(x) \notin [y_3, y_4]$. According to Lemma 3.3-2(b), for all $x \in \mathbb{C}_x$, $Y_0(x) \in G_{\mathscr{L}} \cup G_{\mathscr{L}_{ext}}$ and according to Lemma 3.3-1, $[y_3, y_4] \subset (G_{\mathscr{L}} \cup G_{\mathscr{L}_{ext}})^c$, which confirms $Y_0(x) \notin [y_3, y_4]$. From the above, we have $\pi_2(y)$ is analytic at $Y_0(x_3)$, then the limits in **1.** are immediate results of the analytic properties of $\pi_2(Y_0(x))$.

2. Since both $\pi_1(x)$ and $\pi_2(Y_0(x))$ are meromorphic (proved in **1.**) and $Y_0(x)$ is analytic (Lemma 3.1) in $\tilde{\tilde{\mathbb{C}}}_x$, equation (4.2) in the cut plane $\tilde{\tilde{\mathbb{C}}}_x$ except at the poles of $\pi_1(x)$ or $\pi_2(Y_0(x))$. □

Remark 4.4 *Let us extend the definition of $\pi_1(x)$ to $x = x_3$ by $\pi_1(x_3) = \lim_{x \to x_3} \pi_1(x)$ for x in the cut plane. We say that x_3 is a pole if the limit of $\pi_1(x)$ is infinite as $x \to x_3$ in the cut plane.*

According to the above interlacing property and the Tauberian-like theorem, for exact tail asymptotics of the boundary probabilities $\pi_{n,0}$ and $\pi_{0,n}$, we only need to carry out an asymptotic analysis at the dominant singularities of the functions $\pi_1(x)$ and $\pi_2(y)$, respectively. There are only two possible types of singularities, poles or branch points. We need to answer the following questions:

Q1. How many singularities on the circle of convergence (dominant singularities)?

Q2. What is the multiplicity of a pole?

Q3. Is the branch point also a pole?

For the random walk considered in this paper, we will answer all these questions. We will see that on the convergent circle, there is only one singularity or there are exactly two singularities. For the former, Theorem 4.1 will be applied, and for the latter, Theorem 4.2 will be applied.

4.3 Poles of $\pi_1(x)$

Parallel properties about poles of the function $\pi_2(y)$ can be obtained in the same fashion, which will not be detailed here.

Lemma 4.3 *1. Let $x \in G_{\mathscr{M}} \cap (\overline{D})^c$, then the possible poles of $\pi_1(x)$ in $G_{\mathscr{M}} \cap (\overline{D})^c$ are necessarily zeros of $h_1(x, Y_0(x))$, and $|Y_0(x)| \le 1$.*

2. Let $y \in G_{\mathscr{L}} \cap (\overline{D})^c$, then the possible poles of $\pi_2(y)$ in $G_{\mathscr{L}} \cap (\overline{D})^c$ are necessarily zeros of $h_2(X_0(x), y)$, and $|X_0(y)| \le 1$.

Proof. **1.** When $x \in \mathscr{M}$, then $Y_0(x) = y \in [y_1, y_2]$. From Lemma 3.1, for $|x| = 1$, $|Y_0(x)| \le 1$. For $x \in G_{\mathscr{M}} \cap (\overline{D})^c$, it follows from the maximum modulus principle, we have $|Y_0(x)| \le 1$. Hence, $\pi_2(Y_0(x))$ is analytic in $G_{\mathscr{M}} \cap (\overline{D})^c$. From Theorem 4.3, if $h_1(x, Y_0(x)) \neq 0$, equation (4.3) holds, which implies that the possible poles of $\pi_1(x)$ in $G_{\mathscr{M}} \cap (\overline{D})^c$ are necessarily zeros of $h_1(x, Y_0(x))$.

2. The proof is similar. □

Theorem 4.4 *Let x_p be a pole of $\pi_1(x)$ with the smallest modulus. Assume that $|x_p| \le x_3$. Then, one of the follow two cases must hold:*

1. x_p *is a zero of* $h_1(x, Y_0(x))$*;*

2. $\widetilde{y}_0 = Y_0(x_p)$ *is a zero of* $h_2(X_0(y), y)$ *and* $|\widetilde{y}_0| > 1$.

Parallel results hold for a pole of $\pi_2(y)$.

Proof. Suppose that x_p is not a zero of $h_1(x, Y_0(x))$. According to equation (4.3) in Theorem 4.3, x_p must be a pole of $\pi_2(Y_0(x))$ and $|\widetilde{y}_0| > 1$. Furthermore, by Lemma 4.3, $x_p \notin G_{\mathscr{M}}$. If $\widetilde{y}_0$ is not a zero of $h_2(X_0(y), y)$, according to equation (4.5) in Theorem 4.3, $\widetilde{y}_0$ must be a pole of $\pi_1(X_0(y))$, that is, $\widetilde{x}_0 = X_0(\widetilde{y}_0)$ is a pole of $\pi_1(x)$. It follows from Lemma 4.3 that $\widetilde{x}_0 = X_0(\widetilde{y}_0)$ is a zero of $h_1(x, Y_0(x))$ if $\widetilde{x}_0 \in G_{\mathscr{M}}$. There are two possible cases: $\Delta > 0$ or $\Delta \le 0$. If $\Delta > 0$, by Lemma 3.3-1(a) and 2(c), $\widetilde{x}_0 \in G_{\mathscr{M}}$. In the case of $\Delta \le 0$, according to Lemma 3.3-1(b), 1(c) and 2(b), we also have $\widetilde{x}_0 \in G_{\mathscr{M}}$. However, this case is not possible, since otherwise according to Lemma 3.3-1 we would have $\widetilde{x}_0 = x_p$ or $\widetilde{x}_0 = -x_p$, both leading to a contradiction. This completes the proof. $\square$

Remark 4.5 *We will show in the next subsection that a pole of* $\pi_1(x)$ *with the smallest modulus in the disk* $|x| \le x_3$ *is real.*

4.4 Zeros of $h_1(x, Y_0(x))$

In this subsection, we provide properties on the zeros of the function $h_1(x, Y_0(x))$. The main result is stated in the following theorem.

Theorem 4.5 *For a non-singular random walk having genus 1, consider the following two possible cases:*

1. Either $p_{i,j}$ *or* $p^{(1)}_{i,j}$ *is not X-shaped. In this case, either* $h_1(x, Y_0(x))$ *has no zeros with modulus in* $(1, x_3]$*, or it has only one simple zero, say* x^**, with modulus in* $(1, x_3]$*, and* x^* *is positive.*

2. Both $p_{i,j}$ *and* $p^{(1)}_{i,j}$ *are X-shaped. In this case, either* $h_1(x, Y_0(x))$ *has no zeros with modulus in* $(1, x_3]$*, or it has exact two simple zeros, namely,* $x^* > 0$ *(with modulus in* $(1, x_3]$*) and* $-x^*$*, both are zeros of* $h_1(x, Y_0(x))$ *or both are zeros of* $a(x)h_1(x, Y_1(x))$.

With this theorem and Theorem 4.4, we are able to apply the Tauberian-like theorem to characterize the tail asymptotic properties for the boundary probability sequence $\pi_{n,0}$. To show the above Theorem, we need the following several lemmas and two propositions. Instead of directly considering the function $f_0(x) = h_1(x, Y_0(x))$, we consider a polynomial $f(x)$, which is essentially the product of $f_0(x)$ and $f_1(x) = h_1(x, Y_1(x))$:

$$f(x) = f_0(x)\widetilde{f}_1(x),$$

where $\widetilde{f}_1(x) = a(x)f_1(x)$. It is easy to verify, by noticing

$$Y_0(x)Y_1(x) = \frac{c(x)}{a(x)} \quad \text{and} \quad Y_0(x) + Y_1(x) = -\frac{b(x)}{a(x)},$$

that

$$\begin{aligned} f(x) &= a(x)b_1^2(x) - b(x)b_1(x)a_1(x) + c(x)a_1^2(x) && (4.6)\\ &= d_6x^6 + d_5x^5 + d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0. && (4.7) \end{aligned}$$

Hence, a zero of $f_i(x)$, $i = 0, 1$, has to be a zero of $f(x)$, and any zero of $f(x)$ is either a zero of $f_0(x)$ or a zero of $\widetilde{f}_1(x) = a(x)f_1(x)$.

We can also write

$$f(x) = a(x)[a_1(x)]^2 R_-(x)R_+(x), \tag{4.8}$$

where

$$R_\pm(x) = F(x) \pm \frac{\sqrt{D_1(x)}}{2a(x)} \tag{4.9}$$

with

$$F(x) = \frac{b_1(x)}{a_1(x)} - \frac{b(x)}{2a(x)}. \tag{4.10}$$

Remark 4.6 *1. It can be easily seen that both $f_0(x)$ and $\widetilde{f}_1(x)$ are analytic on the cut complex plan. In fact, the analyticity of $f_0(x)$ is obvious and the analyticity of $\widetilde{f}_1(x)$ is due to the cancellation of the zeros of $a(x)$ and the pole of $f_1(x)$.*

All proofs for Lemmas 4.4–4.7 and for Proposition 4.1 and Proposition 4.2 are organized into Appendix A.

Lemma 4.4 ***1.*** ***(a)*** *$Y_0'(1) = \frac{M_x}{-M_y}$ if $M_y < 0$;* ***(b)*** *$Y_1'(1) = \frac{M_x}{-M_y}$ if $M_y > 0$; and* ***(c)*** *$Y_1(1) = Y_0(1)$ and $x = 1$ is a branch point of $Y_1(x)$ and $Y_0(x)$ if $M_y = 0$. In this case, $Y_1'(1)$ and $Y_0'(1)$ do not exist. Parallel results hold for functions $X_k(y)$.*

2. *If $M_y \neq 0$, then $f(x)$ has at least one non-unit zero in $[x_2, x_3]$ and 1 is a simple zero of $f(x)$. Parallel results holds for the case of $M_x \neq 0$.*

Lemma 4.5 ***1.*** *Let z be a branch point of $Y_0(x)$. If $f(z) = 0$, then z cannot be a repeated root of $f(x) = 0$.*

2. *$f(x)$ (therefore both $f_0(x)$ and $\widetilde{f}_1(x)$) has (have) no zeros on the cuts, except possibly at a branch point. More specifically, $f(x) < 0$ if $a(x) < 0$ and $f(x) > 0$ if $a(x) > 0$.*

3. *$f_0(x)$ and $\widetilde{f}_1(x)$ have no common zeros except possibly at a branch point or at zero.*

4. *Consider the random walk in Theorem 4.5-1. If $f_0(x)$ has a zero in $[-x_3, -1)$, then $f_0(x)$ has an additional (different) zero in $[-x_3, -1)$.*

5. *For the random walk in Theorem 4.5-1, if $|x| \in (1, x_3]$, then $|Y_0(-|x|)| < Y_0(|x|)$.*

Lemma 4.6 *Consider the random walk in Theorem 4.5-1. If $M_y \leq 0$, then $x = 1$ is the only zero of $f_0(x) = h_1(x, Y_0(x))$ on the unit circle $|x| = 1$. If $M_y > 0$, then $f_0(x)$ has no zero on unit circle $|x| = 1$.*

Remark 4.7 *From the proof of Lemma 4.6, we can see that for the random walk considered in Theorem 4.5-2, $f_0(x)$ has no zeros with non-zero imaginary part on the unit circle.*

The proof of Theorem 4.5 is based on detailed properties of the function $f(x)$ and also the powerful continuity argument to connect an arbitrary random walk to a simpler one. For using this continuity argument, we consider the following special random walk.

Special Random Walk. This is the random walk for which $p_{i,j}$ is cross-shaped (or $p_{i,j} = 0$ whenever $|ij| = 1$), and $p^{(1)}_{-1,1} = p^{(1)}_{-1,0} = 0$. We first prove the counterpart result to Theorem 4.5 for the Special Random Walk.

Proposition 4.1 *For the Special Random Walk, the following results hold:*

1. *$f(x) = 0$ has six real roots with exact one non-unit root in $[x_2, x_3]$. More specifically, two roots are zero, two in $[x_2, x_3]$, one in $(-\infty, x_1]$, and one in $[x_4, \infty)$.*

2. *If $f_0(x)$ has a zero, say x^*, in $(1, x_3]$, then x^* is the only zero of $f_0(x)$ with modulus in $(1, x_3]$. Furthermore, $f_0(x)$ has no other zeros with modulus greater than 1 except possibly at $x = x_4$.*

For the random walk considered in Theorem 4.5-2, we first prove the following results.

Lemma 4.7 *For the random walk considered in Theorem 4.5-2 (or both $p_{i,j}$ and $p^{(1)}_{i,j}$ are X-shaped), $f(1) = f(-1) = 0$, and $f(x) = 0$ has two more real roots, say $0 < x_0 \neq 1$ and $-x_0$, and two complex roots.*

Proposition 4.2 *For the random walk considered in Theorem 4.5-2 (or both $p_{i,j}$ and $p^{(1)}_{i,j}$ are X-shaped), either the two complex zeros of $f(x)$ are zeros of $\widetilde{f}_1(x) = a(x)f_1(x)$ or they are inside the unit circle.*

PROOF. of Theorem 4.5. **1.** For the random walk considered here (either $p_{i,j}$ or $p^{(1)}_{i,j}$ is not X-shaped), let

$$\begin{aligned}
\mathbf{p} &= (p_{-1,-1}, p_{0,-1}, p_{1,-1}, p_{-1,0}, p_{0,0}, p_{0,1}, p_{-1,1}, p_{0,1}, p_{1,1}),\\
\mathbf{p}^{(1)} &= (p^{(1)}_{-1,0}, p^{(1)}_{0,0}, p^{(1)}_{0,1}, p^{(1)}_{-1,1}, p^{(1)}_{0,1}, p^{(1)}_{1,1}).
\end{aligned}$$

Define

$$A = \Big\{ \big(\mathbf{p}, \mathbf{p}^{(1)}\big) : 0 \leq p_{i,j}, p^{(1)}_{i,j} \leq 1 \ \text{ and } \ \sum_{i,j} p_{i,j} = \sum_{i,j} p^{(1)}_{i,j} = 1 \Big\}.$$

For an arbitrary random walk for which either $p_{i,j}$ or $p^{(1)}_{i,j}$ is not X-shaped, let ρ be the corresponding point in A. We assume that $M_y \leq 0$ for the random walk ρ (and a similar proof can be found for the case of $M_y > 0$). Let ρ_0 be an arbitrarily chosen point in A corresponding the Special Random Walk. We prove the result by contradiction. Suppose otherwise that the statement were not true. There would be three possible cases: (i) $\mathrm{Im}(x^*) \neq 0$; (ii) $-x_3 \leq x^* < -1$; and (iii) there exists $x_0 \in (1, x_3]$ with $x_0 \neq x^*$ such that $f_0(x_0) = 0$.

Case (i). Clearly, $\overline{x^*}$ is also a root of $f(x) = 0$. Choose a simple connected path ℓ in A to connect ρ to ρ_0 such that on ℓ (excluding ρ, but including ρ_0) $M_y < 0$. The zeros of $f(x)$ as a function of parameters in A are continues on ℓ. There are two possible cases: (a) the zero function $x_0(\theta)$ (with $x_0(\rho) = x^*$) never passes the unit circle when θ travels from ρ to ρ_0; and (b) $x_0(\theta)$ passes the unit circle at some point $\theta \in \ell$.

If (a) occurs, let θ_0 be the first point at which $x_0(\theta) = \overline{x}_0(\theta)$, where $\overline{x}_0(\theta)$ is the zero function with $\overline{x}_0(\rho) = \overline{x^*}$. If $\overline{x^*}$ is a zero of $\widetilde{f}_1$, then f_0 and $\widetilde{f}_1$ would have a common zero $x_0(\theta_0) = \overline{x}(\theta_0)$ at θ_0, which contradicts Lemma 4.5-3. Hence, the only possibility is that $\overline{x^*}$ is also a zero of f_0. From θ_0 on, both $x_0(\theta)$ and $\overline{x}(\theta)$ should always be zeros of f_0, since otherwise only at a branch point a zero of f_0 could be switched to a zero of $\widetilde{f}_1$ and all branch points are real, which means that $x_0(\theta) = \overline{x}(\theta)$ is a branch point and a multiple roots, contradicting to Lemma 4.5-1. As θ_0 approaches ρ_0, it leads to a contradiction that two zeros of f_0 are in $(1, x_3]$.

If (b) occurs, we can assume that when $x_0(\theta)$ pases the unit circle it is a zero of f_0 based on the proof in (a). Then, f_0 has two zeros since 1 is always a zero of f_0 independent of the parameters (or θ) when $M_y < 0$, which is a different zero from $x_0(\theta)$. This contradicts to the fact that f_0 has only one zero at the unit circle.

Case (ii). In this case, $f_0(x)$ would have another zero in $[-x_3, -1)$ at ρ according to Lemma 4.5-4. Consider the same two cases (a) and (b) as in (i). We can then follow a similar proof to show that case (ii) is impossible.

Case (iii). A similar proof will show that the case is impossible.

2. This is a direct consequence of Lemma 4.7 and Proposition 4.2. □

The following Lemma gives a necessary and sufficient condition under which $f_0(x) = h_1(x, Y_0(x))$ has a zero in $(1, x_3]$.

Lemma 4.8 *Assume $M_y \neq 0$. We have following results:*

1. *If $f_0(x_3) \geq 0$, $f_0(x)$ has a zero in $(1, x_3]$;*

2. *If $f_0(x_3) < 0$, $f_0(x)$ has no zeros in $(1, x_3]$.*

Proof. **1.** There are two cases: $M_y > 0$ or $M_y < 0$. If $M_y > 0$, then $f_0(1) < 0$, which leads to the conclusion. If $M_y < 0$, then $f_0'(1) < 0$, which also leads to the conclusion since $f_0(1) = 0$ and $f_0(x_3) \geq 0$.

2. Again there are two cases: $M_y > 0$ or $M_y < 0$. By simple calculus, in either case, we obtain that if $f_0(x) = 0$ had a root in $(1, x_3]$, then it would have another root in $(1, x_3]$ since $f_0(x_3) < 0$. This contradicts to Theorem 4.5. □

4.5 Zeros of $h_2(X_0(y), y)$

Following the same argument in the previous subsection, we have the following result:

Theorem 4.6 *For a non-singular random walk having genus 1, consider the following two possible cases:*

1. *Either $p_{i,j}$ or $p_{i,j}^{(2)}$ is not X-shaped. In this case, either $h_2(X_0(y), y)$ has no zeros with modulus in $(1, y_3]$, or it has only one simple zero, say y^*, with modulus in $(1, y_3]$, and y^* is positive.*

2. *Both $p_{i,j}$ and $p_{i,j}^{(2)}$ are X-shaped. In this case, either $h_2(X_0(y), y)$ has no zeros with modulus in $(1, y_3]$, or it has exact two simple zeros, namely, $y^* > 0$ (with modulus in $(1, y_3]$) and $-y^*$, both are zeros of $g_0(y)$ or both are zeros of $g_1(y)$, where*

$$g_0(y) = h_2(X_0(y), y) \qquad \text{and} \qquad g_1(y) = h_2(X_1(y), y).$$

From the above analysis, we know that if $h_1(x, Y_0(x))$ has a zero in $(1, x_3]$, then such a zero is unique. Similarly, if $h_2(X_0(y), y)$ has a zero in $(1, y_3]$, then such a zero is unique. For convenience, we make the following convention:

Convention 1 *Let x^* be the unique zero in $(1, x_3]$ of the function $h_1(x, Y_0(x))$, if such a zero exists, otherwise let $x^* = \infty$. Similarly Let y^* be the unique zero in $(1, y_3]$ of the function $h_2(X_0(y), y)$ if such a zero exists, otherwise let $y^* = \infty$.*

According to Theorem 4.4, the unique pole in $(1, x_3]$ of $\pi_1(x)$ is either x^*, or the image of the pole under Y_0 is a zero of $h_2(X_0(y), y)$. Our focus in this subsection is on this special case of y^*.

Theorem 4.7 *If the pole in $(1, x_3]$ of $\pi_1(x)$ is not x^*, then, it, denoted by $\widetilde{x}_1$, satisfies:*

1. *$\widetilde{x}_1 = X_1(y^*)$, where y^* is the unique zero in $(1, y_3]$ of the function $h_2(X_0(y), y)$;*

2. *$\widetilde{x}_1$ is the only pole of $\pi_1(x)$ with modulus in $(1, y_3]$, except for the case where both $p_{i,j}$ and $p^{(2)}_{i,j}$ are X-shaped, for which $-\widetilde{x}_1$ is the other pole of $\pi_1(x)$ with modulus in $(1, y_3]$.*

Proof. **1.** Let $\widetilde{x}$ be the solution of $y^* = Y_0(x)$. Then, $\widetilde{x} = \widetilde{x}_0 \stackrel{\triangle}{=} X_0(y^*)$ or $\widetilde{x} = \widetilde{x}_1 \stackrel{\triangle}{=} X_1(y^*)$. If $y^* \in G_{\mathscr{L}}$, then $\widetilde{x} = \widetilde{x}_0$ so that $y^* = Y_0(X_0(y^*))$. In this case, by Lemma 4.3, $\widetilde{x}_0 < 1$. If $y^* \in G^c_{\mathscr{L}}$, then $\widetilde{x} = \widetilde{x}_1$ so that $y^* = Y_0(X_1(y^*)$ and $\widetilde{x}_1 \in G^c_{\mathscr{M}}$.

2. It follows from the fact that the zero, y^*, of $h_2(X_0(y), y)$ in $(1, y_3]$ is unique and the fact that $y^* = Y_0(x)$ has only two possible solutions $\widetilde{x}_0 < 1$ and $\widetilde{x}_1$. In the case where both $p_{i,j}$ and $p^{(2)}_{i,j}$ are X-shaped, $-y^*$ is the other zero of $h_2(X_0(y), y)$ with either $-y^* = Y_0(-\widetilde{x}_1)$ or $-y^* = Y_0(-\widetilde{x}_0)$. □

Corollary 4.1 *Let $\widetilde{x}$ be a solution of $y^* = Y_0(x)$. In order for $\widetilde{x}$ to be in $(1, x_3]$ we need $y^* \in G^c_{\mathscr{L}}$. Furthermore, we have $y^* < y_3$.*

Proof. The first conclusion is directly from the proof to Theorem 4.7 and the second one follows from that fact that by Lemma 3.3-1 and Lemma 3.3-2(b), there exists no $x \in (1, x_3]$ such that $y^* = y_3 = Y_0(x)$. Therefore, we should have $y^* < y_3$. □

Convention 2 *Let $\widetilde{x}_1 = X_1(y^*)$ if the unique zero y^* in $(1, y_3]$ of the function $h_2(X_0(y), y)$ exists, otherwise let $\widetilde{x}_1 = \infty$.*

4.6 Asymptotics behaviour of $\pi_1(x)$ and $\pi_2(y)$

In this subsection, we provide asymptotic behaviour of two unknown functions $\pi_1(x)$ and $\pi_2(y)$. We only provide details for $\pi_1(x)$, since the behaviour for $\pi_2(y)$ can be characterized in the same fashion.

It follows from the discussion so far that:

(1) If $p_{i,j}$ is not X-shaped, then, independent of the properties of $p^{(1)}_{i,j}$ and $p^{(2)}_{i,j}$, there is only one dominant singularity, which is the smallest one of x^*, $\widetilde{x}_1$ and x_3. Here x^*, $\widetilde{x}_1$ and x_3 are not necessarily all different.

(2) If $p_{i,j}$ is X-shaped, then both x_3 and $-x_3$ are branch points.

- **(a)** If $p^{(1)}_{i,j}$ is not X-shaped, then $h_1(x, Y_0(x)$ has either no zero or one zero x^* in $(1, x_3]$; and if $p^{(1)}_{i,j}$ is X-shaped, then $h_1(x, Y_0(x)$ has either no zero or two zeros $x^* \in (1, x_3]$ and $-x^*$.
- **(b)** Similar to (a), $h_2(X_0(y), y)$ has either no zero in $(1, y_3]$ or one zero y^* in it. For the latter, if $p^{(2)}_{i,j}$ is not X-shaped, then $\widetilde{x}_1 = X_1(y^*)$ is the only pole of $\pi_2(Y_0(x))$ with modulus in $(1, x_3]$; and if $p^{(2)}_{i,j}$ is X-shaped, then $\widetilde{x}_1 = X_1(y^*) \in (1, x_3]$ and $-\widetilde{x}_1 = X_1(-y^*)$ are the only two poles of $\pi_2(Y_0(x))$ with modulus in $(1, x_3]$.

Therefore, in case (2), we either have only one dominant singularity or exactly two dominant singularities depending on which of x^*, $\widetilde{x}_1$ and x_3 is smallest and the property of $p_{i,j}^{(k)}$, $k = 1, 2$.

The theorem in this subsection provides detailed asymptotic properties at a dominant singularity for all possible cases. Let x_{dom} be a dominant singularity of $\pi_1(x)$. Clearly, $|x_{dom}| = x^*$, $|x_{dom}| = \widetilde{x}_1$ or $|x_{dom}| = x_3$. To state this theorem for the cases where $x_{dom} = \pm x_3$, notice that through simple calculation we can write

$$h_1(x, Y_0(x)) = p_1(x) + q_1(x)\sqrt{1 - \frac{x}{x_{dom}}}, \tag{4.11}$$

$$Y_0(x) = p(x) + q(x)\sqrt{1 - \frac{x}{x_{dom}}}, \tag{4.12}$$

$$Y_0(x_{dom}) - Y_0(x) = \left(1 - \frac{x}{x_{dom}}\right) p^*(x) - q(x)\sqrt{1 - \frac{x}{x_{dom}}}, \tag{4.13}$$

$$h_1(x, Y_0(x)) - h_1(x_{dom}, Y_0(x_{dom})) = \left(1 - \frac{x}{x_{dom}}\right) p_1^*(x) + q_1(x)\sqrt{1 - \frac{x}{x_{dom}}}, \tag{4.14}$$

where

$$p(x) = \frac{-b(x)}{2a(x)}, \quad p^*(x) = \frac{\frac{b(x)}{2a(x)} - \frac{b(x_{dom})}{2a(x_{dom})}}{\frac{1}{x_{dom}}(x - x_{dom})}, \quad p_1(x) = \frac{-b(x)a_1(x)}{2a(x)} + b_1(x),$$
$$p_1^*(x) = x_{dom}\left(\frac{a_1(x) - a_1(x_{dom}) + b_1(x) - b_1(x_{dom})}{x_{dom} - x}\right),$$

$$q(x) = \begin{cases} -\frac{1}{2a(x)}\sqrt{\frac{D_1(x)}{1 - \frac{x}{x_{dom}}}}, & \text{if } x_{dom} = x_3, \\ \frac{1}{2a(x)}\sqrt{\frac{D_1(x)}{1 - \frac{x}{x_{dom}}}}, & \text{if } x_{dom} = -x_3, \end{cases}$$

and $q_1(x) = a_1(x)q(x)$.

Define

$$\begin{aligned} L(x) &= \frac{[h_2(x, Y_0(x))\pi_2(Y_0(x)) + h_0(x, Y_0(x))\pi_{0,0}]h_1(x, Y_1(x))a(x)}{xf'(x)}, \\ \widetilde{L}(y) &= \frac{[h_1(X_0(y), y)\pi_1(X_0(y)) + h_0(X_0(y), y)\pi_{0,0}]h_2(X_1(y), y)\widetilde{a}(y)}{yg'(y)}, \end{aligned}$$

where $f(x) = a(x)h_1(x, Y_0(x))h_1(x, Y_1(x))$ is a polynomial defined in Section 4.4 and $g(y) = \widetilde{a}(y)h_2(X_1(y), y)$ $h_2(X_0(y), y)$ is the counterpart polynomial for function h_2.

The following Theorem shows the behaviour of $\pi_1(x)$ at x_{dom}. Recall that $\widetilde{y}_0 = Y_0(x^*)$.

Theorem 4.8 *Assumed that both $h_2(x^*, Y_0(x^*))\pi_2(Y_0(x^*)) + h_0(x^*, Y_0(y^*))\pi_{0,0} \neq 0$ and $h_1(X_0(\widetilde{y}_0), \widetilde{y}_0)$ $\pi(X_0(\widetilde{y}_0)) + h_0(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi_{0,0} \neq 0$. For the function $\pi_1(x)$, a total of four types of asymptotics exist as x approaches to a dominant singularity of $\pi_1(x)$, based on the detailed property of the dominant singularity.*

***Case 1**: If $|x_{dom}| = x^* < \min\{\widetilde{x}_1, x_3\}$, or $|x_{dom}| = \widetilde{x}_1 < \min\{x^*, x_3\}$, or $|x_{dom}| = x^* = \widetilde{x}_1 = x_3$, then*

$$\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)\pi_1(x) = c_{0,1}(x_{dom}),$$

where

$$c_{0,1}(x_{dom}) = \begin{cases} L(x_{dom}), & \text{if } x^* < \min\{\widetilde{x}_1, x_3\}; \\ \dfrac{-h_2(x_{dom}, \widetilde{y}_0)\widetilde{y}_0 \widetilde{L}(\widetilde{y}_0)}{h_1(x_{dom}, \widetilde{y}_0)Y_0'(x_{dom})x_{dom}}, & \text{if } \widetilde{x}_1 < \min\{x^*, x_3\}; \\ \dfrac{h_2(x_{dom}, \widetilde{y}_0)\widetilde{L}(\widetilde{y}_0)\widetilde{y}_0}{q_1(x_{dom})q(x_{dom})}, & \text{if } x^* = \widetilde{x}_1 = x_3, \end{cases}$$

with $\widetilde{y}_0 = Y_0(x_{dom})$.

Case 2: *If* $|x_{dom}| = x^* = x_3 < \widetilde{x}_1$ *or* $|x_{dom}| = \widetilde{x}_1 = x_3 < x^*$, *then*

$$\lim_{\frac{x}{x_{dom}} \to 1} \sqrt{1 - x/x_{dom}}\,\pi_1(x) = c_{0,2}(x_{dom}),$$

where

$$c_{0,2}(x_{dom}) = \begin{cases} \dfrac{h_2(x_{dom}, \widetilde{y}_0)\pi_2(\widetilde{y}_0) + h_0(x_{dom}, \widetilde{y}_0)\pi_{0,0}}{-q_1(x_{dom})}, & \text{if } x^* = x_3 < \widetilde{x}_1; \\ \dfrac{h_2(x_{dom}, \widetilde{y}_0)\widetilde{y}_0 \widetilde{L}(\widetilde{y}_0)}{h_1(x_{dom}, \widetilde{y}_0)q(x_{dom})}, & \text{if } \widetilde{x}_1 = x_3 < x^*, \end{cases}$$

with $\widetilde{y}_0 = Y_0(x_{dom})$.

Case 3: *If* $|x_{dom}| = x_3 < \min\{\widetilde{x}_1, x^*\}$, *then*

$$\lim_{x \to x_{dom}} \sqrt{1 - x/x_{dom}}\,\pi_1'(x) = c_{0,3}(x_{dom}),$$

where

$$c_{0,3}(x_{dom}) = -\frac{q(x_{dom})x_{dom}^2}{2}\frac{d}{dy}\left[\frac{h_2(x_{dom}, y)\pi_2(y) + h_0(x_{dom}, y)\pi_{0,0}}{-h_1(x_{dom}, y)}\right]\Bigg|_{y=Y_0(x_{dom})}.$$

Case 4: *If* $|x_{dom}| = x^* = \widetilde{x}_1 < x_3$, *then*

$$\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^2 \pi_1(x) = c_{0,4}(x_{dom}),$$

where

$$c_{0,4}(x_{dom}) = \frac{h_2(x_{dom}, \widetilde{y}_0)[h_1(\widetilde{x}_0, \widetilde{y}_0)\pi_1(\widetilde{x}_0) + h_0(\widetilde{x}_0, \widetilde{y}_0)]\pi_{0,0}]}{x^{*2}h_1'(x_{dom}, \widetilde{y}_0)Y_0'(x_{dom})h_2'(X_0(\widetilde{y}_0), \widetilde{y}_0)},$$

with $\widetilde{y}_0 = Y_0(x_{dom})$ *and* $\widetilde{x}_0 = X_0(\widetilde{y}_0)$.

Proof. **Case 1.** If $x^* < \widetilde{x}_1$, then x_{dom} is not a pole of $\pi_2(Y_0(x))$. According to Theorem 4.5, x_{dom} is a simple pole of $\pi_1(x)$. From equation (4.3) in Theorem 4.3 and Lemmas 4.4 and 4.5, we have

$$\begin{aligned} \pi_1(x) &= \frac{-h_1(x, Y_0(x))\pi_2(Y_0(x)) - h_0(x, Y_0(x))\pi_{0,0}}{h_1(x, Y_0(x))} \\ &= \frac{-[h_1(x, Y_0(x))\pi_2(Y_0(x)) + h_0(x, Y_0(x))\pi_{0,0}]h_1(x, Y_1)a(x)}{f(x)} \\ &= \frac{-[h_1(x, Y_0(x))\pi_2(Y_0(x)) + h_0(x, Y_0(x))\pi_{0,0}]h_1(x, Y_1)a(x)}{(x - x_{dom})f^*(x)}, \end{aligned}$$

where $f^*(x_{dom}) = f'(x_{dom}) \neq 0$. It follows that

$$\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right) \pi_1(x) = L(x_{dom}).$$

Similarly, if $\widetilde{x}_1 < x^*$, following the same argument used in the above, we have $\lim_{y\to\widetilde{y}_0}\left(1-\frac{y}{\widetilde{y}_0}\right)\pi_2(y) = \widetilde{L}(\widetilde{y}_0)$ and

$$\begin{aligned}
&\lim_{x\to x_{dom}}\left(1-\frac{x}{x_{dom}}\right)\pi_1(x) \\
= &\lim_{x\to x_{dom}}\frac{-h_2(x,Y_0(x))\left(1-\frac{Y_0(x)}{\widetilde{y}_0}\right)\pi_2(Y_0(x)) - (1-\frac{Y_0(x)}{y_{dom}})h_0(x,Y_0(x))\pi_{0,0}}{\frac{1-\frac{Y_0(x)}{\widetilde{y}_0}}{1-\frac{x}{x_{dom}}}h_1(x,Y_0(x))} \\
= &\frac{-h_2(x_{dom},\widetilde{y}_0)\widetilde{L}(\widetilde{y}_0)\widetilde{y}_0}{h_1(x_{dom},\widetilde{y}_0)Y_0(x)'(x_{dom})x_{dom}}.
\end{aligned}$$

In the case of $x^* = \widetilde{x}_1 = x_3 = |x_{dom}|$, we first have $\lim_{x\to x_{dom}}\left(1-\frac{Y_0(x)}{\widetilde{y}_0}\right)\pi_2(Y_0(x)) = \widetilde{L}(\widetilde{y}_0)$. Then, using equations (4.13), (4.14) and the expression for $h_1(x_3,\widetilde{y}_0)$, we obtain

$$\begin{aligned}
&\lim_{x\to x_{dom}}\left(1-\frac{x}{x_{dom}}\right)\pi_1(x) \\
= &\lim_{x\to x_{dom}}\frac{-h_2(x,Y_0(x))\frac{\sqrt{1-\frac{x}{x_{dom}}}}{1-\frac{Y_0(x)}{\widetilde{y}_0}}\left[\left(1-\frac{Y_0(x)}{\widetilde{y}_0}\right)\pi_2(Y_0(x))\right] - \sqrt{1-\frac{x}{x_{dom}}}h_0(x,Y_0(x))\pi_{0,0}}{h_1(x,Y_0(x))/\sqrt{1-\frac{x}{x_{dom}}}} \\
= &\frac{\widetilde{L}(\widetilde{y}_0)h_2(x_{dom},\widetilde{y}_0)\widetilde{y}_0}{q_1(x_{dom})q(x_{dom})}.
\end{aligned}$$

Case 2. If $x^* = x_{3,}$ then $h_1(x_{dom},\widetilde{y}_0) = p_1(x_{dom}) = 0$, using equations (4.3), (4.11), (4.13) and (4.14), we can rewrite $\pi_1(x)$ as

$$\pi_1(x) = \frac{-h_2(x,Y_0(x))\pi_2(Y_0(x)) - h_0(x,Y_0(x))\pi_{0,0}}{\sqrt{1-x/x_{dom}}\left[\sqrt{1-x/x_{dom}}p_1^*(x)+q_1(x)\right]}.$$

It follows that

$$\lim_{x\to x_{dom}}\sqrt{1-x/x_{dom}}\pi_1(x) = \lim_{x\to x_{dom}}\frac{-h_2(x,Y_0(x))\pi_2(Y_0(x)) - h_0(x,Y_0(x))\pi_{0,0}}{\left[\sqrt{1-x/x_{dom}}p_1^*(x)+q_1(x)\right]} = c_{0,2}(x_{dom}).$$

Note that $q_1(x_{dom}) \neq 0$.

Similarly, if $\widetilde{x}_1 = x_3$, then $\widetilde{y}_0$ is a pole of $\pi_2(y)$, which gives $\lim_{y\to\widetilde{y}_0}\left(1-\frac{y}{\widetilde{y}_0}\right)\widetilde{\pi}(y) = \widetilde{L}(\widetilde{y}_0)$. Again, using equations (4.3), (4.11), (4.13) and (4.14), we obtain

$$\begin{aligned}
&\lim_{x\to x_{dom}} \sqrt{1-\frac{x}{x_{dom}}}\pi_1(x) \\
= &\lim_{x\to x_{dom}} \frac{-h_2(x,Y_0(x))\frac{\sqrt{1-\frac{x}{x_{dom}}}}{1-\frac{Y_0(x)}{\widetilde{y}_0}}\left(1-\frac{Y_0(x)}{\widetilde{y}_0}\right)\pi_2(Y_0(x)) - \sqrt{1-\frac{x}{x_{dom}}}h_0(x,Y_0(x))\pi_{0,0}}{h_1(x,Y_0(x))} \\
= & -\frac{h_2(x_{dom},\widetilde{y}_0)\widetilde{L}(\widetilde{y}_0)}{h_1(x_{dom},\widetilde{y}_0)} \lim_{x\to x_{dom}} \frac{\widetilde{y}_0\sqrt{1-\frac{x}{x_{dom}}}}{(1-x/x_{dom})p^*(x) - q(x)\sqrt{1-x/x_{dom}}} \\
= & \frac{h_2(x_{dom},\widetilde{y}_0)\widetilde{L}(\widetilde{y}_0)\widetilde{y}_0}{h_1(x_{dom},\widetilde{y}_0)q(x_{dom})}.
\end{aligned}$$

Case 3. Let

$$T(x,y) = \frac{h_2(x,y)\pi_2(y) + h_0(x,y)\pi_{0,0}}{-h_1(x,y)}.$$

Then,

$$\pi_1'(x) = \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}\frac{dY_0(x)}{dx}$$

with

$$\frac{dY_0(x)}{dx} = p'(x) + q'(x)\sqrt{1-x/x_{dom}} - \frac{q(x)}{2x_{dom}\sqrt{1-x/x_{dom}}},$$

$$\frac{\partial T}{\partial x} = \frac{\widetilde{a}_2(y)\pi_2(y) + \widetilde{a}_0(y) + [a_1'(x)y + b_1'(x)]T(x,y)}{-h_1(x,y)}$$

and

$$\frac{\partial T}{\partial y} = \frac{\frac{\partial h_2(x,y)}{\partial y}\pi_2(y) + h_2(x,y)\pi_2'(y) + \frac{\partial h_0(x,y)\pi_{0,0}}{\partial y} + \frac{\partial h_1(x,y)}{\partial y}T(x,y)}{-h_1(x,y)},$$

where $p(x)$ and $q(x)$ are defined by equation (4.12). Since $\lim_{x\to x_{dom}} \sqrt{1-x/x_{dom}}\frac{\partial T}{\partial x} = 0$, $\lim_{x\to x_{dom}} \sqrt{1-x/x_{dom}}\frac{dY_0(x)}{dx} = -\frac{q(x_{dom})}{2x_{dom}}$ and $\frac{\partial T}{\partial y}$ is continuous at $(x_{dom}, Y_0(x_{dom}))$,

$$\lim_{x\to x_{dom}} \sqrt{1-x/x_{dom}}\pi_1'(x) = -\frac{q(x_{dom})}{2x_{dom}}\frac{\partial T}{\partial y}|_{(x_3,\widetilde{y}_0)} \tag{4.15}$$

$$= -\frac{q(x_{dom})}{2x_{dom}}\frac{dT(x_{dom},y)}{dy}|_{y=\widetilde{y}_0} = c_{0,3}(x_{dom}). \tag{4.16}$$

It is easy to see $c_{3,0}(x_{dom}) \neq 0$, since otherwise $\pi_1'(x_{dom}) < \infty$, which contradicts the fact that x_3 is a branch point of $\pi_1(x)$.

Case 4. From equation (4.3) and (4.5) in Theorem 4.3, we have

$$\pi_1(x) = \frac{h_2(x,Y_0(x))h_1(X_0(Y_0(x)),Y_0(x))\pi_1(X_0(Y_0(x))) + N(x)}{h_1(x,Y_0(x))h_2(X_0(Y_0(x)),Y_0(x))},$$

where

$$N(x) = [h_2(x,Y_0(x))h_0(X_0(Y_0(x)),Y_0(x)) - h_2(X_0(Y_0(x)),Y_0(x))h_0(x,Y_0(x))]\pi_{0,0}.$$

Since

$$\lim_{x\to x_{dom}} \frac{h_1(x,Y_0(x))}{x-x_{dom}} = \lim_{x\to x_{dom}} \frac{h_1(x,Y_0(x)) - h_1(x_{dom},Y_0(x_{dom}))}{x-x_{dom}} = h_1'(x_{dom},\widetilde{y}_0)$$

and

$$\begin{aligned}
\lim_{x \to x_{dom}} \frac{h_2(X_0(Y_0(x)), Y_0(x))}{x - x_{dom}} &= \lim_{x \to x_{dom}} \frac{h_2(X_0(Y_0(x)), Y_0(x)) - h_2(X_0(\widetilde{y}_0), \widetilde{y}_0)}{x - x_{dom}} \\
&= Y_0'(x_{dom}) h_2'(X_0(\widetilde{y}_0), \widetilde{y}_0),
\end{aligned}$$

we obtain

$$\lim_{x \to x_{dom}} \frac{\left(1 - \frac{x}{x_{dom}}\right)^2}{h_1(x, Y_0(x)) h_2(X_0(Y_0(x)), Y_0(x))} = \frac{1}{x_{dom}^2 h_1'(x_{dom}, \widetilde{y}_0) Y_0'(x_{dom}) h_2'(X_0(\widetilde{y}_0), \widetilde{y}_0)},$$

which yields

$$\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^2 \pi_1(x) = c_{0,4}(x_{dom}).$$

□

Remark 4.8 *It should be noted that the above theorem provides the asymptotic behaviour at a dominant singularity, either positive or negative.*

Corollary 4.2 *If* $h_2(x^*, Y_0(x^*))\pi_2(Y_0(x^*)) + h_0(x^*, Y_0(y^*))\pi_{0,0} = 0$ *or* $h_1(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi(X_0(\widetilde{y}_0)) + h_0(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi_{0,0} = 0$, *then the function* $\pi_1(x)$, *as* x *approaches to its dominant singularity, has one of the three types of asymptotic properties shown in Case 1 to Case 3 of Theorem 4.8.*

Proof. First suppose that $h_2(x^*, Y_0(x^*))\pi_2(Y_0(x^*)) + h_0(x^*, Y_0(x^*))\pi_{0,0} = 0$, but $h_1(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi_1(X_0(\widetilde{y}_0)) + h_0(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi_{0,0} \neq 0$. We then have the following four cases:

1. If $x^* < \widetilde{x}_1 < x_3$, then $\widetilde{x}_1$ is a pole and the dominant singular point of $\pi_1(x)$ since x^* is a removable singular point of $\pi_1(x)$, which leads to $\lim_{x \to \widetilde{x}} \left(1 - \frac{x}{\widetilde{x}_1}\right) \pi_1(x) = C$, the same type in Case 1.

2. If $x^* < \widetilde{x}_1 = x_3$, the same type of asymptotic result as in Case 2 can be obtained.

3. If $x^* = x_3 < \widetilde{x}_1$, then the factor $\sqrt{1 - x/x_{dom}}$ is cancelled out from both the denominator and the numerator in the expression for $\pi_1(x)$. By considering $\pi_1'(x)$, we obtain the same type of asymptotic result as that given in Case 3.

4. If $x^* = \widetilde{x}_1$, then x^* would be a pole of $\pi_2(Y_0(x))$. This would imply $\pi_2(Y_0(x^*)) = \infty$, which contradict to $h_2(x^*, Y_0(x^*))\pi_2(Y_0(x^*)) + h_0(x^*, Y_0(x^*))\pi_{0,0} = 0$ since $h_2(X_0(Y_0(x^*), Y_0(x^*)) = 0$ implies $h_2(x^*, Y_0(x^*)) \neq 0$. Hence this case is impossible.

Next, assume that both $h_2(x^*, Y_0(x^*))\pi_2(Y_0(x^*)) + h_0(x^*, Y_0(x^*))\pi_{0,0} = 0$ and $h_1(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi_1(X_0(\widetilde{y}_0)) + h_0(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi_{0,0} = 0$. We then have the following two cases:

1. If $\max\{\widetilde{x}_1, x^*\} < x_3$, then both $\widetilde{x}_1$ and x^* are removable poles of $\pi_1(x)$. By considering $\pi_1'(x)$, we obtain the same type of asymptotic result as that given in Case 3.

2. If $\max\{\widetilde{x}_1, x^*\} = x_3$, then the factor $\sqrt{1 - x/x_{dom}}$ is cancelled out from both the denominator and the numerator in the expression for $\pi_2(Y_0(x))$ if $\widetilde{x}_1 = x_3$ and for $\pi_1(x)$ if $x^* = x_3$. By considering $\pi_1'(x)$, we obtain the same type of asymptotic result as that given in Case 3.

Finally, the case in which $h_1(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi_1(X_0(\widetilde{y}_0)) + h_0(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi_{0,0} = 0$, but $h_2(x^*, Y_0(x^*))\pi_2(Y_0(x^*)) + h_0(x^*, Y_0(x^*))\pi_{0,0} \neq 0$ can be similarly considered. □

Remark 4.9 *We believe that both $h_2(x^*, Y_0(x^*))\pi_2(Y_0(x^*)) + h_0(x^*, Y_0(x^*))\pi_{0,0} \neq 0$ and $h_1(X_0(\widetilde{y}_0), \widetilde{y}_0)$ $\pi_1(X_0(\widetilde{y}_0)) + h_0(X_0(\widetilde{y}_0), \widetilde{y}_0)\pi_{0,0} \neq 0$ always hold, though at this moment we could not find a proof. However, no new type of asymptotic property will appear without this condition as shown in Corollary 4.2. In the rest of the paper, the analysis will be carried out with this condition, which is also valid without this condition.*

Remark 4.10 *When $x_{dom} = |x_3| < \min\{x^*, \widetilde{x}_1\}$, the numerator in the expression for $\pi_1(x)$ is not zero at x_3.*

5 Tail Asymptotics of Boundary Probabilities $\pi_{n,0}$ and $\pi_{0,n}$

Since $\pi_1(x)$ and $\pi_2(y)$ are symmetric, properties for $\pi_1(x)$ can be easily translated to the counterpart properties for $\pi_2(y)$. Therefore, tail asymptotics for the boundary probabilities $\pi_{0,n}$ can be directly obtained by symmetry.

The exact tail asymptotics of the boundary probabilities $\pi_{n,0}$ is a direct consequence of Theorem 4.8 and a Tauberian-like theorem applied to the function $\pi_1(x)$. Specifically, if $\pi_1(x)$ has only one dominant singularity, then Theorem 4.1 is applied; and if $\pi_1(x)$ has two dominant singularity, then Theorem 4.2 is applied.

The following theorem shows that there are four types of exact tail asymptotics, for large n, together with a possible periodic property if $\pi_1(x)$ has two dominant singularities that have the same asymptotic property.

In the theorem, let x_{dom} be the positive dominant singularity of $\pi_1(x)$. Consider the following four cases regarding which of x^*, $\widetilde{x}_1$ and x_3 will be x_{dom}:

Case 1. $x_{dom} = \min\{x^*, \widetilde{x}_1\} < x_3$ with $x^* \neq \widetilde{x}_1$, or $x_{dom} = \widetilde{x}_1 = x^* = x_3$;

Case 2. $x_{dom} = x_3 = \min\{x^*, \widetilde{x}_1\}$ with $x^* \neq \widetilde{x}_1$;

Case 3. $x_3 = x_{dom} < \min\{x^*, \widetilde{x}_1\}$;

Case 4. $x_{dom} = x^* = \widetilde{x}_1 < x_3$.

Theorem 5.1 *Consider the stable non-singular genus 1 random walk. Corresponding to the above four cases, we have the following tail asymptotic properties for the boundary probabilities $\pi_{n,0}$ for large n. In all cases, $c_{0,i}(x_{dom})$ $(1 \leq i \leq 4)$ are given in Theorem 4.8.*

1. *If $p_{i,j}$ is not X-shaped, then there are four types of exact tail asymptotics:*

Case 1: *(Exact geometric decay)*

$$\pi_{n,0} \sim c_{0,1}(x_{dom}) \left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.1}$$

Case 2: *(Geometric decay multiplied by a factor of $n^{-1/2}$)*

$$\pi_{n,0} \sim \frac{c_{0,2}(x_{dom})}{\sqrt{\pi}} n^{-1/2} \left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.2}$$

Case 3: *(Geometric decay multiplied by a factor of $n^{-3/2}$)*

$$\pi_{n,0} \sim \frac{c_{0,3}(x_{dom})}{\sqrt{\pi}} n^{-3/2} \left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.3}$$

Case 4: *(Geometric decay multiplied by a factor of n)*

$$\pi_{n,0} \sim c_{0,4}(x_{dom}) n \left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.4}$$

2. *If $p_{i,j}$ is X-shaped, but both $p_{i,j}^{(1)}$ and $p_{i,j}^{(2)}$ are not X-shaped, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) It is given by (5.1);*

Case 2: *(Geometric decay multiplied by a factor of $n^{-1/2}$) It is given by (5.2);*

Case 3: *(Geometric decay multiplied by a factor of $n^{-3/2}$)*

$$\pi_{n,0} \sim \frac{\left[c_{0,3}(x_{dom}) + (-1)^{n-1} c_{0,3}(-x_{dom})\right]}{\sqrt{\pi}} n^{-3/2} \left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.5}$$

Case 4: *(Geometric decay multiplied by a factor of n) It is given by (5.4);*

3. *If $p_{i,j}$ and $p_{i,j}^{(1)}$ are X-shaped, but $p_{i,j}^{(2)}$ is not, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) When $x^* \geq \widetilde{x}_1$, it is given by (5.1); when $x_{dom} = x^* < \widetilde{x}_1$, it is given by*

$$\pi_{n,0} \sim \left[c_{0,1}(x_{dom}) + (-1)^{n-1} c_{0,1}(-x_{dom})\right] \left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.6}$$

Case 2: *(Geometric decay multiplied by a factor of $n^{-1/2}$) When $x^* > \widetilde{x}_1$, it is given by (5.2); when $x_{dom} = x^* < \widetilde{x}_1$, it is given by*

$$\pi_{n,0} \sim \frac{\left[c_{0,2}(x_{dom}) + (-1)^{n-1} c_{0,2}(-x_{dom})\right]}{\sqrt{\pi}} n^{-1/2} \left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.7}$$

Case 3: *(Geometric decay multiplied by a factor of $n^{-3/2}$) It is given by (5.5).*

Case 4: *(Geometric decay multiplied by a factor of n) It is given by (5.4).*

4. *If $p_{i,j}$ and $p_{i,j}^{(2)}$ are X-shaped, but $p_{i,j}^{(1)}$ is not, then it is the symmetric case to **3**. All expression in **3** are valid after switching x^* and $\widetilde{x}_1$.*

5. *If all $p_{i,j}$, $p_{i,j}^{(1)}$ and $p_{i,j}^{(2)}$ are X-shaped, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) When $x^* \leq \widetilde{x}_1$, it is given by (5.6); when $x^* > \widetilde{x}_1$, it is also given by (5.6) by replacing the dominant singularity x^* by $\widetilde{x}_1$.*

Case 2: *(Geometric decay multiplied by a factor of $n^{-1/2}$) When $x^* < \widetilde{x}_1$, it is given by (5.7); when $x^* > \widetilde{x}_1$, it is also given by (5.7) by replacing the dominant singularity x^* by $\widetilde{x}_1$.*

Case 3: *(Geometric decay multiplied by a factor of $n^{-3/2}$) It is given by (5.5).*

Case 4: *(Geometric decay multiplied by a factor of n) It is given by*

$$\pi_{n,0} \sim \left[c_{0,4}(x_{dom}) + (-1)^{n-1}c_{0,4}(-x_{dom})\right] n \left(\frac{1}{x_{dom}}\right)^{n-1}. \tag{5.8}$$

Proof. **1.** Since $p_{i,j}$ is not X-shaped, all $-x_3$, $-x^*$ and $-\widetilde{x}_1$ are not dominant singularities according to Corollary 3.1, Theorem 4.5 and Theorem 4.6. Therefore, there is only one dominant singularity for $\pi_1(x)$. The tail asymptotic properties of $\pi_{n,0}$ follow from Theorem 4.8 and the direct application of the Tauberian-like theorem (Theorem 4.1).

2. We only provide a proof to the cases, which are not identical to that in 1.

Case 1. For the case that $x_{dom} = \widetilde{x}_1 = x^* = x_3$, we notice that $-x_3$ is also a dominant singularity (Corollary 3.1). In this case, the Tauberian-like theorem (Theorem 4.2) is used to have a tail asymptotic expression consisting of two terms, one, corresponding to the positive dominant singularity, with the exact geometric decay rate and the other, corresponding to the negative dominant singularity, with the geometric decay rate multiplied by a factor of $n^{-3/2}$. Therefore, the term with the geometric decay rate is the dominant (decay slower) term leading to the same tail asymptotic property given in (5.1).

Case 2. Similar to Case 1, $-x_3$ is also a dominant singularity. The Tauberian-like theorem (Theorem 4.2) leads to a tail asymptotic expression consisting of two terms, one with the geometric rate multiplied by a factor of $n^{-1/2}$ (dominant term) and the other by $n^{-3/2}$.

Case 3. In this case, both x_3 and $-x_3$ are dominant singularities having the same asymptotic property according to Theorem 4.8. The tail asymptotic expression follows from the application of the Tauberian-like theorem (Theorem 4.2).

3. In this case, $-x_3$ and $-x^*$ are singularities, but $-\widetilde{x}_1$ is not. We only provide a proof to the cases, which are not identical to that in 1 or in 2.

Case 1. For the case when $x^* = \widetilde{x}_1 = x_3$, there are two dominant singularities. The Tauberian-like theorem (Theorem 4.2) leads to a tail asymptotic expression consisting of two terms, one (corresponding to the positive singularity) with a geometric decay rate, and the other (corresponding to the negative singularity) with the same geometric decay rate multiplied by a factor of $n^{-1/2}$ that is dominated by the geometric decay.

When $x^* < \widetilde{x}_1$, both x^* and $-x^*$ are dominant singularities with the same asymptotic property, which leads to the tail asymptotic expression by using Theorem 4.2.

Case 2. For case when $x_3 = x^*$, there are two dominant singularities having the same asymptotic property. The tail asymptotic expression follows from Theorem 4.2.

Case 4. In this case, there are two dominant singularities, but the contribution from the positive dominant singularity dominates that from the negative dominant singularity. The tail asymptotic expression follows from Theorem 4.2.

4. The symmetric case to 3.

5. In this case, all $-x^*$, $-x^*$ and $-x_3$ are singularities. We only provide a proof to the cases, which are not considered in the above.

Case 1. The only new situation here is the case when $x^* = x^* = x_3$. In this case, we have the same asymptotic property at both dominant singularities, which leads to (5.6).

Case 4. In this case, we have the same asymptotic property at both dominant singularities, which leads to (5.8).

□

From the above theorem, it is clear that if there is only one dominant singularity, then the boundary probabilities $\pi_{n,0}$ have the following four types of astymptotics: **1.** exact geometric; **2.** geometric multiplied by a factor of $n^{-1/2}$; **3.** geometric multiplied by a factor of $n^{-3/2}$; and **4.** geometric multiplied by a factor of n. If there are two dominant singularities, but with different asymptotic properties, $\pi_{n,0}$ also has one of the above four types of tail asymptotic properties. Finally, if we have the same asymptotic property at both dominant singularities, then $\pi_{n,0}$ reveals a periodic property with the above four types of tail asymptotics, which is a new discovery.

6 Tail Asymptotics of the Marginal Distributions

In the previous section, we have seen that the asymptotic behaviour of the function $\pi_1(x)$ ($\pi_2(y)$) at its dominant singularity or singularities determines the tail asymptotic property of the boundary probabilities $\pi_{n,0}$ ($\pi_{0,n}$). According the the fundamental form of the random walk, it, together with the property of the kernel function $h(x, y)$, also determines the tail asymptotic property of the marginal distribution $\pi_n^{(1)} = \sum_j \pi_{n,j}$ (and $\pi_n^{(2)} = \sum_i \pi_{i,n}$).

In this section, we provide details for the exact tail asymptotics of the marginal distribution $\pi_n^{(1)}$. The exact tail asymptotics of $\pi_n^{(2)}$ can be easily obtained by symmetry. First, based on the fundamental form, we have

$$\pi(x, y) = \frac{h_1(x, y)\pi_1(x) + h_2(x, y)\pi_2(y) + h_0(x, y)\pi_{0,0}}{-h(x, y)}$$

and therefore,

$$\begin{aligned}\pi(x, 1) &= \frac{h_1(x, 1)\pi_1(x) + h_2(x, 1)\pi_2(1) + h_0(x, 1)\pi_{0,0}}{-h(x, 1)} \\ &= \frac{h_1(x, 1)\pi_1(x) + h_2(x, 1)\pi_2(1) + h_0(x, 1)\pi_{0,0}}{-\widetilde{a}(1)[x - X_0(1)][x - X_1(1)]}.\end{aligned}$$

If $M_x \geq 0$, then $X_1(1) = 1$, which implies that the denominator of the expression for $\pi(x, 1)$ does not have any zero outside the unit circle. In this case, $\pi_n^{(1)}$ has the same tail asymptotics as $\pi_{n,0}$. The only difference is the expression for the coefficient, which can be obtained from straight forward calculations.

If $M_x < 0$, then $X_0(1) = 1$ and $X_1(1) > 1$. If $p_{i,j}$ is not X-shaped, the analysis is so-called standard, details of which will be provided here. If $p_{i,j}$ is X-shaped, then there are four subcases based on if $p_{i,j}^{(k)}$ is X-shaped or not. For these cases, detailed analysis varies, but similar. We provide details here for the case where both $p_{i,j}^{(k)}$ for $k = 1, 2$ are not X-shaped. Let $z = \min\{x^*, \widetilde{x}_1\}$. and consider the following four cases:

1. $\min(X_1(1), z) < x_3$ and $X_1(1) \neq z$. In this case, $\pi_n^{(1)}$ has an exact geometric decay with the decay rate equal to $x_{dom} = \min(X_1(1), z)$:

$$\pi_n^{(1)} \sim c_1^{(x)} \left(\frac{1}{x_{dom}}\right)^{n-1},$$

where

$$c_1^{(x)} = \begin{cases} \frac{[h_1(X_1(1),1)\pi_1(X_1(1))+h_2(X_1(1),1)\pi_2(1)+h_0(X_1(1),1)\pi_{0,0}]X_1(1)}{\widetilde{a}(1)(X_1(1)-1)}, & X_1(1) < z, \\ \frac{h_1(z,1)c_{0,1}(z)+h_2(z,1)\pi_2(1)+h_0(z,1)\pi_{0,0}}{-\widetilde{a}(1)[z-X_0(1)][z-X_1(1)]}, & X_1(1) > z, \end{cases}$$

with $c_{0,1}(z)$ being given in Theorem 4.8.

2. $X_1(1) = z < x_3$. In this case, $X_1(1) = \widetilde{x}_1$ is impossible, since otherwise $h(X_1(1), 1) = 0$, which implies $1 = Y_0(\widetilde{x}_1)$ or $1 = Y_1(\widetilde{x}_1)$. This is contradiction to $Y_0(\widetilde{x}_1) > 1$. Hence, only $X_1(1) = x^*$ may hold. There are two subcases:

2(a): $1 = Y_0(x^*)$. In this case, $h_1(x^*, 1) = h_1(x^*, Y_0(x^*)) = 0$. We can write $h_1(x, 1)$ as $h_1(x, 1) = a_1(x) + b_1(x) = (x - X_1(1))h_1^*(x)$ with $\frac{h_1(x,1)}{x-X_1(1)}$ being a linear function of x, which yields

$$\begin{aligned} \pi(x,1) &= \frac{h_1(x,1)\pi_1(x) + h_2(x,1)\pi_2(1) + h_0(x,1)\pi_{0,0}}{-\widetilde{a}(1)(x-1)[x-X_1(1)]} \\ &= \frac{h_1^*(x)\pi_1(x)}{-\widetilde{a}(1)(x-1)} + \frac{h_2(x,1)\pi_2(1) + h_0(x,1)\pi_{0,0}}{-\widetilde{a}(1)(x-1)[x-X_1(1)]}. \end{aligned}$$

Therefore, $\pi(x, 1)$ has a single pole $X_1(1)$, which leads to an exact geometric decay (recalling $\pi(1, 1) \neq 1$):

$$\pi_n^{(1)} \sim c_{2,1}^{(x)} \left(\frac{1}{X_1(1)}\right)^{n-1}$$

with the coefficient given by

$$c_{2,1}^{(x)} = \lim_{x \to X_1(1)} \left(1 - \frac{x}{X_1(1)}\right) \pi(x,1) = \frac{[h_2(X_1(1),1)\pi_2(1) + h_0(X_1(1),1)\pi_{0,0}]X_1(1)}{\widetilde{a}(1)(X_1(1)-1)}.$$

2(b): $1 = Y_1(x^*)$. In this case, $h_1(x^*, Y_0(x^*)) = 0$ and $Y_0(x^*) < Y_1(x^*)$, we obtain $h_1(x^*, 1) = h_1(x^*, Y_1(x^*)) > 0$, which implies that x^* is a double pole of $\pi(x, 1)$ (noting that $h_2(x^*, 1)\pi_2(1) + h_0(x^*, 1)\pi_{0,0} > 0$ since $h_2(x^*, 1) > h_2(X_0(1), 1) = 0$ and $h_0(x^*, 1)\pi_{0,0} > 0$). The corresponding tail asymptotic is given by

$$\pi_n^{(1)} \sim c_{2,2}^{(x)} n \left(\frac{1}{X_1(1)}\right)^{n-1},$$

where

$$\begin{aligned} c_{2,2}^{(x)} &= \lim_{x \to X_1(1)} \left(1 - \frac{x}{X_1(1)}\right)^2 \pi(x,1) \\ &= \frac{X_1(1)[h_1(X_1(1),1)c_{0,1}(x^*) + h_2(X_1(1),1)\pi_2(1) + h_0(X_1(1),1)\pi_{0,0}]}{\widetilde{a}(1)[X_1(1)-1]} \end{aligned}$$

with $c_{0,1}(x^*)$ given in Theorem 4.8.

3. $\min(X_1(1), z) = x_3$. In this case, there are four possible subcases, for which proofs are omitted since they are similar to that for the previous cases:

3(a): $X_1(1) = z = x_3$ leading to an exact geometric decay:

$$\pi_n^{(1)} \sim c_{3,1}^{(x)} \left(\frac{1}{x_3}\right)^{n-1},$$

where

$$c_{3,1}^{(x)} = \lim_{x \to x_3} \left(1 - \frac{x}{x_3}\right) \pi(x,1) = \frac{h_2(x_3,1)\pi_2(1) + h_0(x_3,1)\pi_{0,0}}{x_3 \widetilde{a}(1)(x_3 - 1)}.$$

3(b): $X_1(1) = x_3 < z$ leading to an exact geometric decay:

$$\pi_n^{(1)} \sim c_{3,2}^{(x)} \left(\frac{1}{x_3}\right)^{n-1},$$

where

$$c_{3,2}^{(x)} = \lim_{x \to x_3} \left(1 - \frac{x}{x_3}\right) \pi(x,1) = \frac{h_1(x_3,1)\pi(x_3) + h_2(x_3,1)\pi_2(1) + h_0(x_3,1)\pi_{0,0}}{x_3 \widetilde{a}(1)(x_3 - 1)}.$$

3(c): $z = x_3 < X_1(1)$ with $x^* \neq \widetilde{x}_1$ leading to a geometric decay multiplied by the factor $n^{-1/2}$:

$$\pi_n^{(1)} \sim c_{3,3}^{(x)} n^{-1/2} \left(\frac{1}{x_3}\right)^{n-1},$$

where

$$c_{3,3}^{(x)} = \lim_{x \to x_3} \left(1 - \frac{x}{x_3}\right)^{1/2} \pi(x,1) = \frac{h_1(x_3,1)c_{0,2}(x_3)}{\widetilde{a}(1)(X_1(1) - 1)[X_1(1) - x_3]}$$

with $c_{0,2}(x_3)$ given in Theorem 4.8.

3(d): $z = x^* = \widetilde{x}_1 = x_3 < X_1(1)$ leading to an exact geometric decay

$$\pi_n^{(1)} \sim c_{3,4}^{(x)} \left(\frac{1}{x_3}\right)^{n-1},$$

where

$$c_{3,4}^{(x)} = \lim_{x \to z} \left(1 - \frac{x}{z}\right) \pi(x,1) = \frac{h_1(z,1)c_{0,1}(z) + h_2(z,1)\pi_2(1) + h_0(z,1)\pi_{0,0}}{-\widetilde{a}(1)[z - X_0(1)][z - X_1(1)]}.$$

4. $x_3 < \min(z, X_1(1))$ leading to a geometric decay multiplied by the factor $n^{-3/2}$:

$$\pi_n^{(1)} \sim c_4^{(x)} n^{-3/2} \left(\frac{1}{x_3}\right)^{n-1},$$

where

$$c_4^{(x)} = \lim_{x \to x_3} \left(1 - \frac{x}{x_3}\right)^{1/2} \pi'(x,1) = \frac{h_1(x_3,1)c_{0,3}(x_3)}{\widetilde{a}(1)(x_3 - 1)[X_1(1) - x_3]}$$

with $c_{0,3}(x_3)$ given in Theorem 4.8.

For the completeness, we provide a summary of tail asymptotic properties for the marginal distribution $\pi_n^{(1)}$ for all possible cases. For this purpose, let x_{dom} be the positive dominant singularity of $\pi(x,1)$. Note that $X_1(1) \neq \widetilde{x}_1$. The following are the all possible cases according to which of $\widetilde{x}_1$, x^*, x_3 and $X_1(1)$ is x_{dom}.

Case A. $x_{dom} = \min\{\widetilde{x}_1, x^*, x_3\} < X_1(1)$;

Case B. $x_{dom} = X_1(1) < \min\{\widetilde{x}_1, x^*, x_3\}$;

Case C. $x_{dom} = X_1(1) = x^* < \min\{\widetilde{x}_1, x_3\}$;

Case D. $x_{dom} = X_1(1) = x_3 < x^*$;

Case E. $x_{dom} = X_1(1) = x_3 = x^*$.

Remark 6.1 *The cases here are different from the cases classified in the previous section and the next section.*

The exact tail asymptotic properties are obtained according to the expression of $\pi(x, 1)$ and the Taubarian-like theorem.

Theorem 6.1 *For the stable non-singular genus 1 random walk, the exact tail asymptotic properties for the marginal distribution $\pi_n^{(1)}$, as n is large, are summarized as:*

Case A: *This case includes Cases 1–4 in the previous section. $\pi_n^{(1)}$ has the same types of asymptotic properties as $\pi_{n,0}$ given in Theorem 5.1, respectively, with possible different expressions for the coefficients.*

Case B: *$\pi_n^{(1)}$ has an exact geometric decay.*

Case C: *$\pi_n^{(1)}$ has an exact geometric decay if $Y_0(x^*) = 1$ and a geometric decay multiplied by a factor of n if $Y_1(x^*) = 1$, respectively.*

Case D: *$\pi_n^{(1)}$ has an exact geometric decay.*

Case E: *$\pi_n^{(1)}$ has an exact geometric decay.*

7 Tail Asymptotics for Joint Probabilities

In the previous sections, we have seen how we can derive exact tail asymptotic properties for the boundary probabilities and for the marginal distributions based on the asymptotic property of $\pi_1(x)$ ($\pi_2(y)$) and the kernel function. However, the exact tail asymptotic behaviour for joint probabilities cannot be obtained directly from them. Further tools are needed for this purpose. Our goal is to characterize the exact tail asymptotics for $\pi_{n,j}$ for each fixed j and $\pi_{i,n}$ for each fixed i. Due to the symmetry, in this section, we provide details only for the former.

The relevant balance equations of the random walk are given by

$$\begin{aligned}
(1 - p_{0,0}^{(0)})\pi_{0,0} &= p_{-1,0}^{(1)}\pi_{1,0} + p_{0,-1}^{(2)}\pi_{0,1} + p_{-1,-1}\pi_{1,1}, \\
(1 - p_{0,0}^{(1)})\pi_{1,0} &= p_{1,0}^{(0)}\pi_{0,0} + p_{-1,0}^{(1)}\pi_{2,0} + p_{-1,-1}\pi_{2,1} + p_{1,-1}^{(2)}\pi_{0,1} + p_{0,-1}\pi_{1,1}, \\
(1 - p_{0,0}^{(1)})\pi_{i,0} &= p_{1,0}^{(1)}\pi_{i-1,0} + p_{-1,0}^{(1)}\pi_{i+1,0} + p_{-1,-1}\pi_{i+1,1} + p_{1,-1}\pi_{i-1,1} + p_{0,-1}\pi_{i,1}, \quad i \geq 2, \\
(1 - p_{0,0})\pi_{i,j} &= p_{1,-1}\pi_{i-1,j+1} + p_{-1,-1}\pi_{i+1,j+1} + p_{0,-1}\pi_{i,j+1} + p_{1,0}\pi_{i-1,j} + p_{-1,0}\pi_{i+1,j} \\
&\quad + p_{1,1}\pi_{i-1,j-1} + p_{0,1}\pi_{i,j-1} + p_{-1,1}\pi_{i+1,j-1}, \quad j \geq 2.
\end{aligned}$$

Let

$$\begin{aligned}\varphi_j(x) &= \sum_{i=1}^{\infty} \pi_{i,j} x^{i-1}, \quad j \geq 0,\\ \psi_i(y) &= \sum_{j=1}^{\infty} \pi_{i,j} y^{i-1}, \quad i \geq 0.\end{aligned}$$

From the above definition, it is clear that $\varphi_0(x) = \pi_1(x)$ and $\psi_0(y) = \pi_2(y)$. From the relevant balance equations, we obtain

$$c(x)\varphi_1(x) + b_1(x)\varphi_0(x) = a_0^*(x), \tag{7.1}$$

$$c(x)\varphi_2(x) + b(x)\varphi_1(x) + a_1(x)\varphi_0(x) = a_1^*(x), \tag{7.2}$$

$$c(x)\varphi_{j+1}(x) + b(x)\varphi_j(x) + a(x)\varphi_{j-1}(x) = a_j^*(x), \quad j \geq 2, \tag{7.3}$$

or

$$\varphi_{j+1}(x) = \frac{-b(x)\varphi_j(x) - a(x)\varphi_{j-1}(x) + a_j^*(x)}{c(x)}, \quad j \geq 0, \tag{7.4}$$

where

$$\begin{aligned}a_0^*(x) &= -c_2(x)\pi_{0,1} - b_0(x)\pi_{0,0},\\ a_1^*(x) &= -c_2(x)\pi_{0,2} - b_2(x)\pi_{0,1} - a_0(x)\pi_{0,0},\\ a_j^*(x) &= -c_2(x)\pi_{0,j+1} - b_2(x)\pi_{0,j} - a_2(x)\pi_{0,j-1}, \quad j \geq 2.\end{aligned}$$

First, we establish the fact that a zero of $c(x)$ is not a pole of $\varphi_j(x)$ for all $j \geq 0$. Therefore $\varphi_j(x)$ has the same singularities as $\varphi_0(x)$.

Let $y = Y_0(x)$ be in the cut plane $\widetilde{\mathbb{C}}_x$, and let y_{dom} and x_{dom} be the positive dominating singular points of $\psi_0(y)$ and $\varphi_0(x)$, respectively. Let

$$f_k(x) = -a_2(x) \sum_{j=k-1}^{\infty} \pi_{0,j} y^{j-(k-1)} - b_2(x) \sum_{j=k}^{\infty} \pi_{0,j} y^{j-k} - c_2(x) \sum_{j=k+1}^{\infty} \pi_{0,j} y^{j-(k+1)}, \quad k \geq 1,$$

then,

$$\begin{aligned}f_1(x) &= y f_2(x) - c_2(x)\pi_{0,2} - b_2(x)\pi_{0,1},\\ f_k(x) &= y f_{k+1}(x) + a_k^*(x), \quad k \geq 2.\end{aligned} \tag{7.5}$$

According to Theorem 4.3, when $|x| < x_{dom}$, we obtain

$$\begin{aligned}h_1(x,y)\varphi_0(x) &= -h_2(x,y)\psi_0(y) - h_0(x,y)\pi_{0,0}\\ &= y[f_1(x) - a_0(x)\pi_{0,0}] + a_0^*(x) \qquad (7.6)\\ &= y^2 f_2(x) + y a_1^*(x) + a_0^*(x) \qquad (7.7)\\ &= y^3 f_3(x) + y^2 a_2^*(x) + y a_1^*(x) + a_0^*(x). \qquad (7.8)\end{aligned}$$

Let $u = \frac{y}{c(x)} = \frac{Y_0(x)}{c(x)}$. Since a zero of $c(x)$ is a zero of $Y_0(x)$, u is analytic on the cut plane $\widetilde{C}_x$. Using $\frac{b(x)}{a(x)} = -Y_1(x) - Y_0(x)$ and $\frac{c(x)}{a(x)} = Y_1(x)Y_0(x)$, we obtain

$$1 + b(x)u = -yua(x) \text{ and } \frac{1 + b(x)u}{c(x)} = -a(x)u^2. \tag{7.9}$$

Write $a = a(x)$, $b = b(x)$, $c = c(x)$, $a_i = a_i(x)$, $b_i = b_i(x)$, $a_j^* = a_j^*(x)$, $f_j = f_j(x)$ and $\varphi_j = \varphi_j(x)$. We have following Lemma, which confirms that a zero of $c(x)$ is not a pole of $\varphi_j(x)$ for all $j \geq 0$. Therefore, $\varphi_j(x)$ has the same singularities as $\varphi_0(x)$.

Lemma 7.1 *Let*

$$w_{-1} = -\frac{a_1}{au}, \quad w_0 = -b_1, \quad w_j = buw_{j-1} + (1+bu)w_{j-2}. \tag{7.10}$$

Then,

$$(-1)^j \left[buw_{j-1} + (1+bu)w_{j-2}\right] + b_1 + a_1 y = (-1)^j (1+bu)w_{j-1}, \quad j \geq 1, \tag{7.11}$$

and

$$h_1\varphi_1 = yuf_2w_0 + ug_1, \tag{7.12}$$

$$h_1\varphi_j = (-1)^{j+1}yuf_{j+1}w_{j-1} + u\sum_{k=0}^{j-2}(-1)^{j+1-k}a_{j-k}^*w_{j-1-k}(au)^k + ug_1(au)^{j-1}, \quad j \geq 2, \tag{7.13}$$

where $g_1 = a_0^*(x)a_1(x) - b_1(x)a_1^*(x)$.

Proof. By applying (7.9) and (7.10), we easily obtain equation (7.11) for $j = 1$. Assume that equation (7.11) is true for $j \leq k$, we show

$$(-1)^{k+1} \left[buw_k + (1+bu)w_{k-1}\right] + b_1 + a_1 y = (-1)^{k+1}(1+bu)w_k. \tag{7.14}$$

From the inductive assumption and the definition of w_j, we have

$$\begin{aligned} b_1 + a_1 y &= (-1)^k(1+bu)w_{k-1} - (-1)^k\left[buw_{k-1} + (1+bu)w_{k-2}\right], \\ &= (-1)^k(1+bu)w_{k-1} + (-1)^{k+1}w_k, \end{aligned} \tag{7.15}$$

which yields equation (7.14). Equation (7.12) is obtained by the direct substitutions of equations (7.1) and (7.6).

Next, we show equation (7.13). We use the induction again. According to equations (7.2), (7.7) and (7.12),

$$\begin{aligned} c(x)h_1\varphi_2 &= -bh_1\varphi_1 - a_1h_1\varphi_0 + h_1a_1^* \\ &= -b[yuf_2w_0 + ug_1] - a_1[y^2f_2 + ya_1^* + a_0^*] + a_1^*[a_1y + b_1]. \end{aligned}$$

It follows from equations (7.9) and (7.10) that

$$\begin{aligned} h_1\varphi_2 &= -bu^2f_2w_0 - a_1yuf_2 + u(au)g_1 = -uf_2\left[buw_0 + a_1\frac{1+bu}{-au}\right] + u(au)g_1 \\ &= -uf_2[buw_0 + (1+bu)w_{-1}] + u(au)g_1 = -yuf_3w_1 - a_2^*uw_1 + u(au)g_1, \end{aligned}$$

which gives equation (7.13) for $j = 2$. Assume that equation (7.13) is true for $j \leq n$. We prove the result for $j = n+1$. From equations (7.9), (7.11), (7.5) and the inductive assumption, we have

$$\begin{aligned}
& c(x)h_1\varphi_{n+1} = -bh_1\varphi_n - ah_1\varphi_{n-1} + h_1 a_n^* \\
= \quad & (-1)^{n+2} ybuf_{n+1}w_{n-1} + bu\sum_{k=0}^{n-2}(-1)^{n+2-k}a_{n-k}^* w_{n-1-k}(au)^k - bug_1(au)^{n-1} \\
& +(-1)^{n+1} yauf_n w_{n-2} + au\sum_{k=0}^{n-3}(-1)^{n+1-k}a_{n-1-k}^* w_{n-2-k}(au)^k - g_1(au)^{n-1} + a_n^*[a_1 y + b_1] \\
= \quad & (-1)^{n+2} yf_{n+1}[buw_{n-1} - yauw_{n-2}] + a_n^*\left\{(-1)^{n+2}buw_{n-1} + (-1)^{n+1}yauw_{n-2} + a_1 y + b_1\right\} \\
& +bu\sum_{k=1}^{n-2}(-1)^{n+2-k}a_{n-k}^* w_{n-1-k}(au)^k + \sum_{k=0}^{n-3}(-1)^{n+1-k}a_{n-1-k}^* w_{n-2-k}(au)^{k+1} - (1+bu)g_1(au)^{n-1} \\
= \quad & (-1)^{n+2} yf_{n+1}[buw_{n-1} + (1+bu)w_{n-2}] + (-1)^{n+2}a_n^*(1+bu)w_{n-1} \\
& +(1+bu)\sum_{k=1}^{n-2}(-1)^{n+2-k}a_{n-k}^* w_{n-1-k}(au)^k - (1+bu)g_1(au)^{n-1},
\end{aligned}$$

which yields

$$\begin{aligned}
h_1\varphi_{n+1} \quad = \quad & (-1)^{n+2}uf_{n+1}w_n + (-1)^{n+1}a_n^* au^2 w_{n-1} + au^2\sum_{k=1}^{n-2}(-1)^{n+1-k}a_{n-k}^* w_{n-1-k}(au)^k + ug_1(au)^n \\
= \quad & (-1)^{n+2}yuf_{n+1}w_n + u\sum_{k=0}^{n-1}(-1)^{n+2-k}a_{n+1-k}^* w_{n-k}(au)^k + ug_1(au)^n.
\end{aligned}$$

This completes the proof. □

Corollary 7.1 *$\varphi_0(x)$ and $\varphi_j(x)$, $j \geq 1$, have the same singularities.*

The following Lemma is useful in characterizing the tail asymptotics of $\pi_{n,j}$ for a fixed j.

Lemma 7.2 *If $\min\{x^*, \widetilde{x}_1\} > x_3$, then*

$$\lim_{x \to x_{dom}} \sqrt{1 - \frac{x}{x_{dom}}}\varphi_j'(x) = c_{3,j}(x_{dom}),$$

where $c_{3,0}(x_{dom})$ is given in Theorem 4.8 and

$$c_{3,j+1}(x_{dom}) = [A_3(x_{dom}) + B_3(x_{dom})j]\left(\frac{1}{Y_1(x_{dom})}\right)^j, \quad j \geq 0, \tag{7.16}$$

with

$$A_3(x_{dom}) = -\frac{c_{3,0}(x_{dom})b_1(x_{dom})}{c(x_{dom})}, \tag{7.17}$$

$$B_3(x_{dom}) = \frac{-h_1(x_{dom}, Y_0(x_{dom}))c_{3,0}(x_{dom})}{c(x_{dom})}. \tag{7.18}$$

Proof. When $\min\{x^*, \widetilde{x}_1\} > x_3$, we have $x_{dom} = \pm x_3$. Without lose of generality, we assume $x_{dom} = x_3$ in the proof. Since $\varphi_j(x)$, $j \geq 0$, is continuous at x_3, $\lim_{x\to x_3} \sqrt{1-\frac{x}{x_3}}\varphi_j(x) = 0$. Let $j = 1$. Then,

$$\varphi_1'(x) = \frac{-c'(x)\varphi_1(x) - b_1(x)\varphi_0'(x) - b_1(x)\varphi_0'(x) + a_0^{*\prime}(x)}{c(x)}$$

and

$$\lim_{x\to x_3} \sqrt{1-\frac{x}{x_3}}\varphi_1'(x) = \frac{-b_1(x_3)c_{3,0}(x_3)}{c(x_3)} = c_{3,1}(x_3).$$

Assume that $\lim_{x\to x_3} \sqrt{1-\frac{x}{x_3}}\varphi_k'(x)$ exists for $k \leq j$ and

$$\lim_{x\to x_3} \sqrt{1-\frac{x}{x_3}}\varphi_k'(x) = c_{3,k}(x_3),$$

we obtain

$$\varphi_{k+1}'(x) = \frac{-c'(x)\varphi_{k+1}(x) - b(x)\varphi_k'(x) - b'(x)\varphi_k(x) - a(x)\varphi_{k-1}'(x) - a'(x)\varphi_{k-1}(x) + a_k^{*\prime}(x)}{c(x)},$$

and

$$\lim_{x\to x_3} \sqrt{1-\frac{x}{x_3}}\varphi_{k+1}'(x) = \frac{-b(x_3)c_{3,k}(x_3) - a(x_3)c_{3,k-1}(x_3)}{c(x_3)} = c_{3,k+1}(x_3).$$

Therefore, we can inductively have

$$\begin{aligned} c_{3,1}(x_3)c(x_3) + c_{3,0}(x_3)b_1(x_3) &= 0, && (7.19)\\ c_{3,2}(x_3)c(x_3) + c_{3,1}(x_3)b(x_3) + c_{3,0}(x_3)a_1(x_3) &= 0, && (7.20)\\ c_{3,j+1}(x_3)c(x_3) + b(x_3)c_{3,j}(x_3) + a(x_3)c_{3,j-1}(x_3) &= 0, \quad j \geq 2. && (7.21) \end{aligned}$$

It follows that $\{c_{3,k}(x_3)\}$ is the solution of the second order recursive relation determined by equations (7.19)–(7.21). Since $b^2(x_3) - 4a(x_3)c(x_3) = 0$, $c_{3,j}(x_3)$ takes the form given by equation (7.16). $A_3(x_3)$ and $B_3(x_3)$ are obtained by using the initial equations:

$$\begin{aligned} A_3(x_3)c(x_3) + c_{3,0}(x_3)b_1(x_3) &= 0,\\ \frac{[A_3(x_3) + B_3(x_3))c(x_3)]}{Y_1(x_3)} + A_3(x_3)b(x_3) + c_{3,0}(x_3)a_1(x_3) &= 0. \end{aligned}$$

□

We are now ready to prove the main theorem of this section, in which

$$\begin{aligned} A_1(x_{dom}) &= -B_1(x_{dom}) + \frac{-c_{1,0}(x_{dom})b_1(x_{dom})}{c(x_{dom})} && (7.22)\\ &= \left(\frac{h_1(x_{dom}, Y_0(x_{dom}))}{a(x_{dom})[Y_1(x_{dom}) - Y_0(x_{dom})]Y_0(x_{dom})} - \frac{b_1(x_{dom})}{c(x_{dom})}\right) c_{1,0}(x_{dom}), \end{aligned}$$

$$A_2(x_{dom}) = -\frac{c_{2,0}(x_{dom})b_1(x_{dom})}{c(x_{dom})}, \tag{7.23}$$

$A_3(x_{dom})$ is given in (7.17),

$$A_4(x_{dom}) = -\frac{b_1(x_{dom})c_{0,4}(x_{dom})}{c(x_{dom})}, \tag{7.24}$$

$$B_1(x_{dom}) = \frac{-h_1(x_{dom}, Y_0(x_{dom}))c_{1,0}(x_{dom})}{a(x_{dom})[Y_1(x_{dom}) - Y_0(x_{dom})]Y_0(x_{dom})}, \tag{7.25}$$

$$B_2(x_{dom}) = \frac{-c_{2,0}(x_{dom})h_1(x_{dom}, Y_0(x_{dom}))}{aY_0(x_{dom})^2}, \tag{7.26}$$

and $B_3(x_{dom})$ is given in (7.18).

Theorem 7.1 *Consider the stable non-singular genus 1 random walk. Corresponding to the four case, we then have the following tail asymptotic properties for the joint probabilities $\pi_{n,j}$ for large n.*

1. *If $p_{i,j}$ is not X-shaped, then there are four types of exact tail asymptotics:*

Case 1: *(Exact geometric decay)*

$$\pi_{n,j} \sim \left[A_1(x_{dom})\left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} + B_1(x_{dom})\left(\frac{1}{Y_0(x_{dom})}\right)^{j-1}\right]\left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1; \tag{7.27}$$

Case 2: *(Geometric decay multiplied by a factor of $n^{-1/2}$)*

$$\pi_{n,j} \sim \frac{[A_2(x_{dom}) + (j-1)B_2(x_{dom})]}{\sqrt{\pi}}\left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n^{-1/2}\left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1; \tag{7.28}$$

Case 3: *(Geometric decay multiplied by a factor of $n^{-3/2}$)*

$$\pi_{n,j} \sim \frac{[A_3(x_{dom}) + (j-1)B_3(x_{dom})]}{\sqrt{\pi}}\left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n^{-3/2}\left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1; \tag{7.29}$$

Case 4: *(Geometric decay multiplied by a factor of n)*

$$\pi_{n,j} \sim \left[A_4(x_{dom})\left(\frac{1}{Y_1(x_{dom})}\right)^{j-1}\right] n \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1. \tag{7.30}$$

2. *If $p_{i,j}$ is X-shaped, but both $p^{(1)}_{i,j}$ and $p^{(2)}_{i,j}$ are not X-shaped, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) It is given by (7.27);*

Case 2: *(Geometric decay multiplied by a factor of $n^{-1/2}$) It is given by (7.28);*

Case 3: *(Geometric decay multiplied by a factor of $n^{-3/2}$) It is given by*

$$\pi_{n,j} \sim \frac{[A_3(x_{dom}) + (-1)^{n+j}A_3(-x_{dom})]}{\sqrt{\pi}}\left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n^{-3/2}\left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1; \tag{7.31}$$

Case 4: *(Geometric decay multiplied by a factor of n) It is given by (7.30).*

3. *If $p_{i,j}$ and $p^{(1)}_{i,j}$ are X-shaped, but $p^{(2)}_{i,j}$ is not, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) When $\widetilde{x}_1 < x^*$, it is given by (7.27); when $\widetilde{x}_1 = x^* = x_3$, it is also given by (7.27); when $x^* < \widetilde{x}_1$, it is given by*

$$\pi_{n,j} \sim \left[A_1(x_{dom}) + (-1)^{n+j} A_1(-x_{dom})\right] \left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1; \tag{7.32}$$

Case 2: *(Geometric decay multiplied by a factor of $n^{-1/2}$) When $x^* > \widetilde{x}_1$, it is given by (7.28); when $x^* < \widetilde{x}_1$, it is given by*

$$\pi_{n,j} \sim \frac{\left[A_2(x_{dom}) + (-1)^{n+j} A_2(-x_{dom})\right]}{\sqrt{\pi}} \left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n^{-1/2} \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1; \tag{7.33}$$

Case 3: *(Geometric decay multiplied by a factor of $n^{-3/2}$) It is given by*

$$\pi_{n,j} \sim \frac{\left[A_3(x_{dom}) + (-1)^{n+j} A_3(-x_{dom})\right]}{\sqrt{\pi}} \left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n^{-3/2} \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1;$$

Case 4: *(Geometric decay multiplied by a factor of n) It is given by (7.30).*

4. *If $p_{i,j}$ and $p^{(2)}_{i,j}$ are X-shaped, but $p^{(1)}_{i,j}$ is not, then it is the symmetric case to **3**. All expression in **3** are valid after switching x^* and $\widetilde{x}_1$.*

5. *If all $p_{i,j}$, $p^{(1)}_{i,j}$ and $p^{(2)}_{i,j}$ are X-shaped, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) When $x^* \leq \widetilde{x}_1$, it is given by (7.32); when $x^* > \widetilde{x}_1$, it is also given by (7.32) by replacing the dominant singularity x^* by $\widetilde{x}_1$;*

Case 2: *(Geometric decay multiplied by a factor of $n^{-1/2}$) When $x^* < \widetilde{x}_1$, it is given by (7.33); when $x^* > \widetilde{x}_1$, it is also given by (7.33) by replacing the dominant singularity x^* by $\widetilde{x}_1$;*

Case 3: *(Geometric decay multiplied by a factor of $n^{-3/2}$) It is given by (7.31);*

Case 4: *(Geometric decay multiplied by a factor of n) It is given by*

$$\pi_{n,j} \sim \left[A_4(x_{dom}) + (-1)^{n+j} A_4(-x_{dom})\right] \left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1.$$

PROOF. **1.**

Case 1: It follows from Section 4.6 that $\lim_{x\to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right) \varphi_0(x) = c_{0,1}(x_{dom})$. By the induction and equations (7.1)–(7.3), $\lim_{x\to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right) \varphi_j(x) = c_{1,j}(x_{dom})$ with

$$\begin{aligned}
c_{1,1}(x_{dom})c(x_{dom}) + c_{1,0}(x_{dom})b_1(x_{dom}) &= 0, \\
c_{1,2}(x_{dom})c(x_{dom}) + c_{1,1}(x_{dom})b(x_{dom}) + c_{1,0}(x_{dom})a_1(x_{dom}) &= 0, \\
c_{1,j+1}(x_{dom})c(x_{dom}) + c_{1,j}(x_{dom})b(x_{dom}) + c_{1,j-1}(x_{dom})a(x_{dom}) &= 0, \quad j \geq 2.
\end{aligned}$$

Since $c_{1,j}(x_{dom})$, $j \geq 0$, satisfies the second order recursive relation above, it takes the form of

$$c_{1,j+1}(x_{dom}) = A_1(x_{dom})\left(\frac{1}{Y_1(x_{dom})}\right)^j + B_1(x_{dom})\left(\frac{1}{Y_0(x_{dom})}\right)^j, \quad j \geq 0.$$

To determine $A_1 = A_1(x_{dom})$ and $B_1 = B_1(x_{dom})$, we use the initial equations:

$$(A_1 + B_1)c(x_{dom}) + c_{1,0}(x_{dom})b_1(x_{dom}) = 0, \tag{7.34}$$

$$\left[A_1\left(\frac{1}{Y_1(x_{dom})}\right) + B_1\left(\frac{1}{Y_0(x_{dom})}\right)\right]c(x_{dom}) + (A_1 + B_1)b(x_{dom}) + c_{1,0}(x_{dom})a_1(x_{dom}) = 0. \tag{7.35}$$

Multiplying both sides of equation (7.35) by $Y_0(x_{dom})$, adding the resulting one to (7.34), and taking into account $a(x_{dom})Y_0^2(x_{dom}) + b(x_{dom})Y_0(x_{dom}) + c(x_{dom}) = 0$, $h_1(x_{dom}, Y_0(x_{dom})) = a_1(x_{dom})Y_0(x_{dom}) + b_1(x_{dom})$ and $c(x_{dom}) = Y_0(x_{dom})Y_1(x_{dom})a(x_{dom})$ yield:

$$\begin{aligned}
(A_1 + B_1)c(x_{dom}) + c_{1,0}(x_{dom})b_1(x_{dom}) &= 0,\\
A_1\frac{Y_0(x_{dom})}{Y_1(x_{dom})}c(x_{dom}) + B_1c(x_{dom}) + (A_1 + B_1)b(x_{dom})Y_0(x_{dom}) + c_{1,0}(x_{dom})a_1(x_{dom})Y_0(x_{dom}) &= 0,
\end{aligned}$$

which gives (7.25) and (7.22). So, $B_1(x_{dom}) = 0$ if $x_{dom} = x^*$ and $B_1(x_{dom}) \neq 0$ if $x_{dom} = \widetilde{x}_1$. By the Tauberian-like theorem, we obtain (7.27).

Case 2: Similar to that for 1-Case 1. From the proof, we have (7.26) and (7.23).

Case 3: Write

$$\varphi_j'(x) = \sum_{n=0}^{\infty}(n+1)\pi_{n+2,j}x^n = \sum_{n=0}^{\infty}(n+1)x_3^n\pi_{n+2,j}\left(\frac{x}{x_3}\right)^n.$$

According Lemma 7.2 and the Tauberian-like theorem, we have

$$(n+1)x_3^n\pi_{n+2,j} \sim \frac{c_{3,j}(x_3)}{\sqrt{\pi}}n^{-1/2},$$

which is equivalent to (7.29).

Case 4: The results can be proved in the same fashion as in Case 1 and Case 2.

The proofs of the other cases are omitted due to the similarity to **1** and Theorem 5.1. □

8 Examples and Concluding Remarks

In this paper, for a non-singular genus 1 random walk, we proposed a kernel method to study the exact tail asymptotic behaviour of the joint stationary probabilities along a coordinate direction, when the value of the other coordinate is fixed, and also the exact tail asymptotic behaviour for the two marginal distributions. A total of four different types of exact tail asymptotics exists. The fourth one, a geometric decay multiplied by a factor n, was not reported before for this discrete-time model (the same type was reported recently for a continuous-time random walk model by Dai and Miyazawa [8]). In this study, we also revealed a new periodic phenomena for all four types of exact tail asymptotics when there are two dominant singularities for the unknown generating function, say $\pi_1(x)$, with the same asymptotic property at them.

The key idea of this kernel method is simple and the use of the Tauberian-like theorem greatly simplifies the analysis, which, unlike in the situation when a standard Tauberian theorem is used, is also rigorous. Under the assumption that there is only one dominant singularity, this method provides a straightforward routine analysis for the exact tail asymptotic behaviour. However, without this assumption, the analysis is not simple, at least to our best effort, for telling how many dominant singularities and when a pole is simple. It is also challenging to characterize the exact tail asymptotic along a coordinate direction when the value of the other coordinate is not zero, since it is not a direct consequence of the kernel method.

This kernel method can also be used for characterizing the exact tail asymptotics for the non-singular genus 0 case and the singular random walks (see Li, Tavakoli and Zhao [32]). With the detailed analysis provided in this paper, we expect further research in applying this kernel method to more general models.

The complete characterization of the exact tail asymptotic behaviour provided in this paper does not necessarily imply that for any specific model, a characterization explicitly in terms of the system parameters exists. However, we are confident that for any specific model, if using a different method could lead to a such characterization, in terms of system parameters, then it can be done using the kernel method. Finally, we mention two examples, which have been analyzed by using the proposed kernel method.

Example 1. A generalized two-demand model was considered in Li and Zhao [36] using the same idea proposed in this paper. For this model, let λ and λ_k $(k = 1, 2)$ be the Poisson arrival rate with two demands and the arrival rate of the two dedicated Poisson arrivals, respectively. Furthermore, let μ_k $(k = 1, 2)$ be the exponential service rates of the two independent parallel servers. For a detailed description of the model, one may refer to [36]. For this model, the three regions, on which the joint probabilities along a coordinate direction, say queue 1, have an exact geometric decay, a geometric decay multiplied by a factor $n^{-1/2}$ and a geometric decay multiplied by a factor $n^{-3/2}$ are extremely simple, which are: (a) $\frac{\mu_1}{\lambda+\lambda_1} < \frac{\mu_2-\lambda_2}{\lambda}$; (b) $\frac{\mu_1}{\lambda+\lambda_1} = \frac{\mu_2-\lambda_2}{\lambda}$; and (c) $\frac{\mu_1}{\lambda+\lambda_1} > \frac{\mu_2-\lambda_2}{\lambda}$, respectively.

Example 2. Consider the simple random walk, or a random walk for which $p_{i,j}$ and both $p_{i,j}^{(k)}$ $(k = 1, 2)$ are cross-shaped. We then can follow the general results obtained in this paper to have refined properties. For example, consider the case of $M_y > 0$ and $M_x < 0$ and assume that the system is stable. Then, along the x-direction, $\pi_{n,j}$ has three types exact asymptotics in the following respective regions:

1. Exact geometric:

$$\frac{x_3}{x_3 - 1}\left[\sqrt{\frac{p_{0,-1}}{p_{0,1}}} - 1\right] p_{0,1}^{(1)} + p_{1,0}^{(1)} x_3 > p_{-1,0}^{(1)};$$

2. Geometric with a factor $n^{-1/2}$:

$$\frac{x_3}{x_3 - 1}\left[\sqrt{\frac{p_{0,-1}}{p_{0,1}}} - 1\right] p_{0,1}^{(1)} + p_{1,0}^{(1)} x_3 = p_{-1,0}^{(1)};$$

3. Geometric with a factor $n^{-3/2}$:

$$\frac{x_3}{x_3 - 1}\left[\sqrt{\frac{p_{0,-1}}{p_{0,1}}} - 1\right] p_{0,1}^{(1)} + p_{1,0}^{(1)} x_3 < p_{-1,0}^{(1)}.$$

When $M_y < 0$ and $M_x < 0$, this example also reveals the fourth type of exact tail asymptotic property, or a geometric decay multiplied by the factor n along the x-coordinate direction in the region defined by the following conditions:

$$\frac{x_3}{x_3 - 1}\left[\sqrt{\frac{p_{0,-1}}{p_{0,1}}} - 1\right] p_{0,1}^{(1)} + p_{1,0}^{(1)} x_3 \geq p_{-1,0}^{(1)}, \tag{8.1}$$

$$\frac{y_3}{y_3 - 1}\left[\sqrt{\frac{p_{-1,0}}{p_{1,0}}} - 1\right] p_{1,0}^{(2)} + p_{0,1}^{(2)} y_3 \geq p_{0,-1}^{(2)}, \tag{8.2}$$

$$h_1(x^*, \widetilde{y}_0) = 0, \tag{8.3}$$

$$\frac{p_{-1,0}}{p_{1,0}} < \frac{p_{-1,0}^{(1)}}{p_{1,0}^{(1)}}, \tag{8.4}$$

and

$$\frac{(x^* - 1)p_{0,-1}^{(2)} p_{1,0} + p_{1,0}^{(2)} p_{0,-1}}{(x^* - 1)p_{0,1}^{(2)} p_{1,0} + p_{1,0}^{(2)} p_{0,1}} = 1 + \frac{(x^* - 1)[p_{-1,0}^{(1)} - p_{1,0}^{(1)} x^*]}{p_{0,1}^{(1)} x^*}. \tag{8.5}$$

Here, $x^* \in (1, x_3]$ and $y^* \in (1, y_3]$ are the zero $h_1(x, Y_0(x))$ and $h_2(X_0(y), y)$, respectively, whose existence is guaranteed by Lemma 4.8 under conditions (8.1) and (8.2); $\widetilde{y}_0 = Y_0(x^*)$ and in this case we have $\widetilde{y}_0 = y^*$; and $\widetilde{x}_0 = X_0(Y_0(x^*))$.

It is not very difficult to see this is not an empty region. The last thing which we need to check is the coefficient

$$c_{0,4}(x_{dom}) = \frac{h_2(x_{dom}, y^*)[h_1(\widetilde{x}_0, y^*)\pi(\widetilde{x}_0) + h_0(\widetilde{x}_0, y^*)]\pi_{0,0}]}{x^{*2} h_1'(x_{dom}, y^*) Y_0'(x_{dom}) h_2'(X_0(y^*), y^*)} \neq 0, \tag{8.6}$$

or

$$h_1(\widetilde{x}_0, y^*)\pi(\widetilde{x}_0) + h_0(\widetilde{x}_0, y^*)\pi_{0,0} \neq 0,$$

which is true since $h_2(x_{dom}, y^*) = h_2(X_1(y^*), y^*) > h_2(X_0(y^*), y^*) = 0$.

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A Proof to Lemmas 4.4–4.7 and Propositions 4.1–4.2

<u>PROOF. of Lemma 4.4.</u> **1.** From $h(x,y)=0$, we have

$$y' = -\frac{a'(x)y^2 + b'(x)y + c'(x)}{2a(x)y + b(x)}.$$

Using $a(1)+b(1)+c(1)=0$, the property in (2.2) and the expression for $Y_k(1)$ in Lemma 3.1, we obtain (a) and (b). (c) is obvious.

2. There are two possible cases: $M_y<0$ and $M_y>0$. If $M_y<0$, according to the ergodicity condition in Theorem 2.1, $M_y^{(1)}M_x - M_yM_x^{(1)} < 0$ must hold, which yields

$$\begin{aligned} f'(1) &= a(1)h_1'(1,Y_0(1))h_1(1,Y_1(1)) \\ &= a(1)\left[a_1'(1) + a_1(1)Y_0'(1) + b_1'(1)\right] h_1(1,Y_1(1)) \\ &= \frac{a(1)h_1(1,Y_1(1))}{-M_y}\left[M_y^{(1)}M_x - M_yM_x^{(1)}\right] < 0. \end{aligned}$$

From equation (4.8), $f(x_3)\geq 0$, it follows that $f(x)=0$ has a root in $(1,x_3]$ since $f(1)=0$ and $f'(1)<0$.

If $M_y>0$, we have

$$\begin{aligned} f'(1) &= a(1)h_1(1,Y_0(1))h_1'(1,Y_1(1)) \\ &= \frac{-a(1)h_1(1,Y_0(1))[M_xM_y^{(1)} - M_yM_x^{(1)}]}{M_y}. \end{aligned}$$

If $M_xM_y^{(1)} - M_yM_x^{(1)} < 0$, from $f(x_2)\geq 0$, $f(1)=0$ and $f'(1)>0$, $f(x)=0$ has a root in $[x_2,1)$. Similarly, if $M_xM_y^{(1)} - M_yM_x^{(1)} > 0$, we have $f'(1)<0$, which implies that $f(x)=0$ has a root in $(1,x_3]$. Also, 1 is not a repeated root of $f(x)=0$ since $f'(1)\neq 0$ when $M_y\neq 0$. □

<u>PROOF. of Lemma 4.5.</u> **1.** Suppose $f(z)=0$. From equation (4.8), we have $F(z)=0$. So we can write $F(x)=(x-z)G(z)$. Similarly, since $D_1(z)=0$ (Recall $D_1(x)=b^2(x)-4a(x)c(x)$), we can write $D_1(x)=(x-z)D^*(x)$, where $D^*(x)$ is a polynomial. It follows that $f(x)=(x-z)T(x)$, where

$$T(x) = a(x)[a_1(x)]^2\left\{(x-z)[G(z)]^2 - \frac{D^*(x)}{4a^2(x)}\right\}.$$

Since the random walk has genus 1, z is not a repeated root of $D_1(x)=0$, which implies $a(x)[a_1(x)]^2\frac{D^*(z)}{4a^2(x)} \neq 0$ (note that $a(z)\neq 0$ since $D_1(z)=0$ and $b(z)>0$ when $z<0$). It follows that $T(z)\neq 0$, that is, z is not a repeated root of $f(x)=0$.

2. This is a direct consequence of equation (4.8).

3. Suppose x' is a common root. If $a(x')\neq 0$, it is easy to obtain that x' is a branch point. Assume $a(x')=0$. Clearly, x' cannot be a positive number. Since $\widetilde{f_1}(x') = a_1(x')[-2b(x')]=0$, $f_0(x')=\frac{a_1(x')c(x')}{-b(x')}+b_1(x')=0$ and $b(x')\neq 0$, we obtain $a_1(x')=0$ and $b_1(x')=0$, which implies that $x'=0$ since $b_1(x)$ has only nonnegative zeros.

4. Let $-|z|$ be a negative root of $f_0(x)=0$ in $[-x_3,-1)$. From the definition of $f_0(x)$, we have $\sum_{i\geq -1, j\geq 0} p_{i,j}^{(1)}[-|z|]^iY_0^j(-|z|)=1$, which implies $f_0(|z|)>0$ since $Y_0(|z|)>|Y_0(-|z|)|$. According to

$f_0(1) \le 0$ and Lemma 4.4-1, $f_0(x) = 0$ has a root, say z' in $(1, |z|)$. Again, from $Y_0(|z'|) > |Y_0(-|z'|)|$, $f_0(-|z'|) < 0$, which implies $f_0(x) = 0$ has a root in $(-|z'|, -1)$ since $f_0(-1) > 0$. Clearly, this root is greater than $-|z|$.

5. Let $|x| \in (1, x_3]$. Since $-b(-|x|) < 0$, $Y_0(-|x|) = \frac{-b(-|x|)+\sqrt{D_1(x)}}{a(-|x|)} = \frac{2c(-|x|)}{-b(-|x|)-\sqrt{D_1(-|x|)}}$. From $b(-|x|) \ge b(|x|)$, $\sqrt{D_1(-|x|)} > \sqrt{D_1(|x|)}$ and $|c(-|x|)| \le c(|x|)$, we obtain $|Y_0(-|x|)| < \frac{2c(|x|)}{b(|x|)+\sqrt{D_1(|x|)}} = Y_0(|x|)$. □

PROOF. of Lemma 4.6. Assume $|z| = 1$ and $z \neq 1$ or -1. From Lemma 3.2-1, $|Y_0(z)| < 1$. Since

$$h_1(x, y) = a_1(x)y + b_1(x) = x \left(\sum_{i \ge -1, j \ge 0} p_{i,j} x^i y^j - 1 \right)$$

and when $|z| = 1$, $\left|\frac{1}{z}\right| = 1$ as well, we obtain

$$\left| \sum_{i \ge -1, j \ge 0} p_{i,j} z^i Y_0(z)^j \right| \le \sum_{i \ge -1, j \ge 0} p_{i,j} |z^i| |Y_0(z)|^j < 1,$$

which yields $f_0(x) = h_1(z, Y_0(z)) \neq 0$.

For $z = -1$, $|Y_0(-1)| < 1$ if $p_{i,j}$ is not X-shaped, and $Y_0(-1) = -1$ if $p_{i,j}$ is X-shaped and $p_{i,j}^{(1)}$ is not X-shaped. It follows that $f_0(-1) > 0$ in both cases since $b_1(-1) \ge |a_1(-1)|$ and $b_1(-1) > 0$ in the first case and $b_1(-1) > |a_1(-1)|$ in the second case. □

For special ransom walk 1, we have

$$a_1(x) = p_{0,1}^{(1)} x + p_{1,1}^{(1)} x^2 \quad \text{and} \quad b_1(x) = -x + p_{1,0}^{(1)} x^2, \tag{A.1}$$

$$a(x) = p_{0,1} x, b(x) = p_{-1,0} - x + p_{1,0} x^2 \quad \text{and} \quad c(x) = p_{0,-1} x. \tag{A.2}$$

In this case, $f(x)$ becomes

$$f(x) = x^2 f^*(x), \tag{A.3}$$

where

$$\begin{aligned} f^*(x) &= d_4^* x^4 + d_3^* x^3 + d_2^* x^2 + d_1^* x + d_0^* \\ &= p_{0,1} x [1 - p_{1,0}^{(1)} x]^2 + [p_{-1,0} - x + p_{1,0} x^2][1 - p_{1,0}^{(1)} x][p_{0,1}^{(1)} + p_{1,1}^{(1)} x] + p_{0,-1} x [p_{0,1}^{(1)} + p_{1,1}^{(1)} x]^2 \end{aligned}$$

with

$$d_0^* = p_{-1,0} p_{0,1}^{(1)} \text{ and } d_4^* = -p_{1,0} p_{1,0}^{(1)} p_{1,1}^{(1)}.$$

PROOF. of Proposition 4.1. **1.** Obviously, From equation (A.3) and Lemma 4.4, $f(x) = 0$ has at least four real roots with two in $[x_2, x_3]$ and two equal to zero. The facts that $f(x_1) \ge 0$, $f(x_4) \ge 0$ and $f(\pm\infty) = -\infty$ yield one root in $(-\infty, x_1]$ and another root in $[x_4, +\infty)$.

2. It is a direct result of Proposition 4.1-1 and Lemma 4.5-4. □

For the random walk considered in Theorem 4.5-2 (or both $p_{i,j}$ and $p_{i,j}^{(1)}$ are X-shaped, we have

$$a_1(x) = p_{-1,1}^{(1)} + p_{1,1}^{(1)} x^2, \quad b_1(x) = -x, \tag{A.4}$$

$$a(x) = p_{-1,1} + p_{1,1}x^2, \quad b(x) = -x, \quad c(x) = p_{-1,-1} + p_{1,-1}x^2. \tag{A.5}$$

Therefore, $f(x)$ becomes

$$\begin{aligned} f(x) &= a(x)b_1^2(x) - b(x)b_1(x)a_1(x) + c(x)a_1^2(x) \\ &= x^2[p_{-1,1} + p_{1,1}x^2] - x^2[p^{(1)}_{-1,1} + p^{(1)}_{1,1}x^2] + [p_{-1,-1} + p_{1,-1}x^2][p^{(1)}_{-1,1} + p^{(1)}_{1,1}x^2]^2 \\ &= d_6x^6 + d_4x^4 + d_2x^2 + d_0, \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} d_6 &= \left[p^{(1)}_{1,1}\right]^2 p_{1,-1}, \\ d_4 &= p_{1,1} - p^{(1)}_{1,1} + 2p_{1,-1}p^{(1)}_{-1,1}p^{(1)}_{1,1} + p_{-1,-1}\left[p^{(1)}_{1,1}\right]^2, \\ d_2 &= p_{-1,1} - p^{(1)}_{-1,1} + 2p_{-1,-1}p^{(1)}_{-1,1}p^{(1)}_{1,1} + p_{1,-1}\left[p^{(1)}_{-1,1}\right]^2, \\ d_0 &= \left[p^{(1)}_{-1,1}\right]^2 p_{-1,-1}. \end{aligned}$$

PROOF. of Lemma 4.7 $f(1) = f(-1) = 0$ follows from $Y_i(1) = -Y_i(-1)$, $a_1(1) = a_1(-1)$ and $b_1(1) = -b_1(-1)$. From Lemma 4.5, there exists an $x_0 \neq 1$, $x_0 \in [x_2, x_3]$ such that $f(x_0) = 0$. We provide details for the case of $f_1(z) = 0$ and a similar proof can be found for the other case. Since $x_2 < x_0 \leq x_3$, $-b(z) = -b(-z) > 0$. It follows that

$$Y_1(x_0) = \frac{-b(x_0)}{2a(x_0)} + \frac{\sqrt{b^2(x_0) - 4a(x_0)c(x_0)}}{2a(x_0)} \tag{A.7}$$

and

$$f_1(x_0) = a_1(x_0)Y_1(x_0) + b_1(x_0) = 0. \tag{A.8}$$

On the other hand, from $-x_3 \leq -x_0 < -1$ we have $-b(-x_0) < 0$, which yields

$$Y_1(-x_0) = \frac{b(x_0)}{2a(x_0)} - \frac{\sqrt{b^2(x_0) - 4a(x_0)c(x_0)}}{2a(x_0)} = -Y_1(x_0) \tag{A.9}$$

and

$$f_1(-x_0) = a_1(-x_0)Y_1(-x_0) + b_1(-x_0) = -f_1(x_0) = 0.$$

It follows from equation (A.6) that $f(x)$ can be written as

$$f(x) = d_6(x^2 - 1)(x^2 - x_0^2)(x^2 + \eta).$$

Since $\frac{d_0}{d_6} > 0$, we have $\eta > 0$, which indicates that $f(x) = 0$ has two complex roots. □

PROOF. of Proposition 4.2. Suppose that one of the two complex roots is a root of $f_0(x) = 0$. First assume $\frac{d_0}{d_6} \leq 1$. Then, $z^2\eta = \frac{d_0}{d_6} \leq 1$ implies $|\eta| < 1$. In the case of $\frac{d_0}{d_6} > 1$, we choose a path ℓ to connect the random walk here to the one with $\frac{d_0}{d_6} \leq 1$. Then on ℓ, the two complex roots of $f(x) = 0$ have to pass through the unit circle, which is impossible according to Remark 4.7 and Lemma 4.6.