A matrix geometric approach for random walks in the quadrant

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[Full Paper]

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ABSTRACT
In this manuscript, the authors consider a sub-class of the two-dimensional homogeneous nearest neighbor (simple) random walk restricted on the lattice. In particular, the sub-class of random walks with equilibrium distributions given as series of product-forms is considered and the derivations for the calculations of the terms involved in the equilibrium distribution representation, as well as the eigenvalues and the corresponding eigenvectors of the matrix $\boldsymbol{R}$ are presented. The above results are obtained by connecting three existing approaches available for such an analysis: the matrix geometric approach, the compensation approach and the boundary value problem method.

Keywords
Random walks; Equilibrium distribution; Matrix geometric approach; Compensation approach; Boundary value problem method.

1. INTRODUCTION
The objective of this work is to demonstrate how to obtain the equilibrium distribution of the state of a two-dimensional homogeneous nearest neighbor (simple) random walk restricted on the lattice using the matrix geometric approach. This type of random walk can be modeled as a Quasi-Birth-Death (QBD) process with the characteristic that both the levels and the phases are countably infinite. Then, based on the matrix geometric approach, if $\boldsymbol{\pi}_n = \begin{pmatrix}\pi_{n,0} & \pi_{n,1} & \cdots\end{pmatrix}$ denotes the vector of the equilibrium distribution at level n, $n = 0, 1, \ldots$, it is known that $\boldsymbol{\pi}_{n+1} = \boldsymbol{\pi}_n \boldsymbol{R}$. This is a very well known result, but the complexity of the solution lies in the calculation of the infinite dimension matrix $\boldsymbol{R}$. In this manuscript, the authors develop a new methodological approach for the calculation of the eigenvalues and eigenvectors of matrix $\boldsymbol{R}$. Moreover, this approach can be numerically used for the approximation of the matrix $\boldsymbol{R}$.

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As a first step and for illustration purposes, as well as for reasons of simplicity, we restrict our analysis to random walks whose equilibrium distribution away from the origin $(0, 0)$ can be written as a series (finite or infinite) of product-forms. In particular, under the following sufficient conditions, referred to as *conditions for meromorphicity*, cf. [1, 4],

- Step size: Only transitions to neighboring states are allowed;
- Forbidden steps: No transitions from interior states to the North, North-East, and East are allowed;
- Homogeneity: All transitions in the same direction occur according to the same rate;

the equilibrium distribution of the simple random walk restricted on the lattice can be written as a series of product-forms for all states $n, m > 0$, say

$$\pi_{n,m} = \sum_{k=0}^{\infty} \tilde{c}_k \tilde{\alpha}_k^n \tilde{\beta}_k^m, \ n, m > 0, \tag{1}$$

cf. [1]. Note that the above conditions do not necessarily imply that the transitions on the boundaries are identical to the transitions in the interior, thus we allow for $\pi_{0,m}$ and $\pi_{n,0}$ to exhibit a slightly different structural pattern, i.e.

$$\pi_{n,0} = \sum_{k=0}^{\infty} \tilde{e}_k \tilde{\alpha}_k^n, \ n > 0, \tag{2}$$

$$\pi_{0,m} = \sum_{k=0}^{\infty} \tilde{d}_k \tilde{\beta}_k^m, \ m > 0, \tag{3}$$

while $\pi_{0,0}$ can be obtained as a function of (1)-(3) by solving the system of equilibrium equations at the origin.
Some examples of queueing systems that satisfy the above conditions are the 2×2 switch and the join the shortest queue, see e.g., [2]. Also, we would like to remark that the conditions for meromorphicity are not necessary for a random walk to have an equilibrium distribution in the form of a series of product-forms, e.g. two-node Jackson networks although violate the above conditions have equilibrium distributions with a product-form representation.
In this manuscript, the authors illustrate the connection between the form of the equilibrium distribution depicted in Equation (1) and the derivation of the eigenvalues and eigenvectors of the infinite matrix $\boldsymbol{R}$. Moreover, this work sets

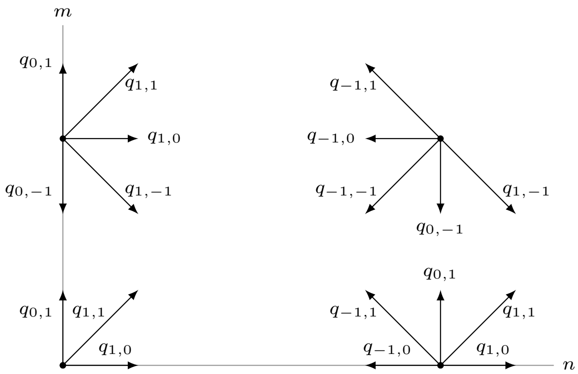

Figure 1: Transition rate diagram of the homogeneous simple random walk on the state space (n, m) with no transitions in the interior to the North, North-East, and East. Only the transitions at a few selected states are depicted as an indication.

the groundwork for the probabilistic interpretation of the terms α and β appearing in the series of product-forms.
The paper is organized as follows: in Section 2 the model is described and in Section 3 the three relevant methods are sketched; more concretely, the matrix geometric approach is presented in Section 3.1, the compensation approach in Section 3.2 and the boundary value problem method in Section 3.3. In Section 4 the derivations for the calculations of the terms involved in the equilibrium distribution representation are presented and in Section 4.1 the eigenvalues and the eigenvectors of matrix $\boldsymbol{R}$ are derived. Finally, in Section 5 conclusions and future work is discussed.

2. MODEL DESCRIPTION

To investigate the scope of applicability of the method described in this manuscript we study a class of Markov processes on the lattice in the positive quadrant of $\mathbb{R}^2$. We consider random walks (processes) for which the transition rates are constant, i.e. do not depend on the state, and we further assume that transitions are restricted to neighboring states. The transition rates are depicted in Figure 1.

3. RELATED WORK

3.1 QBD processes and matrix geometric approach

For the model described in the section above, we define n to be the *level* and m the *phase*. Thus, the generator of the random walk can be written as follows

$$\boldsymbol{G} = \begin{bmatrix} \tilde{\boldsymbol{A}}_0 & \tilde{\boldsymbol{A}}_1 & 0 & 0 & \cdots \\ \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & 0 & \cdots \\ 0 & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots \\ 0 & 0 & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with

$$\boldsymbol{A}_{-1} = \begin{bmatrix} q_{-1,0} & q_{-1,1} & 0 & 0 & \cdots \\ q_{-1,-1} & q_{-1,0} & q_{-1,1} & 0 & \cdots \\ 0 & q_{-1,-1} & q_{-1,0} & q_{-1,1} & \cdots \\ 0 & 0 & q_{-1,-1} & q_{-1,0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\boldsymbol{A}_{0} = \begin{bmatrix} -\tilde{q} & q_{0,1} & 0 & 0 & \cdots \\ q_{0,-1} & -q & 0 & 0 & \cdots \\ 0 & q_{0,-1} & -q & 0 & \cdots \\ 0 & 0 & q_{0,-1} & -q & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\boldsymbol{A}_{1} = \begin{bmatrix} q_{1,0} & q_{1,1} & 0 & 0 & \cdots \\ q_{1,-1} & 0 & 0 & 0 & \cdots \\ 0 & q_{1,-1} & 0 & 0 & \cdots \\ 0 & 0 & q_{1,-1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\tilde{\boldsymbol{A}}_{1} = \begin{bmatrix} q_{1,0} & q_{1,1} & 0 & 0 & \cdots \\ q_{1,-1} & q_{1,0} & q_{1,1} & 0 & \cdots \\ 0 & q_{1,-1} & q_{1,0} & q_{1,1} & \cdots \\ 0 & 0 & q_{1,-1} & q_{1,0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\tilde{\boldsymbol{A}}_{0} = \begin{bmatrix} -\tilde{q} & q_{1,0} & 0 & 0 & \cdots \\ q_{-1,0} & -q & q_{1,0} & 0 & \cdots \\ 0 & q_{-1,0} & -q-q_{1,0} & q_{1,0} & \cdots \\ 0 & 0 & q_{-1,0} & -q-q_{1,0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $\tilde{q} = q_{1,0} + q_{1,1} + q_{0,1} + q_{-1,1} + q_{-1,0}$ and $q = q_{1,-1} + q_{0,-1} + q_{-1,-1} + q_{-1,0} + q_{-1,1}$.

3.1.1 Stability condition

Let $\boldsymbol{A} = \boldsymbol{A}_{-1} + \boldsymbol{A}_0 + \boldsymbol{A}_1$ and $\boldsymbol{x}$ be the unique solution to

$$\boldsymbol{x}\boldsymbol{A} = 0$$

such that $x\mathbb{1} = 1$, with $\mathbb{1}$ a column vector of ones. For the random walk at hand $\boldsymbol{x}$ corresponds to the vector of the equilibrium distribution of a Birth-Death process with birth rates $\lambda_0 = q_{-1,1} + q_{0,1} + q_{1,1}$, $\lambda_n = q_{-1,1}$, $n \geq 1$, and death rates $\mu_n = q_{-1,-1} + q_{0,-1} + q_{1,-1}$, $n \geq 1$. Then, it was shown in Theorem 1.7.1 [13] that the QBD is positive recurrent if and only if

$$\boldsymbol{x}\boldsymbol{A}_{-1}\mathbb{1} < \boldsymbol{x}\boldsymbol{A}_1\mathbb{1}.$$

3.1.2 Structure of the QBD solution

Let $\pi_{n,m}$, $n, m \geq 0$ denote the equilibrium distribution of the QBD. Then, if $\boldsymbol{\pi}_n = \begin{pmatrix} \pi_{n,0} & \pi_{n,1} & \cdots \end{pmatrix}$ denotes the equilibrium vector of level n, $n = 0, 1, \ldots$, it is known that

$$\boldsymbol{\pi}_{n+1} = \boldsymbol{\pi}_n \boldsymbol{R}, \tag{4}$$

where the infinite dimensional matrix $\boldsymbol{R}$ is obtained as the minimal non-negative solution to the matrix quadratic equation

$$\boldsymbol{A}_1 + \boldsymbol{R}\boldsymbol{A}_0 + \boldsymbol{R}^2\boldsymbol{A}_{-1} = 0,$$

cf. [13]. Solving this last equation in terms of $\boldsymbol{R}$ we obtain recursively the equilibrium vector $\boldsymbol{\pi}_{n+1}$, $n \geq 0$, in terms of the matrix $\boldsymbol{R}$ and the vector of the equilibrium distribution corresponding to level 1. However, the structure of the random walk is overly generic and thus does not permit the calculation of the infinite matrix $\boldsymbol{R}$. This will be achieved by combining the two other approaches used in the analysis of random walks on the lattice: the compensation approach and the boundary value problem method.

3.2 Compensation approach

The compensation approach is developed by Adan et al. in a series of papers [1, 2, 3] and aims at a direct solution for the sub-class of two-dimensional random walks on the lattice of the first quadrant that obey the conditions for meromorphicity. The compensation approach can also be effectively used in cases that the random walk at hand does not satisfy the aforementioned conditions, but the equilibrium distribution still can be written in the form of series of product-forms. This is due to the fact that this approach exploits the structure of the equilibrium equations in the interior of the quarter plane by imposing that linear (finite or infinite) combinations of product-forms satisfy them. This leads to a kernel equation for the terms appearing in the product-forms. Then, it is required that these linear combinations satisfy the equilibrium equations on the boundaries as well. As it turns out, this can be done by alternatingly compensating for the errors on the two boundaries, which eventually leads to a (potentially) infinite series of product-forms.

For the model described in Section 2 one can easily show, cf. [1], that

Step 1: $\pi_{n,m} = \tilde{\alpha}^n \tilde{\beta}^m$, $m, n > 0$, is a solution to the equilibrium equations in the interior if and only if $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy

$$\begin{aligned}&\tilde{\alpha}\tilde{\beta}(q_{-1,1} + q_{1,-1} + q_{0,-1} + q_{-1,-1} + q_{-1,0})\\ &= \tilde{\alpha}^2 q_{-1,1} + \tilde{\beta}^2 q_{1,-1} + \tilde{\alpha}\tilde{\beta}^2 q_{0,-1}\\ &\quad + \tilde{\alpha}^2\tilde{\beta}^2 q_{-1,-1} + \tilde{\alpha}^2\tilde{\beta} q_{-1,0}.\end{aligned} \tag{5}$$

Step 2: Consider a product-form $c_0\tilde{\alpha}_0^n\tilde{\beta}_0^m$ that satisfies the kernel equation (5) and also satisfies the equilibrium equations of the horizontal boundary. Without loss of generality we can assume that $c_0 = 1$. If the product-form $c_0\tilde{\alpha}_0^n\tilde{\beta}_0^m$ also satisfies the equilibrium equations of the vertical boundary then this constitutes the solution of the equilibrium equations up to a multiplicative constant that can be obtained using the normalizing equation. Otherwise, consider a linear combination of two product-forms, say $\tilde{\alpha}_0^n\tilde{\beta}_0^m + c_1\tilde{\alpha}_1^n\tilde{\beta}_1^m$, $m, n > 0$, such that this combination satisfies now the equilibrium equations of the vertical boundary. For this to happen it must be that $\tilde{\beta}_1 = \tilde{\beta}_0$ and that $\tilde{\alpha}_1$ satisfies the kernel equation (5) for $\tilde{\beta} = \tilde{\beta}_0$.

Step 3: Finally, as long as our expression of linear combinations of product-forms violates one of the two equilibrium equations on the boundary, we continue by adding new product-form terms satisfying the kernel equation (5). This will eventually lead to Equation (1). Of course, one still needs to show that the series expression of Equation (1) converges for all $n, m > 0$.

3.3 Boundary value problem method

The boundary value problem method is an analytic method which is applicable to some two-dimensional random walks restricted to the first quadrant. The bivariate probability generating function (PGF), say

$$\Pi(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \pi_{n,m} x^n y^m, \ |x|, |y| \le 1,$$

of the position of a homogeneous nearest neighbor random walk satisfies a functional equation of the form

$$\begin{aligned}&K(x,y)\Pi(x,y) + A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y)\\ &+ C(x,y)\Pi(0,0) = 0,\end{aligned} \tag{6}$$

with $K(x,y)$, $A(x,y)$, $B(x,y)$ and $C(x,y)$ known bivariate polynomials in x and y, depending only on the parameters of the random walk. In particular,

$$\begin{aligned}K(x,y) = {}& y^2 q_{-1,1} + x^2 q_{1,-1} + x q_{0,-1}\\ &+ q_{-1,-1} + y q_{-1,0}\\ &- xy(q_{-1,1} + q_{1,-1} + q_{0,-1} + q_{-1,-1} + q_{-1,0})\end{aligned}$$

and hence $K(1/\tilde{\alpha}, 1/\tilde{\beta}) = 0$ reduces to exactly Equation (5).

The boundary value problem method consists of the following steps:

i) First, define the zero tuples (x, y) such that $K(x, y) = 0$, $|x|, |y| < 1$.

ii) Then, along the curve $K(x, y) = 0$ (and provided that $\Pi(x, y)$ is defined on this curve), Equation (6) reads

$$\begin{aligned}&A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y)\\ &+ C(x,y)\Pi(0,0) = 0.\end{aligned} \tag{7}$$

iii) Finally, in same instances Equation (7) can be solved as a Riemann (Hilbert) boundary value problem.

Malyshev pioneered this approach of transforming the functional equation to a boundary value problem in the 1970's. The idea to reduce the functional equation for the generating function to a standard Riemann-Hilbert boundary value problem stems from the work of Fayolle and Iasnogorodski [7] on two parallel M/M/1 queues with coupled processors (the service speed of a server depends on whether or not the other server is busy). Extensive treatments of the boundary value technique for functional equations can be found in Cohen and Boxma [6, Part II] and Fayolle, Iasnogorodski and Malyshev [8]. The model depicted in Figure 1 can be analyzed by the approach developed by Fayolle and Iasnogorodski [7, 8] and Cohen and Boxma [6], however this approach does not lead to the direct determination of the equilibrium distribution, since it requires inverting the PGF, and the existing numerical approaches for this method are oftentimes tedious and case specific.

4. ANALYSIS

In this paper, we connect for the first time the three approaches: the matrix geometric approach, the compensation approach and the boundary value problem method. We will demonstrate now how to easily compute recursively these $\tilde{\alpha}$'s and $\tilde{\beta}$'s.

$$(\tilde{\alpha}_0, \tilde{\beta}_0) \longrightarrow (\tilde{\alpha}_1, \tilde{\beta}_1 = \tilde{\beta}_0) \longrightarrow (\tilde{\alpha}_2 = \tilde{\alpha}_1, \tilde{\beta}_2) \longrightarrow (\tilde{\alpha}_3, \tilde{\beta}_3 = \tilde{\beta}_2) \quad \cdots$$

Figure 2: The recursive structure of the product-form terms.

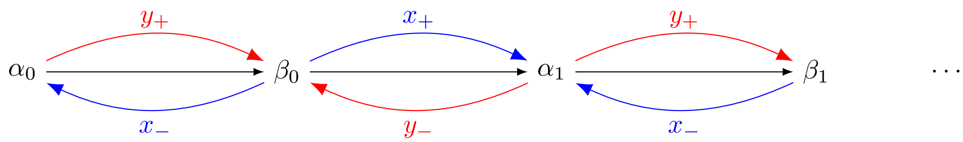

Figure 3: Evolutions of α's and β's.

Step 1: Rewrite the PGF as

$$\begin{aligned}
\Pi(x,y) &= \sum_{n=0}^{\infty} x^n \pi_n \begin{pmatrix} 1 & y & y^2 & \cdots \end{pmatrix}^T \\
&= \pi_0 \begin{pmatrix} 1 & y & y^2 & \cdots \end{pmatrix}^T \\
&\quad + \sum_{n=1}^{\infty} x^n \pi_1 \boldsymbol{R}^{n-1} \begin{pmatrix} 1 & y & y^2 & \cdots \end{pmatrix}^T \\
&= \pi_0 \begin{pmatrix} 1 & y & y^2 & \cdots \end{pmatrix}^T \\
&\quad + \pi_1 (x^{-1}\boldsymbol{I} - \boldsymbol{R})^{-1} \begin{pmatrix} 1 & y & y^2 & \cdots \end{pmatrix}^T.
\end{aligned} \tag{8}$$

In the last equation the term $(x^{-1}\boldsymbol{I} - \boldsymbol{R})^{-1}$ should be interpreted as an operator instead of the inverse of a matrix. Then, Equation (6) reduces to

$$\begin{aligned}
&\pi_1 (x^{-1}\boldsymbol{I} - \boldsymbol{R})^{-1} \Big(K(x,y) \begin{pmatrix} 1 & y & y^2 & \cdots \end{pmatrix}^T \\
&\quad + A(x,y) \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix}^T \Big) \\
&= -\pi_0 \Big((K(x,y) + B(x,y)) \begin{pmatrix} 1 & y & y^2 & \cdots \end{pmatrix}^T \\
&\quad + (A(x,y) + C(x,y)) \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix}^T \Big).
\end{aligned} \tag{9}$$

Note that due to (1) we can meromorphically continue the PGF on the entire complex domain, i.e. the PGF is holomorphic on the entire complex domain except for a set of isolated points (the poles of the function) $x^{-1} = \tilde{\alpha}_k$ and $y^{-1} = \tilde{\beta}_k$, $k \geq 0$. More concretely, it can be shown that these PGFs are meromorphic functions, i.e., they have a finite number of poles in every finite domain, cf. [5].

Step 2: For the isolated point $x^{-1} = \tilde{\alpha}_0$ the right hand side of (9) is well defined, which implies that

$$\begin{aligned}
&K(x,y) \begin{pmatrix} 1 & y & y^2 & \cdots \end{pmatrix}^T \\
&\quad + A(x,y) \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix}^T = 0,
\end{aligned}$$

or equivalently that $K(x,y) = 0$ and $A(x,y) = 0$. This reveals the starting solution $\tilde{\alpha}_0$, with $|\tilde{\alpha}_0| < 1$ for the iterative calculation of the sequences $\{\tilde{\alpha}_k\}_{k\geq 0}$ and $\{\tilde{\beta}_k\}_{k\geq 0}$.

For the starting solution $x^{-1} = \tilde{\alpha}_0$ we calculate recursively $y^{-1} = \tilde{\beta}_0$ by the kernel equation $K(x,y) = 0$. This will produce a single $\tilde{\beta}$ with $|\tilde{\beta}| \leq |\tilde{\alpha}|$. We can proceed in an analogous manner and construct the entire set of product-form terms. Moreover, one can easily show, cf. [1], that

$$\tilde{\beta}_{2k+1} = \tilde{\beta}_{2k}, \ k \geq 0$$

and $\tilde{\alpha}_{2k+1}$, $k \geq 0$, is obtained as the root of

$$K(1/\tilde{\alpha}_{2k+1}, 1/\tilde{\beta}_{2k+1}) = 0$$

with $|\tilde{\alpha}_{2k+1}| < |\tilde{\beta}_{2k+1}|$. Also,

$$\tilde{\alpha}_{2k} = \tilde{\alpha}_{2k-1}, \ k \geq 1,$$

and $\tilde{\beta}_{2k}$, $k \geq 1$, is obtained as the root of

$$K(1/\tilde{\alpha}_{2k}, 1/\tilde{\beta}_{2k}) = 0$$

with $|\tilde{\beta}_{2k}| < |\tilde{\alpha}_{2k}|$. Figure 2 displays the way in which the product-form terms are generated.

Step 3: We now consider a new representation of $\pi_{n,m}$ so as to avoid repetitions in the family of roots of the kernel. To this purpose, for $k \geq 0$ we denote:

$$\alpha_k = \tilde{\alpha}_{2k}, \qquad \beta_k = \tilde{\beta}_{2k}.$$

Then, Equations (1), (2) and (3) are re-written, for $n, \ m > 0$

$$\pi_{n,m} = c_0 \alpha_0^n \beta_0^m + \sum_{k=1}^{\infty} c_k \alpha_k^n (\beta_{k-1}^m + f_k \beta_k^m), \tag{10}$$

and

$$\pi_{n,0} = \sum_{k=0}^{\infty} e_k \alpha_k^n, \qquad \pi_{0,m} = \sum_{k=0}^{\infty} d_k \beta_k^m, \tag{11}$$

with $c_0 = \tilde{c}_0$, $e_0 = \tilde{e}_0$ and $d_0 = \tilde{d}_0$, and, for $k \geq 1$, $c_k = \tilde{c}_{2k-1}$, $f_k = \tilde{c}_{2k}/\tilde{c}_{2k-1}$, $e_k = \tilde{e}_{2k} + \tilde{e}_{2k-1}$ and $d_k = \tilde{d}_{2k} + \tilde{d}_{2k-1}$. Equivalently, we can also consider the following representation, for $n, m > 0$,

$$\pi_{n,m} = \sum_{k=0}^{\infty} c_k^* \beta_k^m (\alpha_k^n + f_{k+1}^* \alpha_{k+1}^n), \tag{12}$$

with $c_0^* = c_0$, $c_k^* = c_k f_k$ for $k > 0$ and $f_{k+1}^* = c_{k+1}/c_k^*$.

Now we have the sequence of the zero-tuples of the kernel equation, cf. Figure 3, with 'forward' operators y_+ and x_+ and 'backward' operators y_- and x_- constructed as the solutions of the kernel equation $K(x,y) = 0$ with respect to y and x. More concretely, by re-writing the kernel equation $K(x,y)=0$ as a quadratic function in either y or x

$$K(x,y) = a_x(y-y_+(x))(y-y_-(x)),$$
$$K(x,y) = \tilde{a}_y(x-x_+(y))(x-x_-(y))$$

for some functions a_x and $\tilde{a}_y$, yields

$$y_+\left(\frac{1}{\alpha_k}\right) = \frac{1}{\beta_k}, \qquad x_+\left(\frac{1}{\beta_k}\right) = \frac{1}{\alpha_{k+1}}, \tag{13}$$
$$y_-\left(\frac{1}{\alpha_k}\right) = \frac{1}{\beta_{k-1}}, \qquad x_-\left(\frac{1}{\beta_k}\right) = \frac{1}{\alpha_k}. \tag{14}$$

Step 4: It remains to show how to calculate the coefficients of the product-form terms. Observe that the representations (10)-(11) for the $\pi_{n,m}$ (or equivalently (12)-(11)) yield

$$\begin{aligned}\Pi(x,y) =& \Pi(x,0) + (\Pi(0,y) - \pi_{0,0}) \\ &+ c_0 \frac{\alpha_0\beta_0 xy}{(1-\alpha_0 x)(1-\beta_0 y)} \\ &+ \sum_{k=1}^{\infty} c_k \frac{\alpha_k x}{1-\alpha_k x}\left(\frac{\beta_{k-1}y}{1-\beta_{k-1}y} + f_k \frac{\beta_k y}{1-\beta_k y}\right),\end{aligned} \tag{15}$$

with

$$\Pi(x,0) = \pi_{0,0} + \sum_{k=1}^{\infty} e_k \frac{\alpha_k x}{1-\alpha_k x}, \tag{16}$$
$$\Pi(0,y) = \pi_{0,0} + \sum_{k=1}^{\infty} d_k \frac{\beta_k y}{1-\beta_k y}. \tag{17}$$

We first identify the sequences $\{e_k\}_{k\ge 0}$ and $\{d_k\}_{k\ge 0}$ appearing in (11). Next, we set $y = y_+(x)$ in (9) (for which $K(x, y_+(x)) = 0$) and substitute the representations for $\Pi(x,0)$ and $\Pi(0,y)$. We multiply the resulting equation with $1-\alpha_i x$ and take the limit as $x \to 1/\alpha_i$. This yields

$$\begin{aligned}0 =& A\left(\frac{1}{\alpha_i}, y_+\left(\frac{1}{\alpha_i}\right)\right) e_i \\ &+ B\left(\frac{1}{\alpha_i}, y_+\left(\frac{1}{\alpha_i}\right)\right) d_i \lim_{x\to\frac{1}{\alpha_i}} \frac{1-\alpha_i x}{1-\beta_i y_+(x)}.\end{aligned} \tag{18}$$

Similarly, repeating the above procedure for $y = y_-(x)$ produces

$$\begin{aligned}0 =& A\left(\frac{1}{\alpha_i}, y_-\left(\frac{1}{\alpha_i}\right)\right) e_i \\ &+ B\left(\frac{1}{\alpha_i}, y_-\left(\frac{1}{\alpha_i}\right)\right) d_{i-1} \lim_{x\to\frac{1}{\alpha_i}} \frac{1-\alpha_i x}{1-\beta_{i-1} y_-(x)}.\end{aligned} \tag{19}$$

Similarly, we could have chosen $x = x_\pm(y)$, but the resulting equations would be identical to the ones derived above.

Now starting from e_0 all the coefficients $\{e_k\}_{k\ge 0}$ and $\{d_k\}_{k\ge 0}$ are obtained recursively as follows: for a given e_k using Equation (18) one can derive d_k, next Equation (19) produces e_{k+1}.

Step 5: Having $\{e_k\}_{k\ge 0}$ and $\{d_k\}_{k\ge 0}$ recursively identified in terms of e_0, we show in this paragraph how to obtain the sequence $\{c_k\}_{k\ge 0}$. Multiplying Equation (9) with $1-\alpha_i x$, then substituting (15)-(17) therein, and afterwards taking the limit as $x \to 1/\alpha_i$ and setting $y=1$ gives

$$\begin{aligned}0 =& \left(A\left(\frac{1}{\alpha_i},1\right) + K\left(\frac{1}{\alpha_i},1\right)\right) e_i \\ &+ K\left(\frac{1}{\alpha_i},1\right) c_0 \frac{\beta_0}{1-\beta_0}\end{aligned} \tag{20}$$

and

$$\begin{aligned}0 =& \left(A\left(\frac{1}{\alpha_i},1\right) + K\left(\frac{1}{\alpha_i},1\right)\right) e_i \\ &+ K\left(\frac{1}{\alpha_i},1\right) c_i \left(\frac{\beta_{i-1}}{1-\beta_{i-1}} + f_i \frac{\beta_i}{1-\beta_i}\right),\end{aligned} \tag{21}$$

for $i > 0$. Equivalently, using representation (12) yields

$$\begin{aligned}\Pi(x,y) =& \Pi(x,0) + (\Pi(0,y) - \pi_{0,0}) \\ &+ c_0 \frac{\alpha_0\beta_0 xy}{(1-\alpha_0 x)(1-\beta_0 y)} \\ &+ \sum_{k=0}^{\infty} c_k^* \frac{\beta_k y}{1-\beta_k y}\left(\frac{\alpha_k x}{1-\alpha_k x} + f_{k+1}^* \frac{\alpha_{k+1} x}{1-\alpha_{k+1} x}\right).\end{aligned} \tag{22}$$

Now using (22), multiplying Equation (9) by $1-\beta_i y$, and afterwards taking the limit as $y \to 1/\beta_i$ and setting $x=1$ yields, for $i \ge 0$,

$$\begin{aligned}0 =& \left(B\left(1,\frac{1}{\beta_i}\right) + K\left(1,\frac{1}{\beta_i}\right)\right) d_i \\ &+ K\left(1,\frac{1}{\beta_i}\right) c_i^* \left(\frac{\alpha_i}{1-\alpha_i} + f_{i+1}^* \frac{\alpha_{i+1}}{1-\alpha_{i+1}}\right)\end{aligned} \tag{23}$$

which is equivalent to

$$\begin{aligned}0 =& \left(B\left(1,\frac{1}{\beta_0}\right) + K\left(1,\frac{1}{\beta_0}\right)\right) d_0 \\ &+ K\left(1,\frac{1}{\beta_0}\right) \left(c_0 \frac{\alpha_0}{1-\alpha_0} + c_1 \frac{\alpha_1}{1-\alpha_1}\right)\end{aligned} \tag{24}$$

and

$$\begin{aligned}0 =& \left(B\left(1,\frac{1}{\beta_i}\right) + K\left(1,\frac{1}{\beta_i}\right)\right) d_i \\ &+ K\left(1,\frac{1}{\beta_i}\right) \left(c_i f_i \frac{\alpha_i}{1-\alpha_i} + c_{i+1} \frac{\alpha_{i+1}}{1-\alpha_{i+1}}\right),\end{aligned} \tag{25}$$

for $i \ge 1$. Now the iterative procedure is as follows: Starting from c_0, we derive e_0 from Equation (20) (and hence from Step 4 all coefficients $\{e_k\}_{k\ge 0}$ and $\{d_k\}_{k\ge 0}$ are produced in terms of c_0). Then, from Equation (24) we calculate c_1. Having c_1 and using (21) we identify f_1, which allows us to derive c_2 from Equation (25). Continuing this procedure permits the identification of the sequences $\{c_k\}_{k\ge 0}$ and $\{f_k\}_{k\ge 1}$. The starting constant c_0 is uniquely identified by the normalization equation.

Note that from the construction of Equation (6) it follows that we can take any $|y| \le 1$ instead of $y = 1$ (respectively any $|x| \le 1$) since the rhs of the above equation does not in practice depend on y.

4.1 The matrix $\boldsymbol{R}$

We are now in position to present the main result of the manuscript, that connects the derivation of the matrix $\boldsymbol{R}$ with Equations (10) and (11), and hence the boundary value problem with the matrix geometric approach.

THEOREM 1. *The terms* $\{\alpha_k\}_{k\geq 0}$ *constitute the different eigenvalues of the matrix* $\boldsymbol{R}$. *For eigenvalue* α_k *the corresponding eigenvector of the matrix* $\boldsymbol{R}$ *is* $\boldsymbol{h}_k = (h_{k,0}, h_{k,1}, k_{k,2}\ldots)$, *with* $h_{k,0} = e_k$ *and* $h_{k,m} = c_k(\beta_{k-1}^m + f_k\beta_k^m)$ $(m = 1, 2, \ldots)$, *if and only if* $c_k \neq 0$.

PROOF. From (10) and (11) note that, for $n > 0$,

$$\begin{aligned}\pi_n &= \begin{pmatrix}\pi_{n,0} & \pi_{n,1} & \pi_{n,2} & \cdots\end{pmatrix}\\ &= \sum_{k=0}^{\infty}\alpha_k^n\boldsymbol{h}_k.\end{aligned}$$

Plugging this last result into (4) and after straightforward manipulations yields

$$\sum_{k=0}^{\infty}\alpha_k^n\boldsymbol{h}_k(\alpha_k\boldsymbol{I} - \boldsymbol{R}) = 0,\ \forall n > 0.$$

From this last equation it is needed that

$$\boldsymbol{h}_k(\alpha_k\boldsymbol{I} - \boldsymbol{R}) = 0,\ \forall k \geq 0,$$

which implies the statement of the Theorem, cf. [11]. □

REMARK 1. *Note that one could use in the above proof the representations of Equations* (1) *and* (2)*, instead of* (10) *and* (11)*. Such a choice, would reveal that the sequence* $\{\tilde{\alpha}_k\}_{k\geq 0}$ *constitutes the eigenvalues of matrix* $\boldsymbol{R}$*, with eigenvalues* $\tilde{\alpha}_k$*,* $k \geq 1$*, having an algebraic multiplicity of 2, since* $\tilde{\alpha}_{2k} = \tilde{\alpha}_{2k-1}$*,* $k \geq 1$*, and geometric multiplicity equal to 2, since for eigenvalue* $\tilde{\alpha}_{2k}$ *there are two eigenvectors*

$$\begin{matrix}(\tilde{e}_{2k-1} & \tilde{c}_{2k-1}\tilde{\beta}_{2k-1} & \tilde{c}_{2k-1}\tilde{\beta}_{2k-1}^2 & \cdots)\\ (\tilde{e}_{2k} & \tilde{c}_{2k}\tilde{\beta}_{2k} & \tilde{c}_{2k}\tilde{\beta}_{2k}^2 & \cdots)\end{matrix}$$

which if added together and taking into account the connections between the various representations produce exactly the eigenvector $\boldsymbol{h}_k$*,* $k \geq 1$*, appearing in Theorem 1.*

4.1.1 Numerical evaluation of matrix $\boldsymbol{R}$

It is known, see [1, 4], that the sequences $\{\alpha_k\}_{k\geq 0}$ and $\{\beta_k\}_{k\geq 0}$ decrease exponentially fast to 0. Based on this fact, we suggest to truncate the dimension of the matrix $\boldsymbol{R}$, say at phase N, and obtain its approximation, say $\boldsymbol{R}_N$, as

$$\boldsymbol{R}_N = \boldsymbol{H}_N^{-1}\boldsymbol{D}_N\boldsymbol{H}_N,$$

with $\boldsymbol{D}_N = \text{diag}(\alpha_0, \alpha_1, \ldots, \alpha_N)$ and the matrix $\boldsymbol{H}_N = (\boldsymbol{h}_1^N, \ldots \boldsymbol{h}_N^N)$, where $\boldsymbol{h}_k^N = (h_{k,0}, \ldots, h_{k,N})$. Then, as $N \to \infty$ the matrix $\boldsymbol{R}_N = \boldsymbol{H}_N\boldsymbol{D}_N\boldsymbol{H}_N^{-1}$ converges to the infinite matrix $\boldsymbol{R}$, cf. [11]. Furthermore, closely inspecting the structure of the matrix $\boldsymbol{H}_N = (h_{k,m})_{0\leq k,m\leq N}$ we observe that it can be written as a generalized Vandermonde matrix for which the inverse can be easily calculated.

5. CONCLUSIONS AND FUTURE WORK

In this manuscript, the authors present a methodological approach that on the one hand permits a straightforward derivation of the equilibrium distribution of random walks in the quadrant satisfying the conditions for meromorphicity and on the other hand connects three existing techniques: the matrix geometric approach, the compensation approach, and the boundary value problem method. Furthermore, the derivation of the eigenvalues and eigenvectors of matrix $\boldsymbol{R}$ sets the groundwork for the probabilistic interpretation of the terms α and β appearing in the expression of the equilibrium distribution. This work can be easily extended to cover a wider spectrum of random walks with meromorphic probability generating functions.

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