Journal of Combinatorial Theory, Series A 116 (2009) 460–477

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Classifying lattice walks restricted to the quarter plane

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ARTICLE INFO

Article history:
Received 31 August 2007
Available online 1 October 2008

Keywords:
Enumeration
Lattice walks
Generating functions
Holonomic
D-finite

ABSTRACT

This work considers the nature of generating functions of random lattice walks restricted to the first quadrant. In particular, we find combinatorial criteria to decide if related series are algebraic, transcendental holonomic or otherwise. Complete results for walks taking their steps in a maximum of three directions of restricted amplitude are given, as is a well-supported conjecture for all walks with steps taken from a subset of $\{0, \pm 1\}^2$. New enumerative results are presented for several classes, each obtained with a variant of the kernel method.

0. Introduction

Lattice walks are a combinatorial object with a long history, however lately there has been a rise in interest in their generating functions (for example, [5,7,9,15]) and global approaches and results. A few such studies have been performed. For example, Banderier and Flajolet [1] examine the nature of their generating functions, and provide general results of asymptotic analysis for one-dimensional lattice walks. Two-dimensional walks in the quarter plane have been treated by a collection of case analyzes [7,9,14,23]. The goal of the present work is to determine general characterizations of the generating functions, based strictly on combinatorial properties of the allowable steps. For example, can we characterize the walks whose generating function is algebraic, transcendental holonomic (that is, which satisfies a linear differential equations with polynomial coefficients) or neither? The answer has important repercussions for sequence generation, asymptotics, amongst other consequences.

We focus our attention to those walks in the quarter plane which take steps from some subset of $\{0, \pm 1\}^2$. These can be either viewed as the eight allowable moves for a king in chess, or one of the eight standard compass directions. These are a superset of the *nearest neighbor walks*, and a subset of the *next nearest neighbor walks*. We give explicit complete generating function expressions and classifications when this subset is of cardinality three, and we are able to formulate well-supported

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0097-3165/$ – see front matter
doi:10.1016/j.jcta.2008.06.011

conjectures in the general case. Conjectures 1 and 2 in Section 3 propose necessary and sufficient combinatorial conditions on the set of allowable steps which guarantee the generating function for the walks restricted to the quarter plane is holonomic.

An extended abstract of this work appeared in the 2007 Formal Power Series and Algebraic Combinatorics conference proceedings [19].

1. Walks and their generating functions

1.1. Walks in the lattice

Henceforth we refer to the class of walks considered here *compass walks*. Precisely, each movement on the integer lattice is a step from some fixed set $\mathcal{Y} \subseteq \{\pm 1, 0\}^2 \setminus \{(0,0)\}$, which we also specify by the corresponding compass directions $\{\mathsf{N}, \mathsf{NE}, \ldots, \mathsf{W}, \mathsf{NW}\}$. Such a set $\mathcal{Y}$ is called a *step set*. A *walk in the quarter plane of length n* is a sequence of steps w in $\mathcal{Y}^*$, $w = w_1, w_2, \ldots, w_n$, such that for each $k \leqslant n$, the vector sum $(x_k, y_k) = \sum_{i=1}^{k} w_i$ satisfies $x_k \geqslant 0$, $y_k \geqslant 0$, that is, it remains in the first quadrant. These two inequalities define the walk in the sense that isomorphic walks will satisfy comparable inequalities. We denote the set of all valid walks with steps from $\mathcal{Y}$ by $\mathcal{L}(\mathcal{Y})$. We can consider this as a formal language (in the sense of theoretical computer science) over the alphabet $\mathcal{Y}$ with the horizontal and vertical conditions as prefix conditions on any word in the language.

1.2. Complete and counting generating functions

Fix some step set $\mathcal{Y}$. We associate to $\mathcal{L}(\mathcal{Y})$ two formal power series: $W(t)$, a (univariate) counting generating function and $Q(x, y; t)$, a (multivariate) generating function which refines $W(t)$. The series $W(t)$ is the ordinary generating function for the number of walks, that is, the coefficient of t^n is the number of walks of length n. The complete generating function, $Q(x, y; t)$, encodes more information. The coefficient of $x^i y^j t^n$ in $Q(x, y; t)$ is the number of walks of length n ending at the point (i, j). Note that the specialization $x = y = 1$ in the complete generating function is precisely the counting series, i.e. $Q(1, 1; t) = W(t)$. If the choice of step set is not clear by context, we add a subscript.

In part, our interest in the complete generating function stems from the fact that if it is in a particular functional class, then generally so is $W_{\mathcal{Y}}$. Furthermore, we can determine a useful functional equation that it satisfies. The *fundamental equation* satisfied by a complete generating function is determined from the recursive definition that a walk of length n is a walk of length $n - 1$ plus a step. For a walk ending on the x- or y-axis, it is possible that not all of the directions from $\mathcal{Y}$ will be permissible for the next step, and thus we subtract out these cases.

Definition 1.1 *(Fundamental equation).* The fundamental equation of the complete generating function of walks with step set $\mathcal{Y}$ is given by[1]

$$\begin{aligned} Q(x, y; t) = 1 + \sum_{(i,j)\in\mathcal{Y}} t x^i y^j Q(x, y; t) - t\bar{y} \sum_{i:\,(i,-1)\in\mathcal{Y}} x^i Q(x, 0; t) - t\bar{x} \sum_{j:\,(-1,j)\in\mathcal{Y}} y^j Q(0, y; t) \\ + \chi\big[(-1,-1)\in\mathcal{Y}\big] t\bar{x}\bar{y} Q(0, 0; t). \end{aligned} \tag{1.1}$$

1.3. Classifying formal power series

Ideally, we would like to have explicit expressions for these two series. In many cases this is possible, however, we are also interested in understanding the analytic nature of these series with respect to three classes: algebraic, transcendental holonomic, and non-holonomic.

[1] $\chi[P] = 1$ if P is true and 0 otherwise; $\bar{x} = \frac{1}{x}$; $\bar{y} = \frac{1}{y}$.

Let $\underline{x} = x_1, x_2, \ldots, x_n$. A multivariate generating function $G(\underline{x})$ is *algebraic* if there exists a multivariate polynomial $P(\underline{x}, y)$ such that $P(\underline{x}, G(\underline{x})) = 0$. Flajolet [12] summarizes a number of criteria which imply the transcendence of a series. One which we shall use here is a consequence of his Theorem D.

Theorem 1.1. *If the Taylor coefficient of z^n of a function $f(z)$ (analytic at the origin) is asymptotically equivalent to $\gamma\beta^n n^r$, and further if r is irrational or a negative integer; or if β is transcendental; or if $\gamma\Gamma(r+1)$ transcendental, then $f(z)$ is transcendental.*

A multivariate series $G(\underline{x})$ is *holonomic* if the vector space generated by its partial derivatives (and their iterates), over rational series of $\underline{x}$ is finite-dimensional. This is equivalent to the existence of n partial differential equations of the form

$$0 = p_{0,i} f(\underline{x}) + p_{1,i} \frac{\partial f(\underline{x})}{\partial x_i} + \cdots + p_{d_i,i} \frac{\partial^{d_i} f(\underline{x})}{\partial x_i^{d_i}},$$

for i satisfying $1 \leqslant i \leqslant n$, and where the $p_{j,i}$ are all polynomials in $\underline{x}$. Holonomic series are also known as *D-finite* series, and the sequences of coefficients are said to be *P-recursive*.

Algebraic series are always holonomic, but as $\exp(x)$ is both holonomic and transcendental, this containment is strict. The closure properties of these two classes are presented by Stanley [25, Chapter 6]. In particular, we use the fact that if $F(\underline{x})$ is holonomic with respect to the x_i, and if the algebraic substitution $x_i = y_i(z_1, \ldots, z_k) \equiv y_i$ makes sense as a power series substitution, then $f(y_1, \ldots, y_n)$ is holonomic with respect to $z_1, \ldots, z_k$.

The goal of this work is to determine when $W_{\mathcal{Y}}$ and $Q_{\mathcal{Y}}$ are algebraic, transcendental holonomic, or non-holonomic. The next two theorems are model examples of the kind of result we aspire to emulate, from the classification point of view.

Theorem 1.2 *(Half-plane condition (folklore, and [1])). Let $\mathcal{Y}$ be a subset of $\{\pm 1, 0\}^2$. The complete generating series $Q(x, y; t)$ for walks that start at $(0, 0)$, take their steps in $\mathcal{Y}$ and stay in a half plane is algebraic.*

A step set $\mathcal{Y}$ is *symmetric with respect to the y-axis* (respectively x-axis) if $(i, j) \in \mathcal{Y}$ implies that $(i, -j) \in \mathcal{Y}$ (respectively $(-i, j) \in \mathcal{Y}$).

Theorem 1.3 *(Bousquet-Mélou [6,7]). Let $\mathcal{Y}$ be a finite subset of $\{\pm 1, 0\} \times \mathbb{Z} \setminus \{(0, 0)\}$ that is symmetric with respect to the y-axis. Then the complete generating function $Q(x, y; t)$ for walks that start from $(0, 0)$, take their steps in $\mathcal{Y}$, and stay in the first quadrant is holonomic.*

For example, $\mathcal{Y} = \{\mathsf{E}, \mathsf{NW}, \mathsf{SW}\}$ satisfies the conditions of Theorem 1.3, and thus $G_{\mathcal{Y}}(x, y; t)$ is a holonomic function. Naturally, an analogous result for step sets from $\mathbb{Z} \times \{\pm 1, 0\} \setminus \{(0, 0)\}$ which are symmetric with respect to the x-axis is also true.

1.4. Combinatorial operations on step sets

There are two operations which act on step sets and which play an important role in our classification. The vector operator $\mathsf{reflect}(x, y) = (y, x)$ switches the coordinates, effectively flipping a step across the line $x = y$. When applied to a step set, this operator preserves both algebraicity and holonomicity of the generating function, since it amounts to a simple variable switch in $Q(x, y; t)$.

A second useful operator is $\mathsf{rev}(x, y) = (-x, -y)$, which reverses the direction of a step. For example, $\mathsf{rev}(\mathsf{N}) = \mathsf{S}$, and $\mathsf{rev}(\mathsf{SW}) = \mathsf{NE}$. The reverse of a set $\mathsf{rev}(\mathcal{Y})$, is the result when rev is applied to each element of $\mathcal{Y}$. The reverse of a walk $w = w_1 w_2 \ldots$, is the application of rev to each step, taken in the reverse order: $\mathsf{rev}(w) = \mathsf{rev}(w_n)\mathsf{rev}(w_{n-1}) \ldots$. Geometrically this is reflection in the line $x = -y$.

The reverse of a step set may result in a step set which has only the trivial walk in the quarter plane. Aside from these, the effect of this operation on the generating function can be determined, but as it is much less direct than in the reflect case, it is not entirely clear whether it preserves either holonomicity or algebraicity. Evidence would seem to indicate that in the case when the walk is non-trivial it does preserve both properties. Section 3 offers intuition as to why this could be the case.

1.5. Techniques for enumerating walks

Lattice walks have a great deal of structure that can be exploited for enumeration, and there do exist general techniques. The Kernel method and its variants are very powerful family of tools, and could treat each case listed here in a unified fashion. The Holonomy Ansatz is another technique which has the advantage of being strongly automated.

1.5.1. The kernel method

A *kernel method approach* has as a departure point an equation resembling the fundamental equation (1.1). It uses different specializations of x and y which fix or annihilate the coefficient of $Q(x, y; t)$. The way we have written the equation, this is $K_r := 1 - t\sum_{(i,j)\in\mathcal{Y}} x^i y^j$. The coefficient in this form is called *the rational kernel*, and is denoted $K_r(x, y)$. In the course of our classification we use three different variants: The first, classic, approach (see [6,21]) specializes y as a function of x which annihilate a polynomial form of the kernel (here, $xyK_r(x, y)$), and the problem is reduced to a simpler one. This gives us explicit expressions for two examples in Section 2.4

Bousquet-Mélou, in her study of Kreweras' walks [7] takes a slightly different approach, the *algebraic kernel method*, which first determines a group of actions on the pair (x, y) which fixes the rational kernel $K_r(x, y)$. These pairs are used to generate more functional equations, from which we can deduce properties such as algebraicity, and occasionally, explicit enumerative results. We use this method to treat a related class in Section 2.3.2.

The group that arises in this approach can be defined for all step sets and turns out to be fundamental for our understanding of the general case. We define it in the next section.

A second generalization, described as the *iterated kernel method* in [15], uses more or less the same approach as the classic kernel method, but in the case when the aforementioned group is infinite. This leads to explicit expressions which we can show are not holonomic. Section 2.5 offers two examples.

1.5.2. The group of the walk

For each step set we define a group of actions which fixes the corresponding rational kernel. For the (non-degenerate) walks we consider here, it will always be a dihedral group, generated by two involutions. In our restricted setting we can give an explicit description of the generators. For step sets containing steps with larger amplitudes a generalized definition is possible, and we consider this at the end of the section.

Definition 1.2 *(The group of the walk ($G(\mathcal{Y})$)).* Let $\mathcal{Y}$ be a fixed step set which is not singular. The *group of the walk* $\mathcal{Y}$, denoted $G(\mathcal{Y})$, is the group of transformations which map $\mathbb{R}^2$ to itself, and is defined by the two generators τ_x and τ_y defined as follows. Set

$$K(x, y) := xyK_r(x, y) = xy - t\sum_{(i,j)\in\mathcal{Y}} x^{i+1}y^{j+1} = a(x)ty^2 + b(t, x)y + c(x)t.$$

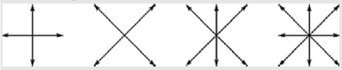
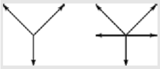
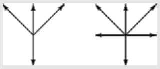
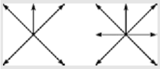
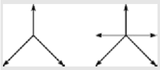
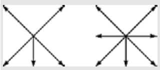
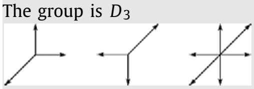
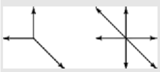
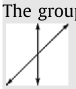
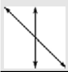
If $a(x) \neq 0$, then define τ_x as the transformation

$$\tau_x : (x, y) \mapsto \bigl(x, a(x)^{-1}c(x)\bar{y}\bigr).$$

Switch the roles of x and y, and likewise define τ_x.

The generators for *all* compass step sets with finite groups (up to symmetry in the line $x = y$) are given in Table 1. These mappings are not invertible when the step set is singular.

Table 1
Generators for compass step sets $\mathcal{Y}$ with finite groups $G(\mathcal{Y})$

$\mathcal{Y}$	Generators of $G(\mathcal{Y})$
The group is D_2	
	$\tau_x(x,y)=(\frac{1}{x},y),\ \tau_y(x,y)=(x,\frac{1}{y})$
	$\tau_x(x,y)=(\frac{1}{x},y),\ \tau_y(x,y)=(x,\frac{x}{y(x^2+1)})$
	$\tau_x(x,y)=(\frac{1}{x},y),\ \tau_y(x,y)=(x,\frac{x}{y(x^2+1+x)})$
	$\tau_x(x,y)=(\frac{1}{x},y),\ \tau_y(x,y)=(x,\frac{x^2+1}{y(x^2+1+x)})$
	$\tau_x(x,y)=(\frac{1}{x},y),\ \tau_y(x,y)=(x,\frac{x^2+1}{xy})$
	$\tau_x(x,y)=(\frac{1}{x},y),\ \tau_y(x,y)=(x,\frac{x^2+1+x}{xy})$
	$\tau_x(x,y)=(\frac{1}{x},y),\ \tau_y(x,y)=(x,\frac{x^2+1+x}{y(x^2+1)})$
The group is D_3	
	$\tau_x(x,y)=(\frac{1}{xy},y),\ \tau_y(x,y)=(x,\frac{1}{xy})$
	$\tau_x(x,y)=(\frac{y}{x},y),\ \tau_y(x,y)=(x,\frac{x}{y})$
The group is D_4	
	$\tau_x(x,y)=(\frac{1}{xy^2},y),\ \tau_y(x,y)=(x,\frac{1}{xy})$
	$\tau_x(x,y)=(\frac{y^2}{x},y),\ \tau_y(x,y)=(x,\frac{x}{y})$

Both generators of $G(\mathcal{Y})$ fix the corresponding rational kernel. If $y=Y_0(x)$ and $y=Y_1(x)$ are the two roots of the quadratic polynomial in y given by $xyK_r(x,y)=0$, then $\tau_x(x,y)=(x,\frac{Y_1(x)Y_0(x)}{y})$. We can write

$$xK_r=ta(x)\left(1-\frac{Y_0}{y}\right)(y-Y_1),$$

and thus

$$\begin{aligned}xK_r\big(\tau_x(x,y)\big)=xK_r\left(x,\frac{Y_1Y_0}{y}\right)&=ta(x)\left(1-\frac{Y_0y}{Y_0Y_1}\right)\left(\frac{Y_0Y_1}{y}-Y_1\right)\\&=ta(x)\left(1-\frac{y}{Y_1}\right)Y_1\left(\frac{Y_0}{y}-1\right)\\&=xK_r(x,y).\end{aligned}$$

It should be noted that this group does not necessarily give the full group of transformations which fix the kernel of a given step set.

In a later section we discuss conditions on $\mathcal{Y}$ which assure finiteness of the group generated by τ_x and τ_y.

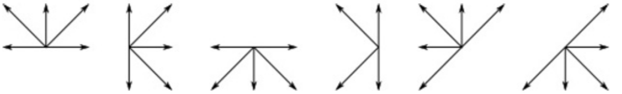


Fig. 1. Singular walks are subsets of these sets of directions.

This approach is inspired by the Galois automorphisms and the group of the random walk describe by Fayolle et al. in [11], and their treatment of the random walks.

1.5.3. The Quasi-Holonomic Ansatz

Kauers and Zeilberger [16] developed an automatic technique (the Quasi-Holonomic Ansatz) which is useful to determine enumerative expressions for excursions, that is, walks which return to their starting point. Their algorithm is based on finding, and proving recurrences satisfied by the counting sequences. They determine compact enumerative expressions for excursions for all of the cases treated in the next section, their results are tabulated in [16], and the automatic proofs are available on the corresponding website.

2. Step sets of cardinality three

Next we consider all the step sets of cardinality three. The following sections group step sets by the analytic complexity of their generating functions.

Luckily, to classify all of the walks with step sets of cardinality three, it is not necessary to consider all $\binom{8}{3} = 56$ possibilities. Any step set which is a subset of $\{\mathsf{SE}, \mathsf{S}, \mathsf{SW}, \mathsf{W}, \mathsf{NW}\}$ has no valid walk in the quarter plane. These are 10 in total. Reflecting a step set in the line $x = y$ yields an isomorphic class of step sets. There are only four step sets which are invariant under this action, leaving 21 pairs of duplicates with respect to this action, and thus, there are 25 classes of walks, up to symmetry in the line $x = y$. We can reduce this even further by determining other isomorphic classes. We show next that there are a total of eleven classes of walks, which are mutually non-isomorphic, and give explicit generating functions for each class.

2.1. The singular algebraic class

As we remarked in Section 1, each step set defines a family of walks equivalent to a set of words with restrictions on the numbers of certain letters in each prefix. These restrictions take the form of two sets of inequalities: one restricts the walks to the upper half plane, and the other restricts walks to the right-hand plane. In many cases one of the sets implies the other, or one is trivial. For example, the vertical constraint on any prefix of a valid walk from the set $\{\mathsf{NE}, \mathsf{E}, \mathsf{SW}\}$, is $\#\mathsf{NE} \geqslant \#\mathsf{SW} \geqslant 0$, and implies the horizontal constraint $\#\mathsf{NE} + \#\mathsf{E} \geqslant \#\mathsf{SW} \geqslant 0$. This is an example of a *singular* step set. The singular step sets correspond with the notion of degeneracy for the group of the walk.

Definition 2.1 *(Singular step set).* A step set is *singular* if it is a subset of any of the following sets:

1. $\mathcal{S}_1 = \{\mathsf{W}, \mathsf{NW}, \mathsf{N}, \mathsf{NE}, \mathsf{E}\}$, $\mathsf{reflect}(\mathcal{S}_1)$, $\mathsf{rev}(\mathcal{S}_1)$, $\mathsf{reflect}(\mathcal{S}_1)$;
2. $\mathcal{S}_2 = \{\mathsf{NE}, \mathsf{N}, \mathsf{NW}, \mathsf{W}, \mathsf{SW}\}$, $\mathsf{reflect}(\mathcal{S}_2)$.

These sets are pictured in vector format in Fig. 1.

Proposition 2.1. *If $\mathcal{Y}$ is a singular step set, then the complete generating function $Q_{\mathcal{Y}}(x, y; t)$ is an algebraic function.*

The algebraicity of the counting generating function for walks with a singular step set can be deduced from the observation that the quarter plane condition of singular walks is equivalent to the half plane condition, and these walks are algebraic by Theorem 1.2. Furthermore, it is straightforward to construct a simple pushdown automaton which recognizes the language of the walk. The language is thus unambiguously context-free and it follows that the counting generating function is algebraic [10], and is easy to determine. Essentially, these are generalized Motzkin paths.

Proof of Proposition 2.1. In order to prove the algebraicity of the complete generating function, we give an explicit, unambiguous grammar satisfied by the walks. A well-chosen weighting on the certain variables gives a system of algebraic equations satisfied by the complete generating function, and thus proves the algebraicity.

If the walk is singular, then there is only one governing, non-trivial, inequality. Without loss of generality suppose it is the vertical condition. Divide the directions of the step set into three subsets: Let $\mathcal{A} = \{a_i\}_1^k = \{(j,1) \in \mathcal{Y}\}$, $\mathcal{B} = \{b_i\}_1^l = \{(j,-1) \in \mathcal{Y}\}$, $\mathcal{C} = \{c_i\}_1^m = \{(j,0) \in \mathcal{Y}\}$. Thus, the vertical condition is given by $\sum_{i=1}^k \#a_i \geqslant \sum_{j=1}^l \#b_j$.

If $\mathcal{Y}$ is singular, then $\mathcal{L}(\mathcal{Y})$ is generated by $\mathcal{S}$ in the following grammar, which assures that the number of $\mathcal{A}$s is always greater than the number of $\mathcal{B}$s for any prefix. The class $\mathcal{S}$ is decomposed in the classic manner by the last step up at a certain height:

$$\mathcal{S} \to (\mathcal{M}\mathcal{A})^*\mathcal{M}, \qquad \mathcal{M} \to \epsilon|\mathcal{C}\mathcal{M}|\mathcal{A}\mathcal{M}\mathcal{B}\mathcal{M},$$
$$\mathcal{A} \to a_1|\dots|a_k, \qquad B \to b_1|\dots|b_l,$$
$$\mathcal{C} \to c_1|\dots|c_m.$$

There is a direct correspondence between these grammars and the functional equations satisfied by the generating function [10,13]. Each step is weighted by a monomial corresponding to its direction, namely (i,j) is assigned to $x^i y^j t$. This gives a solvable algebraic system for the complete generating system:

$$\left\{\begin{array}{l} S(x,y,t) = \dfrac{M(x,y,t)^2 A(x,y,t)}{1 - M(x,y,t)A(x,y,t)} \\ M(x,y,t) = 1 + C(x,y,t)M(x,y,t) + A(x,y,t)B(x,y,t)M(x,y,t)^2 \\ A(x,y,t) = \displaystyle\sum_{i=1}^k a_i(x,y,t), \qquad B(x,y,t) = \sum_{i=1}^l b_i(x,y,t) \\ C(x,y,t) = \displaystyle\sum_{i=1}^m c_i(x,y,t) \end{array}\right\}.$$

We can then solve for $Q_{\mathcal{Y}}(x,y;t) = S(x,y,t)$. $\square$

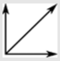
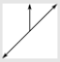
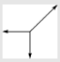
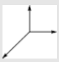
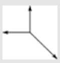
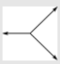
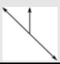
To illustrate the process from the proof of Proposition 2.1, we determine algebraic equations satisfied by the generating system for walks given by $\mathcal{Y} = \{\mathsf{N}, \mathsf{NE}, \mathsf{SW}\}$. The dominating constraint is $\#\mathsf{NE} \geqslant \#\mathsf{SW}$. Set $\mathcal{A} = \{\mathsf{NE}\}$, $\mathcal{B} = \{\mathsf{SW}\}$, and $\mathcal{C} = \{\mathsf{N}\}$. The language $\mathcal{L}(\mathcal{Y})$ are generated by $\mathcal{S}$ in the following grammar:

$$\mathcal{S} \to (\mathcal{M}\mathcal{A})^*\mathcal{M}, \qquad \mathcal{M} \to \epsilon|\mathcal{C}\mathcal{M}|\mathcal{A}\mathcal{M}\mathcal{B}\mathcal{M}.$$

In this case, the algebraic system is determined from the three substitutions $A(x,y,t) = xyt$, $B(x,y,t) = \bar{x}\bar{y}t$, and $C(x,y,t) = xt$ into the above system. The solution $S(x,y;t)$ of this system gives an expression for $Q(x,y;t)$:

$$Q(x,y;t) = S(x,y,t) = -\frac{-1 + yt + \sqrt{1 - 2yt + t^2y^2 - 4t^2}}{t(2t - yx + y^2xt + yx\sqrt{1 - 2yt + t^2y^2 - 4t^2})}.$$

Table 2
Generating function data for walks whose step set has cardinality three

#	$\mathcal{Y}$	$W_{\mathcal{Y}}(t)$	Initial counting sequence	Nature of series	Cf.
1		$(1-3t)^{-1}$	1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683	Rational	Section 2.1
2		$\frac{1-3t-\sqrt{-3t^2-2t+1}}{2t(3t-1)}$	1, 2, 5, 13, 35, 96, 267, 750, 2123, 6046	Algebraic	Section 2.1
3		$\frac{1-4t-\sqrt{1-8t^2}}{4t(3t-1)}$	1, 2, 6, 16, 48, 136, 408, 1184, 3552, 10432	Algebraic	Section 2.1
4		$\frac{1-2t-\sqrt{1-8t^2}}{2t(3t-1)}$	1, 1, 3, 5, 15, 29, 87, 181, 543, 1181	Algebraic	Section 2.1
5		$\frac{2(1-1/T)\sqrt{1-T^2}}{3t-1}-\frac{1}{t}$	1, 1, 3, 7, 17, 47, 125, 333, 939, 2597	Algebraic	Section 2.3.1, [7]
6		$\frac{(T+1-2/T)\sqrt{1-T+T^2/4-T^4/4}}{3t-1}-\frac{1}{t}$	1, 2, 4, 10, 26, 66, 178, 488, 1320, 3674	Algebraic	Section 2.3.2
7		$\frac{1-t-\sqrt{(1+t)(1-3t)}}{2t^2}$	1, 1, 2, 4, 9, 21, 51, 127, 323, 835	Algebraic*	Section 2.3.3, [23]
8		$\frac{1-H(1)-M+H(M)}{1-3t}$	1, 1, 2, 3, 8, 15, 39, 77, 216, 459	Holonomic	Section 2.4.1
9		$\frac{1-S(1)-M+S(M)}{1-3t}$	1, 1, 3, 7, 19, 49, 139, 379, 1079, 3011	Holonomic	Section 2.4.2
10			1, 1, 3, 7, 21, 55, 165, 457, 1371, 3909	Not holonomic	Section 2.5, [20]
11			1, 1, 2, 4, 10, 23, 61, 153, 418, 1100	Not holonomic	Section 2.5, [20]

* The complete series is transcendental holonomic. T satisfies $T=t(2+T^3)$; $M=\frac{1-t-\sqrt{1-2t-3t^2}}{2t}$, $H(x)=\sum \chi[k+1\equiv n \pmod 2]\binom{2n-2}{n-1}\binom{n+1}{\frac{1}{2}(n+k+1)}\frac{1}{n(n+1)}x^k t^{2n-1}$, $S(y)=\sum \chi[k+1\equiv n \pmod 2]\binom{2n+2}{n+1}\binom{n+1}{\frac{1}{2}(n+k+1)}\frac{1}{2n(2n+1)}y^k t^{2n+1}$.

In total there are 18 singular step sets. However, the walks which are true two-dimensional walks are governed by one of the following four inequality types, and they are all equivalent to one of four different classes[2] (where α, β, and γ are each replaced by the number of steps in a fixed direction):

$$(\#1)\quad \alpha \geqslant 0,\ \beta \geqslant 0,\ \gamma \geqslant 0, \qquad (\#2)\quad \alpha \geqslant \beta \geqslant 0,\ \gamma \geqslant 0,$$
$$(\#3)\quad \alpha+\beta \geqslant \gamma \geqslant 0, \qquad (\#4)\quad \alpha \geqslant \beta+\gamma \geqslant 0.$$

2.2. Summary of non-singular walks

The remaining non-singular walks, (seven in total) break down as follows. (Again, the numbers refer to those in Table 2.) Step set #5 is known as Kreweras' walks after Kreweras' study [17]. The algebraicity of $W_5(t)$ is surprising and well studied. Step set #6 is rev(#5), and is solved by applying the same algebraic kernel method [7] which gives the effective algebraicity results for step set #5. The set #7 is examined in detail in Section 2.3.3. Step sets #8 and #9 satisfy the criteria given in Theorem 1.3, and are thus holonomic, and one can exploit their symmetry in the kernel method to determine explicit expressions for $Q(x,y;t)$. As we shall see from calculations in Section 2.4, these walks are not algebraic. The final two, step sets #10 and #11, are not holonomic, and the proof from [20] is outlined in Section 2.5.

2 Examples are given by a step set with the corresponding number in Table 2.

2.3. Algebraic walks

2.3.1. Kreweras' walks: Step set #5

The step set $\mathcal{Y} = \{\mathsf{NE}, \mathsf{S}, \mathsf{W}\}$ is interesting since its generating functions are algebraic, but $\mathcal{L}(\mathcal{Y})$ is *not* context-free. (One can demonstrate this with the pumping lemma.) As we mentioned above, it has been studied in several different contexts, and a direct (i.e. combinatorial) explanation of the algebraicity of the excursions was offered by Bernardi [2]. He defines a bijection with a family of planar maps. Here we shall stick to just reporting enumerative data. In the next section, however, we follow the same method as Bousquet-Mélou in [7], in order to determine a new family of walks with an algebraic generating function, namely, $\mathsf{rev}(\mathcal{Y})$.

The fundamental equation expresses the complete generating function for walks given by the set $\{\mathsf{NE}, \mathsf{S}, \mathsf{W}\}$ in terms of the walks which returns to the x-axis. Theorem 1 of [7] gives an explicit formulation of this generating function.

Theorem 2.2 *(Bousquet-Mélou [7]). Let $T \equiv T(t)$ be the power series in t defined by $T = t(2 + T^3)$. The generating function for Kreweras' walks ending on the x-axis is*

$$Q_5(x, 0, t) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{T} - \frac{1}{x}\right)\sqrt{1 - xT^2}\right).$$

We have that $Q_5(y, 0; t) = Q_5(0, y; t)$ and thus we have the following expression for the complete generating function:

$$Q_5(x, y; t) = \frac{xy - tQ_5(x, 0; t) - tQ_5(0, y; t)}{xy - t(x + x + x^2y^2)}.$$

2.3.2. Reverse Kreweras walks: Step set #6

A second non-singular algebraic class is obtained from the reverse of the Kreweras' walks. The language $\mathcal{L}(\mathcal{Y})$ is similarly not context-free. As Bousquet-Mélou suggests [3], the algebraic kernel method can be applied in a straightforward manner to this set. This results in explicit information about step set #6, given by $\{\mathsf{N}, \mathsf{E}, \mathsf{SW}\}$. (There is an (albeit, longer) derivation that does not use results from the Kreweras walks [3].)

We have the following fundamental equation:

$$Q_6(x, y; t) = 1 + t(x + y + \bar{x}\bar{y})Q_6(x, y; t) - \frac{t}{xy}\big(Q_6(0, y; t) + Q_6(x, 0; t) - Q_6(0, 0; t)\big).$$

We drop the index, rearrange the terms, and re-write this

$$xyK_r Q(x, y; t) = xy - t\big(R_0(x) + R_0(y) - R_{00}(t)\big), \tag{2.1}$$

where the kernel of the fundamental equation is $K_r = (1 - t(x + y + \bar{x}\bar{y}))$, $R_0(x) = Q_6(x, 0; t) = Q_6(0, x; t)$ (by symmetry of the step set), and $R_{00}(t) = Q_6(0, 0; t)$. First, observe that the set of excursions starting from the origin with steps from #6 are isomorphic to those with steps from #5 via rev. Thus, the generating function $R_{00}(t)$ can be deduced from Theorem 2.2. The x in the denominator of the expression for $Q_5(x, 0; t)$ is a removable singularity, and hence we can determine $R_{00} = Q_5(0, 0; t)$ by taking the limit as x tends to 0. This results in the expression

$$R_{00}(t) = \frac{T(1 - T^3/4)}{2t}, \quad T = t\big(2 + T^3\big).$$

Next, we determine the group of the walk. It is generated by $\tau_y : (x, y) \mapsto (x, \frac{1}{xy})$ and $\tau_x : (x, y) \mapsto (\frac{1}{xy}, y)$. We remark that this is the same as the group of the walk for the Kreweras' walks. Thus, we have that $G(\mathcal{Y})$ is D_3, the dihedral group on six elements. We apply the invariance to obtain the following equalities

$$K_r(\bar{x}, \bar{y}) = K_r(\bar{x}, xy) = K_r(xy, \bar{y}), \tag{2.2}$$

and note that $K_r(\bar{x}, \bar{y})$ is the rational kernel for Kreweras' walks. Denote this function by $\bar{K}_r \equiv K_r(\bar{x}, \bar{y})$. Next, generate three equations substituting different values for x and y (using Eq. (2.2)), into Eq. (2.1):

$$\bar{x}\bar{y}\bar{K}_r Q(\bar{x}, \bar{y}; t) = \bar{x}\bar{y} - tR_0(\bar{x}) - tR_0(\bar{y}) + tR_{00}(t),$$
$$y\bar{K}_r Q(\bar{x}, xy; t) = y - tR_0(\bar{x}) - tR_0(xy) + tR_{00}(t),$$
$$x\bar{K}_r Q(xy, \bar{y}; t) = x - tR_0(xy) - tR_0(\bar{y}) + tR_{00}(t).$$

We form a composite equation taking the sum of the first two equations and subtracting the third:

$$\bar{x}\bar{y}R(\bar{x}, \bar{y}) + yQ(\bar{x}, xy) - xQ(xy, \bar{y}) = \frac{1}{\bar{K}_r}\big(\bar{x}\bar{y} + y - x - 2tR_0(\bar{x}) + tR_{00}(t)\big). \tag{2.3}$$

The explicit expressions for Y_0 and Y_1 are

$$Y_0(x) = \frac{1 - t\bar{x} - \sqrt{(1 - t\bar{x})^2 - 4t^2 x}}{2tx} \quad \text{and} \quad Y_1(x) = \frac{1 - t\bar{x} + \sqrt{(1 - t\bar{x})^2 - 4t^2 x}}{2tx}.$$

Set Δ to be the common discriminant $(1 - t\bar{x})^2 - 4t^2 x$ of Y_0 and Y_1.

The partial fraction expansion of $\bar{K}_r^{-1}$ is given by

$$\frac{1}{\bar{K}_r} = \frac{1}{(1 - Y_0\bar{y})(y - Y_1)} = \frac{1}{\sqrt{\Delta(x)}}\left(\sum_{n\geqslant 0} \bar{y}^n Y_0^n + \sum_{n\geqslant 1} y^n Y_1^{-n}\right).$$

The series Δ factors into three power series, respectively in $C[x][[t]], C[[t]], C[\bar{x}][[t]]$, which we shall call $\Delta_+(x)$, $\Delta(t)$, and $\Delta_-(\bar{x})$. This is an instance of a canonical factorization of a power series which is useful in many enumeration problems. We now make direct use of some of Bousquet-Mélou's intermediary calculations. She determined that $\Delta_-(\bar{x}) = 1 - \bar{x}(T(1 + T^3/4) + \bar{x}T^2/4)$ (which we later denote $1 - \bar{x}\mathcal{X}$), with T the unique power series in t defined by $T = t(2 + T^3)$.

Next we extract the constant term with respect to y from both sides of Eq. (2.3), and express this using $Q_d(x)$, the generating series for walks that end on the diagonal:

$$-xQ_d(x) = \sqrt{\Delta(x)}^{-1}\big(2Y_0 - x - 2tR_0(\bar{x}) + tR_{00}(t)\big). \tag{2.4}$$

Here, we have made use of the fact that $Y_1^{-1}\bar{x} = Y_0$. Now, both the left- and right-hand sides are series in $\mathbb{C}[x, \bar{x}][[t]]$. For any such series $f(x, \bar{x}, t)$ denote the sub-series whose coefficients of t^n are polynomials in $\mathbb{C}[\bar{x}]$ by $f^{\leqslant}$. We now isolate these sub-series from either side of Eq. (2.4). This gives

$$0 = \frac{-2tR_0(\bar{x}) + tR_{00}(t)}{\sqrt{\Delta_0\Delta_-(\bar{x})}} - \left(\frac{x - 2Y_0}{\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leqslant}. \tag{2.5}$$

We begin with the calculation

$$\frac{x}{\sqrt{\Delta_0\Delta_-(\bar{x})}} = \frac{x}{\sqrt{\Delta_0}}\frac{1}{\sqrt{1 - \bar{x}\mathcal{X}}} = \frac{x}{\sqrt{\Delta_0}}\left(1 - \frac{\bar{x}\mathcal{X}}{2} + O\big(\bar{x}^2\big)\right),$$

where $\mathcal{X} \in k[[t]]$, leading to

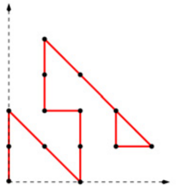
$$\left(\frac{x}{\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leqslant}\big(\sqrt{\Delta_0\Delta_-(\bar{x})}\big) = \left(\frac{x}{\sqrt{\Delta_0\Delta_-(\bar{x})}} - \frac{x}{\sqrt{\Delta_0}}\right)\big(\sqrt{\Delta_0\Delta_-(\bar{x})}\big) = x\big(1 - \sqrt{\Delta_-(\bar{x})}\big).$$

The remaining term is slightly more delicate to compute. We have

$$\left(\frac{2Y_0}{\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leqslant} = \frac{\bar{x}(\frac{1-t\bar{x}}{t})}{\sqrt{\Delta_0\Delta_-(\bar{x})}} - \left(\frac{\bar{x}}{t}\sqrt{\Delta_+}\right)^{\leqslant}.$$

Next we develop $\frac{\bar{x}}{t}\sqrt{\Delta_+}$ to determine the negative part

$$\left(\frac{\bar{x}}{t}\sqrt{\Delta_+(x)}\right)^{\leqslant} = \left(\frac{\bar{x}}{t}\sqrt{1 - xT^2}\right)^{\leqslant} = \left(\frac{\bar{x}}{t}\big(1 - xT^2/2 - \big(xT^2\big)^2/4 - O\big(x^6\big)\big)\right)^{\leqslant} = \frac{\bar{x}}{t} - \frac{T^2}{2t}.$$



7	13					
3	4	10	11	12		
1	2	5	6	8	9	14

Fig. 2. A walk from step set $\mathcal{Y} = \{\mathsf{N}, \mathsf{SE}, \mathsf{W}\}$ and its corresponding Young tableau.

Putting it all together,

$$\left(\frac{2Y_0}{\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leqslant}\sqrt{\Delta_0\Delta_-(\bar{x})} = \frac{1}{t}\left(\bar{x}\left(1-t^2\right) - \left(\bar{x} - \frac{T^2}{2}\right)\sqrt{\Delta_0\Delta_-(\bar{x})}\right).$$

We now clear the denominator in Eq. (2.5) to get the explicit expression given in the following theorem. We can indeed conclude the algebraicity of $Q(x,0;t)$ from this expression. Using this we reconstruct an expression for the complete generating series demonstrating its algebraicity.

Theorem 2.3. *Let $\mathcal{Y} = \{\mathsf{N}, \mathsf{E}, \mathsf{SW}\}$. Then the generating series for walks that return to the x-axis satisfies*

$$2Q_6(x,0;t) = Q_6(0,0;t) + \left(\frac{-2x}{Tt}\left(1 - \frac{T^2}{2x}\right) + \frac{1}{tx}\right)\sqrt{U} + \left(1 - tx - \frac{t}{x^2}\right)xt^{-2},$$

where T is the power series in t defined by $T = t(2 + T^3)$, and U is the power series in t defined by $U(x) = 1 - xT(1 + T^3/4) + x^2T^2/4$, and $Q_6(0,0;t) = T(1 - T^3/4)/2t$.

Theorem 2.4. *Let $\mathcal{Y} = \{\mathsf{N}, \mathsf{E}, \mathsf{SW}\}$. Then the complete generating series for walks with steps from $\mathcal{Y}$ satisfies*

$$Q_6(x,y;t) = \frac{xy - S(x) - S(y) + S(0)}{xy - t(x^2y + xy^2 + 1)},$$

where $S(x) = tQ_6(x,0;t)$ from the previous theorem.

2.3.3. Tableaux walks: Step set #7

The walk set $\mathcal{Y} = \{\mathsf{N}, \mathsf{SE}, \mathsf{W}\}$ is an interesting example because the nature of the complete and the counting generating functions differ. Regev [23] proved that the number of such walks of length n is equal to the nth Motzkin number, thus admitting the counting generating function

$$W_7(t) = \frac{1 - t - \sqrt{(1+t)(1-3t)}}{2t^2}.$$

We can deduce this, and an expression for the complete generating function, by exploiting a bijective correspondence between these walks and standard Young tableaux of height at most 3. Standard Young tableaux are labelled Ferrer's diagrams of a partition such that the boxes are labelled in a strictly increasing manner from left to right and from bottom to top. We construct a standard Young tableau of height three of size n from a walk $w = w_1, w_2, \ldots, w_n$ as follows. If $w_i = \mathsf{N}$ (respectively SE, W), then place label i in the next available space to the right on the bottom row (respectively second, top). The final tableau is increasing from left to right by construction, and the prefix condition $\#\mathsf{N} \geqslant \#\mathsf{SE} \geqslant \#\mathsf{W}$ ensures that it is increasing along the columns. Fig. 2 gives an example of such a correspondence.

There is a well-known formula for counting standard Young tableaux, known as the hook formula [25]. We apply this formula to count the number of tableaux of form (n_1, n_2, n_3) from which we deduce the number of walks of length n that end at co-ordinate $(i,j) = (n_1 - n_2, n_2 - n_3)$. The number of such tableaux are

$$a(n_1,n_2,n_3) = (n_1 - n_2 + 1)(n_2 - n_3 + 1)(n_1 - n_3 + 2)\frac{(n_1+n_2+n_3)!}{(n_1+2)!(n_2+1)!n_3!}.$$

Now, the total length of the corresponding walk is the size of the tableau, $n = n_1 + n_2 + n_3$ and to end at (i, j), we have $i = n_1 - n_2$ and $j = n_2 - n_3$. Thus, when we make the substitution,

$$a_{ij}(n) = \frac{(i+1)(j+1)(i+j+2)n!}{(\frac{n-i-2j}{3})!(\frac{n-i+j+3}{3})!(\frac{n+2i+j+6}{3})!}.$$

We can verify that this is *P-finite* [18], and thus the complete generating function is holonomic.

In order to establish transcendence, consider the generating function of walks that return to the origin. The counting sequence is given by

$$a_{00}(n) = \frac{2(3n)!}{n!(n+1)!(n+2)!},$$

whose asymptotic form $a_{00}(n) \sim \frac{\sqrt{3}}{\pi} 27^n n^{-4}$ excludes it from the algebraic class, by Theorem 1.1. Thus, as there is a non-algebraic series which results from a specialization of the complete generating function, it is not algebraic. Thus, $Q_7(x, y; t)$ is a transcendental holonomic series.

2.4. Holonomic walks

In the next example we use the classic kernel method to determine enumerative results for walks with steps from {N, SE, SW} and {W, NE, SE}. These results allow us to conclude that the series, although holonomic, are transcendental. The property we exploit is their axis symmetry.

2.4.1. Step set #8

Set $\mathcal{Y} = \{\mathsf{N}, \mathsf{SE}, \mathsf{SW}\}$. The functional equation for $Q_{\mathcal{Y}} = Q$ is

$$K(x, y) Q(x, y; t) = xy - (x^2 + 1) t Q(x, 0; t) - t Q(0, y; t) + t Q(0, 0; t), \tag{2.6}$$

with kernel $K(x, y) = xy(1 - ty - t(x + \bar{x})\bar{y})$. Let $Y_1(x)$ satisfy $K(x, Y_1(x)) = 0$, and vanish at $t = 0$. The quadratic formula gives

$$Y_1 = Y_1(x) = \frac{x - \sqrt{x^2 - 4x^3t^2 - 4xt^2}}{2xt} = \frac{1 - \sqrt{1 - 4t^2(x + \bar{x})}}{2t} = (x + \bar{x})t + (x + \bar{x})^2 t^3 + O(t^5).$$

Next, we generate two equations based upon the observations that $K(\bar{x}, y) = K(x, y)$ and $Y_1(\bar{x}) = Y_1(x)$:

$$0 = xY_1 - t(x^2 + 1) Q(x, 0; t) - t Q(0, Y_1; t) + t Q(0, 0; t),$$
$$0 = \bar{x}Y_1 - t(\bar{x}^2 + 1) Q(\bar{x}, 0; t) - t Q(0, Y_1; t) + t Q(0, 0; t).$$

The difference of these two equations yields a simpler third equation,

$$(x - \bar{x})Y_1 = t(x^2 + 1) Q(x, 0; t) - t(\bar{x}^2 + 1) Q(\bar{x}, 0; t). \tag{2.7}$$

As we did in the case of the reverse Kreweras walks, we view both sides of the equation as elements of $\mathbb{Q}[x, \bar{x}][[t]]$, and isolate the sub-series (the positive part, denoted by the $\geqslant$ in the exponent) which is an element of $\mathbb{Q}[x][[t]]$. We have that $Q(\bar{x}, 0; t)^{\geqslant} = Q(0, 0; t)$, and thus the positive part of the right-hand side of Eq. (2.7) is $t(x^2 + 1) Q(x, 0; t) - t Q(0, 0; t)$. Define

$$H(x) := t(x^2 + 1) Q(x, 0; t) - t Q(0, 0; t) = ((x - \bar{x})Y_1)^{\geqslant}.$$

We use the series expansion of $\sqrt{1 - 4t^2(x + \bar{x})}$ to determine the expression

$$Y_1 = \sum_{n>0} (x + \bar{x})^n \frac{1}{n} \binom{2n-2}{n-1} t^{2n-1}.$$

The positive series $((x + \bar{x})^n)^{\geqslant} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} x^{n-2k}$.

Thus, when we combine these two, we can describe $a(n,k)$, the coefficient of $t^{2n-1}x^k$ in $((x-\bar{x})Y_1)^{\geqslant}$ for $k \leqslant n+1$:

$$a(n,k) = \begin{cases} 0 & k \equiv n \bmod 2, \\ \binom{2n-2}{n-1}\binom{n+1}{\frac{n+k+1}{2}}\frac{k}{n(n+1)} & \text{otherwise.} \end{cases}$$

We substitute this result into the original equation:

$$K(x,y)Q(x,y;t) = xy - H(x) - tQ(0,y;t). \tag{2.8}$$

Next, we define $X_1 = X_1(t)$ as the unique power series in t satisfying $K(X_1, y) = 0$ and $X_1(0) = 0$. Again, this is easily determined using the quadratic formula. Substituting this into Eq. (2.8), we have that $tQ(0,y) = X_1 y - H(X_1)$. This gives the following theorem.

Theorem 2.5. *Let $\mathcal{Y} = \{\mathsf{N}, \mathsf{SE}, \mathsf{SW}\}$. Then the complete generating series for walks with steps from $\mathcal{Y}$ is*

$$Q_8(x,y;t) = \frac{xy - H(x) - X + H(X)}{xy - txy^2 - t(x^2+1)},$$

where $H(x)$ is defined

$$H(x) = \sum_{\substack{n,k \\ k+1 \equiv n \pmod 2}} \binom{2n-2}{n-1}\binom{n+1}{\frac{1}{2}(n+k+1)}\frac{k}{n(n+1)}x^k t^{2n-1}$$

and

$$X = \frac{y - ty^2 - \sqrt{y^2 - 2y^3t + t^2y^4 - 4t^2}}{2t}.$$

To show that the generating function is transcendental suppose, for the purpose of illustrating a contradiction, that $Q_8(x,y;t)$ were algebraic. Were it so, this implies the algebraicity of $Q_8(x,0;t)$ and $Q_8(0,0;t)$, which then in turn implies the algebraicity of $H(1)$ from the statement of the theorem. However, if $H(1) = \sum h_n t^{2n-1}$, we can show that asymptotically, as n tends to infinity h_n tends to $\frac{8^n\sqrt{2}}{4\pi n^2}$. Thus by Theorem 1.1, the series $H(1,t)$ is transcendental. Thus, we have established a contradiction, and $Q_8(x,y;t)$ is not algebraic. Recall, however, that it is holonomic, by Theorem 1.3.

2.4.2. Step set #9

A similar calculation gives the following theorem.

Theorem 2.6. *Let $\mathcal{Y} = \{\mathsf{NE}, \mathsf{SE}, \mathsf{W}\}$. Then the generating series for walks with steps from $\mathcal{Y}$ returning to the y-axis satisfies*

$$tyQ_9(0,y;t) = \sum_{\substack{n,k \\ k+1 \equiv n \pmod 2}} \frac{k}{(2n+1)(2n+2)}\binom{2n+2}{n+1}\binom{n+1}{\frac{1}{2}(n+k+1)}.$$

If $S(y) = tyQ_9(0,y;t)$ and $Y = \frac{(x-t)-\sqrt{(x-t)^2-4t^2x^4}}{2tx^2}$, then the complete generating series is given by

$$Q_9(x,y;t) = \frac{xy - Y + S(Y) - S(y)}{xy - tx^2(y^2+1) - ty}.$$

2.5. Non-holonomic walks

Given that most walks in the half plane and many of the walks in a slit plane are algebraic [4,8, 24], one may be surprised to discover that there are classes with generating functions which are not holonomic. This was first demonstrated by Bousquet-Mélou and Petkovšec with the knight's walks [9] ($\mathcal{Y}=\{(2,-1),(-1,2)\}$) which are not holonomic. In fact, evidence would seem to indicate that many walks in the quarter plane are not holonomic.

The final two step sets we consider have non-holonomic complete generating functions. This is established in [20], and is based on a similar problem of self-avoiding walks in wedges [15]. The general argument is to show that there are an infinite number of singularities in the counting generating function. This is similar to the argument of Bousquet-Mélou and Petkovšek, whose proof also uses an adaptation of a kernel method argument in order to demonstrate a source for an infinite number of singularities. Our construction is slightly more direct since in our case the kernel is a quadratic, not a cubic. However, we also use the fact that the group of the walk is infinite to generate an infinite set of singularities.

We pull from [20] the following two results which completes our classification. In the next section we present a small summary of the argument.

Theorem 2.7. *Let $Q_{10}(x,y;t)$ be the complete generating function for random walks on the first quadrant of the integer lattice with steps taken from* $\{\mathsf{NW},\mathsf{SE},\mathsf{NE}\}$ *and let $W_{10}(t)=Q_{10}(1,1;t)$ be the corresponding counting generating function. Neither of these functions are holonomic.*

Theorem 2.8. *Let $Q_{11}(x,y;t)$ be the complete generating function for random walks on the first quadrant of the integer lattice with steps taken from* $\{\mathsf{NW},\mathsf{SE},\mathsf{N}\}$ *and let $W_1(t)=Q_{11}(1,1;t)$ be the corresponding counting generating function. Neither of these functions are holonomic.*

2.5.1. Step set #10

To begin, we write the fundamental equation of these walks in its kernel form:

$$\left(xy-tx^2y^2-tx^2-ty^2\right)Q_{10}(x,y)=xy-tx^2Q_{10}(x,0)-ty^2Q_{10}(y,0). \tag{2.9}$$

Here, for brevity we did abbreviate $Q(x,y;t)$ as $Q(x,y)$, and have used the $x\leftrightarrow y$ symmetry to rewrite $Q(0,y)$ as $Q(y,0)$. There are two solutions for the kernel $K(x,y)=xy-tx^2y^2-tx^2-ty^2$ as a function of y

$$Y_{\pm1}(x)=\frac{x}{2t(1+x^2)}\left(1\mp\sqrt{1-4t^2\left(1+x^2\right)}\right). \tag{2.10}$$

We write $Y_n(x)=(Y_1\circ)^n(x)$ and remark that the set $\{Y_n \mid n\in\mathbb{Z}\}$ forms an infinite group, under the operation $Y_n\circ Y_m=Y_{n+m}$, with identity $Y_0=x$. We can also show the useful relation,

$$\frac{1}{Y_1(x)}+\frac{1}{Y_{-1}(x)}=\frac{1}{tx}.$$

Using the fact that the composition (as algebraic functions of x) $Y_{-1}(Y_{+1}(x))=x=Y_0(x)$, this relation can generalize upon substitution of $x=Y_n$ as

$$\frac{1}{Y_n(x)}=\frac{1}{tY_{n-1}(x)}-\frac{1}{Y_{n-2}(x)}. \tag{2.11}$$

By construction we have $K(x,Y_1(x))=0$, thus substituting $y=Y_1(x)$ into Eq. (2.9) gives (after a little tidying)

$$Q(x,0)=\frac{Y_1(x)}{x}\frac{1}{t}-\frac{Y_1(x)^2}{x^2}Q\left(Y_1(x),0\right).$$

Now we substitute $x=Y_n(x)$ into this equation to obtain

$$Q\left(Y_n(x),0\right)=\left(\frac{Y_{n+1}(x)}{Y_n(x)}\right)\frac{1}{t}-\left(\frac{Y_{n+1}(x)}{Y_n(x)}\right)^2 Q\left(Y_{n+1}(x),0\right).$$

Using this expression for $Q(Y_n(x),0)$ for various n, we can iteratively generate a new expression for $Q(x,0)$:

$$Q(x,0)=\frac{1}{x^2t}\sum_{n=0}^{N-1}(-1)^n Y_n(x)Y_{n+1}(x)+(-1)^N\left(\frac{Y_N(x)}{x}\right)^2 Q\left(Y_N(x),0\right).$$

Since $Y_n(x)=xt^n+o(xt^n)$ we have that $\lim_{N\to\infty}Y_N(x)=0$ as a formal power series in t and consequently

$$Q(x,0)=\frac{1}{x^2t}\sum_{n\geqslant 0}(-1)^n Y_n(x)Y_{n+1}(x). \tag{2.12}$$

We now specialize the fundamental equation:

$$(1-3t)Q(1,1;t)=1-2tQ(1,0;t)=1-2\sum_{n\geqslant 0}(-1)^n Y_n(1)Y_{n+1}(1). \tag{2.13}$$

The result follows from the following lemma, which is proved in [20].

Lemma 2.9. *Suppose q_c is a zero of $\bar{Y}_N(q):=Y_N(1;\frac{q}{1+q^2})^{-1}$, and that $q_c\neq 0$. Then*

1. (*Distinct poles*) *For all $k\neq N$, $\bar{Y}_k(q_c)\neq 0$; and*
2. (*Non-cancellation*) *The function $Q(1,0;\frac{q}{1+q^2})$ also has a pole at the $q=q_c$.*

We now have all the components in place to prove the main result.

Proof of Theorem 2.7. The function $Q(1,1;\frac{q}{1+q^2})$ has a set of poles given by the zeroes of the $\bar{Y}_n(q)$, by the preceding lemma. The set of such poles form an infinite set. Thus, $Q(1,1,\frac{q}{1+q^2})$ is not holonomic. For a multivariate series to be holonomic, any of its algebraic specializations must be holonomic, and as $Q(1,1,\frac{q}{1+q^2})$ is an algebraic specialization of both $Q(x,y;t)$ and $Q(1,1;t)$, neither of these two functions are holonomic. □

A natural extension of this work would be to complete the story, and use the functional equations to find explicit enumerative expressions.

2.5.2. Step set #11

This set does not have the same symmetry, and thus $Q(1,0)$ and $Q(0,1)$ are treated separately, although by essentially the same strategy.

3. Prospects for general criteria

3.1. Two conjectures

The group of the walk is useful for determining additional equations in the kernel method. However, an examination of all compass walks and their groups reveals a surprising fact about when this group is finite. Table 1 lists all compass walks with finite groups (up to symmetry in the line $x=y$).

Condition 3.1. *Suppose that $\mathcal{Y}$ is a subset of $\{0,\pm 1\}^2\setminus(0,0)$. One of the following is true*:

1. *$\mathcal{Y}$ is symmetric across either the x- or y-axis*;

2. $\mathcal{Y} = \mathsf{rev}(\mathcal{Y})$;
3. $\mathcal{Y} = \mathsf{reflect}(\mathsf{rev}(\mathcal{Y}))$;
4. $\mathcal{Y} = \{\mathsf{NE}, \mathsf{S}, \mathsf{W}\}$ (*Kreweras*) *or* $\mathcal{Y} = \{\mathsf{N}, \mathsf{E}, \mathsf{SW}\}$ (rev(*Kreweras*)).

Proposition 3.2. *Suppose that $\mathcal{Y}$ is a non-singular subset of* $\{0, \pm 1\}^2 \setminus (0, 0)$. *Then $G(\mathcal{Y})$ is finite if and only if $\mathcal{Y}$ satisfies Condition* 3.1.

Proof. Inspection of Table 1. □

The following conjecture is true in the case of $|\mathcal{Y}| = 3$, and all other known cases.

Conjecture 1. *Suppose the group of the step set $\mathcal{Y}$ is not singular. Then $W_{\mathcal{Y}}$ is holonomic if and only if the group of its step set is finite.*

It was the infinite sets of singularities arising from the infiniteness of the group that rendered the final two examples non-holonomic. A generalization or formalization of this might offer a possible path to prove one direction of the conjecture.

Combining the above proposition and conjecture, we have potential conditions on $\mathcal{Y}$ for it to have a holonomic complete generating function.

Conjecture 2. *The generating function $W_{\mathcal{Y}}$ is holonomic if and only if either $\mathcal{Y}$ is singular, or Condition* 3.1 *holds.*

This conjecture also implies that rev preserves holonomicity. If $K_r(x, y)$ is the kernel related to $\mathcal{Y}$, then $K_r(\bar{x}, \bar{y})$ is the kernel related to $\mathsf{rev}(\mathcal{Y})$. By taking the viewpoint that $\bar{x}$ and $\bar{y}$ are mere variables, we see that if the $\mathcal{Y}$-step set group is finite, then so is the one for $\mathsf{rev}(\mathcal{Y})$-step set as the groups will be the same, as observed in the Kreweras, rev(Kreweras) case.

There may be a better way to formulate these conditions. Some walks are better described using a different lattice. For example, the Kreweras and the reverse Kreweras walks have a rotational symmetry in the triangular lattice which may be important to understand these surprisingly algebraic classes.

We note that step set #10, whose generating function is not holonomic, does have some symmetry since $\mathcal{Y} = \mathsf{reflect}(\mathcal{Y})$, but this is not sufficient to offer holonomicity.

If true, this conjecture would respond positively to a conjecture of Gessel on the nature of the walks given by the step set $\mathcal{Y} = \{\mathsf{N}, \mathsf{SE}, \mathsf{S}, \mathsf{NW}\}$. Indeed this step set seems to resist most classical approaches and the holonomicity of its counting generating function is a subject of active debate. The present evidence [3,22] seems to imply that the counting generating function is holonomic, but the complete one is *not*! Indeed, it is on account of this case that we do not propose similar conjectures for the complete generating functions. Given this, and our earlier observations on the rotational symmetry of the Kreweras walks in the triangular lattice, it seems clear that the next step is to better understand how to make an argument using rotational symmetry. This kind of understanding would be useful in order to combinatorially explain the aesthetically pleasing algebraic equation satisfied by $W(t)$ for $\mathcal{Y} = \{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{S}, \mathsf{SW}, \mathsf{W}\}$:

$$\frac{1}{1-6t} = W(t) + 3t^2 W(t)^2 + 4t^2 W(t)^3 + 2t^3 W(t)^4.$$

This equation was output of the Maple gfun package, and can be confirmed by the obstinate kernel method [3].

It would also be interesting to see if there exists a direct way to incorporate the properties of the group directly into either the Holonomy Ansatz of Zeilberger [26], or the Quasi-Holonomic Ansatz of Kauers and Zeilberger [16].

3.2. Other questions

Aside from proving the two conjectures, there are other natural questions that arise. Does much change if instead we use the group of all transformations that fix the kernel, instead of the group defined by the given generators? Several natural generalizations are possible, such as what is a meaningful definition for group in 3 dimensions? Step set #7 has a natural counterpart in bijection with Young Tableaux of height 4, and this class is known to be holonomic by similar reasoning. However, the addition of a dimension corresponds to an additional variable, and the techniques of the kernel method become much more difficult. Is there an analogous definition still yield dihedral groups? What about step sets which are not of the compass type; what kinds of groups come out of that? Fayolle et al. give a characterization for when the group that they define for similar structures is finite. Can this be translated into straightforward combinatorial terms? Are these walks all at least differentiably algebraic?

4. Note added in proof

During the revision of this paper, it appears that a computational proof of the holonomy of the so-called Gessel walks has been completed by Kauers, Koutschan and Zeilberger. Once verified, it will add further strength to the conjectures.

Acknowledgments

This work was completed, in part, during an NSERC (Natural Sciences and Engineering Research Council of Canada) Postdoc at LaBRI, Université Bordeaux I, under the wise guidance of Mireille Bousquet-Mélou. The author is particularly grateful for the careful comments from two anonymous referees for this version and also to earlier readers that commented on an earlier version [19] which appeared in the proceedings of the 19th International Conference on Formal Power Series and Combinatorics (FPSAC).

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