AE 3613

Orbits 1

Dr. Jennings

HW #03

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FS 2024

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Problem 1

Assuming Dr. P's Corvette had no maximum speed and could thus reach escape velocity, the required speed (magnitude of velocity) to "escape" Earth's gravity could be modeled very simply by ignoring the rotation of Earth. The equation for escape velocity would then be $v_{esc} = \sqrt{\frac{2\mu}{r^2}}$, where μ is the gravitational parameter, and r is the distance of Dr. P's Corvette to the center of the Earth. It should be noted that for ease of calculation the Earth is assumed perfectly spherical, and that the mass of Dr. P's Corvette is negligible compared to the mass of Earth; meaning that rather than $\mu = G(m_1 + m_2)$ can be further simplified to $\mu = Gm_{\oplus}$, where G is the universal gravitational constant, $G = 6.67430 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$. By substituting with $m_{\oplus} = 5.9722 \cdot 10^{24}\ kg$ and $r = r_{\oplus}$, where $r_{\oplus} = 6378.137 \cdot 10^3\ m$ is the equatorial radius of Earth, it is found that $v_{esc} = 11.179905\ km/s$, or that $v_{esc} = 25008.797182\ mph$.

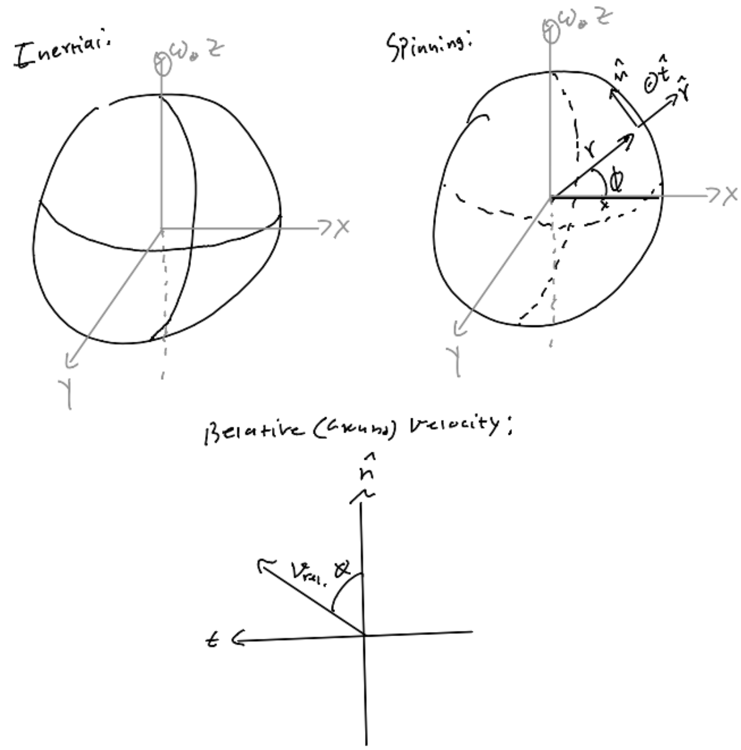

Figure 1

Now if the Earth is no longer assumed to be still, meaning it has a spin, it becomes much more difficult to calculate the magnitude of the ground velocity Dr. P would need to reach for "escape". It's still possible using the Basic Kinematic Equation (BKE) or Transport Theorem, $\frac{{}^{I}d\vec{\boldsymbol{r}}}{dt} = \frac{{}^{R}d\vec{\boldsymbol{r}}}{dt} + {}^{I}\vec{\boldsymbol{\omega}}^{R} \times \vec{\boldsymbol{r}}$, where $\frac{d\vec{\boldsymbol{r}}}{dt}$ is the escape velocity relative to the inertial frame, or the absolute escape velocity, $\frac{{}^{R}d\vec{\boldsymbol{r}}}{dt}$ is the escape velocity relative to the rotating frame, or local escape velocity, ${}^{I}\vec{\boldsymbol{\omega}}^{R}$ is the angular velocity of the rotating frame relative to the inertial frame, and $\boldsymbol{r}$ is the position vector of the car relative to the rotating frame. If the coordinate frames are designated as they are in figure 1, then the BKE can be rewritten and rearranged as $\vec{\boldsymbol{v}}_{esc} = \vec{\boldsymbol{v}}_{local} + \vec{\boldsymbol{\omega}} \times \vec{r}$, where $\vec{\boldsymbol{v}}_{local} = |v|cos(\theta)\hat{\boldsymbol{n}} +$

$|v|sin(\theta)\hat{\boldsymbol{t}}$, $\boldsymbol{\omega} = |\omega_{\oplus}|\sin(\phi)\,\hat{\boldsymbol{r}} + |\omega_{\oplus}|\cos(\phi)\,\hat{\boldsymbol{n}}$, and $\vec{\boldsymbol{r}} = r_{\oplus}\hat{\boldsymbol{r}}$, where θ is the angle counterclockwise from the $\hat{\boldsymbol{n}}$ axis to the local position vector, $\vec{\boldsymbol{r}}$, and ϕ is the latitude of Rolla. To solve for ground speed, the magnitude of $\vec{\boldsymbol{v}}_{local}$, the equation is evaluated and rearranged to become a quadratic set equal to zero (figure 2), $v^2(a) + v(b) + c = 0$, where $a = 1$, $b = 2\omega_{\oplus}rsin(\theta)\cos(\phi)$, and $c = \omega_{\oplus}{}^2r^2cos^2(\phi) - \boldsymbol{v}_{esc}{}^2$. By using quadratic formula, $v = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and substituting values for $\omega_{\oplus} = 7.2921159 \cdot 10^{-5}\,\frac{rads}{s}$, $r_{\oplus} = 6378.137\ km$, $\phi = 0.662377304619\ rads$, and $|\vec{\boldsymbol{v}}_{esc}| = 11.179905\ km/s$ (from part a), the quadratic equation for v can be plotted as a function of $\theta\{[0,2\pi]\}$ using MATLAB (figure 3).

$$^{I}\vec{V}_{esc} = V_{loc}\cos\theta\,\hat{n} + V_{loc}\sin\theta\,\hat{t} + \omega_{\oplus} r\cos\phi\,\hat{t}$$

↳ take magnitude, square, & subtract V_{esc}^2 from both sides

$$0 = \left(V_{loc}\cos\theta\,\hat{n}\right)^2 + \left(\left(V_{loc}\sin\theta + \omega_{\oplus} r\cos\phi\right)\hat{t}\right)^2 - V_{esc}^2$$

$$\Rightarrow 0 = V_{loc}^2\left(\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{1}\right) + V_{loc}\left(2\omega_{\oplus} r\sin\theta\cos\phi\right) + \left(\omega_{\oplus}^2 r^2\cos^2\phi - V_{esc}^2\right)$$

so...

let $a = 1$; $b = 2\omega_{\oplus} r\sin\theta\cos\phi$; $c = \omega_{\oplus}^2 r^2\cos^2\phi - V_{esc}^2$

quadratic formula...

$$V_{loc} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Figure 2

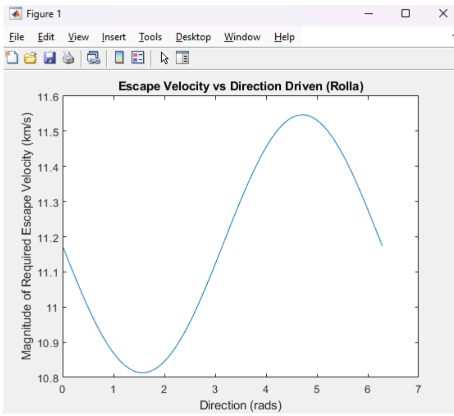

Figure 3

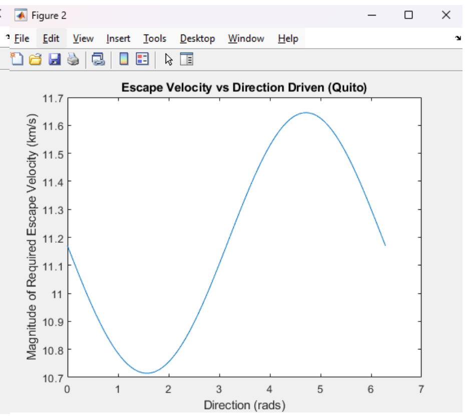

Figure 4

If Dr. P decides to start from Quito, Ecuador instead of Rolla, phi in the equation from part B is simply replaced with λ, the latitude of Quito, where $\lambda = -0.0031529896536\, rads$. The new quadratic simply changes to where $a = 1$, $b = 2\omega_{\oplus} r sin(\theta)\cos(\lambda)$, and $c = \omega_{\oplus}{}^{2} r^{2} cos^{2}(\lambda) - \boldsymbol{v}_{esc}{}^{2}$, and the new values are used to plot $v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ as a function of $\theta\{[0,2\pi]\}$ using MATLAB (figure 4). As is evident from comparing the graphs (figure 5), the minimum escape velocity is not much less than it is in Rolla at this scale. It should be noted however that while the difference may seem rather small, the required velocity is still around 360 kilometers per hour slower in Quito than it is in Rolla.

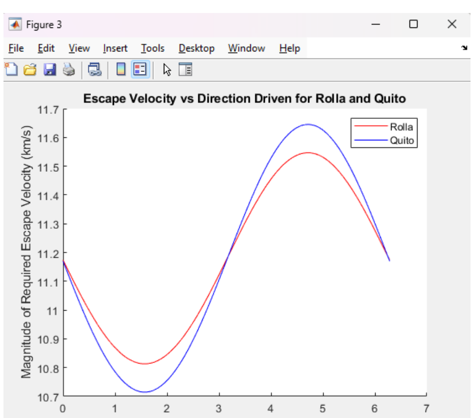

Figure 5

Problem 2

Dr. P has somehow started heading directly East from Rolla at a ground velocity of 30 thousand miles per hour according to his speedometer ($v_{grnd} = 13.4112\ \frac{km}{s}$), and assuming atmospheric effects are negligible upon the assumptions for problem 1, the specific energy, magnitude of specific angular momentum, the semi-major axis, and the cars speed while passing the Moons orbital distance can all be calculated. The specific energy of Dr. P's Corvette can be modeled as $\varepsilon = \frac{|\vec{v}|^2}{2} - \frac{\mu}{r}$, where v is the magnitude of the total velocity relative to the inertial frame previously established in problem 1. This means that the effect of Earth's angular velocity needs to be accounted for in the absolute velocity. Since Dr. P is driving East, or in the positive $\hat{\boldsymbol{\iota}}$ direction, the ground velocity and velocity due to spin are in the same direction and can simply be added together, so $|\vec{v}| = v_{grnd} + v_{spin}$, where v_{spin} is the same value from part A of problem 1 ($v_{spin} = \omega_{\oplus} r cos\phi$). Substituting this into the specific energy equation, $\varepsilon = \frac{v_{grnd} + \omega_{\oplus} r cos\phi}{2} - \frac{\mu}{r}$, the cars energy is found to be $\varepsilon = 32.420780\ \frac{km^2}{s^2}$. The specific angular momentum of Dr. P's Corvette is $\vec{\boldsymbol{h}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}$, and its magnitude is simply $h = rv$, where v is still $v = v_{grnd} + v_{spin}$, and $r = r_{\oplus}$; so $h = 87877.635689 \frac{km^2}{s}$. The semi-major axis of Dr P's Corvette is found by utilizing the equation $\varepsilon = \frac{-\mu}{2a}$, where a is the semi-major axis. The equation is then rearranged, $a = \frac{-\mu}{2\varepsilon}$, and it is found that $a = -6147.331280\ km$. It should be noted that since $a < 0$, the orbit of Dr. P's Corvette is hyperbolic, and that it will drift further and further away from the Earth never to be seen again. Eventually the car reaches the Moons orbital distance. The speed (magnitude of velocity) of Dr. P's Corvette at this point can be found by rearranging the specific energy equation and replacing $r_\oplus$ with $r_☾$, where $r_☾ = 384378.472\ km$ is the average orbital distance of the moon from the Earth. The resulting equation is $\varepsilon = \frac{v_☾{}^2}{2} - \frac{\mu}{r_☾}$, which can be rearranged to $v_☾ = \sqrt{\mu(\frac{2}{r_☾} - \frac{1}{a})}$, and it's found that $v_☾ = 8.180194\ \frac{km}{s}$.

Citations

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MATLAB

Problem 1

```
% Clear old data, plots, and console

clear

close all

clc


% Define constant variables
```

```
G = 6.67430e-11; % (N * m^2 ./ kg) or (m^3 ./ kg * s^2)

m_earth = 5.9722e24; % (kg)

m_car = 0; % Negligable compared to m_earth

mu = G * (m_earth + m_car);

radius = 6378.137e3; % Equatorial radius (m)


% Define "theta"

resolution = 1e5;

theta = linspace(0,2*pi, resolution);


% Spinning constants


angVel = 7.2921159e-5; % (rads/s)

phi_rolla = 0.662377304619; % (rads)

phi_quito = -0.0031529896536; % (rads)



% Part A
```

V_esc_ns = sqrt(2 * mu ./ radius) ./ 1000; % Escape velocity if the Earth is not spinning (km/s)

ns_mph = V_esc_ns * 3600 ./ 1.60934; % No spin escape velocity (m)

ns_mps = V_esc_ns * 1000;

fprintf('The escape velocity if the earth is NOT spinning is %.6f km/s, or %.6f mph \n', V_esc_ns, ns_mph);

% Part B

% Escape velocity with a spinning Earth

b_a = 1;

b_b = 2 * angVel * radius * sin(theta) * cos(phi_rolla);

b_c = angVel.^2 * radius.^2 * (cos(phi_rolla)).^2 - ns_mps.^2;

bDiscriminant = b_b.^2 - 4*b_a*b_c;

bRoot1 = (-b_b + sqrt(bDiscriminant)) / (2*b_a); % Can ignore negative solution because of changing theta

```
V_esc_rolla = bRoot1 / 1000;

figure;
plot(theta, V_esc_rolla);
title('Escape Velocity vs Direction Driven (Rolla)');
xlabel('Direction (rads)');
ylabel('Magnitude of Required Escape Velocity (km/s)');
hold off;

% Part C

% Escape velocity with a spinning Earth
c_a = 1;
c_b = 2 * angVel * radius * sin(theta) * cos(phi_quito);
c_c = angVel.^2 * radius.^2 * (cos(phi_quito)).^2 - ns_mps.^2;

cDiscriminant = c_b.^2 - 4*c_a*c_c;

cRoot1 = (-c_b + sqrt(cDiscriminant)) / (2*c_a); % Can ignore negative solution
because of changing theta
```

```
V_esc_quito = cRoot1 / 1000;

figure;
plot(theta, V_esc_quito);
title('Escape Velocity vs Direction Driven (Quito)');
xlabel('Direction (rads)');
ylabel('Magnitude of Required Escape Velocity (km/s)');
hold off;

figure;
hold on;
plot(theta, V_esc_rolla, 'r'); % Plot Rolla in red
plot(theta, V_esc_quito, 'b'); % Plot Quito in blue

% Add legend to differentiate between the two curves
legend('Rolla', 'Quito');

% Add title and axis labels
title('Escape Velocity vs Direction Driven for Rolla and Quito');
```

```
xlabel('Direction (rads)');

ylabel('Magnitude of Required Escape Velocity (km/s)');


hold off;
```

Problem 2

```
% Clear old data, plots, and console

clear

close all

clc


% Define constant variables


G = 6.67430e-11; % (N * m^2 ./ kg) or (m^3 ./ kg * s^2)

m_earth = 5.9722e24; % (kg)

m_car = 0; % Negligable compared to m_earth

mu = G * (m_earth + m_car);

radius = 6378.137e3; % Equatorial radius (m)


% Spinning constants
```

```
angVel = 7.2921159e-5; % (rads/s)
phi_rolla = 0.662377304619; % (rads)

vGrnd = 13411.2; % (m/s)
vAbs = vGrnd + (radius * cos(phi_rolla) * angVel);

% Part A

energy = (vAbs^2 / 2) - (mu / radius);
energy_km = energy / 1000000;

fprintf('The specific energy of Dr. P’s Corvette driving due East from Rolla at a
local speed of 30,000 mph is %.6f km^2./s^2\n', energy_km);

% Part B

h_mag = radius * vAbs;
h_mag_km = h_mag / 1000000;
```

```
fprintf('The specific angular momentum of Dr. P's Corvette driving due East from Rolla at a local speed of 30,000 mph is %.6f km^2/s\n', h_mag_km);

% Part c

a = - mu ./ (2 * energy);
a_km = a / 1000;

fprintf('The semi-major axis of Dr. P's Corvette driving due East from Rolla at a local speed of 30,000 mph is %.6f km\n', a_km);

% Part D

r_moon = 378000000 + radius; % (m)

vMoon = sqrt(mu * ((2 / r_moon) - (1 / a)));
vMoon_km = vMoon / 1000;
```

```
fprintf('The velocity of Dr. P's Corvette driving due East from Rolla at a local speed of 30,000 mph when it reaches the orbital distance of the moon is %.6f km/s\n', vMoon_km);
```