phys 4200 final

Gravesteijn, Wouter

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1 Part 1, Differential equation

Calculating properties of rockets in flight is notoriously difficult. Hence the term "it's not rocket science" when referring to trivial tasks.

The difficulty, even in an ideal case, is that everything is changing all the time. The rocket burns fuel and thereby loses mass, it accelerates which causes air resistance to increase, it gains altitude and encounters less dense air, as it goes up the gravitational pull will decrease. In real life there would be a gravity turn involved to gain the horizontal velocity needed to attain an orbit.

In this case a few simplifications are implemented, there is no orbit attained, let's assume we're shooting down something and can go straight up. Another simplification is the gravitational pull, the rocket remains fairly close to earth so the pull remains the same. To keep the differential equation transparent enough for intuition the drag component is also eliminated. As the highest velocity is attained at altitudes where the air is very thin this is less of a stretch that it might seem.

2 Theory

The impulse based rocket equation:

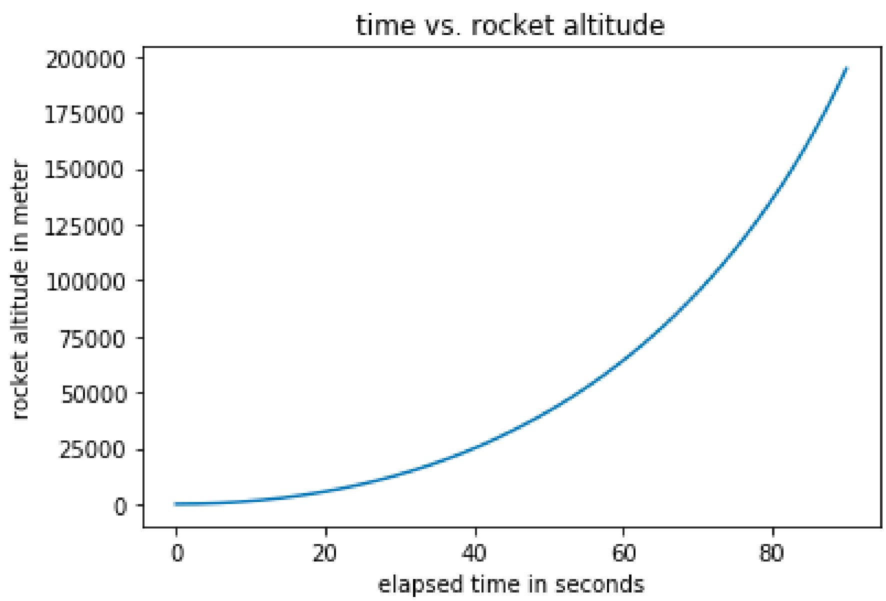
$$v = -c * ln(\frac{m_0 + dmdt * t}{m_0}) - g * t \tag{1}$$

where :
m_0 = initial total rocket mass in kg
dmdt = change in mass (burn rate) in kg/s, negative as mass is expelled (depends on size and number of engines)
t = time in seconds
v = velocity in m/second
c = exhaust velocity in m/s (determined by fuel type and nozzle shape)
g = gravitational acceleration = 9.8 m/s2.

This equation uses the Delta-V concept:

The burn rate (*dmdt*) of fuel applied for x fraction of a second gives a specific impulse.
When multiplied by the exhaust escape velocity this yields the specific thrust in newtons which in turn gives the applied force on the remaining rocket mass.

The resulting differential equation thus depends on mass versus change in mass at time t which gives the total accrued velocity



3 Conclusion

The above graph shows the data for a rocket of mass 2000kg with 1800 kg of fuel burning 20 kg/s with an exhaust velocity of 3500 m/s. The total burn time is 90 seconds. The attained altitude at end of burn is 194.7km.

After this the rocket would enter the unpowered part of the flight which is governed by simple kinematic formula's and not part of the explored differential equations. The maximum attained altitude is further explored in the next part of this final.

4 Part 2, Optimizing

We're going to optimize for maximum altitude based of rate of fuel burn. Engines are expensive and complicated and the burn rate depends on amount and size of engines.

Both drag and gravity are used this time.

First the equations are written out for clarity, these are then combined for easier optimization. Then constants are defined.

Although drag is used, a constant air density is used without regard for altitude for simplicity and the atmosphere extends forever. The highest velocities are attained well above the atmosphere to this is fine for a first calculation.

5 Theory

Let's start with the appropriate formula's:

Drag is a function of area, steam lining and air density: A, C_d,ρ

The drag equation:

$$drag = 0.5 C_d A \rho \tag{2}$$

Thrust is a function of exhaust velocity and burn rate

$$T = c(-dmdt) \tag{3}$$

A handy dandy constant for keeping things clear:

$$k = (2\sqrt{\frac{(T - m * g)drag}{m}} \tag{4}$$

Another handy dandy constant for keeping things clear:

$$q = \sqrt{\frac{T - mg}{drag}}) \tag{5}$$

The velocity equation at time t:

$$v = q\frac{1 - e^{-kt}}{1 + e^{-kt}} \tag{6}$$

The attained altitude at time t during thrust:

$$y_{thrust} = \frac{-m}{2 * drag} * ln[\frac{T - mg - drag * v^2}{T - mg}] \tag{7}$$

The attained altitude at time t during coasting:

$$y_{coast} = \frac{m}{2 * drag} * ln[\frac{mg + drag * v^2}{mg}] \tag{8}$$

The attained altitude would be the $y_{thrust} + y_{coast}$

C_d = drag coefficient = 0.05
A = area of rocket= 0.7 m^2
ρ = average air density = 0.88 kg/m^3
dmdt = change in mass (burn rate) in kg/s, negative as mass is expelled (depends on size and number of engines) initially set at 30kg/s
t = time in seconds
v = velocity in m/s
c = exhaust velocity in m/s (determined by fuel type) set at 3500 m/s which is average
m = average mass of the rocket during burn, considering a constant burn rate this is simply the empty mass plus half the fuel mass. Set at $200kg + \frac{1800kg}{2}$
g = gravitational acceleration = 9.8 m/s^2

6 The trick

The trick here is to balance the altitude versus the velocity at the moment of burnout. To maximize altitude at burnout, the lowest amount of thrust would be best, resulting in a low final velocity. To maximize the altitude from coasting, a high velocity is preferred. These two are thus at odds and there is a sweet spot that maximizes the combination.

7 Conclusion

All the above formula's are combined into a single monster equation giving the maximum attained altitude based on burn rate.

Upon inspection, it can be seen that the thrust is dependent on the burn rate which in turn dictates the burn time given a certain amount of fuel.

If the rocket size and payload are fixed, the burn rate can be tweaked to yield a certain altitude. This mega-formula is then put through optimization cycles to see which burn rate yield the highest altitude.

Below is an image of the final output of the calculation giving the optimal fuel burn rate in kg/s, which turn out to be 16.8 kg/s, the attained altitude is 1358494,4 meter or 1358 km.

```
     fun: -1358494.4773317748
     jac: array([-976.40625])
 message: 'Optimization terminated successfully.'
    nfev: 20
     nit: 3
    njev: 3
  status: 0
 success: True
       x: array([-16.8])
```

References

works cited

For the differential equation:

https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-07-dynamics-fall-2009/lecture-notes/MIT16$_0$7F09$_L$ec14.*pdf*

14.2*TheRocketEquation*, *web.mit.edu*/16.*unified*/*www*/*FALL*/*thermodynamics*/*notes*/*node*103.*html*.

For the optimization part:

Rocket Equations, www.rocketmime.com/rockets/rckt$_e$*qn.html*.

RocketEquationsQuickReference, *www.rocketmime.com*/*rockets*/*qref.html*.