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解7：泊松分布有线性可加性 且X_i与X同分布

$\Rightarrow T \sim P(n\lambda)$

解9：正态分布有线性可加性 且X_i与X同分布

$\Rightarrow T \sim N(n\mu, n\sigma^2)$

解10：(1) $X_1-2X_2+3X_3 \sim N(0, 14)$

$4X_4-5X_5 \sim N(0, 41)$

$\Rightarrow \frac{1}{\sqrt{14}}(X_1-2X_2+3X_3) \sim N(0,1)$，$\frac{1}{\sqrt{41}}(4X_4-5X_5) \sim N(0,1)$

$\Rightarrow$ 若 $a(X_1-2X_2+3X_3)^2+b(4X_4-5X_5)^2$ 服从 χ 分布

$\Rightarrow$ $a=\frac{1}{14}$ $b=\frac{1}{41}$ 自由度为2

(2) $\sum_{i=1}^{3} X_i \sim N(0,3)$，$\sum_{i=4}^{5} X_i^2 \sim \chi^2(2)$

$\Rightarrow \pm\frac{1}{\sqrt{3}}\sum_{i=1}^{3} X_i \sim N(0,1)$

$\Rightarrow \pm\frac{1}{\sqrt{3}}\sum_{i=1}^{3} X_i \Big/ \sqrt{\sum_{i=4}^{5} X_i^2/2} = \pm\frac{\sqrt{2}}{\sqrt{3}}\sum_{i=1}^{3} X_i \Big/ \sqrt{\sum_{i=4}^{5} X_i^2} \sim t(2)$

$\Rightarrow$ 对比可知 $c=\pm\frac{\sqrt{6}}{3}$，自由度为2

(3) $\sum_{i=1}^{3} X_i^2 \sim \chi^2(3)$，$\sum_{i=4}^{5} X_i^2 \sim \chi^2(2)$

$\sum_{i=1}^{3} X_i^2/3 \Big/ \sum_{i=4}^{5} X_i^2/2$

$=\frac{2}{3}\sum_{i=1}^{3} X_i^2 \Big/ \sum_{i=4}^{5} X_i^2 \sim F(3,2)$

$\Rightarrow d=\frac{2}{3}$，自由度为$(3,2)$

证明11：$\bar{X} \sim N(\mu_1, \frac{\sigma^2}{n})$ $\bar{Y} \sim N(\mu_2, \frac{\sigma^2}{m})$

其中 $n=8$，$m=9$ $S_w^2=\frac{1}{15}\left(\sum_{i=1}^{8}(X_i-\bar{X})^2+\sum_{j=1}^{9}(Y_j-\bar{Y})^2\right)$

$=\frac{(n-1)S_1^2+(m-1)S_2^2}{8+9-2}$

$S_1=\frac{1}{n-1}\sum_{i=1}^{8}(X_i-\bar{X})^2$ $S_2=\frac{1}{m-1}\sum_{i=1}^{9}(Y_i-\bar{Y})^2$.

满足定理条件.

$\Rightarrow \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_w\sqrt{\frac{1}{8}+\frac{1}{9}}} \sim t(m+n-2)=t(15)$. 证毕.

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解12: $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\cdot n\frac{1}{\lambda} = \frac{1}{\lambda}$

$D(\bar{X}) = D\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} D(X_i) = \frac{1}{n^2}\, n\frac{1}{\lambda^2} = \frac{1}{n\lambda^2}$

$ES^2 = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right) = \frac{1}{n-1} E\sum_{i=1}^{n}(X_i-\bar{X})^2$

$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right) = \frac{1}{n-1}\left[\sum_{i=1}^{n} E(X_i^2) - E(n\bar{X}^2)\right]$

$= \frac{1}{n-1}\left[\sum_{i=1}^{n}\left(E^2(X_i)+D(X_i)\right) - n\left(E^2(\bar{X})+D(\bar{X})\right)\right]$

$= \frac{1}{n-1}\left[n\frac{1}{\lambda^2} + n\frac{1}{\lambda^2} - n\frac{1}{\lambda^2} - \frac{1}{\lambda^2}\right] = \frac{1}{\lambda^2}$

故 $E(\bar{X}) = \frac{1}{\lambda}$ $D(\bar{X}) = \frac{1}{n\lambda^2}$ $ES^2 = \frac{1}{\lambda^2}$

解16: $X_{(1)} = \min(X_1, X_2 \cdots X_n)$

$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} \ge x)$

$= 1 - (1-F_{X_1}(x))(1-F_{X_2}(x))\cdots(1-F_{X_n}(x))$

又 $X \sim U(a,b)$ $\Rightarrow X_i \sim U(a,b)$

$\Rightarrow f_{X_i}(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & 其他 \end{cases} \Rightarrow F_{X_i}(x) = \begin{cases} 1 & x>b \\ \frac{x-a}{b-a} & a<x<b \\ 0 & x<a \end{cases}$

$\Rightarrow F_{X_{(1)}}(x) = 1 - \left(1-\frac{x-a}{b-a}\right)^n \quad a<x<b$

$\Rightarrow f_{X_{(1)}}(x) = n\left(1-\frac{x-a}{b-a}\right)^{n-1}\cdot\frac{1}{b-a} = n\frac{(b-x)^{n-1}}{(b-a)^n} \quad a<x<b$

$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = F_{X_1}(x)\,F_{X_2}(x)\cdots F_{X_n}(x) = \left(\frac{x-a}{b-a}\right)^n$

$\Rightarrow f_{X_{(n)}}(x) = n\left(\frac{x-a}{b-a}\right)^{n-1}\cdot\frac{1}{b-a} = n\frac{(x-a)^{n-1}}{(b-a)^n}$

故 $X_{(1)}(x)$ 与 $X_{(n)}(x)$ 的概率密度函数为

$f_{X_{(1)}}(x) = \begin{cases} n\frac{(b-x)^{n-1}}{(b-a)^n} & a<x<b \\ 0 & 其他 \end{cases} \qquad f_{X_{(n)}}(x) = \begin{cases} n\frac{(x-a)^{n-1}}{(b-a)^n} & a<x<b \\ 0 & 其他 \end{cases}$

解19:(1) $\bar{X} \sim N(1,1) \Rightarrow P(\bar{X}>2) = 1 - P(\bar{X}<2)$

$= 1 - \phi\left(\frac{2-1}{1}\right) = 1 - \phi(1) = 0.1587$

$P(1<\bar{X}<2) = \phi\left(\frac{2-1}{1}\right) - \phi\left(\frac{1-1}{1}\right) = \phi(1) - 0.5 = 0.3413$

(2) $P(X_{(1)}>4) = P(X_1>4, X_2>4 \cdots X_9>4) = \left[1-\phi\left(\frac{4-1}{3}\right)\right]^9$

$= [1-\Phi(1)]^9 = 0.1587^9 = 6.3854\times 10^{-8}$

$p(X_{(9)}<4) = P(X_1<4,\ X_2<4 \cdots X_9<4)$

$= \Phi^9(\frac{4-1}{3}) = \Phi^9(1) = 0.8413^9 = 0.2111$

故 (1) $P(\bar{X}>2) = 0.1587 \quad P(1<\bar{X}<2) = 0.3413$

$P(X_{(1)}>4) = 0.1587^9 \quad P(X_{(9)}<4) = 0.8413^9$

解1: (1) $E(X) = \int_0^1 (\theta+1)x^{\theta+1}dx = \frac{\theta+1}{\theta+2}x^{\theta+2}\Big|_0^1 = \frac{\theta+1}{\theta+2}$

$\Rightarrow$ 令 $\frac{\theta+1}{\theta+2} = \bar{X} \Rightarrow \hat{\theta} = \frac{1-2\bar{X}}{\bar{X}-1},\quad \bar{X} = \sum_{i=1}^n X_i$

(2) $E(\ln X) = \int_0^{+\infty} \frac{\ln x}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}dx = \mu$

$E(\ln^2 X) = \int_0^{+\infty} \ln^2 x \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}dx = E^2(\ln X) + D(\ln X) = \sigma^2+\mu^2$

$=$ 令 $\begin{cases} \mu = \frac{1}{n}\sum_{i=1}^n \ln X_i \\ \sigma^2+\mu^2 = \sum_{i=1}^n (\ln X_i)^2 \cdot \frac{1}{n} \end{cases} \Rightarrow \begin{cases} \mu = \frac{1}{n}\sum_{i=1}^n \ln X_i = \overline{\ln X_i} \\ \sigma^2 = \frac{1}{n}\sum_{i=1}^n (\ln X_i - \overline{\ln X_i})^2 \end{cases}$

故 $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n \ln X_i \quad \hat{\sigma} = \frac{1}{n}\sum_{i=1}^n (\ln X_i - \frac{1}{n}\sum_{i=1}^n \ln X_i)^2$

(3) $E(X) = \frac{1}{\lambda} = \bar{X} \Rightarrow \hat{\lambda} = \frac{1}{\bar{X}}$

(4) 令 $X = T+\mu \Rightarrow x-\mu = t \Rightarrow f(t+\mu,\mu,\lambda) = \begin{cases} \lambda e^{-t} & t\ge 0 \\ 0 & t<0 \end{cases}$

$\Rightarrow E(X) = E(T)+\mu = \frac{1}{\lambda}+\mu = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}$

$D(X) = D(T+\mu) = D(T) = \frac{1}{\lambda^2} = \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2$

$\Rightarrow \hat{\lambda} = \sqrt{\frac{1}{\frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2}} \quad \hat{\mu} = \bar{X} - \sqrt{\frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2}$

(5) $E(X) = \int_{\theta-0.5}^{\theta+0.5} x\,dx = \theta = \bar{X}$

$\Rightarrow \hat{\theta} = \bar{X}$

解2: 令 $x-\mu = t \quad X-\mu = T \Rightarrow f(t) = \frac{1}{2}\lambda e^{-\lambda|t|} \quad -\infty<t<+\infty$.

$E(X) = E(T+\mu) = E(T)+\mu = \int_{-\infty}^{+\infty} \frac{1}{2}t\lambda e^{-\lambda|t|}dt + \mu = \mu = \bar{X}$

$D(X) = D(T+\mu) = D(T) = E(T^2) - E^2(T) = E(T^2) = \int_{-\infty}^{+\infty} t^2\frac{1}{2}\lambda e^{-\lambda|t|}dt$

$= \int_0^{+\infty} t^2\lambda e^{-\lambda|t|}dt = \frac{2}{\lambda^2} = \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2$

$\Rightarrow \hat{u} = \bar{x} \quad \hat{\lambda} = 1 / \sqrt{\frac{1}{2n}\sum_{i=1}^{n}(x_i-\bar{x})^2}$

解5： $E(X) = \lambda = \bar{x} = \frac{1}{12}\sum_{i=1}^{12} x_i = \frac{1}{12} \times 21 = \frac{7}{4} = 1.75$

故矩估计值为 1.75

解1：(1)

似然函数为 $L(\theta) = (\theta+1)^n (x_1 x_2 \cdots x_n)^\theta$

对数似然函数为 $\ln L(\theta) = n\ln(\theta+1) + \theta[\ln x_1 + \ln x_2 \cdots \ln x_n]$

该函数对θ求导得 $\frac{d\ln L(\theta)}{d\theta} = \frac{n}{\theta+1} + \sum_{i=1}^{n} \ln x_i$

令导数得0 $\Rightarrow \theta = -\frac{n}{\sum_{i=1}^{n}\ln x_i} - 1$

极大估计量为 $\hat{\theta} = -\frac{n}{\sum_{i=1}^{n}\ln x_i} - 1$

(2) 似然函数为

$$L(u,\sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n x_1 \cdots x_n} e^{-\left[\frac{(\ln x_1-u)^2}{2\sigma^2} + \frac{\ln(x_2-u)^2}{2\sigma^2} \cdots \frac{\ln(x_n-u)^2}{2\sigma^2}\right]}$$

$$\ln L(u,\sigma) = -n\ln\sqrt{2\pi}\sigma - \sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\frac{(\ln x_i-u)^2}{2\sigma^2}$$

$$= -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \sum_{i=1}^{n}\ln x_i - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i-u)^2$$

令 $\begin{cases} \frac{\partial \ln L(u,\sigma)}{\partial u} = \frac{1}{\sigma^2} 2\sum_{i=1}^{n}(\ln x_i - u)\cdot\frac{1}{x_i} = 0 \\ \frac{\partial \ln L(u,\sigma)}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(\ln x_i - u)^2 = 0 \end{cases}$

解得 $\hat{u} = \frac{1}{n}\sum_{i=1}^{n}\ln x_i \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(\ln x_i - \frac{1}{n}\sum_{i=1}^{n}\ln x_i)^2$

(3) 似然函数为 $L(\lambda) = \lambda^n e^{-\lambda\sum_{i=1}^{n}x_i}$

对数似然函数为 $\ln L(\lambda) = n\ln\lambda - \lambda\sum_{i=1}^{n}x_i$

$\frac{\partial \ln(\lambda)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}x_i = 0 \Rightarrow \hat{\lambda} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}x_i} = \frac{1}{\bar{x}}$

(4) 似然函数：$L(\lambda,u) = \lambda^n e^{-\lambda\sum_{i=1}^{n}(x_i-u)}$

对数似然函数：$\ln L(\lambda,u) = n\ln\lambda - \lambda\sum_{i=1}^{n}(x_i-u)$

$\begin{cases} \frac{\partial \ln L(\lambda,u)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}(x_i-u) \\ \frac{\partial \ln L(\lambda,u)}{\partial u} = -n\lambda \end{cases}$

可以看出 $\ln L(\lambda,u)$ 关于 u 递减 $\Rightarrow \hat{u} = \min(x_1, x_2 \cdots x_n)$

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令 $\frac{\partial \ln L(u\lambda)}{\partial \lambda}=0 \Rightarrow \lambda = \bar{x}-u$

$\Rightarrow \hat{\lambda} = 1/(\bar{x}-\min(x_1 x_2 \cdots x_n))$

(5) 似然函数为 $L(\theta)=1$ $\theta-0.5<x_i<\theta+0.5$.

由 $\theta-0.5 \le \min(x_1 x_2 \cdots x_n)$ $\theta+0.5 \ge \max(x_1 x_2 \cdots x_n)$

$\Rightarrow \max(x_1 x_2 \cdots x_n)-0.5 \le \hat{\theta} \le 0.5+\min(x_1 x_2 \cdots x_n)$

其中任意一个 $\hat{\theta}$ 均可当作最大似然估计值

解4：X的密度函数为 $f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-u)^2}{2\sigma^2}}$

(1) 似然函数为 $L(u)=\frac{1}{(\sqrt{2\pi}\sigma)^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-u)^2}$

对数似然函数为 $\ln(L(u)) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-u)^2$

$\frac{\ln L(u)}{du} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-u)=0 \Rightarrow \hat{u}=\frac{1}{n}\sum_{i=1}^{n}x_i=\bar{x}$

(2) 对数似然函数为 $\ln L(\sigma^2) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-u)^2$

$\frac{\ln L(\sigma^2)}{d\sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2}+\frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-u)^2=0$

$\Rightarrow \hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-u)^2$

解7：设捕到带标记鱼的概率为P，$x_i \sim B(1,P)$

似然函数为 $L(P)=C_{150}^{10}P^{10}(1-P)^{140}$

对数似然函数为 $\ln L(P)=\ln C_{150}^{10}+10\ln P+140\ln(1-P)$

$\frac{d\ln L(P)}{dP}=\frac{10}{P}-\frac{140}{1-P}=0 \Rightarrow P=\frac{1}{15}$

$n=1000/P=15000$

故有15000条鱼使出现10条带标记的可能性最大

解9：证明(1)：$E\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}ix_i\right)=\frac{2}{n(n+1)}\sum_{i=1}^{n}iE(x_i)$

$=\frac{2}{n(n+1)}\sum_{i=1}^{n}iu = \frac{2u}{n(n+1)}\cdot\sum_{i=1}^{n}i = \frac{2u}{n(n+1)}\cdot\frac{n(n+1)}{2}=u$

故 $\hat{u}=\frac{2}{n(n+1)}\sum_{i=1}^{n}ix_i$ 是u的无偏估计

证明(2)：$D\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}ix_i\right)=\frac{4}{n^2(n+1)^2}\sum_{i=1}^{n}i^2D(x_i)$

$=\frac{4}{n^2(n+1)^2}\sigma^2\sum_{i=1}^{n}i^2 = \frac{4\sigma^2}{n^2(n+1)^2}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{2(2n+1)}{3n(n+1)}\sigma^2$

由切比雪夫不等式：，$\forall \varepsilon>0$

$$P\left(\left|\frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i - u\right|<\varepsilon\right) > 1-\frac{D\left(\frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i\right)}{\varepsilon^2}$$

$$= 1-\frac{1}{\varepsilon^2}\frac{2}{3}\frac{(2n+1)}{(n+1)n}\sigma^2$$

$$\lim_{n\to\infty} P\left(\left|\frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i-u\right|<\varepsilon\right) = \lim_{n\to\infty}\left(1-\frac{1}{\varepsilon^2}\frac{2}{3}\frac{2n+1}{(n+1)n}\sigma^2\right)=1$$

故 $\hat{u}=\frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i$ 是相合估计.

解10：$E(u_1) = \frac{1}{5}u+\frac{3}{10}u+\frac{1}{2}u = u$

$E(u_2) = \frac{1}{3}u+\frac{1}{4}u+\frac{5}{12}u = u$

$E(u_3) = \frac{1}{3}u+\frac{1}{6}u+\frac{1}{2}u = u$

故 $\hat{u}_1, \hat{u}_2, \hat{u}_3$ 都是 u 的无偏估计量

$D(u_1) = \frac{1}{25}\sigma^2+\frac{9}{100}\sigma^2+\frac{1}{4}\sigma^2 = \frac{9+25+4}{100}\sigma^2 = \frac{38}{100}\sigma^2 = \frac{19}{50}\sigma^2$

$D(u_2) = \frac{1}{9}\sigma^2+\frac{1}{16}\sigma^2+\frac{25}{144}\sigma^2 = \frac{50}{144}\sigma^2$

$D(u_3) = \frac{1}{9}\sigma^2+\frac{1}{36}\sigma^2+\frac{1}{4}\sigma^2 = \frac{14}{36}\sigma^2 = \frac{7}{18}\sigma^2$

由于 $D(u_2)$ 最小 $\Rightarrow$ $\hat{u}_2$ 更有效.

解11：$f(x) = \frac{1}{\theta}$ $\quad 0<x<\theta$ $\quad F(x) = \frac{x}{\theta}$ $\quad 0<x<\theta$

$\Rightarrow F_{X(n)} = P(X_{(n)}\le x) = P(X_1\le x, X_2\le x \cdots X_n\le x)$

$= \frac{x^n}{\theta^n} \Rightarrow f_{X(n)}(x) = \frac{n x^{n-1}}{\theta^n}$ $\quad 0<x<\theta$

$\Rightarrow E(X_{(n)}) = \int_0^\theta \frac{n}{\theta^n}x^n\,dx = \frac{n\,\theta^{n+1}}{\theta^n(n+1)}$

$\Rightarrow E(\hat{\theta}_1) = \frac{n+1}{n}E(X_{(n)}) = \frac{n+1}{n}\cdot\frac{n}{\theta^n}\cdot\frac{\theta^{n+1}}{n+1} = \theta$

$D(X_{(n)}) = E(X_{(n)}^2)-E^2(X_{(n)}) = \int_0^\theta \frac{n}{\theta^n}x^{n+1}dx - \left(\frac{n\theta}{n+1}\right)^2 = \frac{n}{(n+2)(n+1)^2}\theta^2$

$F_{X(1)} = 1-P(X_{(1)}\ge x) = 1-(1-F_{X_1}(x))(1-F_{X_2}(x))\cdots(1-F_{X_i}(x))$

$= 1-(1-\frac{x}{\theta})^n \Rightarrow f_{X(1)}(x) = \frac{n}{\theta}(1-\frac{x}{\theta})^{n-1}$

$E(\hat{\theta}_2) = (n+1)E(X_{(1)}) = n+1\int_0^\theta \frac{n}{\theta}(1-\frac{x}{\theta})^{n-1}x\,dx = n+1\cdot\frac{n\theta}{(n+1)n} = \theta$

$D(\hat{\theta}_2) = (n+1)^2 D(X_{(1)}) = (n+1)^2[E(X_{(1)}^2)-E^2(X_{(1)})] = (n+1)^2\left[\int_0^\theta \frac{n}{\theta}(1-\frac{x}{\theta})^{n-1}x^2dx - \left(\frac{\theta}{n+1}\right)^2\right]$

$= \frac{n}{n+2}\theta^2$ $\quad D(\hat{\theta}_1) = \frac{(n+1)^2}{n^2}D(X_{(n)}) = \frac{1}{n(n+2)}\theta^2$

故 $E(\hat{\theta}_1) = E(\hat{\theta}_2) = \theta$, $D(\hat{\theta}_1) < D(\hat{\theta}_2)$

故 $\hat{\theta}_1, \hat{\theta}_2$ 都是 θ 的无偏估计，且 $\hat{\theta}_1$ 更有效 $\quad$ 6

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