(2012-2013)工科数学分析第二学期期末试题(A 卷)解答（2013.6）

一. 1. $\arcsin\dfrac{5}{2\sqrt{21}}$

2. $\dfrac{1}{3}(2\sqrt{2}-1)$

3. -6

4. $-\dfrac{4}{25\pi}$

5. $2ydydz+(y-x^2)dxdy$

二. 两方程两端分别对 x 求导，得

$$\begin{cases} u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}=0 \\ y\dfrac{\partial u}{\partial x}+v+x\dfrac{\partial v}{\partial x}=0 \end{cases},$$ ………………(3 分)

解得 $\dfrac{\partial u}{\partial x}=-\dfrac{xu+yv}{x^2+y^2}$ ………………(4 分)

两方程两端分别对 y 求导，得

$$\begin{cases} x\dfrac{\partial u}{\partial y}-v-y\dfrac{\partial v}{\partial y}=0 \\ u+y\dfrac{\partial u}{\partial y}+x\dfrac{\partial v}{\partial y}=0 \end{cases},$$ ………………(7 分)

解得 $\dfrac{\partial u}{\partial y}=\dfrac{xv-yu}{x^2+y^2}$ ………………(8 分)

三. $I=\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_{0}^{2\cos\theta}\rho^2 d\rho$ ………………(4 分)

$=\dfrac{8}{3}\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^3\theta d\theta$ ………………(6 分)

$=\dfrac{8}{3}\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-\sin^2\theta)d\sin\theta$

$=\dfrac{2}{9}(8-5\sqrt{2})$ ………………(9 分)

四. $\qquad \dfrac{\partial z}{\partial x} = 3x^2 + 3y^2 - 12 \qquad \dfrac{\partial z}{\partial y} = 6xy \qquad\qquad$ …………(2 分)

令 $\dfrac{\partial z}{\partial x} = 0 \quad \dfrac{\partial z}{\partial y} = 0$ 解得 $\quad x = 0, \quad y = \pm 2 \quad$ 或 $\quad x = \pm 2, \quad y = 0$ …………(3 分)

$\qquad\qquad \dfrac{\partial^2 z}{\partial x^2} = 6x \qquad \dfrac{\partial^2 z}{\partial x \partial y} = 6y \qquad \dfrac{\partial^2 z}{\partial y^2} = 6x \qquad\qquad$ …………(4 分)

在点 $P_1(0,2)$，$\quad A = 0 \qquad B = 12 \qquad C = 0$

$\qquad AC - B^2 = -144 < 0 \qquad$ 故 $P_1(0,2)$ 不是极值点 $\qquad$ …………(5 分)

在点 $P_2(0,-2)$，$\quad A = 0 \qquad B = -12 \qquad C = 0$

$\qquad AC - B^2 = -144 < 0 \qquad$ 故 $P_2(0,-2)$ 不是极值点 $\qquad$ …………(6 分)

在点 $P_3(2,0) \qquad A = 12 \qquad B = 0 \qquad C = 12$

$\qquad AC - B^2 = 144 > 0 \qquad$ 且 $A > 0$

故 $P_3(2,0)$ 是极小值点, 极小值 $z\big|_{(2,0)} = -16 \qquad$ …………(8 分)

在点 $P_4(-2,0) \qquad A = -12 \qquad B = 0 \qquad C = -12$

$\qquad AC - B^2 = 144 > 0 \qquad$ 且 $A < 0$

故 $P_4(-2,0)$ 是极大值点, 极大值 $z\big|_{(-2,0)} = 16 \qquad$ …………(10 分)

五. $\qquad$ 设切点 $M(x,y,z)$, 在切点处，$xyz = \lambda$ 的法向量为 $\vec{n}_1 = \{yz, xz, xy\}$ ….(2 分)

$\qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \quad$ 的法向量为 $\quad \vec{n}_2 = \{\dfrac{x}{a^2}, \dfrac{y}{b^2}, \dfrac{z}{c^2}\} \quad$ …………(4 分)

$\qquad$ 由 $\vec{n}_1 /\!/ \vec{n}_2$, $\qquad$ 得 $\quad \dfrac{\frac{x}{a^2}}{yz} = \dfrac{\frac{y}{b^2}}{xz} = \dfrac{\frac{z}{c^2}}{xy} \qquad\qquad$ …………(6 分)

$\qquad$ 因此有 $\quad \dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$

$\qquad$ 代入椭球面方程得 $\quad x^2 = \dfrac{a^2}{3} \qquad y^2 = \dfrac{b^2}{3} \qquad z^2 = \dfrac{c^2}{3} \qquad$ …………(8 分)

$\qquad\qquad \lambda = xyz = \left| \dfrac{a}{\sqrt{3}} \ \dfrac{b}{\sqrt{3}} \ \dfrac{c}{\sqrt{3}} \right| = \dfrac{abc}{3\sqrt{3}} \qquad\qquad$ …………(9 分)

六.
$$I_z = \iiint\limits_V (x^2 + y^2)\sqrt{x^2 + y^2 + z^2}\,dV \qquad \dots\dots\dots\dots\dots(2\,分)$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^1 r^5 \sin^3 \varphi\, dr \qquad \dots\dots\dots\dots\dots(5\,分)$$

$$= 2\pi \cdot \frac{1}{6} \int_0^{\frac{\pi}{4}} \sin^3 \varphi\, d\varphi \qquad \dots\dots\dots\dots\dots(7\,分)$$

$$= \frac{\pi}{3} \int_0^{\frac{\pi}{4}} (\cos^2 \varphi - 1)\, d\cos \varphi$$

$$= \frac{\pi}{9}\left(2 - \frac{5\sqrt{2}}{4}\right) \qquad \dots\dots\dots\dots\dots(9\,分)$$

七.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2n+1}{2n+3} = 1 \qquad R = 1 \qquad \dots\dots\dots\dots\dots(1\,分)$$

$x = \pm 1$时，级数为 $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{2n+1}$ 收敛，收敛域为 $[-1,1]$ $\quad\dots\dots\dots(2\,分)$

$$令 \quad S(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{2n+1}$$

$$S'(x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n} \qquad \dots\dots\dots\dots\dots(4\,分)$$

$$= \frac{x^2}{1+x^2} \qquad \dots\dots\dots\dots\dots(5\,分)$$

$$S(x) = \int_0^x \frac{x^2}{1+x^2}\,dx = x - \arctan x \qquad \dots\dots\dots\dots\dots(7\,分)$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{2n+1} = \begin{cases} 1 - \dfrac{1}{x}\arctan x & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \dots\dots\dots\dots\dots(9\,分)$$

八. 设 $X = (6y + x^2 y^2) y^3 f(x)$，$Y = (8x + x^3 y) y^3 f(x)$，

由题设，$X = \dfrac{\partial u}{\partial x}$　$Y = \dfrac{\partial u}{\partial y}$　　故有 $\dfrac{\partial Y}{\partial x} = \dfrac{\partial X}{\partial y}$，即　　…………………(1 分)

$(8y^3 + 3x^2 y^4) f(x) + (8xy^3 + x^3 y^4) f'(x) = (24y^3 + 5x^2 y^4) f(x)$　………(3 分)

比较 y 的同次幂系数，得　　$xf'(x) = 2f(x)$　　…………………(4 分)

$$\frac{df(x)}{f(x)} = \frac{2dx}{x}，\qquad f(x) = Cx^2$$　　…………………(6 分)

由 $f(1) = 1$　得 $C = 1$　$f(x) = x^2$　　…………………(7 分)

$$u(x,y) = \int_{(0,0)}^{(x,y)} (6x^2 y^4 + x^4 y^5) dx + (8x^3 y^3 + x^5 y^4) dy + C$$　………(8 分)

$$= 0 + \int_0^y (8x^3 y^3 + x^5 y^4) dy + C$$　　…………………(9 分)

$$= 2x^3 y^4 + \frac{1}{5} x^5 y^5 + C$$　　…………………(10 分)

九.　　　$$f(x) = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$　　…………………(1 分)

$$= \frac{1}{2+(x-1)} - \frac{1}{3+(x-1)}$$

$$= \frac{1}{2} \frac{1}{1+\dfrac{x-1}{2}} - \frac{1}{3} \frac{1}{1+\dfrac{x-1}{3}}$$　　…………………(3 分)

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (\frac{x-1}{2})^n - \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n (\frac{x-1}{3})^n$$　　…………………(6 分)

$$= \sum_{n=0}^{\infty} (-1)^n (\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}})(x-1)^n$$　　…………………(7 分)

收敛域为　$(-1,3) \cap (-2,4) = (-1,3)$　　…………………(9 分)

十. (1) $S: x^2 + y^2 = 2z$ ....(1分)

(2) 设 $S_1: z = 2 \quad (x^2 + y^2 \le 4)$

$$I = \oiint\limits_{S+S_1^-} - \iint\limits_{S_1^-} xy^2 dydz + x^2 ydzdx + (z-1)dxdy \quad \text{..................(2 分)}$$

$$\oiint\limits_{S+S_1^-} xy^2 dydz + x^2 ydzdx + (z-1)dxdy = -\iiint\limits_{V} (y^2 + x^2 + 1)dV \quad \text{.............(4 分)}$$

$$= -\int_0^{2\pi} d\theta \int_0^2 \rho(\rho^2+1)d\rho \int_{\frac{\rho^2}{2}}^2 dz \quad \text{..................(6 分)}$$

$$= -2\pi \int_0^2 \rho(\rho^2+1)(2 - \frac{\rho^2}{2})d\rho$$

$$= -\frac{28}{3}\pi \quad \text{..................(7 分)}$$

$$-\iint\limits_{S_1^-} xy^2 dydz + x^2 ydzdx + (z-1)dxdy = \iint\limits_{S_1^+} (z-1)dxdy$$

$$= \iint\limits_{D_{xy}} dxdy = 4\pi \quad \text{..................(8 分)}$$

$$I = -\frac{28}{3}\pi + 4\pi = -\frac{16}{3}\pi \quad \text{..................(9 分)}$$

十一. 由于 $\sum\limits_{n=1}^{\infty} f(\frac{1}{n})$ 收敛，有 $\lim\limits_{n\to\infty} f(\frac{1}{n}) = 0$ ....(2 分)

由题设，有 $f(0) = \lim\limits_{n\to\infty} f(\frac{1}{n}) = 0$ ....(4 分)

$$\lim\limits_{n\to\infty} \frac{f(\frac{1}{n})}{\frac{1}{n}} = f'(0) \quad \text{..................(6 分)}$$

若 $f'(0) \ne 0$，不妨设 $f'(0) > 0$，则由于 $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ 发散，所以有 $\sum\limits_{n=1}^{\infty} f(\frac{1}{n})$ 发散，

与已知矛盾，故 $f'(0) = 0$ ....(8 分)