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解3：X可能的取值有0，1，2，3

$P(X=0)=\frac{1}{3}\times\frac{C_3^3}{C_5^3}=\frac{1}{3}\times\frac{1}{10}=\frac{1}{30}$

$P(X=1)=\frac{1}{2}\times\frac{C_2^2C_3^1}{C_5^3}+\frac{1}{3}\times\frac{C_3^2C_2^1}{C_5^3}=\frac{1}{2}\times\frac{3}{10}+\frac{1}{3}\times\frac{6}{10}=\frac{7}{20}$

$P(X=2)=\frac{1}{6}\cdot\frac{C_1^1C_4^2}{C_5^3}+\frac{1}{2}\cdot\frac{C_2^1C_3^2}{C_5^3}+\frac{1}{3}\times\frac{C_3^1C_2^2}{C_5^3}=\frac{1}{6}\times\frac{6}{10}+\frac{1}{2}\times\frac{6}{10}+\frac{1}{3}\times\frac{3}{10}=\frac{5}{10}=\frac{1}{2}$

$P(X=3)=\frac{1}{6}\cdot\frac{C_4^3}{C_5^3}+\frac{1}{2}\cdot\frac{C_3^3}{C_5^3}=\frac{1}{6}\cdot\frac{4}{10}+\frac{1}{2}\cdot\frac{1}{10}=\frac{7}{60}$

故X的分布律为：

X	0	1	2	3
P	$\frac{1}{30}$	$\frac{7}{20}$	$\frac{1}{2}$	$\frac{7}{60}$

解5：X可能的取值有3、4、5.

$P(X=3)=\frac{1}{C_5^3}=\frac{1}{10}$ $P(X=4)=\frac{C_3^2}{C_5^3}=\frac{3}{10}$

$P(X=5)=\frac{C_4^2}{C_5^3}=\frac{6}{10}=\frac{3}{5}$

故X的分布律为：

X	3	4	5
P	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{5}$

解8：$P(X\le 0)=P(X=-1)+P(X=0)=a+\frac{1}{8}=\frac{3}{8}\Rightarrow a=\frac{1}{4}$

又 $\frac{1}{8}+a+b+\frac{1}{2}=0\Rightarrow b=\frac{1}{8}$

故 $a=\frac{1}{4}$，$b=\frac{1}{8}$

解9：(1) $\sum_{i=2}^{\infty}\frac{c}{i(i+1)}=C\sum_{i=2}^{\infty}\left(\frac{1}{i}-\frac{1}{i+1}\right)=c\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{i}-\frac{1}{i+1}+\cdots\right)$

$=\frac{c}{2}=1\Rightarrow c=2$

故 $c=2$

(2) $\sum_{k=1}^{\infty}C\frac{2^k}{k!}=c\left(\sum_{k=0}^{\infty}\frac{2^k}{k!}-1\right)=c(e^2-1)=1\Rightarrow c=\frac{1}{e^2-1}$

故 $c=\frac{1}{e^2-1}$

(3) $\sum_{k=3}^{\infty}\frac{c}{3^k}=C\sum_{k=3}^{\infty}\frac{1}{3^k}=c\left(\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^k}+\cdots\right)$

$=C\cdot\frac{1}{27}\frac{1}{1-\frac{1}{3}}=C\cdot\frac{1}{27}\cdot\frac{3}{2}=\frac{c}{18}=1\Rightarrow c=18$

故 $c=18$

解10：(1) $P(X=3)=C_5^3(0.8)^3\cdot(1-0.8)^2=0.2048$

故恰有3个正常工作的概率为 0.2048

(2) $P(X\geqslant 4) = P(X=4) + P(X=5)$

$$= C_5^4 \cdot (0.8)^4 (1-0.8) + (0.8)^5 = 0.73728$$

故至少有4个正常工作的概率为 0.73728

(3) $P(X\leqslant 2) = 1-P(X>2) = 1-P(X=3) - [P(X=4)+P(X=4)]$

$$= 1-0.2048-0.73728 = 0.05792$$

故最多有2个元件正常工作的概率为 0.05792

解16：由题意,设所需要的次数为 n，则 $n\geqslant r$

$$\Rightarrow P(Y=n) = C_{n-1}^{r-1} p^{r-1}(1-p)^{(n-1)-(r-1)} \cdot p = C_{n-1}^{r-1} p^r (1-p)^{n-r}$$

故Y的分布率为 $P(Y=n) = C_{n-1}^{r-1} p^r (1-p)^{n-r}$，$n\geqslant r$

解17：$P(X+Y=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ （k为总人数）

$$P(X=n \mid X+Y=k) = C_k^n p^n (1-p)^{k-n} \quad k\geqslant n$$

$$P(Y=m \mid X+Y=k) = C_k^m p^{k-m} (1-p)^m \quad k\geqslant m$$

$$\Rightarrow P(X=n) = \sum_{k=0}^{\infty} P(X+Y=k) P(X=n \mid X+Y=k) \quad \text{①} (k\geqslant n)$$

$$= \sum_{k=n}^{\infty} P(X+Y=k) P(X=n \mid X+Y=k)$$

$$= \sum_{k=n}^{\infty} \frac{\lambda^k}{k!} \cdot e^{-\lambda} \cdot C_k^n p^n (1-p)^{k-n}$$

$$= \sum_{k=n}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \frac{k!}{n!(k-n)!} \cdot p^n \cdot (1-p)^{k-n}$$

$$= \frac{e^{-\lambda} p^n}{n!} \sum_{k=n}^{\infty} \frac{1}{(k-n)!} \lambda^k (1-p)^{k-n}$$

$$= \frac{1}{n!} e^{-\lambda} p^n \sum_{k=n}^{\infty} \frac{1}{(k-n)!} \lambda^{k-n} \cdot \lambda^n (1-p)^{k-n}$$

$$= \frac{1}{n!} e^{-\lambda} (p\lambda)^n \sum_{k=n}^{\infty} \frac{1}{(k-n)!} \lambda^{k-n} (1-p)^{k-n}$$

$$= \frac{1}{n!} e^{-\lambda} (p\lambda)^n \sum_{j=0}^{\infty} \frac{1}{j!} [\lambda(1-p)]^j \quad (j=k-n)$$

$$= \frac{1}{n!} e^{-\lambda} (p\lambda)^n e^{\lambda(1-p)} = \frac{(p\lambda)^n}{n!} \cdot e^{-\lambda p}$$

即 X服从参数为 $p\lambda$ 的泊松分布，$\Rightarrow X \sim \pi(\lambda p)$.

同理得 $P(Y=m) = \frac{[(1-p)\lambda]^m}{m!} e^{-(1-p)\lambda}$，即 $Y \sim \pi[(1-p)\lambda]$

解18：设A为此药对他有效 $\Rightarrow P(A)=0.75 \quad P(\bar{A})=0.25$

$$P(X=2|A)=\frac{3^2}{2!}e^{-2},\quad P(X=2|\bar{A})=\frac{5^2}{2!}e^{-5}$$

$$P(A|X=2)=\frac{P(A)P(X=2|A)}{P(A)P(X=2|A)+P(\bar{A})P(X=2|\bar{A})}=\frac{0.75\cdot\frac{3^2}{2!}e^{-3}}{0.75\cdot\frac{3^2}{2!}e^{-3}+0.25\cdot\frac{5^2}{2!}e^{-5}}$$

$=0.8886$ 故对他有效的概率为 0.8886

解20：

由5题的分布列可知X的分布函数 $F(x)=\begin{cases}0 & x<3\\ \frac{1}{10} & 3\le x<4\\ \frac{2}{5} & 4\le x<5\\ 1 & x\ge 5\end{cases}$

解21：$P(X=-2)=F(-2)-F(-2-0)=0.1$

$P(X=0)=F(0)-F(0-0)=0.3$

$P(X=2)=F(2)-F(2-0)=0.3$

$P(X=4)=F(4)-F(4-0)=0.3$

故X的分布律为：

X	-2	0	2	4
P	0.1	0.3	0.3	0.3

$P(X<2)=F(2-0)=0.4$，$P(X\le 2)=F(2)=0.7$

$P(X>0)=1-F(0)=0.6$，$P(X\ge 0)=1-F(0-0)=0.9$

$$P(X\ge 0|X\ne 0)=\frac{P(X\ge 0\cap X\ne 0)}{P(X\ne 0)}=\frac{P(X>0)}{P(X\ne 0)}=\frac{0.6}{0.7}=\frac{6}{7}$$

解22：(1) 由 $\lim\limits_{x\to\infty}F(x)=1\Rightarrow A=1$ $\lim\limits_{x\to-\infty}F(x)=0\Rightarrow B=0$

故常数A的值为1，B的值为0

(2) $P(X\le 1)=F(1)=1-(1+1)e^{-1}=1-\frac{2}{e}$

故 $P(X\le 1)=1-\frac{2}{e}$

证明25：由题意，$F'(x)=f(x)$，$f(x)$连续 $\Rightarrow F(x)$处处可导，

由 $F(x)$ 与 $f(x)$ 的连续性 $\Rightarrow n f(x)[F(x)]^{n-1}$ 也连续

$\Rightarrow nf(x)[F(x)]^{n-1}$ 在R上可积，

$\Rightarrow \int_{-\infty}^{x} nf(x)[F(x)]^{n-1}dx = \int_{-\infty}^{x} n F^{n-1}(x)\, dF(x)$

$= F^{n}(x)\Big|_{-\infty}^{x} = [F(x)]^{n} - [F(-\infty)]^{n}$

由题意，$F(+\infty)=1$，$F(-\infty)=0 \Rightarrow \int_{-\infty}^{x} nf(x)[F(x)]^{n-1}dx = [F(x)]^{n}$

令 $x\to+\infty \Rightarrow \int_{-\infty}^{+\infty} nf(x)[F(x)]^{n-1}dx = [F(+\infty)]^{n} = 1$

又 $F(x)\geq 0$，$f(x)\geq 0 \Rightarrow nf(x)[F(x)]^{n-1}\geq 0$.

即 $g(x)=nf(x)[F(x)]^{n-1}$ 满足 $\begin{cases} g(x)\geq 0 \\ \int_{-\infty}^{+\infty} g(x)dx = 1 \end{cases}$

$\Rightarrow g(x)$ 是一个概率密度函数，且 $G(x)=[F(x)]^{n}$，证毕。

解27：(1)由连续性：$F(0)=F(0+0) \Rightarrow 0 = A+B$

又 $\lim\limits_{x\to+\infty} F(x)=1 \Rightarrow A=1$. 解得 $B=-1$

故A的值为1，B的值为-1

(2) $f(x)=F'(x)=\begin{cases} (1-e^{-\frac{x^2}{2}})' & x>0 \\ 0 & x\leq 0 \end{cases} \Rightarrow f(x)=\begin{cases} xe^{-\frac{x^2}{2}} & x>0 \\ 0 & x\leq 0 \end{cases}$

(3) $P(1<X<2)=F(2)-F(1)=(1-e^{-2})-(1-e^{-\frac{1}{2}})=\frac{1}{\sqrt{e}}-\frac{1}{e^2}$

故 $P(1<X<2)=\frac{1}{\sqrt{e}}-\frac{1}{e^2}$

解31：(1) $f(x)$ 为偶函数

$\Rightarrow \int_{0}^{+\infty} f(x)dx = \int_{0}^{+\infty} ce^{-x}dx = ce^{-x}\Big|_{+\infty}^{0} = c = 0.5$

故C的值为0.5

(2) 由 $f(x)$ 为偶函数 $\Rightarrow P(-1<X<2) = \int_{-1}^{0} f(x)dx + \int_{0}^{2} f(x)dx$

$= \int_{0}^{1} f(x)dx + \int_{0}^{2} f(x)dx = \int_{0}^{1}\frac{1}{2}e^{-x}dx + \int_{0}^{2}\frac{1}{2}e^{-x}dx$

$= \frac{1}{2}\left[e^{-x}\Big|_{1}^{0} + e^{-x}\Big|_{2}^{0}\right] = \frac{1}{2}\left[(1-\frac{1}{e})+(1-\frac{1}{e^2})\right] = \frac{1}{2}(2-\frac{1}{e}-\frac{1}{e^2})$

故 $P(-1<X<2)=1-\frac{1}{2}e^{-1}-\frac{1}{2}e^{-2}$

(3) $x<0$ 时，$F(x)=\int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x}\frac{1}{2}e^{x}dx = \frac{1}{2}e^{x}$

$x \geq 0$时 $F(x)=\int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$

$= \int_{-\infty}^{0} \frac{1}{2}e^{x}dx + \int_{0}^{x} \frac{1}{2}e^{-x}dx = \frac{1}{2} + \frac{1}{2}e^{x}\Big|_{x}^{0} = 1-\frac{1}{2}e^{-x}$

故分布函数 $F(x)=\begin{cases} \frac{1}{2}e^{x} & x<0 \\ 1-\frac{1}{2}e^{-x} & x\geq 0 \end{cases}$

解32：(1) $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{+\infty} f(x)dx$

$=\int_{-\infty}^{0} 0dx + \int_{0}^{+\infty} ax^3 e^{-\frac{x^2}{2}}dx = a\int_{0}^{+\infty} x^3 e^{-\frac{x^2}{2}}dx$

$= a\int_{0}^{+\infty} \frac{x^2}{2} e^{-\frac{x^2}{2}}dx^2 = 2a\int_{0}^{+\infty} \frac{x^2}{2} e^{-\frac{x^2}{2}}d\frac{x^2}{2} = 2a\int_{0}^{+\infty} te^{-t}dt$

$=2a\int_{+\infty}^{0} t de^{-t} = 2a\left(te^{-t}\Big|_{+\infty}^{0} - \int_{+\infty}^{0} e^{-t}dt\right)$

$=2a\left(0+e^{-t}\Big|_{+\infty}^{0}\right) = 2a = 1 \Rightarrow a=\frac{1}{2}$

故 a 的值为 $\frac{1}{2}$

(2) $x<0$ 时，$F(x)=\int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} 0dx = 0$

$x\geq 0$时，$F(x)=\int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx = 0+\int_{0}^{x}\frac{1}{2}x^3e^{-\frac{x^2}{2}}dx$

$=\frac{1}{2}\int_{0}^{x} \frac{x^2}{2}e^{-\frac{x^2}{2}}dx^2 = \int_{0}^{x}\frac{x^2}{2}e^{-\frac{x^2}{2}}d\frac{x^2}{2} = \int_{0}^{t} te^{-t}dt \quad (t=\frac{x^2}{2})$

$=-\int_{0}^{t} t de^{-t} = -\left(te^{-t}\Big|_{0}^{t} - \int_{0}^{t} e^{-t}dt\right)$

$=-\left(te^{-t} + e^{-t}\Big|_{0}^{t}\right) = -te^{-t} - (e^{-t}-1) = 1-e^{-t}(t+1)$

$= 1-e^{-\frac{x^2}{2}}\left(\frac{x^2}{2}+1\right)$

故 X 的分布函数 $F(x)=\begin{cases} 0 & x<0 \\ 1-e^{-\frac{x^2}{2}}\left(\frac{x^2}{2}+1\right) & x\geq 0 \end{cases}$

(3) $P(-2<X<4) = F(4)-F(-2) = 1-e^{-8}(8+1)-0 = 1-9e^{-8}$

故 $P(-2<X<4)=1-9e^{-8}$

解35：$x^2+\xi x+1=0$ 有实根 $\Rightarrow \xi^2-4\geq 0 \Rightarrow \xi^2\geq 4$

$\Rightarrow \xi\geq 2$ 或 $\xi\leq -2$. 又 $\xi \sim U(-3,6)$

$\Rightarrow P(\xi\geq 2 \cup \xi\leq -2) = \frac{6-2}{6-(-3)} + \frac{-2-(-3)}{6-(-3)} = \frac{4}{9}+\frac{1}{9}=\frac{5}{9}$

故 $x^2+\xi x+1=0$ 有实根的概率为 $\frac{5}{9}$

解36：(1) $f(x)=\begin{cases}\lambda e^{-\lambda x} & x\geq 0\\ 0 & x<0\end{cases} \Rightarrow F(x)=\begin{cases}1-e^{-\lambda x} & x\geq 0\\ 0 & x<0\end{cases}$

$P(X\geq 4000)=1-P(X<4000)=1-F(4000)=1-(1-e^{-\frac{1}{8000}\cdot 4000})$

$=e^{-\frac{1}{2}}$ 故能正常使用4000h的概率为 $e^{-\frac{1}{2}}$

(2) 由指数分布无记忆性

$\Rightarrow P(X\geq 4000+4000 \mid X\geq 4000)=P(X\geq 4000)=e^{-\frac{1}{2}}$

故已经用了4000h 还能用4000h的概率为 $e^{-\frac{1}{2}}$

(3) X可能的取值为 $0,1,\cdots,20$

$P(X=n)=C_{20}^{n}(e^{-\frac{1}{2}})^n\cdot(1-e^{-\frac{1}{2}})^{20-n}$ 即 $X\sim B(20,e^{-\frac{1}{2}})$

故X的分布律为 $X\sim B(20,e^{-\frac{1}{2}})$

解40 (1) $P(2<X<5)=F(5)-F(2)=\phi(\frac{5-3}{2})-\phi(\frac{2-3}{2})$

$=\phi(1)-\phi(-\frac{1}{2})=\phi(1)-[1-\phi(\frac{1}{2})]=\phi(1)+\phi(\frac{1}{2})-1$

$=0.8413+0.6915-1=0.5328$

故 $P(2<X<5)=0.5328$

(2) $P(|X|>2)=1-P(|X|<2)=1-P(-2<X<2)$

$=1-[F(2)-F(-2)]=1-[\phi(\frac{2-3}{2})-\phi(\frac{-2-3}{2})]$

$=1-[\phi(-\frac{1}{2})-\phi(-\frac{5}{2})]=1-[(1-\phi(\frac{1}{2}))-(1-\phi(\frac{5}{2}))]$

$=1-[\phi(\frac{5}{2})-\phi(\frac{1}{2})]=1-(0.99379-0.6915)=0.69771$

故 $P(|X|>2)=0.69771$

(3) $P(X>C)=1-P(X<C)=P(X<C) \Rightarrow P(X<C)=\frac{1}{2}$

$\Rightarrow C=3$ 故常数c的值为3

(4) $P(X>d)=1-P(X<d)=1-F(d)=1-\phi(\frac{d-3}{2})\geq 0.9$

$\Rightarrow \phi(\frac{d-3}{2})\leq 0.1 \Rightarrow 1-\phi(\frac{3-d}{2})\leq 0.1 \Rightarrow \phi(\frac{3-d}{2})\geq 0.9$

$\Rightarrow \frac{3-d}{2}\geq 1.29 \Rightarrow d\leq 0.42$ 故d至多为0.42.

解42：(1) 由 $P(X\le 70)=0.5 \Rightarrow \mu=70$.

$P(X\le 60)=F(60)=\Phi(\frac{60-70}{\sigma})=1-\Phi(\frac{10}{\sigma})=0.25$

$\Rightarrow \Phi(\frac{10}{\sigma})=0.75 \Rightarrow \sigma=10/0.675=14.81$

故 $\mu=70$，$\sigma=14.81$

(2) $P(X>65)=1-P(X\le 65)=1-F(65)=1-\Phi(\frac{65-70}{\sigma})$

$=\Phi(\frac{5}{\sigma})=\Phi(\frac{5}{14.81})=\Phi(0.3376)=0.6331$

设 A 表示至少有 2 人的体重超过 65 kg，Y 表示体重大于 65 的人数

$P(A)=1-P(Y=0)-P(Y=1)$

$=1-(1-0.6331)^5-C_5^1\,0.6331\,(1-0.6331)^4=0.936$

故至少有2人体重超过 65 kg 的概率为 0.936

解46：$P(|X-\mu|<\sigma)=P(\mu-\sigma<X<\mu+\sigma)=F(\mu+\sigma)-F(\mu-\sigma)$

$=\Phi(\frac{\mu+\sigma-\mu}{\sigma})-\Phi(\frac{\mu-\sigma-\mu}{\sigma})=\Phi(1)-\Phi(-1)=2\Phi(1)-1$

$=2\times 0.8413-1=0.6826$

故随着 σ 的增大，$P(|X-\mu|<\sigma)$ 不变，均为 0.6826