

(2011-2012)工科数学分析第二学期期末试题(A 卷)解答 (2012.6)

一. 1. $\frac{13}{\sqrt{14}}$

2. 0

3. $ye^{xy} + x \cos(xy) + 2xz \cos(xz^2)$

4. $\frac{7}{3}\pi a^4$

5. $\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n} (x-3)^n$

二. $e^z \frac{\partial z}{\partial x} - z - x \frac{\partial z}{\partial x} = 0$ (2 分)

解得 $\frac{\partial z}{\partial x} = \frac{z}{e^z - x}$ (3 分)

$$e^z \frac{\partial z}{\partial y} - x \frac{\partial z}{\partial y} = 1 \quad \dots\dots\dots(5 \text{ 分})$$

解得 $\frac{\partial z}{\partial y} = \frac{1}{e^z - x}$ (6 分)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\frac{\partial z}{\partial y} (e^z - x) - z \cdot e^z \frac{\partial z}{\partial y}}{(e^z - x)^2} \quad \dots\dots\dots(7 \text{ 分})$$

$$= \frac{(e^z - x - ze^z) \frac{\partial z}{\partial y}}{(e^z - x)^2} = \frac{e^z - x - ze^z}{(e^z - x)^3} \quad \dots\dots\dots(8 \text{ 分})$$

三.
$$\begin{cases} 2x - \frac{dz}{dx} = 0 \\ 3 + 2\frac{dy}{dx} = 0 \end{cases} \quad \dots\dots\dots(2 \text{ 分})$$

将点 P 代入解得 $\frac{dy}{dx} = -\frac{3}{2} \quad \frac{dz}{dx} = 2$ (3 分)

曲线的切向量为 $\vec{T} = \{1, -\frac{3}{2}, 2\}$ (4 分)

直线的方向向量为 $\vec{s} = \{3, -5, 5\} \times \{1, 0, 5\} = \{-25, -10, 5\}$ (7 分)

由于 $\vec{T} \cdot \vec{s} = -25 + 15 + 10 = 0 \quad \vec{T} \perp \vec{s}$ 故得证(8 分)

四. $\frac{\partial z}{\partial x} = y(1-2x-y) \quad \frac{\partial z}{\partial y} = x(1-x-2y) \quad \dots\dots\dots(2 \text{ 分})$

令 $\frac{\partial z}{\partial x} = 0 \quad \frac{\partial z}{\partial y} = 0$ 得驻点 $P_1(0,0) \quad P_2(0,1) \quad P_3(1,0) \quad P_4(\frac{1}{3}, \frac{1}{3}) \quad \dots\dots\dots(4 \text{ 分})$

$\frac{\partial^2 z}{\partial x^2} = -2y \quad \frac{\partial^2 z}{\partial x \partial y} = 1-2x-2y \quad \frac{\partial^2 z}{\partial y^2} = -2x \quad \dots\dots\dots(6 \text{ 分})$

在点 $P_1(0,0)$, $A=0, B=1, C=0 \quad AC-B^2 = -1 < 0$

故 $P_1(0,0)$ 不是极值点 $\dots\dots\dots(7 \text{ 分})$

在点 $P_2(0,1)$, $A=-2, B=-1, C=0 \quad AC-B^2 = -1 < 0$

故 $P_2(0,1)$ 不是极值点 $\dots\dots\dots(8 \text{ 分})$

同理, $P_3(1,0)$ 不是极值点 $\dots\dots\dots(9 \text{ 分})$

在点 $P_4(\frac{1}{3}, \frac{1}{3})$, $A=-\frac{2}{3}, B=-\frac{1}{3}, C=-\frac{2}{3} \quad AC-B^2 = \frac{1}{3} > 0$

又 $A < 0$, 故 $P_4(\frac{1}{3}, \frac{1}{3})$ 是极大值点, 极大值为 $z\Big|_{(\frac{1}{3}, \frac{1}{3})} = \frac{1}{27} \quad \dots\dots\dots(11 \text{ 分})$

五. $I = \int_0^{\frac{\pi}{4}} d\theta \int_0^{2\sin\theta} \frac{d\rho}{\sqrt{4-\rho^2}} \quad \dots\dots\dots(4 \text{ 分})$

$= \int_0^{\frac{\pi}{4}} \theta d\theta \quad \dots\dots\dots(7 \text{ 分})$

$= \frac{\pi^2}{32} \quad \dots\dots\dots(9 \text{ 分})$

六. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n+2}{n+3} = 1 \quad R=1 \quad \dots\dots\dots(1 \text{ 分})$

$x=1$ 时, 级数为 $\sum_{n=0}^{\infty} \frac{1}{n+2}$, 发散

$x=-1$ 时, 级数为 $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$, 收敛

收敛域为 $[-1,1)$ $\dots\dots\dots(3 \text{ 分})$

$$\text{设 } S(x) = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n+2}$$

$$S'(x) = \sum_{n=0}^{\infty} x^{n+1} = \frac{x}{1-x} \quad \dots\dots\dots(6 \text{ 分})$$

$$S(x) = -x - \ln(1-x) \quad \dots\dots\dots(8 \text{ 分})$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n+2} = \begin{cases} -\frac{1}{x} - \frac{1}{x^2} \ln(1-x) & x \in [-1,1), x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases} \quad \dots\dots\dots(9 \text{ 分})$$

七.

$$\begin{aligned} I &= 2 \int_0^1 dx \int_{x^2}^1 dy \int_0^{1-y} x^2 dz \quad \dots\dots\dots(4 \text{ 分}) \\ &= 2 \int_0^1 dx \int_{x^2}^1 x^2 (1-y) dy \quad \dots\dots\dots(6 \text{ 分}) \\ &= 2 \int_0^1 \left(\frac{1}{2} x^2 - x^4 + \frac{1}{2} x^6 \right) dx \quad \dots\dots\dots(8 \text{ 分}) \\ &= \frac{8}{105} \quad \dots\dots\dots(9 \text{ 分}) \end{aligned}$$

八.

$$\text{由 } \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y} \quad \dots\dots\dots(1 \text{ 分})$$

$$-\frac{x^2 + y^2 - (x-y+b) \cdot 2x}{(x^2 + y^2)^2} = \frac{x^2 + y^2 - (ax+y) \cdot 2y}{(x^2 + y^2)^2} \quad \dots\dots\dots(3 \text{ 分})$$

$$\text{得 } a=1 \quad b=0 \quad \dots\dots\dots(4 \text{ 分})$$

$$\begin{aligned} u(x,y) &= \int_{(1,0)}^{(x,y)} \frac{x+y}{x^2+y^2} dx - \frac{x-y}{x^2+y^2} dy + C \quad \dots\dots\dots(6 \text{ 分}) \\ &= \int_1^x \frac{1}{x} dx - \int_0^y \frac{x-y}{x^2+y^2} dy + C \quad \dots\dots\dots(8 \text{ 分}) \\ &= -\arctan \frac{y}{x} + \frac{1}{2} \ln(x^2 + y^2) + C \quad \dots\dots\dots(10 \text{ 分}) \end{aligned}$$

九.

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx = \frac{2}{\pi} \int_0^\pi \sin nx dx \quad \dots\dots\dots(3 \text{ 分})$$

$$= \frac{2}{n\pi} (1 - \cos n\pi) = \frac{2}{n\pi} (1 - (-1)^n) \quad \dots\dots\dots(5 \text{ 分})$$

$$S(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ 0 & x = 0, \pm\pi \end{cases} \quad \dots\dots\dots(8 \text{ 分})$$

十.

设曲面 $S_1: z=0 \quad (x^2 + y^2 \leq 1)$

$$I = \oiint_{S+S_1^+} - \iint_{S_1^+} xz^2 dydz + (x^2y - z^3) dzdx + (2xy + y^2z + 3) dxdy \quad \dots\dots\dots(1 \text{ 分})$$

$$\oiint_{S+S_1^+} xz^2 dydz + (x^2y - z^3) dzdx + (2xy + y^2z + 3) dxdy$$

$$= - \iiint_V (z^2 + x^2 + y^2) dV \quad \dots\dots\dots(3 \text{ 分})$$

$$= - \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r^4 \sin \varphi dr \quad \dots\dots\dots(5 \text{ 分})$$

$$= -\frac{2}{5} \pi \quad \dots\dots\dots(6 \text{ 分})$$

$$\iint_{S_1^+} xz^2 dydz + (x^2y - z^3) dzdx + (2xy + y^2z + 3) dxdy$$

$$= \iint_{S_1^+} (2xy + 3) dxdy \quad \dots\dots\dots(7 \text{ 分})$$

$$= \iint_{D_{xy}} (2xy + 3) dxdy = \iint_{D_{xy}} 3 dxdy = 3\pi \quad \dots\dots\dots(8 \text{ 分})$$

$$I = -\frac{2}{5} \pi - 3\pi = -\frac{17}{5} \pi \quad \dots\dots\dots(9 \text{ 分})$$

十一.

$$f(x) = \sin x + x \int_0^x f(u) du - \int_0^x u f(u) du \quad \dots\dots\dots(1 \text{ 分})$$

$$f'(x) = \cos x + \int_0^x f(u) du \quad \dots\dots\dots(2 \text{ 分})$$

$$f(0) = 0 \quad f'(0) = 1 \quad \dots\dots\dots(3 \text{ 分})$$

$$f(x) = f(0) + f'(0)x + o(x) = x + o(x)$$

$$f\left(\frac{1}{n}\right) = \frac{1}{n} + o\left(\frac{1}{n}\right) \sim \frac{1}{n} \quad \dots\dots\dots(5 \text{ 分})$$

$$\text{由于 } \sum_{n=1}^{\infty} \frac{1}{n} \text{ 发散, 故 } \sum_{n=1}^{\infty} f\left(\frac{1}{n}\right) \text{ 发散} \quad \dots\dots\dots(6 \text{ 分})$$

因为 $f'(0) = 1 > 0$, 且 $f'(x)$ 连续, 故在 $x = 0$ 某邻域内 $f'(x) > 0$,

$f(x)$ 单调增加, 因此当 n 充分大时, $f\left(\frac{1}{n}\right)$ 单调减少 $\dots\dots\dots(8 \text{ 分})$

$$\text{又} \quad \lim_{n \rightarrow \infty} f\left(\frac{1}{n}\right) = f(0) = 0$$

$$\text{故} \quad \sum_{n=1}^{\infty} (-1)^n f\left(\frac{1}{n}\right) \text{ 收敛} \quad \dots\dots\dots(9 \text{ 分})$$