Wave-particle duality ellipse and application in quantum imaging with undetected photons

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We present a systematic framework to quantify the interplay between coherence and wave-particle duality in generic two-path interference systems. Our analysis reveals a closed-form duality ellipse (DE) equality, that rigorously unifies visibility (a traditional waveness measure) and predictability (a particleness measure) with degree of coherence, providing a complete mathematical embodiment of Bohr's complementarity principle. Extending this framework to quantum imaging with undetected photons (QIUP), where both path information and photon interference are inherently linked to spatial object reconstruction, we establish an imaging duality ellipse (IDE) that directly connects wave-particle duality to the object's transmittance profile. This relation enables object characterization through duality measurements alone and remains robust against experimental imperfections such as decoherence and misalignment. Our results advance the fundamental understanding of quantum duality while offering a practical toolkit for optimizing coherence-driven quantum technologies, from imaging to sensing.

I. INTRODUCTION

Wave-particle duality sits at the heart of quantum mechanics, directly inspiring Schrödinger's development of his wave equation to account for de Broglie's ad hoc matter wave theory from first principle [1–3]. The quantitative characterization of waveness and particleness by Wootters and Zurek [4] led to the formulation of different versions of a remarkable duality inequality [5–12], $V^2 + D^2 \leqslant 1$, where visibility V and distinguishability (or predictability) D quantify, respectively, the waveness and particleness of a quantum object. However, this inequality does not always reflect the mutually exclusive relationship between wave and particle characteristics, as an increase in one does not necessarily lead to a decrease in the other, which contradicts Bohr's complementarity principle [2]. The reduction from a fully complementary equality to an inequality is primarily due to partial coherence of the two path states [6,7]. Such partial coherence will inevitably induce entanglement between the path and remaining (internal and external) degrees of freedom. Therefore, important advancements have been made to include the role of entanglement into the wave-particle duality picture [13–18]. Particularly, coherence has been treated as an enabler for entanglement in connection with predictability/distinguishability and wave interference visibility [19,20]. The extended role of coherence and entanglement have also been explored in uncertainty re-

lation [21], Lorentz invariance [22], entanglement swapping [23], polarization coherence [24,25], etc. However, in most of these studies, coherence is embedded in other quantities, and the direct role of partial coherence and its precise impact on the duality relation has not been fully explored. This raises the question: what is the fundamental quantitative role of coherence in quantum duality?

Furthermore, wave and particle features, along with their constraints, are recognized as crucial elements in the execution of various quantum measurement, sensing, imaging, and control tasks [26–32]. However, a comprehensive quantitative analysis of the interplay between these dual aspects (as captured by duality relations) in specific quantum tasks remains largely unexplored. This leads to a key follow-up question: what is the operational role of wave-particle duality, quantitatively mediated through coherence, in quantum tasks?

To address these questions, we quantitatively analyze how generic two-path coherence constrains wave-particle duality in the Mach-Zehnder interferometer scenario. By incorporating the degree of coherence (via standard optical cross correlation [33,34]), a compact duality ellipse (DE) relation is obtained. Coherence is shown to be dictating the DE's ellipticity, governing the wave-particle tradeoff. The duality ellipse resolves ambiguities in the duality inequality, ensuring strict adherence to Bohr's complementarity principle by quantitatively enforcing the mutual exclusivity of wavelike (waveness) and particlelike (particleness) properties through the integration of coherence.

We then extend our analysis to account for generic forms of coherence loss and explore the application of the derived DE relation in the recently emerging field of quantum imaging with undetected photons (QIUP) [27,35–40]. In the QIUP scenario, both wave and particle characteristics of the photon are crucial to achieving imaging objectives. Previous studies have
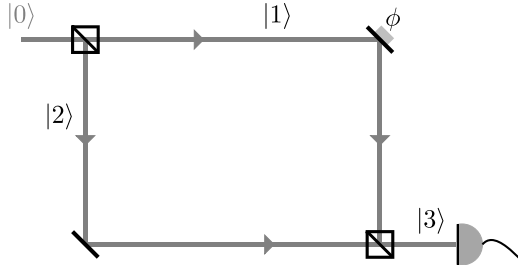
FIG. 1. Schematic of the Mach-Zehnder interference setup for a single photon to analyze wave and particle features.

highlighted the importance of indistinguishability [35,41,42] as a manifestation of waveness in the imaging process. However, the collective influence of particle nature, particularly the role of the quantitative duality relation, remains largely unexplored. Our approach involves a quantitative integration of object information, represented by its transmittance T (a generalized form of coherence), into the dual facets of waveness (quantified by interference visibility V) and particleness (measured by predictability D [6,7]). We demonstrate that when a photon is absorbed by the object, it introduces particle information, leading to the inevitable loss of coherence and a subsequent reduction in interference visibility. This particle information, crucial for object probing, is quantitatively incorporated to construct the image of an object. The incorporation of object information results in an extended imaging duality ellipse (IDE) relation, where the ellipticity (in terms of T) can be used to obtain object information. In essence, QIUP is demonstrated to be quantitatively governed by the photon's fundamental wave-particle duality, suggesting an alternative approach for obtaining object information through the measurements of waveness and particleness of the photon.

II. QUANTUM DUALITY ELLIPSE

We consider the standard two-path scenario by analyzing the typical example of a photon passing through a Mach-Zehnder interferometer (MZI) as illustrated in Fig. 1. While the main results and conclusions hold for any quantum particle, we conduct the following analysis using a photon for convenience, without any loss of generality. A photon starts with the state $|0\rangle|M_0\rangle$, where $|0\rangle$ represents the initial path mode and $|M_0\rangle$ describes the combined state of the remaining internal degrees of freedom of the photon along with all external states of the rest of the universe. Then it enters the MZI and splits into two path states $|1\rangle$ and $|2\rangle$ after the first beamsplitter (BS) with generic probabilities of P_1 and P_2, respectively. When recombined by the second BS, the two path states converge into the same $|3\rangle$ path mode, where ideal interference happens if the marginal state $|M_0\rangle$ remains unchanged during the process. In realistic scenarios, decoherence or other mechanisms such as photon loss can modify the reduced state of the photon's path modes, degrading coherence. Then the (environmental) marginal states in path 1 and path 2 evolve as $|M_0\rangle \rightarrow |M_1\rangle$ in path 1 and $|M_0\rangle \rightarrow |M_2\rangle$ in path 2. Therefore, the most general form of the photon path mode state after the

first BS can always be written as

$$|\Psi_1\rangle = c_1|1\rangle|M_1\rangle + c_2|2\rangle|M_2\rangle, \qquad (1)$$

where $|c_1|^2 = P_1$ and $|c_2|^2 = P_2$ are the normalized coefficients with $P_1 + P_2 = 1$. Then the coherence of the two paths 1 and 2 is determined by the normalized states $|M_1\rangle, |M_2\rangle$. Here we adopt the optical definition of degree of coherence γ in terms of the normalized cross correlation of $|M_1\rangle$ and $|M_2\rangle$ [33], i.e.,

$$\gamma = |\langle M_1 | M_2 \rangle|, \qquad (2)$$

which varies between 0 and 1 with 0 indicating complete incoherence and 1 meaning fully coherent. Note that this degree of coherence is also quantitatively related to entanglement (e.g., the measure concurrence C [43]) between the path and marginal degrees of freedom, i.e., $C = 2|c_1 c_2|\sqrt{1 - \gamma^2}$. Therefore, our following analysis of coherence is naturally also connected to the quantitative links between duality and entanglement [14,16–18,20].

After the second BS (50/50), the two paths converge into the same path mode $|3\rangle$. Then the photon (non-normalized) state becomes

$$|\Psi_2\rangle \propto c_1|3\rangle|M_1\rangle + e^{-i\phi}c_2|3\rangle|M_2\rangle, \qquad (3)$$

where ϕ is the relative phase accumulated between the two paths. The probability P of the photon at the detector is achieved straightforwardly as $P \propto |c_1|^2 + |c_2|^2 + 2|c_1 c_2|\gamma \cos(\phi)$. Then the interference visibility V and predictability D can be computed, respectively [6,8], as $V = (P_{\max} - P_{\min})/(P_{\max} + P_{\min}) = 2|c_1 c_2|\gamma$ and $D = |P_1 - P_2| = ||c_1|^2 - |c_2|^2|$. From the schematic setup in Fig. 1, V can be achieved by registering photon counts when continuously varying the phase ϕ through the piezoelectric transducer in path $|1\rangle$, and D can be obtained by measuring P_1, P_2 when blocking paths $|2\rangle, |1\rangle$, respectively. While these two quantities apparently lead to the conventional duality inequality [6,8], i.e., $V^2 + D^2 \leqslant 1$, it also leads to an exact closed relation

$$\frac{V^2}{\gamma^2} + D^2 = 1, \qquad (4)$$

where the quantitative role of coherence is now apparent via γ.

When V and D are treated as two independent coordinates, this duality relation describes an exact ellipse equation, see illustration in Fig. 2. The degree of coherence γ is the reason causing the elliptical nature of the duality relation as the ellipticity is exactly the complement of coherence $\eta = 1 - \gamma$. Thus, we call the above result (4) the duality ellipse (DE).

Apparently, the duality ellipse relation explicitly singles out the role of coherence γ. More deeply, our relation suggests that the adjusted interference visibility V by coherence γ, i.e., V/γ, serves as a perfect new measure of waveness to ensure a rigorous complementary equality. It avoids the ambiguity of the exclusiveness between waveness and particleness, thus exactly compatible with Bohr's complementarity statement [2].

A critical feature of our duality ellipse is its ability to quantitatively unify coherence effects from both intrinsic (internal) and environmental (external) degrees of freedom, all included
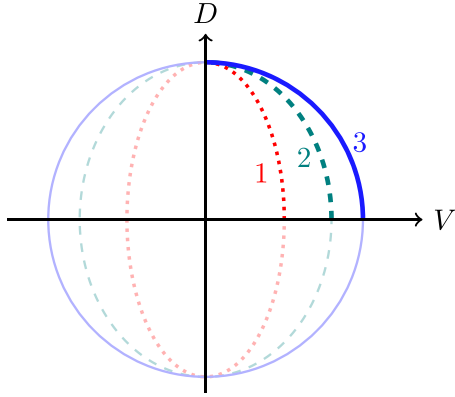
FIG. 2. Illustration of the duality ellipse for ellipticity $\eta = 1 - \gamma = 1 - T = 0$, (blue, curve 3), 0.2 (teal, curve 2), and 0.5 (red, curve 1), respectively. Curve 3 represents the case when the two paths are completely coherent, i.e., $\gamma = 1$ or no object in the QIUP with $T = 1$.

in the marginal states $\gamma = |\langle M_1 | M_2 \rangle|$. This universality ensures that the DE applies equally to systems in pure and mixed states. Notably, in the fully coherent limit $\gamma = 1$, the DE reduces to the conventional pure-state equality $V^2 + D^2 = 1$, thereby recovering the well-established duality relation as a special case of our formalism.

It is worth to note that the partial coherence parameter γ also accounts for photon loss scenarios. For example, if a photon in path 1 is subject to loss (e.g., via interaction with environment), the joint system-environment state becomes $|1\rangle|M_1\rangle \rightarrow a|1\rangle|M_1\rangle + b|l\rangle|M_l\rangle$, where $|l\rangle$ denotes the lost-photon state with probability $|b|^2$. This modifies the coefficient c_1 in Eq. (1) to become $c_1' = c_1 a$. Crucially, our duality analysis remains fully valid under this substitution, as the derivation depends only on the structure of the coefficients and not their specific origin.

III. IMAGING DUALITY ELLIPSE

We now consider the application of the obtained DE (4) to the scenario of quantum imaging with undetected photon (QIUP) [27], which is based on the Zou-Wang-Mandel (ZWM) scheme of induced coherence without induced emission [44], see a schematic illustration in Fig. 3.
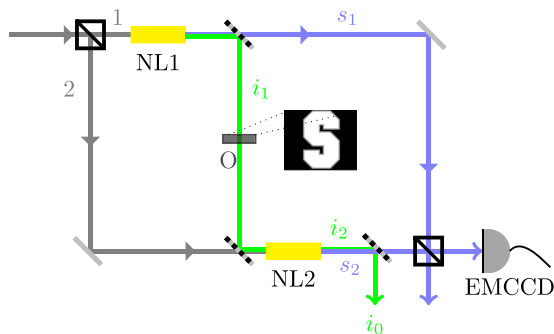

FIG. 3. Schematic illustrations of the QIUP setup in Ref. [27]. Here, object O is represented by the cut-out letter "S."

As is shown in Fig. 3, an input pump beam is split by a general beamsplitter (represented by the cube with the slanted lines) into two, both of which are then down converted by a corresponding identical nonlinear (NL) crystal (illustrated by a yellow box) into a pair of respective entangled photons [45]. The green and blue lines represent the idler and signal photon paths, respectively. The first idler photon mode, i_1 (generated from NL1), is then reflected or absorbed partially by object O. The transmitted portion is then aligned with the idler photon mode i_2 of the second crystal NL2 to become a common idler photon mode i_0 that exits the interferometer. The signal photon modes, s_1, s_2 from the two corresponding crystals are then combined by a 50/50 beamsplitter (BS) for interference detection.

We first analyze the sequence of quantum states along the optical path in this QIUP setup by temporarily assuming that (i) there is no mechanism of coherence loss except for the existence of the object, and (ii) there is perfect alignment of the two idler modes. We will release the two restrictions in the following section.

Under assumption (i), the pump photon path states are completely coherent after being split into two by the first BS, and can be represented as $|\psi_0\rangle = c_1|1\rangle + e^{-i\phi_0}c_2|2\rangle$, where c_1 and c_2 are normalized coefficients, with $|c_1|^2 + |c_2|^2 = 1$. The two path states are denoted as $|1\rangle \equiv |1\rangle_{p_1}|0\rangle_{p_2}$ and $|2\rangle \equiv |0\rangle_{p_1}|1\rangle_{p_2}$ with $|1\rangle_{p_1}$ and $|0\rangle_{p_1}$ indicating, respectively, 1 and 0 photons in path p_1, etc. The relative phase ϕ_0 is induced due to reflection and path length difference.

After spontaneous parametric down conversion (SPDC) [46] in NL1 and NL2, the photon state becomes

$$|\psi_1\rangle = c_1|s_1\rangle|i_1\rangle + e^{-i\phi_1}c_2|s_2\rangle|i_2\rangle, \qquad (5)$$

where ϕ_1 is the new relative phase after the nonlinear crystals. Here $|s_{1,2}\rangle$ and $|i_{1,2}\rangle$ represent the signal and idler photons down converted from the nonlinear crystals NL1,2, respectively. The single-photon pair state is selected with other photon-number states discarded. Note that a more complete description of the state should be $|s_1\rangle|i_1\rangle \equiv |0\rangle_{p_1}|1\rangle_{s_1}|1\rangle_{i_1}|0\rangle_{p_2}|0\rangle_{s_1}|0\rangle_{i_1}$ where $|1\rangle_{s_1}$ denotes 1 photon in the signal mode s_1, etc. The general form of state (5) in terms of photon momentum states can be obtained through standard SPDC analysis, see for example in Ref. [35].

Idler photon 1 then hits the object O, which has a distribution of real transmittance coefficient $T(x, y)$ in the transverse plane and will transmit the idler photon in mode i_1 with probability $|T(x, y)|^2$ and reflect (or absorb) it with probability $|R(x, y)|^2 = 1 - |T(x, y)|^2$. For convenience, we will omit the spatial dependence label (x, y) in the following discussions and simply write the transmittance and reflection coefficients as T and R, respectively. Then the general photon state becomes

$$|\psi_2\rangle = c_1|s_1\rangle(T|i_1\rangle + R|r\rangle) + e^{-i\phi_2}c_2|s_2\rangle|i_2\rangle, \qquad (6)$$

where ϕ_2 is the new relative phase after O, and $|r\rangle$ represents the mode being reflected or absorbed by O.

Under assumption (ii), the perfect alignment of the two idler modes $|i_1\rangle$, $|i_2\rangle$, indicated by the overlapping of the two green lines after NL2, will converge into the same mode $|i_0\rangle$,

which lead to the photon state to become

$$|\psi_3\rangle = c_1|s_1\rangle(T|i_0\rangle + R|r\rangle) + e^{-i\phi_3}c_2|s_2\rangle|i_0\rangle, \qquad (7)$$

where ϕ_3 is the relative phase of the two signal photon modes before the second BS.

Due to partial transmission of the idler mode $|i_1\rangle$ into mode $|i_0\rangle$, the relative probability of the signal photon detection can be computed as $P = |c_1T|^2 + |c_1R|^2 + |c_2|^2 + 2|c_1||c_2|T\cos(\phi_3)$. Then one can obtain the interference visibility as $V = 2|c_1||c_2|T$. Combining with the predictability $D = ||c_1|^2 - |c_2|^2|$, one immediately achieves an imaging duality ellipse relation (IDE)

$$\frac{V^2}{T^2} + D^2 = 1. \qquad (8)$$

Obviously, the restriction between waveness and particleness is now controlled by the object information via transmittance T. The partial transmission of the object serves as a generalized form of partial coherence γ. Hence, it connects directly to the ellipticity $\eta = 1 - |T|$ and the reflection coefficient R is exactly the eccentricity of the ellipse. Here the existence of the object O with nonperfect transmittance $T < 1$ inevitably prevents perfect induction of coherence. Therefore, only partial coherence is restored from the pump photon to the signal photon.

Conversely, the IDE relation above also indicates that the waveness and particleness control the image information in this QIUP scenario. When the visibility V and predictability D can be obtained by measurements, the information or image (via transmittance T) of the object can be determined straightforwardly. In other words, the image of the object can be directly achieved by measuring the ellipticity of the IDE for each point in the transverse plane. As an example, Fig. 4(a) illustrates the simulated image of an object with a cut-out shape of the letter "S." In this case, the transmittance $T(x, y)$ is 1 at the center of the cut-out spaces, and it gradually decreases to 0 as it gets closer to the edges of the shape. This simulated distribution of the transmittance is demonstrated by the variation of the ellipticity $\eta(x, y) = 1 - T(x, y)$ of the IDE in Fig. 4(a). Therefore, the measurement of wave-particle dual nature $V(x, y)$ and $D(x, y)$ for each transverse spatial point is capable to retrieve the object information. It has the advantage of being robust against decoherence, nonperfect alignment, etc., and will be discussed in the following.

IV. ROBUST IMAGING DUALITY ELLIPSE AGAINST PRACTICAL IMPERFECTIONS

In reality, there are many other sources (internal or external) of nonperfections or loss of coherence just as discussed in obtaining the general DE relation (4). Therefore, we will release the two assumptions (i), (ii) adopted in the previous section and show, surprisingly, that although these nonperfections will change the IDE relation (8) quantitatively, they will not affect its form, thus still allowing a reliable robust way of determining the image of the object.

By releasing assumption (i), one needs to take into account possible decoherence of two signal photon modes $|s_1\rangle$ and $|s_2\rangle$, or the initial nonperfect partial coherence of the pump photon modes $|1\rangle$ and $|2\rangle$. Both situations can be modeled
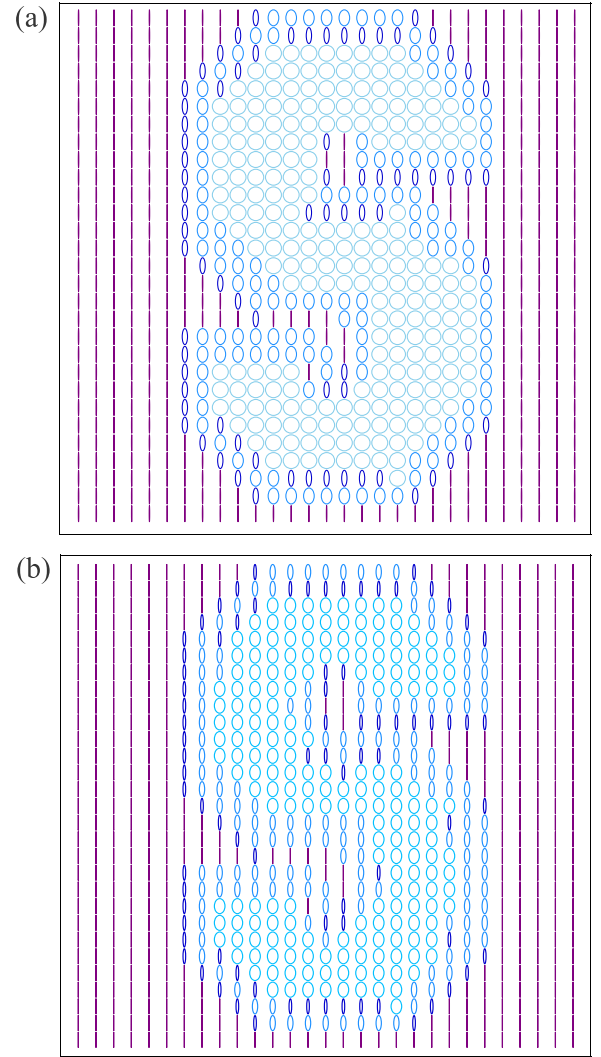

FIG. 4. Illustration of the point by point duality D-V relation in the transverse plane. The blue circles represent full transmission $T = 1$ cut-out parts of the object, and the blue ellipses represent the object's partial transmission $T < 1$. The purple ellipses represent the opaque area of the object with $T = 0$. Here, the top and bottom plots represent the ideal noiseless and the nonperfect alignment practical cases, respectively.

effectively by introducing an external party m with states $|m_1\rangle$, $|m_2\rangle$. As an illustration, here we discuss the case when partial coherence is originated from the pump photon without loss of any generality. That is, we consider a more general pump photon state

$$|\psi_0'\rangle = c_1|1\rangle|m_1\rangle + c_2e^{i\phi_0'}|2\rangle|m_2\rangle, \qquad (9)$$

where $|\langle m_1|m_2\rangle| = \gamma < 1$ is the source of partial coherence due to environmental noise or other types of decoherence mechanisms.

By releasing assumption (ii), we consider practical cases when there is misalignment of the two idler modes i_1, i_2, which will result in partial transfer of coherence from the pump photon to the signal photon [47]. That is, the idler modes after the alignment becomes, $|i_1\rangle \rightarrow |i_0\rangle$ but $|i_2\rangle \rightarrow |i_0'\rangle$, with $|\langle i_0|i_0'\rangle| = \alpha < 1$, a nonunity overlap.

Consequently, the down-converted photon state after idler mode alignment needs to be modified

$$|\psi_2'\rangle = c_1|s_1\rangle(T|i_0\rangle + R|r\rangle)|m_1\rangle$$
$$+ c_2 e^{i\phi_2'}|s_2\rangle|i_0'\rangle|m_2\rangle, \tag{10}$$

where ϕ_2' is the generic two path phase difference. Then the probability of detecting the signal photon can be computed as $P = |c_1|^2 + |c_2|^2 + 2|c_1||c_2|T\gamma\alpha\cos(\phi_2')$. Straightforward calculation leads immediately to the following modified imaging duality ellipse relation

$$\frac{V^2}{\gamma^2 T^2 \alpha^2} + D^2 = 1, \tag{11}$$

which is again an ellipse equation with respect to V and D. The ellipticity, however, is now modified to be $\eta = 1 - T\alpha\gamma$. That is, the wave-particle duality is further controlled by two additional sources of partial coherence, i.e., the nonperfect idler photon alignment α and the original pump photon partial coherence γ.

Although the current ellipticity cannot distinguish the contributions from which of the three sources, T, α, γ, the above duality relation (11) can still be used to determine the image of the object. Apparently, the idler photon nonperfect alignment α and the pump photon's partial coherence γ have a uniform effect across the entire transverse plane of the signal photon. Thus, α and γ are constants for a given pump source and a systematic alignment technique, independent of the signal photon's transverse plane distribution. Consequently, these imperfections will increase the ellipticity $\eta(x, y) = 1 - T(x, y)\alpha\gamma$ everywhere across the transverse plane by a common factor, and $\eta(x, y)$ still has a one to one correspondence to the transmittance $|T(x, y)|$. Therefore, the ellipticity $\eta(x, y)$ distribution is still an image of the object.

Figure 4(b) illustrates the same image of the letter "S" as in the previous section but with the effect of the nonperfections α, γ. The ellipticity of (11) for all points have increased in magnitude, but the relative ellipticity differences among different spatial points remain the same. As is shown, it is still capable to map out the object information even under unknown practical noisy situations.

It is also worth to mention that the overall unknown noise effects $\alpha\gamma$ can be obtained by measuring the ellipticity of (11) without an object $T = 1$ through the relation $\eta(x, y) = 1 - \alpha\gamma$. The exact transmittance distribution $T(x, y)$ can then be achieved when the object is in place.

V. DISCUSSION

In conclusion we have carried out a quantitative analysis of coherence effect on wave-particle duality. It is found that an exact compact duality ellipse relation is achieved where the ellipticity is uniquely controlled by the two-path degree of coherence. This DE clarifies the ambiguity of nonexclusiveness in the previous duality inequality, and is completely compatible with Bohr's complementarity principle.

We further extend our analysis to the application of quantum imaging with undetected photons. One is then led to a compact imaging duality ellipse equation where the object transmittance information T is directly connected to ellipticity $\eta = 1 - T$ of the V-D ellipse. The existence of the object amounts to an equivalent way of detecting the idler photon in mode $|i_1\rangle$, obtaining the particleness information of the signal photon (compromised by reduction of wave information). Thus, achieving the object information is an equivalent way of knowing the dual nature of both wave and particle information as reflected by the imaging duality ellipse (8). The compact IDE relation provides an alternative way of achieving object information through measurements of the photon's waveness and particleness in the image plane. More importantly, as shown in (11), this technique is independent of practical nonperfections of the systems in terms of misalignment and decoherence.

While our analysis of the duality ellipse relation (4) is based on photons, the results are valid for all quantum particles. Further, the DE framework can be generalized to broader quantum technologies, such as metrology, sensing, and communication, where advanced coherence measures and the interplay of wave-particle duality are central. This extension highlights the universality of coherence-modified exact complementarity, offering a unified lens to optimize protocols that leverage quantum coherence and duality.

Finally, we remark that the achieved duality ellipse relations (4), (11) are valid in two-path interference scenarios. It would be very interesting to explore its extension to multipath interference, where generalized fundamental definitions of coherence, waveness and particleness [19,21,23] in the multipath scenario will be crucial.

[1] L. de Broglie, Recherches sur la théorie des quanta, Ann. Phys. **10**, 22 (1925). English translation by A. F. Kracklauer, "On the Theory of Quanta," available from https://fondationlouisdebroglie.org/LDB-oeuvres/De_Broglie_Kracklauer.pdf.

[2] N. Bohr, The Quantum postulate and the recent development of atomic theory, Nature **121**, 580 (1928).

[3] F. Bloch, Heisenberg and the early days of quantum mechanics, Phys. Today **29**(12), 23 (1976).

[4] W. K. Wootters and W. H. Zurek, Complementarity in the double-slit experiment: Quantum nonseparability and a quantitative statement of bohr's principle, Phys. Rev. D **19**, 473 (1979).

[5] R. J. Glauber, Amplifiers, attenuators, and Schrödinger's cat, Ann. New York Acad. Sci. **480**, 336 (1986).

[6] D. M. Greenberger and A. Yasin, Simultaneous wave and particle knowledge in a neutron interferometer, Phys. Lett. A **128**, 391 (1988).

[7] G. Jaeger, M. A. Horne, and A. Shimony, Complementarity of one-particle and two-particle interference, Phys. Rev. A **48**, 1023 (1993).

[8] G. Jaeger, A. Shimony, and L. Vaidman, Two interferometric complementarities, Phys. Rev. A **51**, 54 (1995).

[9] B.-G. Englert, Fringe visibility and which-way information: An inequality, Phys. Rev. Lett. **77**, 2154 (1996).

[10] H.-Y. Liu, J.-H. Huang, J.-R. Gao, M. S. Zubairy, and S.-Y. Zhu, Relation between wave-particle duality and quantum uncertainty, Phys. Rev. A **85**, 022106 (2012).

[11] X.-F. Qian and G. S. Agarwal, Quantum duality: A source point of view, Phys. Rev. Res. **2**, 012031 (2020).

[12] T. H. Yoon and M. Cho, Quantitative complementarity of wave-particle duality, Sci. Adv. **7**, eabi9268 (2021).

[13] M. Jakob and J. A. Bergou, Quantitative complementarity relations in bipartite systems: Entanglement as a physical reality, Opt. Commun. **283**, 827 (2010).

[14] X.-F. Qian, A. Vamivakas, and J. Eberly, Entanglement limits duality and vice versa, Optica **5**, 942 (2018).

[15] A. Norrman, K. Blomstedt, T. Setälä, and A. T. Friberg, Complementarity and polarization modulation in photon interference, Phys. Rev. Lett. **119**, 040401 (2017).

[16] F. De Zela, Hidden coherences and two-state systems, Optica **5**, 243 (2018).

[17] M. L. Basso and J. Maziero, Complete complementarity relations for multipartite pure states, J. Phys. A: Math. Theor. **53**, 465301 (2020).

[18] T. Qureshi, Predictability, distinguishability, and entanglement, Opt. Lett. **46**, 492 (2021).

[19] M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati, Duality of quantum coherence and path distinguishability, Phys. Rev. A **92**, 012118 (2015).

[20] M. L. Basso and J. Maziero, Entanglement monotones from complementarity relations, J. Phys. A: Math. Theor. **55**, 355304 (2022).

[21] S. Dürr, Quantitative wave-particle duality in multibeam interferometers, Phys. Rev. A **64**, 042113 (2001).

[22] M. L. Basso and J. Maziero, Complete complementarity relations and their Lorentz invariance, Proc. R. Soc. A **477**, 20210058 (2021).

[23] M. L. Basso and J. Maziero, Entanglement monotones connect distinguishability and predictability, Phys. Lett. A **425**, 127875 (2022).

[24] X.-F. Qian and M. Izadi, Bridging coherence optics and classical mechanics: A generic light polarization-entanglement complementary relation, Phys. Rev. Res. **5**, 033110 (2023).

[25] Y.-H. Yang, X.-Z. Liu, J.-L. Jiang, H. Chen, X. Yang, X.-F. Qian, S.-M. Fei, and M.-X. Luo, Photonic energy-coherence theorem and experimental validations, Laser & Photon. Rev. e00326 (2025).

[26] A. G. White, J. R. Mitchell, O. Nairz, and P. G. Kwiat, "interaction-free" imaging, Phys. Rev. A **58**, 605 (1998).

[27] G. B. Lemos, V. Borish, G. D. Cole, S. Ramelow, R. Lapkiewicz, and A. Zeilinger, Quantum imaging with undetected photons, Nature (London) **512**, 409 (2014).

[28] X.-F. Qian, K. Konthasinghe, S. Manikandan, D. Spiecker, A. Vamivakas, and J. Eberly, Turning off quantum duality, Phys. Rev. Res. **2**, 012016 (2020).

[29] S. Das, C. Mukhopadhyay, S. S. Roy, S. Bhattacharya, A. S. De, and U. Sen, Wave-particle duality employing quantum coherence in superposition with non-orthogonal pointers, J. Phys. A: Math. Theor. **53**, 115301 (2020).

[30] K. Wang, D. R. Terno, Č. Brukner, S. Zhu, and X.-S. Ma, Controlling wave-particle duality with entanglement between single-photon and bell states, Phys. Rev. A **106**, 053715 (2022).

[31] M. Janovitch, M. Brunelli, and P. P. Potts, Wave-particle duality in a quantum heat engine, Phys. Rev. Res. **5**, L042007 (2023).

[32] J.-K. Li, K. Sun, Y. Wang, Z.-Y. Hao, Z.-H. Liu, J. Zhou, X.-Y. Fan, J.-L. Chen, J.-S. Xu, C.-F. Li, and G.-C. Guo, Experimental demonstration of separating the wave–particle duality of a single photon with the quantum cheshire cat, Light Sci. Appl. **12**, 18 (2023).

[33] M. Born and E. Wolf, *Principle of Optics*, 7th ed. (Cambridge University Press, Cambridge, 1999).

[34] L. Mandel, Coherence and indistinguishability, Opt. Lett. **16**, 1882 (1991).

[35] M. Lahiri, R. Lapkiewicz, G. B. Lemos, and A. Zeilinger, Theory of quantum imaging with undetected photons, Phys. Rev. A **92**, 013832 (2015).

[36] M. Gilaberte Basset, A. Hochrainer, S. Töpfer, F. Riexinger, P. Bickert, J. R. León-Torres, F. Steinlechner, and M. Gräfe, Video-rate imaging with undetected photons, Laser Photon. Rev. **15**, 2000327 (2021).

[37] S. Töpfer, M. Gilaberte Basset, J. Fuenzalida, F. Steinlechner, J. P. Torres, and M. Gräfe, Quantum holography with undetected light, Sci. Adv. **8**, eabl4301 (2022).

[38] E. A. Santos, T. Pertsch, F. Setzpfandt, and S. Saravi, Sub-diffraction quantum imaging with undetected photons, Phys. Rev. Lett. **128**, 173601 (2022).

[39] B. E. Haase, J. Hennig, M. Kutas, E. Waller, J. Hering, G. von Freymann, and D. Molter, Phase-quadrature quantum imaging with undetected photons, Opt. Express **31**, 143 (2023).

[40] D. A. R. Dalvit, T. J. Volkoff, Y.-S. Choi, A. K. Azad, H.-T. Chen, and P. W. Milonni, Quantum frequency combs with path identity for quantum remote sensing, Phys. Rev. X **14**, 041058 (2024).

[41] G. B. Lemos, M. Lahiri, S. Ramelow, R. Lapkiewicz, and W. N. Plick, Quantum imaging and metrology with undetected photons: tutorial, J. Opt. Soc. Am. B **39**, 2200 (2022).

[42] A. Hochrainer, M. Lahiri, M. Erhard, M. Krenn, and A. Zeilinger, Quantum indistinguishability by path identity and with undetected photons, Rev. Mod. Phys. **94**, 025007 (2022).

[43] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett. **80**, 2245 (1998).

[44] X. Zou, L. Wang, and L. Mandel, Induced coherence and indistinguishability in optical interference, Phys. Rev. Lett. **67**, 318 (1991).

[45] Z. J. Ou, *Quantum Optics for Experimentalists* (World Scientific Publishing Company, Singapore, 2017).

[46] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge university press, Cambridge, 1995).

[47] X. Zou, L. Wang, and L. Mandel, Induced coherence without induced emission resulting in nonlocal interference, in *OSA Annual Meeting* (Optica Publishing Group, 1991), p. FC3.