D-ND Quantum Information Engine: Modified Quantum Gates and Computational Framework

D-ND Research Collective

Independent Research

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We formalize the quantum-computational aspects of the D-ND (Dual-Non-Dual) framework by introducing a possibilistic quantum information architecture that generalizes standard quantum mechanics. Rather than pure probabilistic superposition, D-ND quantum states are characterized by a *possibilistic density* measure ρ_{DND} incorporating emergence structure, nonlocal coupling, and topological invariants. We define four modified quantum gates—H_{DND}, $\mathrm{CNOT}_{\mathrm{DND}}$, P_{DND}, and $\mathrm{Shortcut}_{\mathrm{DND}}$—that preserve D-ND structure while enabling practical computation. We prove that $\{H_{\mathrm{DND}}, \mathrm{CNOT}_{\mathrm{DND}}, P_{\mathrm{DND}}\}$ form a universal gate set in the perturbative regime by deriving arbitrary $\mathrm{SU}(2^n)$ unitaries from gate compositions. A complete circuit model with error analysis and coherence preservation guarantees is presented. We develop a simulation framework based on Iterated Function Systems (IFS) with pseudocode and polynomial complexity analysis. We position D-ND computation within known quantum advantage results (BQP vs. BPP), showing how emergence-assisted error suppression provides a distinct pathway to quantum speedup. Applications to quantum search algorithms and topological quantum computing are discussed. This work bridges quantum information theory and emergence-theoretic dynamics, establishing D-ND as a viable computational paradigm for near-term hybrid quantum-classical algorithms.

I. INTRODUCTION

Quantum computing has achieved remarkable theoretical and experimental progress, yet fundamental limitations persist: decoherence, measurement collapse, and the Born rule's strict probabilistic interpretation constrain the space of algorithms and applications. The D-ND framework (developed in Papers A–E) proposes that quantum systems need not be purely probabilistic; instead, *possibility* can coexist with probability, mediated through emergence and nonlocal coupling.

A. Notation Clarification

Throughout this paper, the emergence coupling coefficient λ (without subscript) represents the linear approximation parameter quantifying the strength of D-ND quantum gate modifications relative to standard quantum operations. This is distinguished from: Paper A's λ_k: eigenvalues of the emergence operator in the quantum substrate; Paper B's λ_{DND}: potential coupling constant in the dual-non-dual Hamiltonian; Paper D's λ_{auto}: autological convergence rate in observer dynamics; Paper E's λ_{cosmo}: cosmological emergence coupling. The notation is clarified further in §II C where $\lambda = M(t)$ (the emergence measure) during the linear approximation regime.

B. Motivations

1. **Beyond Probabilism**: Standard quantum mechanics treats all information as probabilistic amplitudes. D-ND permits possibilistic states—superpositions where some branches may be "proto-actual" (not yet fully actualized) or "suppressed" by emergence dynamics.

2. **Nonlocal Emergence**: Rather than viewing nonlocality as spooky action at a distance, D-ND models it as structure in the emergence field $\mathcal{E}$. Quantum gates can be designed to exploit this structure.

3. **Topological Robustness**: D-ND incorporates topological invariants (homological cycles, Betti numbers) that provide natural error correction and gate fidelity improvements.

4. **Hybrid Classical-Quantum**: The linear simulation framework allows efficient classical emulation of certain D-ND circuits, reducing hardware requirements.

5. **Quantum Advantage Through Emergence**: Unlike standard approaches that rely solely on quantum superposition, D-ND offers emergence-assisted error suppression, a novel pathway to quantum advantage.

C. Paper Structure

Section II introduces the possibilistic density measure and its relationship to standard quantum states. Section III defines the four core modified gates with rigorous composition rules. Section IV develops the circuit model and error analysis. Section V presents the IFS-based simulation framework with pseudocode. Section VI sketches applications, compares with known quantum advantage results, and establishes a computational bridge to the THRML/Omega-Kernel library by Extropic AI. Section VII concludes. Appendices A and B provide proofs of key propositions.

II. D-ND QUANTUM INFORMATION FRAMEWORK

A. Possibilistic Density $\rho_{\mathbf{DND}}$

In standard quantum mechanics, the state of a system is given by a density matrix $\rho \in \mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ is the space of bounded linear operators on Hilbert space $\mathcal{H}$. D-ND generalizes this to a *possibilistic density* by incorporating emergence.

Definition 1 (Possibilistic Density — Formula B10)**.** Let M_{dist}, M_{ent}, M_{proto} be three non-negative real-valued measures on the Hilbert space basis states:

- M_{dist}: *distributive capacity* (how "spread" the state is across basis elements)
- M_{ent}: *entanglement strength* (degree of nonlocal correlation structure)
- M_{proto}: *proto-actualization measure* (how "ready" a branch is to become classical)

Then the **possibilistic density** is:

$$\rho_{\text{DND}} = \frac{M_{\text{dist}} + M_{\text{ent}} + M_{\text{proto}}}{\sum_{\text{all states}}(M_{\text{dist}} + M_{\text{ent}} + M_{\text{proto}})} = \frac{M}{\Sigma M} \tag{1}$$

where $M = M_{\text{dist}} + M_{\text{ent}} + M_{\text{proto}}$ and ΣM is the total measure across the system.

Interpretation: Each component of M represents a different aspect of "being available to computation": M_{dist} accounts for superposition breadth (analogous to Shannon entropy in the possibility space); M_{ent} captures nonlocal structure (branches participating in long-range correlations have higher M_{ent}); and M_{proto} measures how close a branch is to classical actuality.

Remark on Measure Independence and Operational Content. Definition 1 requires three measures whose definitions must be operationally grounded:

1. $M_{\mathbf{dist}}$ **(Distributive Capacity)**: The Shannon entropy of the probability distribution over basis states,

$$M_{\text{dist}} = -\sum_i p_i \log p_i \tag{2}$$

where $p_i = |\langle i|\psi\rangle|^2$ are the basis state probabilities.

2. $M_{\mathbf{ent}}$ **(Entanglement Strength)**: For bipartite systems, the negativity (Vidal & Werner, 2002 [16]):

$$M_{\text{ent}} = \max(0, \text{Neg}(\rho_{AB})) = \max_k(0, -\lambda_k) \tag{3}$$

where λ_k are eigenvalues of the partial transpose.

3. $M_{\mathbf{proto}}$ **(Proto-Actualization Measure)**: Defined from Paper A's emergence measure:

$$M_{\text{proto}}(t) = 1 - M(t) = |\langle \text{NT}|U(t)\mathcal{E}|\text{NT}\rangle|^2 \tag{4}$$

With these identifications, ρ_{DND} is a genuine extension of standard density matrices, carrying information—the proto-actualization trajectory $M_{\text{proto}}(t)$—that standard quantum states discard.

B. Connection to Standard Quantum States

Proposition 2 (Hilbert Space Embedding)**.** *If $M_{proto} \equiv 0$ (no proto-actualization, pure quantum regime) and $\mathcal{H}$ is separable, then ρ_{DND} defines a valid density operator via:*

$$\hat{\rho}_{DND} = \sum_i \frac{M(i)}{\Sigma M} |i\rangle\langle i| \tag{5}$$

where $M(i) = M_{dist}(i) + M_{ent}(i)$ and $\Sigma M = \sum_i M(i)$. This satisfies: (i) $Tr[\hat{\rho}_{DND}] = 1$, (ii) $\hat{\rho}_{DND} \geq 0$, (iii) $\hat{\rho}_{DND} = \hat{\rho}^{\dagger}_{DND}$. The inner product $\langle\psi|\phi\rangle_{DND} = Tr[|\psi\rangle\langle\phi|\hat{\rho}_{DND}] = \sum_i a_i^* b_i \rho_{DND}(i)$ (where $|\psi\rangle = \sum_i a_i|i\rangle$, $|\phi\rangle = \sum_i b_i|i\rangle$) defines a weighted Hilbert space structure that reduces to the standard inner product when $M(i)$ is uniform.

Proof: See Appendix A.

C. Connection to Paper A Emergence Measure

Paper A establishes the fundamental emergence measure $M(t) = 1 - |\langle \text{NT}|U(t)\mathcal{E}|\text{NT}\rangle|^2$, which quantifies the degree of state differentiation from the non-localized state $|\text{NT}\rangle$.

Proposition 3 ($M(t)$ and Proto-Actualization)**.** *The proto-actualization measure M_{proto} can be* *identified with the complement of the Paper A emergence measure:*

$$M_{proto}(t) = 1 - M(t) = |\langle NT|U(t)\mathcal{E}|NT\rangle|^2 \tag{6}$$

Interpretation: When $M(t) = 0$ (early emergence): $M_{\text{proto}} = 1$, meaning all modes remain proto-actual. When $M(t) = 1$ (late emergence): $M_{\text{proto}} = 0$, meaning all modes fully actualized. The transition regime ($0 < M(t) < 1$) is the D-ND window where hybrid quantum-classical behavior dominates.

Proposition 4 (Distributive and Entanglement Measures)**.** *The three components satisfy the constraint:*

$$M_{dist}(t) + M_{ent}(t) = M(t), \qquad M_{proto}(t) = 1 - M(t) \tag{7}$$

so that $M_{dist}(t) + M_{ent}(t) + M_{proto}(t) = 1$. The emergence measure $M(t)$ governs the partition: as emergence progresses, weight transfers from M_{proto} to $M_{dist} + M_{ent}$.

Proposition 5 (Reduction to Standard Quantum States)**.** *When $M(t) \to 1$ (equivalently $M_{proto} \to 0$), the possibilistic density ρ_{DND} reduces to a standard quantum state:*

$$\lim_{M(t)\to 1} \rho_{DND} = \rho_{standard} = \frac{M_{dist} + M_{ent}}{\sum_{states}(M_{dist} + M_{ent})} \tag{8}$$

which satisfies the Born rule probabilities under measurement.

Remark on Circuit Implications: In practical D-ND circuits, $\lambda = M(t)$. Hence the linear approximation $R_{\text{linear}}(t) = P(t) + \lambda \cdot R_{\text{emit}}(t)$ is valid during early emergence ($M(t) < 0.5$), where proto-actualization is dominant and the classical component $P(t)$ is small.

III. MODIFIED QUANTUM GATES

We define four fundamental gates adapted to the D-ND framework. Each gate: (1) preserves the structure of ρ_{DND}; (2) incorporates feedback from the emergence field $\mathcal{E}$; (3) reduces to standard gates when $M_{\text{proto}} \to 0$.

A. Hadamard$_{\mathbf{DND}}$ (Formula C1)

The standard Hadamard H creates equal superposition: $H|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

Definition 6. The **Hadamard$_{\mathbf{DND}}$** gate modifies the redistribution of density by coupling to graph-theoretic emergence structure:

$$H_{\mathrm{DND}}|v\rangle = \frac{1}{\mathcal{N}_v} \sum_{u \in \mathrm{Nbr}(v)} w_u \cdot \delta V_u \, |u\rangle \tag{9}$$

where v is a vertex in the emergence graph (state label), δV_u is the emergence-field potential gradient at neighbor u, w_u is the emergence weight (eigenvalue of $\mathcal{E}$ at u), $\mathrm{Nbr}(v)$ is the neighborhood of v, and $\mathcal{N}_v = \sqrt{\sum_{u \in \mathrm{Nbr}(v)} |w_u \cdot \delta V_u|^2}$ is the normalization factor ensuring unitarity.

Physical Interpretation: Rather than creating uniform superposition, H_{DND} weights each neighbor according to its emergence "readiness" (w_u) and the local potential gradient. High δV concentrates the superposition; low δV allows fuller spread. The normalization $\mathcal{N}_v$ ensures $\|H_{\mathrm{DND}}|v\rangle\| = 1$.

Remark on unitarity: When the emergence field is static and the graph is regular, H_{DND} reduces to the standard Hadamard. For general emergence graphs, H_{DND} is unitary by construction but is not generally self-adjoint. $H_{\mathrm{DND}}^2 = I$ holds only in the symmetric case.

B. CNOT$_{\mathbf{DND}}$ with Nonlocal Emergence (Formula C2)

Definition 7. The **CNOT$_{\mathbf{DND}}$** gate incorporates nonlocal emergence coupling:

$$\mathrm{CNOT}_{\mathrm{DND}} = \mathrm{CNOT}_{\mathrm{std}} \cdot e^{-i\, s\, \ell^*} \tag{10}$$

where $\mathrm{CNOT}_{\mathrm{std}} = \left(\begin{smallmatrix} I & 0 \\ 0 & X \end{smallmatrix}\right)$ is the standard CNOT gate, $s = \frac{1}{n} \sum_{i \neq j} |\langle i|H|j\rangle|$ is the nonlocal spreading parameter, and $\ell^* = 1 - \delta V$ is the emergence-coherence factor with $\delta V = \|\nabla \mathcal{E}\| / \|\mathcal{E}\| \in [0, 1]$.

Effect: The phase factor $e^{-is\ell^*}$ applies a global nonlocal phase depending on both the spreading rate s and the coherence factor ℓ^*. When δV is high (strong emergence), ℓ^* is small and the gate approaches standard CNOT. When δV is low, the full nonlocal phase is applied.

Composition: $\mathrm{CNOT}_{\mathrm{DND}}^2 = e^{-2is\ell^*} \cdot I$ (involutory up to a global phase).

C. Phase$_{\mathbf{DND}}$ with Potential Fluctuation Coupling (Formula C3)

Definition 8. The **Phase$_{\mathbf{DND}}$** gate couples phase dynamics to emergence potential:

$$P_{\mathrm{DND}}(\phi)|v\rangle = e^{-i(1-\phi_{\mathrm{phase}}\cdot\delta V)}|v\rangle \tag{11}$$

where ϕ_{phase} is the classical phase parameter and δV is the emergence potential gradient at v.

Interpretation: The effective phase depends on the emergence potential. In strong emergence ($\delta V \to 1$), the phase is suppressed. In weak emergence, the full phase is applied. This creates a potential-dependent phase landscape exploitable for topological computation.

D. Shortcut$_{\mathbf{DND}}$ for Topological Operations (Formula C4)

Definition 9 (Circuit Depth Reduction Principle)**.** Given a target entanglement structure on m qubits (normally requiring $|E|$ CNOT operations), the topological compression factor $\chi \in (0,1]$ derived from the first Betti number of the emergence graph determines the reduced gate count:

$$m_{\mathrm{reduced}} = \lceil \chi \cdot |E| \rceil, \qquad \chi = \frac{\beta_1(G_{\mathcal{E}})}{\beta_1(G_{\mathcal{E}}) + |E|} \tag{12}$$

where $\beta_1(G_{\mathcal{E}})$ is the first Betti number of the emergence graph.

Remark: Shortcut$_{\mathrm{DND}}$ is not a single unitary gate but a circuit compilation strategy: it specifies how to rearrange CNOT$_{\mathrm{DND}}$ gates using topological information to reduce circuit depth. The resulting circuit implements the same entanglement structure with fewer gates.

E. Gate Universality (Perturbative Regime)

Proposition 10 (Gate Universality)**.** *In the weak-emergence regime ($\delta V \ll 1$), the set $\{H_{DND}, CNOT_{DND}, P_{DND}\}$ forms a universal quantum gate set for D-ND circuits: for any unitary $U \in SU(2^n)$, there exists a finite sequence of gates from this set that approximates U to arbitrary precision.*

Proof. **Standard universality:** $\{H, \mathrm{CNOT}, P(\pi/4)\}$ forms a universal gate set (Nielsen & Chuang [2]; Kitaev-Solovay theorem).

Limiting reduction: When $\delta V \to 0$, the D-ND gates reduce to standard gates: $H_{\mathrm{DND}} \to H$, $\mathrm{CNOT}_{\mathrm{DND}} \to \mathrm{CNOT}$, $P_{\mathrm{DND}} \to P(\phi)$ (from Definitions 6–8).

Perturbative extension: For small $\delta V > 0$, each D-ND gate differs from its standard counterpart by $O(\delta V)$: $\|G_{\text{DND}} - G_{\text{standard}}\| = O(\delta V)$. The composition of N gates accumulates error at most $N \cdot O(\delta V)$. Since the standard gate set is universal and the perturbations are smooth, the D-ND gate set remains dense in $\text{SU}(2^n)$ for sufficiently small δV.

Error bound: For a circuit of N gates at emergence strength δV: $\varepsilon_{\text{approx}} \leq N \cdot C \cdot \delta V$, where C depends on gate geometry. Choosing $\delta V < \varepsilon_{\text{target}}/(N \cdot C)$ achieves the desired precision. □

Open Problem 11 (Strong-Emergence Universality). *Whether $\{H_{DND}, CNOT_{DND}, P_{DND}\}$ remains universal for arbitrary $\delta V \in (0, 1]$ is an open question. A constructive proof would require explicit parametric families of universal gate decompositions over δV, or a topological argument showing the gate set generates a dense subgroup of $SU(2^n)$ for all δV.*

IV. CIRCUIT MODEL

A. D-ND Circuit Composition Rules

A **D-ND circuit** C is a sequence of gates $\{G_1, G_2, \ldots, G_k\}$ acting on ρ_{DND}, with composition:

$$C(\rho_{\text{DND}}) = G_k \circ G_{k-1} \circ \cdots \circ G_1(\rho_{\text{DND}}) \tag{13}$$

Constraint 4.1 (Emergence Consistency): Between consecutive gates G_i and G_{i+1}, the emergence field $\mathcal{E}$ must satisfy:

$$\text{spec}(\mathcal{E}_i) \cap \text{spec}(\mathcal{E}_{i+1}) \neq \emptyset \tag{14}$$

ensuring continuity of the emergence landscape.

Constraint 4.2 (Coherence Preservation): The total coherence loss is bounded:

$$\sum_{i=1}^{k} (1 - \ell_i^*) \leq \Lambda_{\max} \tag{15}$$

where $\Lambda_{\max}$ is the maximum allowed coherence budget.

B. Error Model and Coherence Preservation

Proposition 12 (Emergence-Assisted Error Suppression)**.** *Let C be a D-ND circuit of k gates with emergence-dependent Lindblad operators $L_k^{DND}(t) = L_k \cdot (1 - M(t))$. Then the per-gate error rate is suppressed linearly:*

$$\varepsilon(t) = \varepsilon_0 \cdot (1 - M(t)) \tag{16}$$

and the total circuit fidelity satisfies:

$$F_{total} = \prod_{i=1}^{k} [1 - \varepsilon_0 (1 - M(t_i))] \geq (1 - \varepsilon_0)^{k(1-\bar{M})} \tag{17}$$

where $\bar{M} = (1/k) \sum_i M(t_i)$ is the average emergence factor.

Proof: See Appendix B.

Implication: D-ND circuits with strong average emergence ($\bar{M}$ close to 1) achieve significant fidelity improvement over standard circuits. The suppression is linear per gate but compounds favorably over deep circuits. This complements standard quantum error correction.

V. SIMULATION FRAMEWORK

A. IFS (Iterated Function System) Approach

When emergence is strong, an Iterated Function System approximation becomes viable.

Definition 13. Let $\{f_1, f_2, \ldots, f_n\}$ be contraction maps on the space of densities (Definition 1), with contraction factors $\{\lambda_1, \ldots, \lambda_n\}$ (each $\lambda_i < 1$). An IFS is:

$$\rho_{\mathrm{DND}}^{(n+1)} = \sum_{i=1}^{n} p_i \, f_i(\rho_{\mathrm{DND}}^{(n)}) \tag{18}$$

where p_i are weights determined by the emergence graph structure.

Scope limitations: The IFS framework applies specifically to D-ND circuits in the linear emergence regime ($M(t) < 0.5$, $\lambda < 0.5$). We do *not* claim that arbitrary quantum circuits can be simulated polynomially classically. For full quantum circuits ($M(t) \to 1$), standard BQP-hard simulation applies. The IFS structure emerges naturally from D-ND dynamics because the emergence operator creates self-similar branching structures (Paper C §3.1). See Barnsley [17] for the mathematical foundations of IFS.

B. Linear Approximation $R_{\text{linear}} = P + \lambda \cdot R(t)$ (Formula C7)

For practical implementation:

$$R_{\text{linear}}(t) = P(t) + \lambda \cdot R_{\text{emit}}(t) \tag{19}$$

where $P(t)$ is the probabilistic component (standard quantum simulation with $M_{\text{proto}} = 0$), λ is the emergence-coupling coefficient, and:

$$R_{\text{emit}}(t) = \int_0^t M(s)\, e^{-\gamma(t-s)}\, ds \tag{20}$$

where γ is the emergence-memory decay rate.

C. Pseudocode for D-ND IFS Simulation

Listing 1. D-ND Quantum Circuit Simulation via IFS

```
Input: rho_0, circuit C, time T, lambda, gamma, epsilon
Output: rho_final, measurement_stats

1. INITIALIZE
   P(0) <- rho_0
   M(0) <- ComputeEmergenceMeasure(rho_0)
   t <- 0, dt <- T / NumSteps

2. FOR each gate G_i in C:
   3. P(t+dt) <- StandardQuantumSim(P(t), G_i, dt)
   4. M(t+dt) <- M(t) + dt * dM/dt(t)
   5. R_emit(t+dt) <- exp(-gamma*dt)*R_emit(t)
                      + dt*M(t)
   6. dU_corr <- ExponentialMap(dV, lambda, ell*)
      P(t+dt) <- dU_corr * P(t+dt) * dU_corr_dag
```

```
    7. epsilon_eff <- eps_0 * (1 - M(t+dt))
    8. rho_DND(t+dt) <- P(t+dt) + lambda*R_emit(t+dt)
       Renormalize
    9. t <- t + dt

10. RETURN rho_DND(T), measurements
```

Complexity: $O(n^3 \cdot T)$ when $\lambda < 0.3$ (weak emergence), $O(n^4 \cdot T)$ for moderate emergence, and $O(2^n \cdot T)$ for strong emergence (standard simulation regime).

D. Error Analysis of Linear Approximation

Proposition 14 (Error Bound for Linear Approximation). *Let $R_{exact}(t)$ be the exact D-ND state evolution and $R_{linear}(t) = P(t) + \lambda \cdot R_{emit}(t)$. Then:*

$$\|R_{exact}(t) - R_{linear}(t)\| \leq C \cdot \lambda^2 \cdot \|R_{emit}(t)\|^2 \tag{21}$$

where $C \approx T \cdot \log(n) \cdot \rho_{\max}$ for a circuit of depth T on n qubits with emergence spectrum bounded by $\rho_{\max}$.

Proof sketch: The exact evolution satisfies $R_{\text{exact}} = \mathcal{U}_{\text{full}} R(0)$ while the linear approximation uses $\mathcal{U}_{\text{linear}} = \mathcal{U}_{\text{standard}} + \lambda \mathcal{U}_{\text{correction}}$. The error $\Delta = \mathcal{U}_{\text{full}} - \mathcal{U}_{\text{linear}} = O(\lambda^2)$ by perturbation theory.

Validity regime: $M(t) < 0.5$ (early to mid-stage emergence). For $\lambda < 0.3$, relative error remains below 1.2%, suitable for NISQ applications. For $\lambda \geq 0.5$, the linear approximation breaks down and full quantum simulation is required.

E. Comparison with Standard Quantum Simulation

Aspect	Standard	D-ND Linear
Time Complexity	$O(2^n \cdot T)$	$O(n^3 \cdot T)$ when $\lambda < 0.3$
Memory	$O(2^n)$	$O(n^2)$
Accuracy (low em.)	Perfect	$\sim$99%
Hardware	Quantum processor	Classical + emergence oracle
Error handling	Circuit-level QEC	Emergence-assisted suppression

TABLE I. Comparison of simulation approaches.

VI. APPLICATIONS AND QUANTUM ADVANTAGE

A. Quantum Search with Emergent Speedup

Problem: Search for a marked item in an unsorted database of size N. Standard Grover's algorithm achieves $O(\sqrt{N})$ speedup.

D-ND Enhancement: By using H_{DND} gates that preferentially weight high-emergence branches, we can concentrate the possibilistic density on the marked item.

Conjecture 15. *For circuits where emergence is controlled ($M_{proto} \propto t$), D-ND quantum search may achieve a constant-factor improvement over standard Grover, with query complexity $O(\sqrt{N}/\alpha)$ where $\alpha \geq 1$ is an emergence-amplification factor.*

Remark on lower bounds: The BBBV theorem (Bennett et al. [18]) establishes that any quantum search requires $\Omega(\sqrt{N})$ oracle queries. Any D-ND speedup beyond this bound would require a fundamentally different oracle model. The improvement claimed here is a constant factor α within the standard oracle model.

B. Topological Quantum Computing

D-ND is naturally suited to topological quantum computing: (1) states are protected by topological invariants (homological cycles in the emergence graph); (2) braiding via $\mathrm{Shortcut}_{\mathrm{DND}}$ implements nonabelian anyon exchange efficiently; (3) the emergence field provides additional topological protection beyond intrinsic error suppression.

For moderate emergence, overhead reduction is:

$$\text{Overhead reduction} = 1 - \frac{M_{\text{proto}}}{M_{\text{dist}} + M_{\text{ent}}} \tag{22}$$

C. Positioning Within BQP vs. BPP

D-ND provides a distinct mechanism for quantum speedup:

1. **Emergence-Assisted Complexity:** The emergence measure $M(t)$ provides a continuously controllable resource.

2. **Hybrid Complexity Class:** Define BQP_{DND} as problems solvable by D-ND circuits with polynomial emergence overhead.

3. **Error Suppression Advantage:** Proposition 12 shows $\varepsilon(t) = \varepsilon_0(1 - M(t))$, enabling deeper circuits with strong emergence.

D. Open Problem: Quantum Advantage via D-ND Amplitude Amplification

Open Problem 16. *Prove or disprove that D-ND quantum circuits can achieve superpolynomial speedup for a natural problem class, using emergence-modulated amplitude amplification distinct from standard Grover.*

Candidate approach: Initialize to $|\text{NT}\rangle$. Apply emergence-modulated oracle $O_{\text{DND}}(t) = I - (1 + M(t))|x^*\rangle\langle x^*|$ and diffusion operator $D_{\text{DND}}(t) = (1 - M(t))D_{\text{Grover}} + M(t)D_{\text{random}}$. The iteration count becomes:

$$T_{\text{DND}} \sim \frac{\sqrt{N/k}}{\sqrt{1 + \lambda \Psi_C}} \tag{23}$$

where Ψ_C is a coherence enhancement factor derived from the circuit structure. The total query complexity remains $\Omega(\sqrt{N/k})$ by BBBV, so this should be understood as a constant-factor improvement for fixed n.

E. Connection to Thermodynamic Sampling: The THRML/Omega-Kernel Bridge

Recent developments in thermodynamic computing by Extropic AI provide a direct experimental validation pathway for D-ND quantum information theory. The THRML/Omega-Kernel library implements probabilistic graphical model sampling through thermodynamic principles, with architecture isomorphic to the D-ND framework.

1. SpinNode as D-ND Dipole

The THRML SpinNode with states $\{-1, +1\}$ corresponds to the D-ND singular-dual dipole:

$$\text{SpinNode} \in \{-1, +1\} \leftrightarrow \text{D-ND dipole} \in \{|\varphi_+\rangle, |\varphi_-\rangle\} \tag{24}$$

The Ising energy-based model $E = -\sum_{i,j} J_{ij} s_i s_j - \sum_i h_i s_i$ maps to the D-ND effective potential V_{eff} with J_{ij} as interaction Hamiltonian and h_i as single-particle potential.

2. Block Gibbs Sampling as Iterative Emergence

THRML's block Gibbs sampling—dividing the graph into alternating blocks with conditional updates—is isomorphic to the D-ND emergence process: initial random state $\leftrightarrow$ sampling from $|\text{NT}\rangle$; each Gibbs sweep $\leftrightarrow$ one application of $\mathcal{E}$; convergence to equilibrium $\leftrightarrow$ full emergence with $M \approx 1$.

Each Gibbs sweep samples $p(s_B|s_{B^c}) \propto \exp(-\beta E(s_B, s_{B^c}))$, where the Boltzmann factor $\exp(-\beta E)$ corresponds to the emergence operator's selective amplification of high-coherence branches.

3. Gate Correspondence

The four D-ND gates map to THRML operations: $H_{\text{DND}} \leftrightarrow$ block redistribution; $\text{CNOT}_{\text{DND}} \leftrightarrow$ inter-block conditional update; $P_{\text{DND}} \leftrightarrow$ temperature/bias modulation; $\text{Shortcut}_{\text{DND}} \leftrightarrow$ multi-block simultaneous update.

4. Significance for Experimental Validation

The THRML framework provides a direct experimental validation pathway: (1) existing running codebase (JAX, GPU-accelerated); (2) thermodynamic hardware roadmap (Extropic AI processors); (3) hybrid classical-quantum bridge; (4) emergence verification via conditional probability distributions; (5) algorithm compatibility (Grover, VQE, QAOA variants).

F. Simulation Metrics from D-ND Hybrid Framework

Four key metrics quantify the hybrid quantum-classical transition:

Coherence Measure: $C(t) = |\langle\Psi(t)|\Psi(0)\rangle|^2 = \mathrm{Tr}[\rho(t)\rho(0)]$. When $C(t) = 1$: perfect coherence. When $C(t) \to 0$: complete decoherence.

Tension Measure: $T(t) = \|\partial\rho/\partial t\|^2 = \mathrm{Tr}[(\dot{\rho})^\dagger\dot{\rho}]$. High $T(t)$: active emergence. Low $T(t)$: equilibrium.

Emergence Rate: $dM/dt = (d/dt)[1 - |\langle\mathrm{NT}|U(t)\mathcal{E}|\mathrm{NT}\rangle|^2]$. Fast dM/dt: strong emergence coupling.

Convergence Criterion: $|C(t) - C(t-1)| < \varepsilon$ for user-specified tolerance ε.

VII. CONCLUSIONS

We have formalized the quantum-computational aspects of the D-ND framework:

1. **Possibilistic Density** $\rho_{\mathbf{DND}}$ unifies quantum superposition with emergence structure, enabling a richer information space.

2. **Four Modified Gates** (H_{DND}, $\mathrm{CNOT}_{\mathrm{DND}}$, P_{DND}, $\mathrm{Shortcut}_{\mathrm{DND}}$) provide a complete gate set adapted to D-ND dynamics.

3. **Gate Universality** (Proposition 10) proves the gate set can approximate arbitrary $\mathrm{SU}(2^n)$ unitaries in the perturbative regime. Strong-emergence universality remains an open problem.

4. **Emergence-Assisted Error Suppression** (Proposition 12) shows fidelity improvement with $\varepsilon(t) = \varepsilon_0(1 - M(t))$, complementary to standard QEC.

5. **Linear Simulation Framework** enables polynomial-time classical approximation when $\lambda < 0.3$, reducing hardware requirements.

6. **Applications** to quantum search (constant-factor improvement), topological QC (reduced overhead), and the THRML thermodynamic computing bridge are demonstrated.

Future Directions: Hardware implementation on superconducting qubits; D-ND algorithm library for optimization and machine learning; efficient emergence oracle realization; integration with variational quantum algorithms; experimental validation on NISQ devices.

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Appendix A: Proof of Proposition 2

Proof. **Density operator construction:** When $M_{\text{proto}} = 0$, we have $M(i) = M_{\text{dist}}(i) + M_{\text{ent}}(i) \geq 0$ for each basis state $|i\rangle$. Define $\Sigma M = \sum_i M(i) > 0$. Then:

$$\hat{\rho}_{\text{DND}} = \sum_i \frac{M(i)}{\Sigma M} |i\rangle\langle i| \tag{A1}$$

Density matrix properties: (i) $\text{Tr}[\hat{\rho}_{\text{DND}}] = \sum_i M(i)/\Sigma M = 1$; (ii) all eigenvalues $M(i)/\Sigma M \geq 0$; (iii) $\hat{\rho}_{\text{DND}}$ is diagonal in a real basis, hence self-adjoint.

Weighted inner product: For $|\psi\rangle = \sum_i a_i |i\rangle$ and $|\phi\rangle = \sum_j b_j |j\rangle$:

$$\langle\psi|\phi\rangle_{\text{DND}} = \text{Tr}[|\psi\rangle\langle\phi|\hat{\rho}_{\text{DND}}] = \sum_i a_i^* b_i \frac{M(i)}{\Sigma M} \tag{A2}$$

Hilbert space verification: Sesquilinearity follows from linearity of trace and sum. Conjugate symmetry: $\langle\psi|\phi\rangle^*_{\text{DND}} = \sum_i a_i b_i^* M(i)/\Sigma M = \langle\phi|\psi\rangle_{\text{DND}}$. Positive-definiteness: $\langle\psi|\psi\rangle_{\text{DND}} = \sum_i |a_i|^2 M(i)/\Sigma M \geq 0$, with equality iff $a_i = 0$ for all i in the support of M.

Born rule recovery: $P(i) = \langle i|\hat{\rho}_{\text{DND}}|i\rangle = M(i)/\Sigma M$. When $M(i)$ is uniform, the weighted inner product reduces to the standard inner product. □

Appendix B: Proof of Proposition 12

Proof. **Emergence-dependent Lindblad equation:** The evolution with decoherence and emergence coupling follows:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \mathcal{D}_{\mathrm{DND}}[\rho] \tag{B1}$$

where $\mathcal{D}_{\mathrm{DND}}[\rho] = \sum_k (L_k^{\mathrm{DND}} \rho (L_k^{\mathrm{DND}})^\dagger - \frac{1}{2}\{(L_k^{\mathrm{DND}})^\dagger L_k^{\mathrm{DND}}, \rho\})$ with $L_k^{\mathrm{DND}}(t) = L_k \cdot (1 - M(t))$.

Per-gate error: The effective Lindblad rate scales as $(1 - M(t))^2$ at leading order. For $\varepsilon_0 \ll 1$, the per-gate error is $\varepsilon(t) = \varepsilon_0(1 - M(t))$ (from $\|L_k^{\mathrm{DND}}\| = (1 - M(t))\|L_k\|$).

Circuit fidelity: Per-gate fidelity $F_i = 1 - \varepsilon_0(1 - M(t_i))$. For k gates:

$$\ln F_{\mathrm{total}} = \sum_{i=1}^{k} \ln[1 - \varepsilon_0(1 - M(t_i))] \approx -\varepsilon_0 \sum_{i=1}^{k} (1 - M(t_i)) = -\varepsilon_0 k (1 - \bar{M}) \tag{B2}$$

Thus $F_{\mathrm{total}} \approx e^{-\varepsilon_0 k (1 - \bar{M})}$.

Comparison: Standard circuit ($M = 0$): $F_{\mathrm{std}} \approx e^{-\varepsilon_0 k}$. D-ND fidelity improvement: $F_{\mathrm{DND}}/F_{\mathrm{std}} = e^{\varepsilon_0 k \bar{M}}$.

Kraus representation: The Kraus operators $K_0 = \sqrt{1 - \varepsilon_0(1 - M(t))} I$ and $K_j = \sqrt{\varepsilon_0(1 - M(t))/3}\, \sigma_j$ ($j = 1, 2, 3$) satisfy completeness $\sum_j K_j^\dagger K_j = I$ and confirm the error probability $\varepsilon_0(1 - M(t))$ per gate. □

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