Probability and Statistical Inference MATH4001

Cheat Sheet

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Data Type

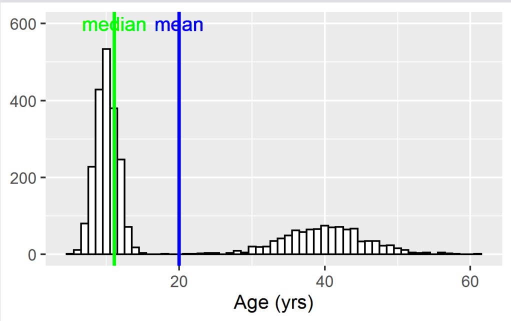
- Categorical
- Interval
- Ordinal
- Ratio

Measures of Location

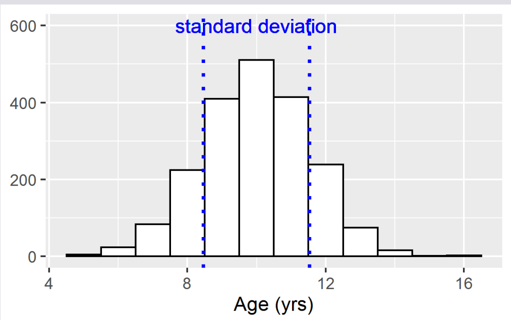
Different aspects of a distribution of data can be summarised by the measures of location:

1. The First Moment: Mean, Mode or Median;
2. The Second Moment: Variance, Standard Deviation;
3. The Third Moment: Skewness.

First Moment: Middle

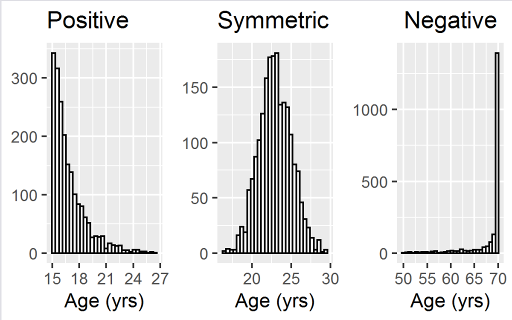


Second Moment: Spread



Measures of Location (cont.)

Third Moment: Symmetry



Mathematical Probability

Definitions

Define some event A that can be the outcome of an experiment.
$\Pr(A)$ is the probability of a given event A will happen.
Rules:

- $\Pr(A)$ is between 0 and 1, $0 \leq Pr(A) \leq 1$;
- $\Pr(A) = 1$, means it will definitely happen;
- $\Pr(A) = 0$, means it will definitely **not** happen;
- $\Pr(A) = 0.05$, is arbitrarily considered unlikely.

Sample Space and Events

The **Sample Space**, S, of an experiment is the universal set of all possible outcomes for that experiment, defined so, no two outcomes can occur simultaneously. For example:

- Throwing a die $S = 1, 2, 3, 4, 5, 6$;
- Tossing two coins $S = HH, TH, HT, TT$.

An event, A, is a subset of the sample space S. For example:

- Throwing a die $S = 3, 4, 6$;
- Tossing two coins $S = TH, TT$.

Axioms of Probabilities

For an event A subset S associated a number $Pr(A)$, the probability of A, which must have the following properties

- $\Pr(A \bigcap B) = 0$; $\Pr(A \bigcup B) = \Pr(A) + \Pr(B)$;
- Probability of the Null Event $\Pr(\emptyset) = 0$;
- The probability of the complement of A, $\Pr(\bar{A}) = 1 - \Pr(A)$;
- $\Pr(A \bigcup B) = \Pr(A) + \Pr(B) - \Pr(A \bigcap B)$.

Conditional Probability

The Conditional Probability $\Pr(A|B)$ denotes the probability of the event A occurring given that the event B has occurred,

$$\Pr(A|B) = \frac{\Pr(A \bigcap B)}{\Pr(B)}.$$

Example: The rain in Ireland

A normal probability would be what is the probability it is going to rain $\Pr(\text{rain})$.
A conditional probability would, be what is the probability it is going to rain **given** that you are in Ireland, $\Pr(\text{rain}|\text{Ireland})$,

$$\Pr(\text{rain}|\text{Ireland}) = \frac{\Pr(\text{rain} \bigcap \text{Ireland})}{\Pr(\text{Ireland})},$$

where the probability of rain is $\Pr(\text{rain}) = 0.3$, the probability of being in Ireland is $\Pr(\text{Ireland}) = 0.4$) and the probability of being in Ireland and it raining is $\Pr(\text{rain} \bigcap \text{Ireland}) = 0.2$,

$$\Pr(\text{rain}|\text{Ireland}) = \frac{0.2}{0.4} = 0.5,$$

You could be interested in the probability that you are in Ireland **given** that it is raining,

$$\Pr(\text{Ireland}|\text{rain}) = \frac{\Pr(\text{rain} \bigcap \text{Ireland})}{\Pr(\text{rain})} = \frac{0.2}{0.4} = 0.75.$$

Bayes Theorem

Bayes Theorem states

$$\Pr(A|B) = \frac{\Pr(B|A)P(A)}{\Pr(B)}.$$

Example: Diagnostic test

The probability that an individual has a rare disease is $\Pr(\text{Disease}) = 0.01$. The probability that a diagnostic test results in a positive (+) test *given you have* the disease is $\Pr(+|\text{Disease}) = 0.95$. On the other hand, the probability that the diagnostic test results in a positive (+) test *given you do not have* the disease is $\Pr(+|\text{No Disease}) = 0.1$. This raises the important question if you are given a positive diagnosis, what is the probability you have the disease $\Pr(\text{Disease}|+)$? From Bayes Theorem we have:

$$\Pr(\text{Disease}|+) = \frac{\Pr(+|\text{Disease})\ \Pr(\text{Disease})}{\Pr(+)}$$

The probability of a positive test is,

$$\Pr(+) = \Pr(+|\text{Disease})\ \Pr(\text{Disease}) + \Pr(+|\text{No Disease})\ \Pr(\text{No Disease}),$$

$$\Pr(+) = 0.1085.$$

$$\Pr(\text{Disease}|+) = \frac{\Pr(+|\text{Disease})\ \Pr(\text{Disease})}{\Pr(+)} = \frac{0.95 \times 0.01}{0.1085} = 0.0875576.$$

This can also be done in a simple table format, by assume a population of 10,000

Group	+ Diagnosis	- Diagnosis	Total
Disease	95	5	100
No Disease	990	8,910	9,900
Total	1,085	8,915	10,000

From the table we can calculate the same answer, $\Pr(\text{Disease}|+) = \frac{95}{1085}$.

Discrete Distribution

Probability Mass Functions

i	0	1	2	3	4
x_i	-1	0	1	2	3
$\Pr(x_i)$	0.3	0.1	0.3	0.1	0.2

The expected value of the distribution is:

$$\mu = E[X] = \Sigma_i x_i p(x_i),$$

$\Sigma_i x_i p(x_i) = -1 \times 0.4 + 0 \times 0.1 + 1 \times 0.3 + 0.1 \times 2 + 0.2 \times 3 = 0.7,$

The variance of the distribution is:

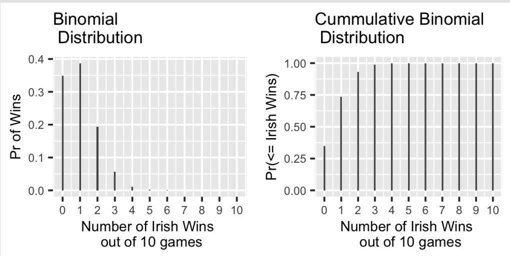
$$Var[X] = \Sigma_i (x_i - \mu)^2 p(x_i) = \Sigma_i (x_i - 0.7)^2 p(x_i).$$

Binomial Distribution

The formula for the Binomial distribution is:

$$\Pr(k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \ldots n,$$

$$E[k] = np, \quad Var[k] = npq.$$

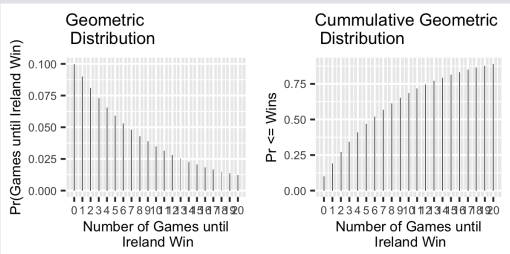


Geometric Distribution

The formula for the Geometric distribution is:

$$\Pr(k) = q^{(k-1)} p, \quad k = 1, 2, \ldots$$

$$E[k] = \frac{1}{p}, \quad Var[k] = \frac{q}{p^2}.$$



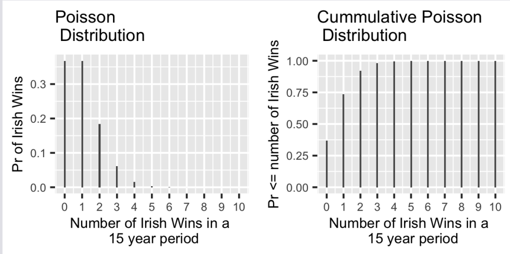
Discrete Distribution

Poisson Distribution

The formula for the Poisson distribution is:

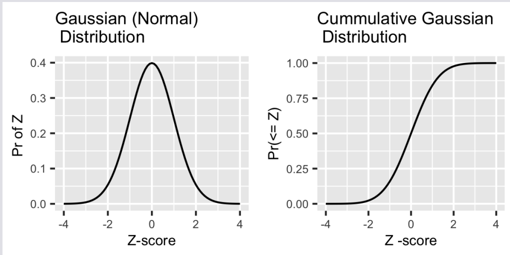
$$\Pr(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

$$E[k] = \lambda, \quad Var[k] = \lambda.$$

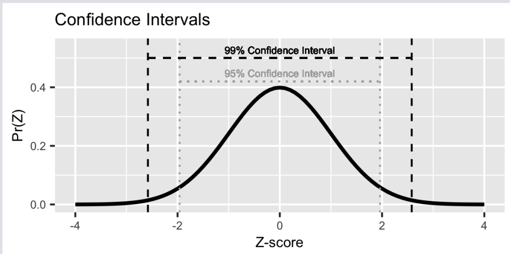


Continuous Distribution

Normal Distribution



Confidence Intervals



Hypothesis Testing

Five steps for Hypothesis testings

1. State the Null Hypothesis H_0;
2. State an Alternative Hypothesis H_{alpha};
3. Calculate a Test Statistic (see below);
4. Calculate a p-value and/or set a rejection region;
5. State your conclusions.

z-test

Continuous Data

The test statistic is given by

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1),$$

where $\bar{x}$ is the observed mean, μ is the historical mean, σ is the standard deviation and n is the number of observations.

Do supplements make you faster?

The effect of a food supplements on the response time in rats is of interest to a biologist. They have established that the normal response time of rats is 1.2 seconds. The $n = 100$ rats were given a new food supplements. The following summary statistics were recorded from the data $\bar{x} = 1.05$ and $\sigma = 0.5$ seconds

1. The rats in the study are the same as normal rats, $H_0 : \bar{x} = 1.2$.
2. The rats are different, $H_\alpha \neq 1.2$.
3. Calculate a Test Statistic $Z = \frac{1.05 - 1.2}{\frac{0.5}{\sqrt{100}}} = -3$
4. Reject the Null hypothesis h_0 if $Z < -1.96$ and $Z > 1.96$
5. The data suggests that rats are faster with the new food.

Proportional Data

The test statistic is given by

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \sim N(0, 1).$$

where $\hat{p}$ is the observed proportion, p is the historical proportion, q is the complement $q = 1 - p$, and n is the number of observations.

t-test

paired t-test

The test statistic is given by

$$t = \frac{\bar{x} - \bar{\mu}_0}{\frac{s}{\sqrt{n}}} \sim t_{\alpha, df}$$

where $\bar{x}$ is the observed mean, μ_0 is the null mean, s is the standard deviation and n is the number of observations.

unpaired t-test

The test statistic is given by

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{2}{n}}} \sim t_{\alpha, df}$$

where $s_p = \sqrt{\frac{s_{x_1}^2 + s_{x_2}^2}{2}}$ is the pooled sample standard deviation, $\bar{x}_1$ and $\bar{x}_2$ are the sample means, n_1 and n_2 are the sample sizes.

χ^2 Independence test

The test statistic to test if data are independent of group is given by:

$$\chi^2_{Ind} = \sum \frac{(O-E)^2}{E} \sim \chi^2_{(r-1)(c-1)}.$$

where O is the observed data, E is the expected data if independent, r is the number of rows and c is the number of columns.

Does ice-cream flavour matter?

An ice-cream company had 500 people sample one of three different ice-cream flavours and asked them to say whether they liked or disliked the ice-cream.

	Vanilla	Chocolate	Strawberry
Liked	130	170	100
Disliked	20	30	50

The χ^2_{Ind} independence test could be used to determine if the enjoyment of the ice-cream depends on the flavour.

χ^2 Goodness of Fit

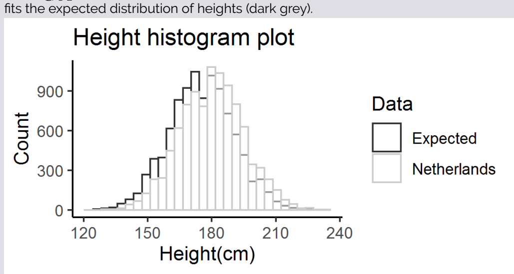
The test statistic to test if data come from a specific distribution is given by:

$$\chi^2_{GoF} = \sum \frac{(O-E)^2}{E} \sim \chi^2_{k-1},$$

where O is the observed data, E is the expected data from a chosen distribution and k is the number of observation bins.

Does it fit?

The χ^2_{GoF} can test if the observed distribution of the height of Dutch people (grey) fits the expected distribution of heights (dark grey).



Linear Regression

A linear regression is used to model a linear relationship of the dependent variable y and the regressors x_1, x_2, ...

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots,$$

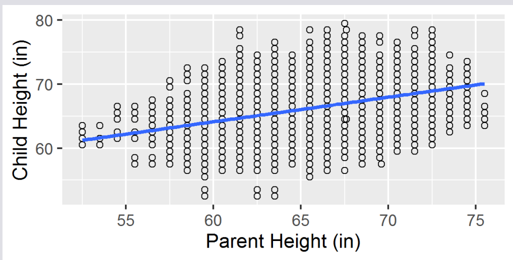
where β_0, β_1 are the slopes of the regressors.

Height Prediction

A simple linear regression (correlation) is used to predict the height of 744 children y using the height of their parent x,

$$y = \beta_0 + \beta_1 x.$$

The plot below shows the fit of the model:



The parents' height x explained 12.7% of the childrens' height y.

Logistic Regression

A logistic regression (or logit model) is used to model the probability of a binary events such as win/lose. The general formula for the Logistic regression is

$$p_i = \frac{e^{\eta}}{1+e^{\eta}},$$

where

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots$$

and β is the slope corresponding to the predictor variable x.

Sexton Conversion Rate

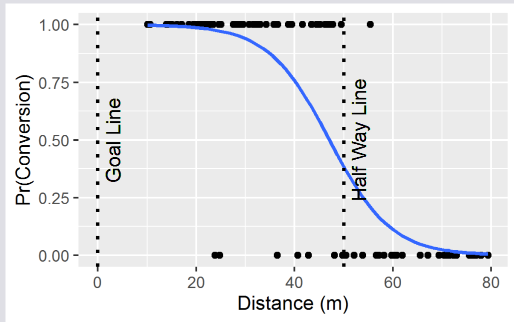
Data from 1000 conversions kicks by Johnny Sexton was acquired; the distance (m) from the goal-line and if the kick was a miss 0 or a conversion 1. The data was fit to a logistic regression. The model was

$$p = \frac{e^{\eta}}{1+e^{\eta}},$$

where

$$\eta = \beta_0 + \beta_1 \text{Distance}$$

and p is the probability of a conversion. The plot below shows the fit of the model:



The model predicts that at the half-way line (50m) Sexton has a 0.375 probability of conversion.

References

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4. Butler, J. S., Course GitHub, https://github.com/john-s-butler-dit/Probability_and _Statistical_Inference
5. Butler, J. S., Probability and Statistical Inference website https://sites.google.com/dit.ie/math4001