Solving Differential Equations representing Simple Harmonic Motion

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Simple Harmonic Motion

1.1 Introduction: Periodic Motion

There are two basic ways to measure time: by duration or periodic motion. Early clocks measured duration by calibrating the burning of incense or wax, or the flow of water or sand from a container. Our calendar consists of years determined by the motion of the sun; months determined by the motion of the moon; days by the rotation of the earth; hours by the motion of cyclic motion of gear trains; and seconds by the oscillations of springs or pendulums. In modern times a second is defined by a specific number of vibrations of radiation, corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

Sundials calibrate the motion of the sun through the sky, including seasonal corrections. A clock escapement is a device that can transform continuous movement into discrete movements of a gear train. The early escapements used oscillatory motion to stop and start the turning of a weight-driven rotating drum. Soon, complicated escapements were regulated by pendulums, the theory of which was first developed by the physicist Christian Huygens in the mid 17th century. The accuracy of clocks was increased and the size reduced by the discovery of the oscillatory properties of springs by Robert Hooke. By the middle of the 18th century, the technology of timekeeping advanced to the point that William Harrison developed timekeeping devices that were accurate to one second in a century.

1.1.1 Simple Harmonic Motion

One of the most important examples of periodic motion is simple harmonic motion (SHM), in which some physical quantity varies sinusoidally. Suppose a function of time has the form of a sine wave function,

$$y\left(t\right) = A\sin\left(\frac{2\,\pi t}{T}\right) \tag{1.1}$$

where $A > 0$ is the amplitude (maximum value). The function $y(t)$ varies between $+A$ and $-A$, because a sine function varies between $+1$ and -1. A plot of $y(t)$ vs. time is shown in Figure 1.1

The sine function is periodic in time. This means that the value of the function at time t will be exactly the same at a later time $t = t + T$, where T is the period. The **frequency**, f, is defined to be

$$f = \frac{1}{T} \tag{1.2}$$

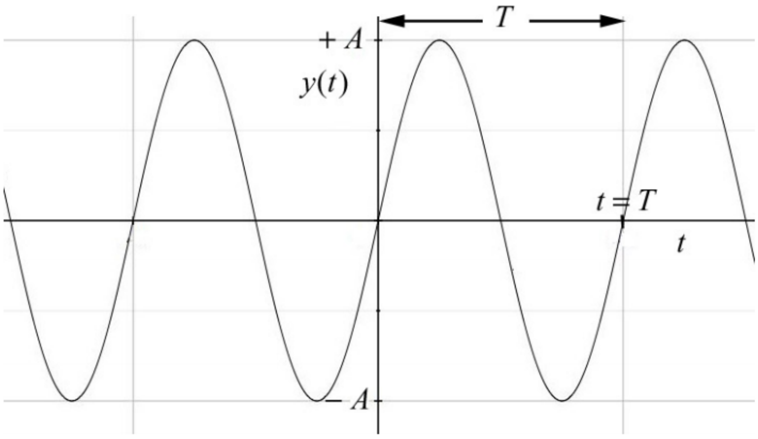

Figure 1.1: Sinusoidal function of time

The SI unit of frequency is inverse seconds, hertz [Hz]. The angular frequency of oscillation is defined to be

$$\omega = \frac{2\,\pi}{T} = 2\,f\pi \tag{1.3}$$

and is measured in radians per second. We therefore have several different mathematical representations for sinusoidal motion

$$y\,(t) = A\sin\left(\frac{2\,\pi t}{T}\right) = A\sin\left(2\,f\pi t\right) = A\sin\left(\omega t\right) \tag{1.4}$$

1.2 Simple Harmonic Motion: Analytic approach

Our first example of a system that demonstrates simple harmonic motion is a spring object system on a frictionless surface, shown in Figure 1.2

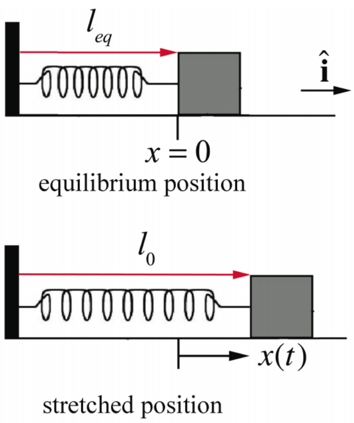

Figure 1.2: Spring-object system

The object is attached to one end of a spring. The other end of the spring is attached to a wall at the left in Figure 1.2. Assume that the object undergoes one-dimensional motion. The spring has a spring constant k and equilibrium length l_{eq} . Choose the origin at the equilibrium position and choose the positive x -direction to the right in the Figure 1.2. In the figure, $x > 0$ corresponds to an extended spring, and $x < 0$ to a compressed spring. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The force acting on the spring is a linear restoring force, $F_x = kx$ (Figure 1.3). The initial conditions are as follows. The spring is initially stretched a distance l_0 and given some initial speed v_0 to the right away from the equilibrium position. The initial position of the stretched spring from the equilibrium position (our choice of origin) is $x_0 = (l_0 - l_{eq}) > 0$ and its initial x -component of the velocity is $v_{x,0} = v_0 > 0$.

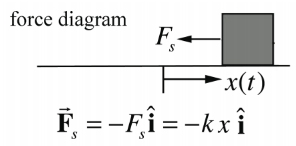

Figure 1.3: Free-body force diagram for spring-object system

Newtons Second law in the x -direction becomes

$$-kx\,(t) = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \tag{1.5}$$

This equation of motion, Eq. 1.5, is called the **simple harmonic oscillator equation** (SHO). Because the spring force depends on the distance x , the acceleration is not constant. Eq. 1.5 is a second order linear differential equation, in which the second derivative of the dependent variable is proportional to the negative of the dependent variable,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{kx\,(t)}{m} \tag{1.6}$$

In this case, the constant of proportionality is k/m,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{kx\,(t)}{m} = 0 \tag{1.7}$$

Since mass and spring constant cannot be negative and zero so, solving the above differential equation, we get,

$$x = K_2 \cos\left(\frac{\sqrt{k}t}{\sqrt{m}}\right) + K_1 \sin\left(\frac{\sqrt{k}t}{\sqrt{m}}\right) \tag{1.8}$$

For $m = 1$ and $k = 1$, then the graph is shown by Figure 1.4. Here red, green, yellow and violet curves represents the initial or boundary conditions [0, 1, 0], [1, 1, 0], [0, 0, 1] and [1, 1, 1] respectively. Putting these initial conditions in Eq. 1.8 one by one. Here t corresponds to first term, x corresponds to second term and $\frac{\mathrm{d}x}{\mathrm{d}t}$ corresponds to third term in the initial conditions.

1.3 Numericals

Example 1. *A spring at rest is suspened from the ceiling wthout mass. A 2 kg weight is then attached to this spring, stretching it 9.8 cm. From a position 2/3 m above equilibrium the weigth is give a downward velocity of 5 m/s. Find the equation of motion. What is the amplitude? At what times the mass first equilibrium?*

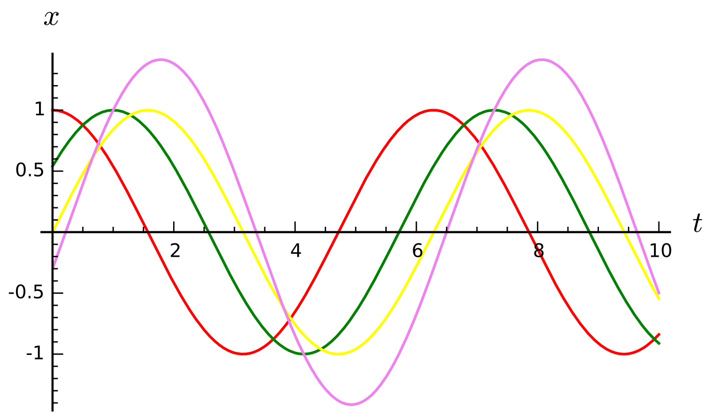

Figure 1.4: Displacement Time graph

Sol. Note $m = 2kg$, $x = 9.8cm$, and $k = mg/x = 200.0$. Therefore, the general solution $x = K_2 \cos(10\,t) + K_1 \sin(10\,t)$. Then by computing the above equation from the initial conditions $x(0) = -2/3$ (down is positive, up is negative), $x'(0) = 5$ we get,

$$x = -\frac{2}{3}\cos(10\,t) + \frac{1}{2}\sin(10\,t)$$

Now we write this in the more compact and useful form

$$x = A\sin(\omega t + \phi) = K_2\cos(\omega t) + K_1\sin(\omega t)$$

where $A = \sqrt{K_1^2 + K_2^2}$ denotes the *amplitude*

$$A = \frac{5}{6}$$

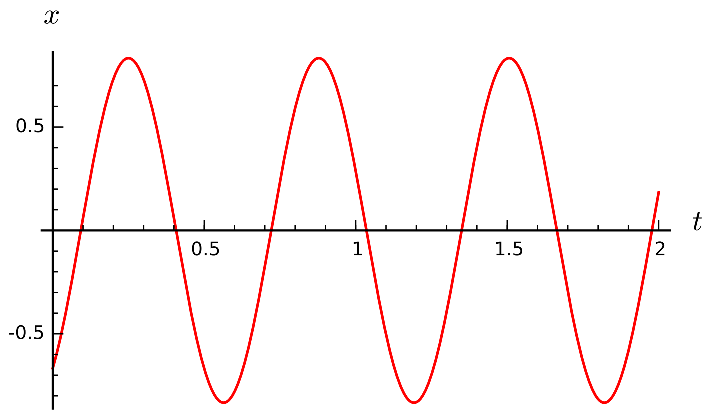

Figure 1.5: Displacement Time graph

Example 2. *A particle of mass m is executing simple harmonic motion along x axis under the action of a force $F = -kx$ with a period of 16 sec. In the course of motion it crosses the equilibrium position at t = 2sec and acquire a velocity of $4m/sec$ at $t = 4sec$. Find the equation of motion and the amplitude of oscillation.*

Sol. Let the equation be $x = a\sin\left(\omega(t-2)\right)$ where a= amplitude of motion. Now

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a\omega\cos\left(\omega(t-2)\right) \tag{1.9}$$

$$4 = a\omega\cos\left(\omega(t-2)\right) \tag{1.10}$$

$$a = \frac{4\,\cos\left(2\,\omega\right)}{\omega} \tag{1.11}$$

$$\omega = \frac{2\,\pi}{T} = \frac{1}{8}\,\pi = \frac{1}{8}\,\pi \tag{1.12}$$

$$\therefore a = \frac{32\,\sqrt{2}}{\pi} \tag{1.13}$$

$$x = \frac{32\,\sqrt{2}\sin\left(\frac{1}{8}\,\pi(t-2)\right)}{\pi} \tag{1.14}$$