Supplementary Material

For the manuscript "Yemin Wu, Shuai Lu, Wei Gu, et al., Distributed Closed-Loop Decision Framework for Unit Commitment".

This material serves as a supplement to the manuscript "Distributed Closed-Loop Decision Framework for Unit Commitment" and provides additional details not included in the main paper due to page limitations. **Appendix A** provides the detailed formulation of the Unit Commitment (UC) model. **Appendix B** details the Re-dispatch (RD) model. **Appendix C** presents the proof for Proposition 1. Finally, **Appendix D** provides the comprehensive proof for Proposition 2, establishing the convergence properties of the proposed algorithm.

Appendix A
Detailed UC model

The objective of the UC model, shown in (A.1), is to minimize the total system operational cost. This cost comprises the generation, no-load, start-up, and shut-down costs of thermal units, as well as penalties for RES curtailment and load shedding. The model is subject to several constraints. Constraints (A.2)–(A.5) define the generation and load limits. Ramping constraints (A.6) limit the rate of change of power output. The logical relationship between a unit's commitment states and its minimum on/off times is enforced by (A.7)–(A.9). System-wide power balance is maintained by (A.10). Transmission network security is enforced by (A.11). Finally, constraints (A.12)–(A.15) ensure sufficient upward and downward spinning reserve is available. The sets for thermal units, RES units, buses, transmission branches, and time periods are denoted by $\mathcal{I}$, $\mathcal{J}$, $\mathcal{Q}$, $\mathcal{B}$, and $\mathcal{T}$, respectively.

$$\begin{aligned}
\min_{\mathbf{x},\mathbf{z}} \sum_{t\in\mathcal{T}} \Bigg[& \sum_{i\in\mathcal{I}} (C_i^g P_{i,t} + C_i^{nl} I_{i,t} + C_i^{su} U_{i,t} + C_i^{sd} D_{i,t}) \\
& + C_t^{curt}(\hat{W}_t^{total} - \sum_{j\in\mathcal{J}} W_{j,t}) + \sum_{q\in\mathcal{Q}} C_{q,t}^{ls} S_{q,t}^{ls} \Bigg] \qquad \text{(A.1)}
\end{aligned}$$

$$\text{wherein} \quad \mathbf{x} = \{P_{i,t}, W_{j,t}, S_{q,t}^{ls}, R_{i,t}^{sru}, R_{i,t}^{srd}\}$$

$$\mathbf{z} = \{I_{i,t}, U_{i,t}, D_{i,t}\}$$

$$\text{s.t.} \quad P_{i,t} + R_{i,t}^{sru} \le \overline{P}_i I_{i,t}, \forall i \in \mathcal{I}, t \in \mathcal{T} \qquad \text{(A.2)}$$

$$P_{i,t} - R_{i,t}^{srd} \ge \underline{P}_i I_{i,t}, \forall i \in \mathcal{I}, t \in \mathcal{T} \qquad \text{(A.3)}$$

$$0 \le W_{j,t} \le \hat{W}_{j,t}, \forall j \in \mathcal{J}, t \in \mathcal{T} \qquad \text{(A.4)}$$

$$0 \le S_{q,t}^{ls} \le L_{q,t}, \forall q \in \mathcal{Q}, t \in \mathcal{T} \qquad \text{(A.5)}$$

$$-RD_i \le P_{i,t} - P_{i,t-1} \le RU_i, \forall i \in \mathcal{I}, t \in \mathcal{T} \qquad \text{(A.6)}$$

$$I_{i,t} - I_{i,t-1} = U_{i,t} - D_{i,t}, \forall i \in \mathcal{I}, t \in \mathcal{T} \qquad \text{(A.7)}$$

$$\sum_{t'=t-T_i^{on}+1}^{t} U_{it'} \le I_{it}, \forall i \in \mathcal{I}, t \in \{T_i^{on}, ..., |\mathcal{T}|\} \qquad \text{(A.8)}$$

$$\sum_{t'=t-T_i^{off}+1}^{t} D_{it'} \le 1 - I_{it}, \forall i \in \mathcal{I}, t \in \{T_i^{off}, ..., |\mathcal{T}|\} \qquad \text{(A.9)}$$

$$\sum_{i\in\mathcal{I}} P_{i,t} + \sum_{j\in\mathcal{J}} W_{j,t} + \sum_{q\in\mathcal{Q}} S_{q,t}^{ls} = \sum_{q\in\mathcal{Q}} L_{q,t}, \forall t \in \mathcal{T} \qquad \text{(A.10)}$$

$$\begin{aligned}
-\overline{P}_b \le & \sum_{i\in\mathcal{I}} f_{i,b} P_{i,t} + \sum_{j\in\mathcal{J}} f_{j,b} W_{j,t} \\
& - \sum_{q\in\mathcal{Q}} f_{q,b}(L_{q,t} - S_{q,t}^{ls}) \le \overline{P}_b, \forall b \in \mathcal{B}, t \in \mathcal{T} \qquad \text{(A.11)}
\end{aligned}$$

$$0 \le R_{i,t}^{sru} \le RU_i I_{i,t}, \forall i \in \mathcal{I}, t \in \mathcal{T} \qquad \text{(A.12)}$$

$$0 \le R_{i,t}^{srd} \le RD_i I_{i,t}, \forall i \in \mathcal{I}, t \in \mathcal{T} \qquad \text{(A.13)}$$

$$\sum_{i\in\mathcal{I}} R_{i,t}^{sru} \ge SRU_t^{req}, \forall t \in \mathcal{T} \qquad \text{(A.14)}$$

$$\sum_{i\in\mathcal{I}} R^{srd}_{i,t} \geq SRD^{req}_t, \forall t \in \mathcal{T} \tag{A.15}$$

APPENDIX B
DETAILED RE-DISPATCH MODEL

The re-dispatch model is subject to several constraints. The real-time output of thermal units must respect their maximum and minimum generation limits (B.2)–(B.3). The upward and downward power adjustments are constrained by the scheduled upward and downward spinning reserves in (B.4)–(B.5). Constraints on RES generation and load shedding are given by (B.6) and (B.7). The model also enforces generator ramping limits (B.8), system-wide power balance (B.9), and transmission network security (B.10).

$$\min_{\mathbf{y}} \sum_{t\in\mathcal{T}} \left[\sum_{i\in\mathcal{I}} (\gamma_i^+ p^+_{i,t} + \gamma_i^- p^-_{i,t}) + \sum_{q\in\mathcal{Q}} C^{ls}_{q,t} S^{ls,RD}_{q,t} \right] \tag{B.1}$$

$$\text{wherein} \quad \mathbf{y} = \{p^+_{i,t}, p^-_{i,t}, W^{RD}_{j,t}, S^{ls,RD}_{q,t}\}$$

$$\text{s.t.} \quad P^*_{i,t} + p^+_{i,t} - p^-_{i,t} \leq \overline{P}_i I^*_{i,t}, \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{B.2}$$

$$P^*_{i,t} + p^+_{i,t} - p^-_{i,t} \geq \underline{P}_i I^*_{i,t}, \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{B.3}$$

$$0 \leq p^+_{i,t} \leq R^{sru,*}_{i,t}, \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{B.4}$$

$$0 \leq p^-_{i,t} \leq R^{srd,*}_{i,t}, \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{B.5}$$

$$0 \leq W^{RD}_{j,t} \leq \tilde{W}_{j,t}, \forall j \in \mathcal{J}, t \in \mathcal{T} \tag{B.6}$$

$$0 \leq S^{ls,RD}_{q,t} \leq L_{q,t}, \forall q \in \mathcal{Q}, t \in \mathcal{T} \tag{B.7}$$

$$\begin{aligned} &-RD_i \leq (P^*_{i,t} + p^+_{i,t} - p^-_{i,t}) \\ &\quad - (P^*_{i,t-1} + p^+_{i,t-1} - p^-_{i,t-1}) \leq RU_i, \forall i \in \mathcal{I}, t \in \mathcal{T} \end{aligned} \tag{B.8}$$

$$\begin{aligned} &\sum_{i\in\mathcal{I}} (P^*_{i,t} + p^+_{i,t} - p^-_{i,t}) + \sum_{j\in\mathcal{J}} W^{RD}_{j,t} + \sum_{q\in\mathcal{Q}} S^{ls,RD}_{q,t} \\ &\quad = \sum_{q\in\mathcal{Q}} L_{q,t}, \forall t \in \mathcal{T} \end{aligned} \tag{B.9}$$

$$\begin{aligned} &-\overline{P}_b \leq \sum_{i\in\mathcal{I}} f_{i,b}(P^*_{i,t} + p^+_{i,t} - p^-_{i,t}) + \sum_{j\in\mathcal{J}} f_{j,b} W^{RD}_{j,t} \\ &\quad - \sum_{q\in\mathcal{Q}} f_{q,b}(L_{q,t} - S^{ls,RD}_{q,t}) \leq \overline{P}_b, \forall b \in \mathcal{B}, t \in \mathcal{T} \end{aligned} \tag{B.10}$$

APPENDIX C
PROOF OF PROPOSITION 1

Proof. Denote the optimal solution of model (15) as $(\boldsymbol{\theta}^*, \{\mathbf{x}^*_i, \mathbf{z}^*_i, \mathbf{y}^*_i, \forall i \in \mathcal{I}\})$. First, we assume that this solution is not feasible for the model (16). Since the constraints on $(\boldsymbol{\theta}, \mathbf{x}_i, \mathbf{z}_i)$ are identical in both models, $(\boldsymbol{\theta}^*, \{\mathbf{x}^*_i, \mathbf{z}^*_i, \forall i \in \mathcal{I}\})$ must be feasible for model (16). Let the corresponding feasible solution of (16) be $(\boldsymbol{\theta}^*, \{\mathbf{x}^*_i, \mathbf{z}^*_i, \mathbf{y}^{**}_i, \forall i \in \mathcal{I}\})$, wherein

$$\mathbf{y}^{**}_i = \underset{\mathbf{y}\in\mathcal{Y}(\mathbf{x}^*_i, \mathbf{z}^*_i, \tilde{\mathbf{w}}_i)}{\text{argmin}} \quad \mathbf{c}_3^\top \mathbf{y}, \forall i \in \mathcal{I}.$$

By the definition of $\arg\min$ we have $\mathbf{c}_3^\top \mathbf{y}^{**}_i \leq \mathbf{c}_3^\top \mathbf{y}^*_i, \forall i \in \mathcal{I}$. Since we assume that $(\boldsymbol{\theta}^*, \{\mathbf{x}^*_i, \mathbf{z}^*_i, \mathbf{y}^*_i, \forall i \in \mathcal{I}\})$ is not feasible for model (16), we have $\exists i \in \mathcal{I}, \mathbf{y}^*_i \neq \mathbf{y}^{**}_i$, and thus:

$$\begin{aligned} &\sum_{i\in\mathcal{I}} \left(\mathbf{c}_4^\top \mathbf{x}^*_i + \mathbf{c}_2^\top \mathbf{z}^*_i + \mathbf{c}_3^\top \mathbf{y}^{**}_i - f^*_{opt,i} \right) + \lambda \|\boldsymbol{\theta}^*\| \\ &< \sum_{i\in\mathcal{I}} \left(\mathbf{c}_4^\top \mathbf{x}^*_i + \mathbf{c}_2^\top \mathbf{z}^*_i + \mathbf{c}_3^\top \mathbf{y}^*_i - f^*_{opt,i} \right) + \lambda \|\boldsymbol{\theta}^*\|. \end{aligned}$$

Note that $(\boldsymbol{\theta}^*, \{\mathbf{x}^*_i, \mathbf{z}^*_i, \mathbf{y}^{**}_i, \forall i \in \mathcal{I}\})$ should also be feasible for model (15) since model (15) is a relaxed problem of the model (16), and thus it is the optimum of the model (15), which contradicts with the fact that $(\boldsymbol{\theta}^*, \{\mathbf{x}^*_i, \mathbf{z}^*_i, \mathbf{y}^*_i, \forall i \in \mathcal{I}\})$ is the optimum of the model (15). Therefore, the assumption that $(\boldsymbol{\theta}^*, \{\mathbf{x}^*_i, \mathbf{z}^*_i, \mathbf{y}^*_i, \forall i \in \mathcal{I}\})$ is not feasible for model (16) does not hold. Furthermore, since model (16) is a tightened edition of model (15), the optimal solution of (15) $(\boldsymbol{\theta}^*, \{\mathbf{x}^*_i, \mathbf{z}^*_i, \mathbf{y}^*_i, \forall i \in \mathcal{I}\})$ must also be the optimum of the former. Therefore, model (16) reduces to its high-point problem (15). ∎

Appendix D
Proof of Proposition 2

Proof. We establish the proof through the following lemmas:

Lemma D.1. *Let $\rho > 0$. If the original optimal solution of $\boldsymbol{\beta}$-fixed problem exists and is finite, then the ADMM converges to its global optima.*

Proof of Lemma D.1. The $\boldsymbol{\beta}$-fixed problem is an LP with finite original optima and $\rho > 0$. Thus, the convergence conclusions of standard ADMM [1] apply directly. ∎

The proof of Lemma D.2–D.4 follows the classical convergence proof of the alternating convex search (ACS) [2].

Lemma D.2. *If the global optimal solutions of the $\boldsymbol{\alpha}$-fixed problem and the $\boldsymbol{\beta}$-fixed problem can always be obtained, the sequence $\{v^{\boldsymbol{P}}_{\boldsymbol{MP}}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)\}_{n\in\mathbb{N}}$ generated by OBCD converges monotonically.*

Proof of Lemma D.2. Solving the $\boldsymbol{\alpha}$-fixed problem, we have

$$\begin{aligned} v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_{n+1}, \hat{\boldsymbol{\gamma}}_{n+1}) \leq v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}, \boldsymbol{\gamma}), \\ \forall(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \Omega^{\boldsymbol{\beta}} \times \Omega^{\boldsymbol{\gamma}}. \end{aligned} \tag{D.1}$$

Solving the $\boldsymbol{\beta}$-fixed problem, we have

$$\begin{aligned} v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{n+1}, \boldsymbol{\beta}_{n+1}, \boldsymbol{\gamma}_{n+1}) \leq v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{n+1}, \boldsymbol{\gamma}), \\ \forall(\boldsymbol{\alpha}, \boldsymbol{\gamma}) \in \Omega^{\boldsymbol{\alpha}} \times \Omega^{\boldsymbol{\gamma}}. \end{aligned} \tag{D.2}$$

By (D.1) and (D.2), we have

$$v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{n+1}, \boldsymbol{\beta}_{n+1}, \boldsymbol{\gamma}_{n+1}) \leq v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) \tag{D.3}$$

Therefore $\{v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)\}_{n\in\mathbb{N}}$ is non-increasing and $v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is bounded from below by its optimal value. Thus below-bounded non-increasing sequence $\{v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)\}_{n\in\mathbb{N}}$ generated by OBCD monotonically converge to a finite value. ∎

Remark D.1. *Although Lemma D.2 guarantees the convergence of the function value sequence $\{v^{\boldsymbol{P}}_{\boldsymbol{MP}}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)\}_{n\in\mathbb{N}}$, the variable sequence $\{\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \boldsymbol{\gamma}_n\}_{n\in\mathbb{N}}$ does not necessarily converge. Lemma D.3–D.4 discusses the convergence results about the variable sequence.*

Definition D.1 (Partial optimum for OBCD-ADMM). *Let $v^{\boldsymbol{P}}_{\boldsymbol{MP}} : \Omega^{\boldsymbol{\alpha}} \times \Omega^{\boldsymbol{\beta}} \times \Omega^{\boldsymbol{\gamma}} \to \mathbb{R}$ be a given function and let $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*) \in \Omega^{\boldsymbol{\alpha}} \times \Omega^{\boldsymbol{\beta}} \times \Omega^{\boldsymbol{\gamma}}$. A point $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ is called a partial optimum of $v^{\boldsymbol{P}}_{\boldsymbol{MP}}$ if it satisfies the following conditions:*

1) *Fixed-$\boldsymbol{\alpha}$ optimality*

$$v^{\boldsymbol{P}}_{\boldsymbol{MP}}(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*) \leq v^{\boldsymbol{P}}_{\boldsymbol{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^*, \boldsymbol{\gamma}), \forall \boldsymbol{\alpha} \in \Omega^{\boldsymbol{\alpha}},\ \boldsymbol{\gamma} \in \Omega^{\boldsymbol{\gamma}}$$

2) *Fixed-$\boldsymbol{\beta}$ optimality*

$$v^{\boldsymbol{P}}_{\boldsymbol{MP}}(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*) \leq v^{\boldsymbol{P}}_{\boldsymbol{MP}}(\boldsymbol{\alpha}^*, \boldsymbol{\beta}, \boldsymbol{\gamma}), \forall \boldsymbol{\beta} \in \Omega^{\boldsymbol{\beta}},\ \boldsymbol{\gamma} \in \Omega^{\boldsymbol{\gamma}}$$

Remark D.2. *Note that in OBCD-ADMM, the definition of partial optimum is stronger than that in traditional ACS [2], and the former is a sufficient condition for the latter. When we define $\mathcal{B} = (\boldsymbol{\beta}, \boldsymbol{\gamma})$ and replace $(\boldsymbol{\beta}^*, \boldsymbol{\gamma})$ in the fixed-$\boldsymbol{\alpha}$ optimality condition with $\mathcal{B}^*$, definition D.1 will degenerate into the definition of partial optimum in [2]. Therefore, the conclusion regarding partial optimum in [2] is also applicable here.*

Lemma D.3. *Let the global optimal solutions of the $\boldsymbol{\alpha}$-fixed problem and the $\boldsymbol{\beta}$-fixed problem can always be obtained. Let the convergence threshold of OBCD-ADMM $\epsilon_2 = 0$. If the optimal solution of $\boldsymbol{\beta}$-fixed problem is unique, then for every convergent function value sequence $\{v^{\boldsymbol{P}}_{\boldsymbol{MP}}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)\}_{n\in\mathbb{N}}$ (assuming convergence at the $\tilde{n}$th iteration), the corresponding point $\boldsymbol{\chi}_{\tilde{n}} \coloneqq (\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}})$ is a partial optimum of $\boldsymbol{MP}^{\mathbf{P}}$ (20) for arbitrary but fixed $\kappa \geq 0$.*

Proof of Lemma D.3. Since $\epsilon_2 = 0$, for every convergent sequence $\{(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)\}_{n\in\mathbb{N}}$, we have $v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\chi}_{\tilde{n}}) = v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\chi}_{\tilde{n}-1})$. By (D.1) and (D.2), we can further get

$$\begin{aligned} v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\chi}_{\tilde{n}}) &= v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}) = v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}-1}, \boldsymbol{\beta}_{\tilde{n}}, \hat{\boldsymbol{\gamma}}_{\tilde{n}}) \\ &= v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}-1}, \boldsymbol{\beta}_{\tilde{n}-1}, \boldsymbol{\gamma}_{\tilde{n}-1}) = v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\chi}_{\tilde{n}-1}) \end{aligned} \tag{D.4}$$

By (D.1), we have

$$\begin{aligned} v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}-1}, \boldsymbol{\beta}_{\tilde{n}}, \hat{\boldsymbol{\gamma}}_{\tilde{n}}) \leq v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}-1}, \boldsymbol{\beta}, \boldsymbol{\gamma}), \\ \forall(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \Omega^{\boldsymbol{\beta}} \times \Omega^{\boldsymbol{\gamma}}. \end{aligned} \tag{D.5}$$

By (D.2), we have

$$v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}) \leq v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}), \forall(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \Omega^{\boldsymbol{\beta}} \times \Omega^{\boldsymbol{\gamma}}. \tag{D.6}$$

Since the optimal solution of the $\boldsymbol{\beta}$-fixed problem is unique, by (D.4), we have

$$\boldsymbol{\alpha}_{\tilde{n}} = \boldsymbol{\alpha}_{\tilde{n}-1},\ \boldsymbol{\gamma}_{\tilde{n}} = \hat{\boldsymbol{\gamma}}_{\tilde{n}}. \tag{D.7}$$

By (D.5), (D.6), and (D.7), we can further obtain

$$\begin{cases} v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}) \le v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}, \boldsymbol{\gamma}), \forall (\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \Omega^{\boldsymbol{\beta}} \times \Omega^{\boldsymbol{\gamma}}, \\ v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}) \le v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}), \forall (\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \Omega^{\boldsymbol{\beta}} \times \Omega^{\boldsymbol{\gamma}}, \end{cases} \tag{D.8}$$

which satisfy the definition of the partial optimum. Thus, $\boldsymbol{\chi}_{\tilde{n}}$ is a partial optimum of $\mathbf{MP}^{\mathbf{P}}$ (20). ■

Remark D.3. *Note that in Lemma D.3, the convergence of the sequence $(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)_{n\in\mathbb{N}}$ is still not theoretically guaranteed. Fortunately, we show that when the objective value converges (assuming convergence at the $\tilde{n}$th iteration), the corresponding point $\boldsymbol{\chi}_{\tilde{n}}$ is a partial minimum. Furthermore, according to (D.7), we can observe that $\boldsymbol{\alpha}$ itself converges. This implies that once the OBCD-ADMM convergence condition is met, the parameter training of meta-learners (i.e., $\boldsymbol{\theta}$) in the current R&D iteration is complete and will no longer update. This level of convergence is sufficient for the practical deployment of the DCLD framework.*

Lemma D.4. *Following the Lemma D.3, if, in addition, $\boldsymbol{\chi}_{\tilde{n}}$ is feasible for **MP** (19), i.e., the penalty term*

$$\kappa \sum_{i\in\mathcal{I}} \sum_{l=1}^{k} \left| \boldsymbol{\phi}_{i,l} - \boldsymbol{\xi}_{i,l}^{\top} \mathbf{d}(\hat{\mathbf{w}}_i(\boldsymbol{\theta})) \right| = 0,$$

*then the convergence point $\boldsymbol{\chi}_{\tilde{n}}$ is a stationary point of the original **MP** (19).*

Proof of Lemma D.4. First, we prove that $\boldsymbol{\chi}_{\tilde{n}} = (\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}})$ is also a partial minimum of model (19). We denote the penalty term as $P(\boldsymbol{\phi}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \kappa)$, and denote the objective of $\mathbf{MP}^{\mathbf{P}}$ (20) as $v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}; \kappa) = v_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) + P(\boldsymbol{\phi}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \kappa)$. Let $\tilde{\Omega}^{\boldsymbol{\alpha}} \subseteq \Omega^{\boldsymbol{\alpha}}, \tilde{\Omega}^{\boldsymbol{\gamma}} \subseteq \Omega^{\boldsymbol{\gamma}}$ are the feasible regions of $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ in model **MP**, respectively. Thus, $(\boldsymbol{\alpha}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}), \boldsymbol{\alpha} \in \tilde{\Omega}^{\boldsymbol{\alpha}}, \boldsymbol{\gamma} \in \tilde{\Omega}^{\boldsymbol{\gamma}}$ is feasible for model (19), then we have

$$\begin{aligned} v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}; \kappa) &= v_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}) + P(\boldsymbol{\phi}, \boldsymbol{\alpha}, \boldsymbol{\beta}_{\tilde{n}}, \kappa) \\ &= v_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}), \forall \boldsymbol{\alpha} \in \tilde{\Omega}^{\boldsymbol{\alpha}},\ \boldsymbol{\gamma} \in \tilde{\Omega}^{\boldsymbol{\gamma}} \end{aligned} \tag{D.9}$$

Since $(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}})$ is a partial optimum of $\mathbf{MP}^{\mathbf{P}}$ (20), then it holds

$$v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}; \kappa) \le v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}; \kappa), \forall \boldsymbol{\alpha} \in \tilde{\Omega}^{\boldsymbol{\alpha}},\ \boldsymbol{\gamma} \in \tilde{\Omega}^{\boldsymbol{\gamma}}. \tag{D.10}$$

Since $(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}})$ is feasible for **MP**, then we have

$$\begin{aligned} v^{\mathbf{P}}_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}; \kappa) &= v_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}) + P(\boldsymbol{\phi}, \boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \kappa) \\ &= v_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}) \end{aligned} \tag{D.11}$$

By (D.9), (D.10), and (D.11), we have

$$v_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}) \le v_{\mathbf{MP}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}), \forall \boldsymbol{\alpha} \in \tilde{\Omega}^{\boldsymbol{\alpha}},\ \boldsymbol{\gamma} \in \tilde{\Omega}^{\boldsymbol{\gamma}}. \tag{D.12}$$

Similarly, we have

$$v_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}}) \le v_{\mathbf{MP}}(\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}, \boldsymbol{\gamma}), \forall \boldsymbol{\beta} \in \tilde{\Omega}^{\boldsymbol{\beta}},\ \boldsymbol{\gamma} \in \tilde{\Omega}^{\boldsymbol{\gamma}}. \tag{D.13}$$

Thus, $\boldsymbol{\chi}_{\tilde{n}} = (\boldsymbol{\alpha}_{\tilde{n}}, \boldsymbol{\beta}_{\tilde{n}}, \boldsymbol{\gamma}_{\tilde{n}})$ is a partial minimum of model (19). Since $v_{\mathbf{MP}}$ is a biconvex and differentiable function, Corollary 4.3 in [2] applies directly. Then $\boldsymbol{\chi}_{\tilde{n}}$ is also a stationary point of the original **MP** (19). ■

Lemma D.5. *Let $\epsilon_1 = 0$ and $\mathbf{Z}$ be the collection of feasible combinations of binary variables. Assume that the BiMIP problem (15) is feasible and has a finite optimum. For the convergence of the outer loop R&D algorithm:*

1) *If the global optimum of **MP** is obtained at each R&D iteration, the algorithm converges to the global optimal solution of the BiMIP (15) within $O(|\mathbf{Z}|)$ iterations.*
2) *If a stationary point of **MP** is obtained at each R&D iteration (via Lemma D.4), R&D converges to a feasible BiMIP solution within $O(|\mathbf{Z}|)$ iterations.*

Proof of Lemma D.5. We first prove Statement 1. Assume the R&D algorithm has not terminated (i.e., $LB < UB$). Let $\boldsymbol{\mathcal{Z}}^{\dagger}_{k} := \{\mathbf{z}^{\dagger}_1, \mathbf{z}^{\dagger}_2, \ldots, \mathbf{z}^{\dagger}_k\}$ be the set of subproblem solutions in the first k R&D iterations. We show that each iteration produces a new $\mathbf{z}^{\dagger}_{k+1} \notin \boldsymbol{\mathcal{Z}}^{\dagger}_{k}$. Consider the two cases for $\mathbf{SP}_2$:

- *Case 1:* $\mathbf{SP}_2$ *is feasible.* By Proposition 8 in [3], if the algorithm generates a $\mathbf{z}^{\dagger}_{k+1}$ previously derived at iteration $k_0 < k+1$, i.e., $\mathbf{z}^{\dagger}_{k+1} \in \boldsymbol{\mathcal{Z}}^{\dagger}_{k}$, then $LB \ge UB$. Thus, when $LB < UB$, $\mathbf{z}^{\dagger}_{k+1}$ must be new, i.e., $\mathbf{z}^{\dagger}_{k+1} \notin \boldsymbol{\mathcal{Z}}^{\dagger}_{k}$.

- *Case 2:* $\mathbf{SP}_2$ *is infeasible.* By Remark 3, $\mathbf{z}^{\dagger}_{k+1}$ is obtained from $\mathbf{SP}_1$. Suppose for contradiction that $\mathbf{z}^{\dagger}_{k+1}$ was generated at a prior iteration $k_0 < k+1$. Then, the optimality cuts for $l = k+1$ are already in the **MP** and its feasible region is unchanged, iteration $k+1$ yields the same solution as iteration k:

$$\begin{aligned}&(\mathbf{x}^*_{k+1}, \mathbf{y}^*_{k+1}, \mathbf{z}^*_{k+1}, \hat{\mathbf{w}}^*_{k+1}, \boldsymbol{\theta}^*_{k+1}, \boldsymbol{\xi}^*_{k+1}) \\ &\qquad = (\mathbf{x}^*_k, \mathbf{y}^*_k, \mathbf{z}^*_k, \hat{\mathbf{w}}^*_k, \boldsymbol{\theta}^*_k, \boldsymbol{\xi}^*_k).\end{aligned} \tag{D.14}$$

At iteration k, $\mathbf{z}^{\dagger}_{k+1}$ is optimal for $\mathbf{SP}_1$, and we have:

$$\begin{aligned}v_{\mathbf{SP1},k} &= \mathbf{c}_1^{\top} \hat{\mathbf{x}}^*_k + \mathbf{c}_2^{\top} \mathbf{z}^{\dagger}_{k+1} \\ &= \mathbf{c}_2^{\top} \mathbf{z}^{\dagger}_{k+1} + \hat{\boldsymbol{\xi}}^{*\top}_k (\mathbf{d}(\hat{\mathbf{w}}(\boldsymbol{\theta}^*_k)) - \mathbf{B}\mathbf{z}^{\dagger}_{k+1}).\end{aligned} \tag{D.15}$$

Since the BiMIP (15) is feasible, the **MP** solution at the $k+1$th iteration must satisfy all constraints, including:

$$\begin{cases}\mathbf{c}_1^{\top} \mathbf{x}^*_{k+1} + \mathbf{c}_2^{\top} \mathbf{z}^*_{k+1} \\ \qquad \le \mathbf{c}_2^{\top} \mathbf{z}^{\dagger}_{k+1} + \boldsymbol{\xi}^{*\top}_{k+1} (\mathbf{d}(\hat{\mathbf{w}}(\boldsymbol{\theta}^*_{k+1})) - \mathbf{B}\mathbf{z}^{\dagger}_{k+1}) \\ \mathbf{y}^*_{k+1} \in \mathcal{Y}(\mathbf{x}^*_{k+1}, \mathbf{z}^*_{k+1}, \tilde{\mathbf{w}}) \\ (\mathbf{x}^*_{k+1}, \mathbf{z}^*_{k+1}) \in \mathcal{M}(\hat{\mathbf{w}}(\boldsymbol{\theta}^*_{k+1})).\end{cases} \tag{D.16}$$

By (D.14)–(D.16), and strong duality conditions, we have

$$\begin{cases}\mathbf{c}_1^{\top} \mathbf{x}^*_{k+1} + \mathbf{c}_2^{\top} \mathbf{z}^*_{k+1} \le v_{\mathbf{SP1},k} \\ \mathbf{y}^*_{k+1} \in \mathcal{Y}(\mathbf{x}^*_{k+1}, \mathbf{z}^*_{k+1}, \tilde{\mathbf{w}}) \\ (\mathbf{x}^*_{k+1}, \mathbf{z}^*_{k+1}) \in \mathcal{M}(\hat{\mathbf{w}}(\boldsymbol{\theta}^*_k)).\end{cases} \tag{D.17}$$

Note that (D.17) is identical to the constraints in $\mathbf{SP}_2$ at iteration k. Thus $\mathbf{SP}_2$ must be feasible at iteration k, contradicting its infeasibility. By the contradiction, $\mathbf{z}^{\dagger}_{k+1} \notin \mathcal{Z}^{\dagger}_k$.

In both cases, if $LB < UB$, a new $\mathbf{z}^{\dagger}_{k+1} \notin \mathcal{Z}^{\dagger}_k$ is generated. Since $|\mathbf{Z}|$ is finite, termination occurs within $O(|\mathbf{Z}|)$ iterations with $LB \ge UB$, achieving global optimality.

Next, we prove Statement 2. Let LB be the global optimum value of **MP**, which is also a valid lower bound to BiMIP (15), and LB' the objective value at a stationary point. It is obvious that $LB' \ge LB$. If $\mathbf{z}^{\dagger}_{k+1} \in \mathcal{Z}^{\dagger}_k$, then we have $UB \le LB \le LB'$ (whether $\mathbf{SP}_2$ is feasible or not), satisfying termination. Thus, a new $\mathbf{z}^{\dagger}_{k+1} \notin \mathcal{Z}^{\dagger}_k$ is generated if the termination condition is not satisfied. Similar to statement 1, the termination occurs within $O(|\mathbf{Z}|)$ iterations, achieving a feasible BiMIP solution. ∎

Lemma D.6. *Let $\mathcal{Z}^{\dagger}_k := \{\mathbf{z}^{\dagger}_1, \mathbf{z}^{\dagger}_2, \ldots, \mathbf{z}^{\dagger}_k\}$ be the set of subproblem solutions obtained in the first k iterations of R&D. Let a stationary point of **MP** is obtained at each R&D iteration (via Lemma D.4). If $\mathbf{z}^{\dagger}_{k+1} \in \mathcal{Z}^{\dagger}_k$, then the R&D in the kth iteration can converge to a stationary point of BiMIP's equivalent single-level strong-duality reformulation.*

Proof of Lemma D.6. Let stationary point of **MP** constructed over $\mathcal{Z}^{\dagger}_k$ be $\zeta(\mathcal{Z}^{\dagger}_k) = (\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \boldsymbol{\theta}^*, \boldsymbol{\xi}^*_1, \ldots, \boldsymbol{\xi}^*_k)$, and let $\mathbf{Z}$ represent the complete set of feasible binary variable combinations. Given $\boldsymbol{\theta}^*$, one can extend this solution to $\zeta(\mathbf{Z}) = \left(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \boldsymbol{\theta}^*, \boldsymbol{\xi}1^*, \ldots, \boldsymbol{\xi}|\mathbf{Z}|^*\right)$ using the KKT conditions. To establish that $\zeta(\mathbf{Z})$ is a stationary point of the full-space reformulation $\mathbf{MP}(\mathbf{Z})$, assume the contrary. Suppose $\zeta(\mathbf{Z})$ is not a stationary point, then some constraints of $\mathbf{MP}(\mathbf{Z})$ are violated. In such a case, the subproblem $\mathbf{SP_2}$ in iteration $k+1$ would obtain $\mathbf{z}'$ as a new feasible solution, implying $\mathbf{z}^{\dagger}_{k+1} \in \mathbf{Z} \setminus \mathcal{Z}^{\dagger}_k$, which contradicts the assumption that $\mathbf{z}^{\dagger}_{k+1} \in \mathcal{Z}^{\dagger}_k$. This completes the proof. ∎

Collectively, these lemmas construct the proof hierarchically. Lemmas D.1–D.4 establish that each master problem within the R&D framework is solved to a stationary point. Building upon this, Lemmas D.5–D.6 further guarantee that the outer R&D algorithm converges to the stationary point of the single-level strong-duality-based reformulation of the model (15) within $O\,|\mathbf{Z}|$ iterations. This completes the proof. ∎

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