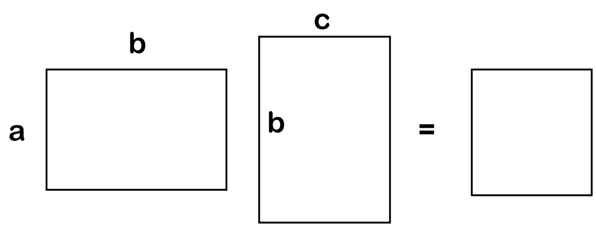
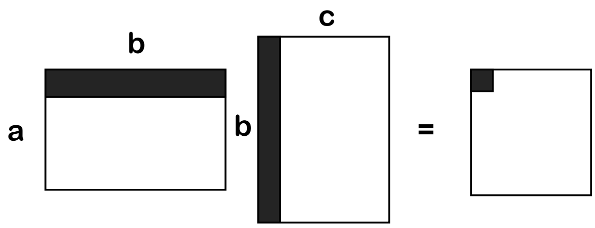
Matrix Chain Multiplication

matrix multiplication



A = a x b matrix
B = b x c matrix

Multiplying the Matrix



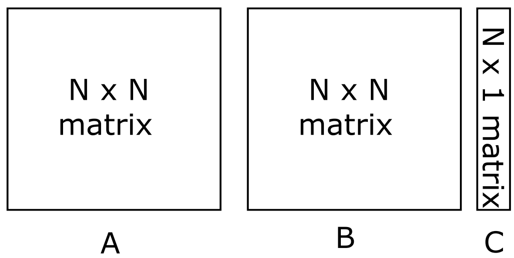
Time used = $\Theta(abc)$

Naïve Method

```
for (i = 1; i <= a;i++) {
  for (j = 1; i <= c;j++) {
     sum = 0;
     for (k = 1;k <= b;k++) {
       sum += A[i][k] * B[k][j];
     }
     C[i][j] = sum;
  }
}
```

O(abc)

Matrix Chain Multiplication



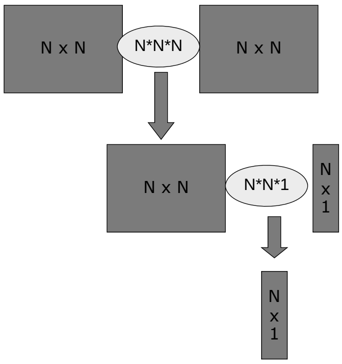
How to compute ABC ?

Matrix Multiplication

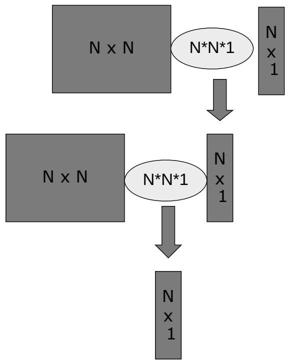
- ABC = (AB)C = A(BC)
- (AB)C differs from A(BC)?
  - Same result, different efficiency

- What is the cost of (AB)C?
- What is the cost of A(BC)?

(AB)C



A(BC)



The Problem

- Input: $a_1, a_2, a_3, \ldots, a_n$
  - n-1 matrices of sizes
  - $a_1 \times a_2$ B_1
  - $a_2 \times a_3$ B_2
  - $a_3 \times a_4$ B_3
  - ....
  - $a_{n-1} \times a_n$ B_{n-1}
- Output
  - The order of multiplication
  - How to parenthesize the chain

Example

INPUT

- a_1 a_2 a_3 a_4 a_5 a_6
- 10 x 5 x 1 x 5 x 10 x 2

B_1 B_2 B_3 B_4 B_5

Possible Output

$((B_1B_2)(B_3B_4))B_5$

$(B_1B_2)((B_3B_4)B_5)$

$(B_1((B_2B_3)B_4))B_5$

And much more…

Consider the Output

What do

$(B_1B_2)((B_3B_4)B_5)$

$(B_1B_2)(B_3(B_4B_5))$

have in common?

What do

$((B_1B_2)(B_3B_4))B_5$

$(((B_1B_2)B_3)B_4))B_5$

have in common?

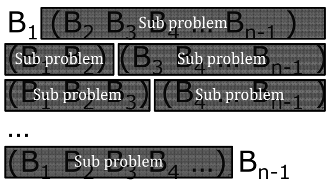
Solving $B_1\ B_2\ B_3\ B_4 \ldots B_{n-1}$

Min cost of

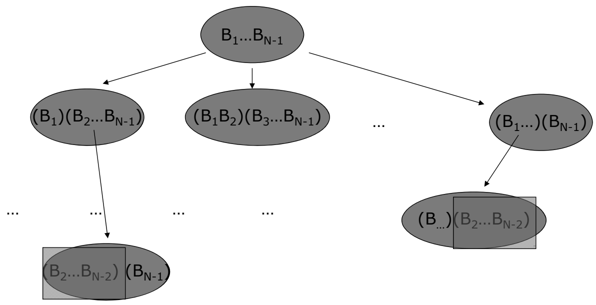
$B_1\ (B_2\ B_3\ B_4 \ldots B_{n-1})$
$(B_1\ B_2)\ (B_3\ B_4 \ldots B_{n-1})$
$(B_1\ B_2\ B_3)\ (B_4 \ldots B_{n-1})$
…
$(B_1\ B_2\ B_3\ B_4 \ldots)\ B_{n-1}$

Solving $B_1\ B_2\ B_3\ B_4 \ldots B_{n-1}$

Min cost of



Matrix Chain Multiplication



Deriving the Recurrent

- mcm(l,r)
  - The least cost to multiply $B_l \ldots B_r$
- The solution is mcm(1,n-1)

The Recurrence

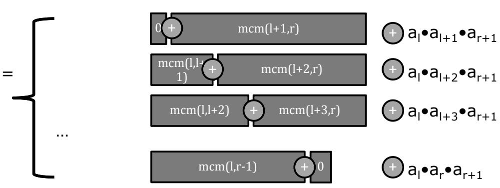
- Initial Case
  - mcm(x,x) = 0
  - mcm(x,x+1) = a[x] * a[x+1] * a[x+2]

The Recurrence

$$\mathrm{mcm}(l,r) = \min \text{ of } \begin{cases} \text{min cost of } B_l\ (B_{l+1}\ B_{l+2}\ B_{l+3} \ldots B_r) & + a_l \bullet a_{l+1} \bullet a_{r+1} \\ \text{min cost of } (B_l\ B_{l+1})\ (B_{l+2}\ B_{l+3} \ldots B_r) & + a_l \bullet a_{l+2} \bullet a_{r+1} \\ \text{min cost of } (B_l\ B_{l+1}\ B_{l+2})\ (B_{l+3} \ldots B_r) & + a_l \bullet a_{l+3} \bullet a_{r+1} \\ \ldots & \\ \text{min cost of } (B_l\ B_{l+1}\ B_{l+2}\ B_{l+3} \ldots)\ B_r & + a_l \bullet a_r \bullet a_{r+1} \end{cases}$$

The Recurrence

mcm(l,r) = min of



Matrix Chain Multiplication

```
int mcm(int l,int r) {
   if (l < r) {
       minCost = MAX_INT;
       for (int i = l;i < r;i++) {
         my_cost = mcm(l,i) + mcm(i+1,r) + (a[l] * a[i+1] * a[r+1]);
         minCost = min(my_cost,minCost);
       }
       return minCost;
   } else {
     return 0;
   }
}
```

Using bottom-up DP

- Design the table
- M[i,j] = the best solution (min cost) for multiplying $B_i \ldots B_j$
- The solution is at M[i,n-1]

What is M[i,j]?

- Trivial case
  - What is m[x,x] ?
  - No multiplication, m[x,x] = 0

What is M[i,j]?

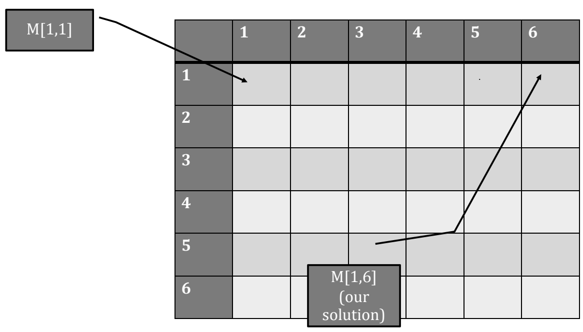
- Simple case
  - What is m[x,x+1] ?
  - $B_x B_{x+1}$
  - Only one solution = $a_x * a_{x+1} * a_{x+2}$

What is M[i,j]?

- General case
  - What is m[x,x+k] ?
  - $B_x B_{x+1} B_{x+2} \ldots B_{x+k}$

$$
\text{min of} \left\{ \begin{array}{ll}
Bx\ (B_{x+1}\ B_{x+2}\ B_{x+3}\ \ldots\ B_{x-k}\ ) & +\ a_x \bullet a_{x+1} \bullet a_{x+k+1} \\
(B_x\ B_{x+1})\ (B_{x+2}\ B_{x+3}\ \ldots\ B_{x+k}\ ) & +\ a_x \bullet a_{x+2} \bullet a_{x-k+1} \\
(B_x\ B_{x+1}\ B_{x+2})\ (B_{x+3}\ \ldots\ B_{x+k}\ ) & +\ a_x \bullet a_{x+3} \bullet a_{x+k+1} \\
\ldots & \\
(B_x\ B_{x+1}\ B_{x+2}\ B_{x+3}\ \ldots)\ B_{x+k} & +\ a_x \bullet a_{x+k} \bullet a_{x+k+1}
\end{array} \right.
$$

Filling the Table



Filling the Table

Trivial case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Plus $a_1 \bullet a_2 \bullet a_6$

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Plus $a_1 \bullet a_3 \bullet a_6$

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Plus $a_1 \bullet a_4 \bullet a_6$

Filling the Table

Arbitrary case

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Plus $a_1 \bullet a_5 \bullet a_6$

Filling the Table

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Filling the Table

	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

1 2 3 4 5

Example

- a_1 a_2 a_3 a_4 a_5 a_6
- 10 x 5 x 1 x 5 x 10 x 2

  B_1 B_2 B_3 B_4 B_5

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0				
2		0			
3			0		
4				0	
5					0

Example

a_1 a_2 a_3 a_4 a_5 a_6
$\boxed{10 \text{ x } 5 \text{ x } 1}$ x 5 x 10 x 2

	1	2	3	4	5
1	0	50			
2		0			
3			0		
4				0	
5					0

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 $\boxed{\text{x } 5 \text{ x } 1 \text{ x } 5}$ x 10 x 2

	1	2	3	4	5
1	0	50			
2		0	25		
3			0		
4				0	
5					0

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50			
2		0	25		
3			0	50	
4				0	
5					0

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50			
2		0	25		
3			0	50	
4				0	100
5					0

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50			
2		0	25		
3			0	50	
4				0	100
5					0

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50			
2		0	25		
3			0	50	
4				0	100
5					0

Option 1 = 0 + 25 + 10 x 5 x 5 = 275

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50			
2		0	25		
3			0	50	
4				0	100
5					0

Option 2 = 50 + 25 + 10 x 1 x 5 = 100 minimal

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50	100 (2)		
2		0	25		
3			0	50	
4				0	100
5					0

(2) means that the minimal solution is by dividing at B_2

Option 2 = 50 + 25 + 10 x 1 x 5 = 100 minimal

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50	100 (2)		
2		0	25		
3			0	50	
4				0	100
5					0

Option 1 = 0+ 50 + 5x 1 x 10 = 100

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50	100 (2)		
2		0	25		
3			0	50	
4				0	100
5					0

Option 1 = 25+ 0 + 5x 5 x 10 = 275

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50	100 (2)		
2		0	25	100 (2)	
3			0	50	
4				0	100
5					0

Option 1 is better

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50	100 (2)		
2		0	25	100 (2)	
3			0	50	70 (4)
4				0	100
5					0

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50	100 (2)	200 (2)	
2		0	25	100 (2)	
3			0	50	70 (4)
4				0	100
5					0

Example

a_1 a_2 a_3 a_4 a_5 a_6
10 x 5 x 1 x 5 x 10 x 2

	1	2	3	4	5
1	0	50	100 (2)	200 (2)	
2		0	25	100 (2)	80 (2)
3			0	50	70 (4)
4				0	100
5					0

Example

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 10 \times & 5 \times & 1 \times & 5 \times & 10 \times & 2 \end{array}$$

	1	2	3	4	5
1	0	50	100 (2)	200 (2)	140 (2)
2		0	25	100 (2)	80 (2)
3			0	50	70 (4)
4				0	100
5					0