Homework #6: wave equations

Problem 1: Assume that $u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ is the unique smooth solution to

$$\partial_t^2 u - \Delta u = 0, \quad x \in \mathbb{R}^3,\ t > 0$$

with the initial condition

$$u|_{t=0} = f \in C_c^\infty(\mathbb{R}^3), \qquad \partial_t u|_{t=0} = g \in C_c^\infty(\mathbb{R}^3).$$

Show that

$$\lim_{t\to+\infty} \int_{\mathbb{R}^3} \left[|\partial_t u(x,t)|^2 - \left| \frac{x}{|x|} \cdot \nabla u(x,t) \right|^2 \right] dx = 0.$$

Problem 2: Assume that $\lambda \in (0, \infty), \epsilon \in (0,1)$ and that $f \in C_c^\infty(\mathbb{R}^3)$. Solve the Helmholtz equation

$$-\Delta u - \lambda u - i\epsilon u = f$$

for $x \in \mathbb{R}^3$.

Problem 3: Find a suitable boundary condition on $\{(x,t) : x = 0, t > 0\}$ such that solutions to the linear wave equation

$$\partial_t^2 u - \partial_x^2 u = 0,$$

for $x > 0, t > 0$, with $u|_{t=0} = f \in C_c^\infty((0,\infty)), \partial_t u|_{t=0} \equiv 0$, satisfy the "non-reflection" condition that

$$u(x,t) \equiv 0, \quad \text{for sufficiently large } t > 1.$$

Problem 4: Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. Define $A : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ as follows. For any $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$Au = -\mathrm{div}\,\big(a(x)\nabla u\big) \in L^2(\Omega),$$

where $a \in C^\infty(\overline{\Omega})$ and $1 \le a(x) \le 10$ for $x \in \Omega$. Find and prove a min-max characterization for the eigenvalues of the operator A and try to understand the corresponding Weyl's law of asymptotics for $N(\lambda)$ as $\lambda \to +\infty$, which is the number of eigenvalues of A that are smaller than or equal to λ.