Post Calculus

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1 Linear Algebra

§1.1 Eigenvectors and Eigenvalues

Definition 1.1.1. A **homogeneous linear system** is one where $\mathbf{Ax} = \mathbf{0}$.

The trivial solution is when $\mathbf{x} = \mathbf{0}$. Non-trivial solutions exist iff $\det(\mathbf{A}) = 0$.

Theorem 1.1.2

Let $\mathbf{A}$ be a square matrix.

$$\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$$

if and only if $\vec{\mathbf{v}}$ is an eigenvector and λ is an eigenvalue.

We notice that

$$\begin{aligned} \mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} &\implies \mathbf{A}\vec{\mathbf{v}} = \lambda\mathbf{I}\vec{\mathbf{v}} \\ &\implies \vec{\mathbf{v}}\left(\mathbf{A} - \lambda\mathbf{I}\right) = \mathbf{0} \\ &\implies \det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0. \end{aligned}$$

The result above is known as the **characteristeric equation**.

Example 1.1.3

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix},$$

find the eigenvalues and eigenvectors of $\mathbf{A}$.

We first notice that $\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$. Meaning that

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix}\right) = 0,$$

implying that $\lambda^2 + 4 = 0$. Thus, $\lambda = \pm 2i$. For $\lambda_1 = 2i$, we have

$$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix}\vec{\mathbf{v}} = \mathbf{0}.$$

By gaussian elimination we arrive at the first eigenvector

$$\vec{\mathbf{v}_1} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Similarly

$$\vec{\mathbf{v}_2} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

for $\lambda_2 = -2i$.

§1.2 Diagonalization

For $\mathbf{y} = \mathbf{A}\mathbf{x}$, assume $\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$$

be a transformation matix. Let

$$\mathbf{y} = \mathbf{P}\mathbf{y}' \text{ and } \mathbf{x} = \mathbf{P}\mathbf{x}'.$$

We can then write a new linear system relating $\mathbf{y}'$ and $\mathbf{x}'$, giving

$$\begin{aligned} \mathbf{P}\mathbf{y}' = \mathbf{A}\mathbf{P}\mathbf{x}' &\implies \mathbf{P}^{-1}\mathbf{P}\mathbf{y}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{P}\mathbf{x}' \\ &\implies \mathbf{y}' = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\mathbf{x}', \end{aligned}$$

where $\mathbf{D} = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)$ is a diagonal matrix such that the diagonals turn out to be the eigenvalues of $\mathbf{A}$.

> **Remark 1.2.1.** In order to find the diagonalization matrix, begin by finding the eigenvalues and eigenvectors of $\mathbf{A}$. Then find $\mathbf{P}$ from the eigenvetors. The diagonlization matrix follows.

Example 1.2.2

Diagonalize

$$\begin{bmatrix} 7/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 5/4 \end{bmatrix}.$$

§1.3 Rotating Conics

Let Q be a quadratic form such that

$$Q = ax_1^2 + (b+c)x_1x_2 + dx_2^2.$$

We can express this as

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We can convert the quadratic form to the canonical form by using the diagonalized matrix of $\mathbf{A}$:

$$C = \begin{bmatrix} x_1' & x_2' \end{bmatrix} \mathbf{D} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

Additionally, note that the transformation matrix is the rotation matrix, meaning that

$$\mathbf{P} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

Theorem 1.3.1

Let Q_n be a quadratic form with n dimensions and C_n the corresponding canonical form. Then, for an $n \times n$ matrix $\mathbf{A}$ and its diagonalized matrix $\mathbf{D}$,

$$Q_n = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$C_n = \begin{bmatrix} x'_1 & x'_2 & \cdots & x'_n \end{bmatrix} \mathbf{D} \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix}.$$

If any linear terms are present, they can be expressed as the product of the coefficient matrix and the matrix with each variable. So,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Exercise 1.3.2.

2 3D Optimization

§2.1 Partial Derivatives

Let $f(x, y, z) = 2x^2 + 3y^2 + z^2$. Partial derivatives treat the other variables as constants. Thus,

$$\frac{\partial f}{\partial x} = 4x, \ \frac{\partial f}{\partial y} = 6y, \ \frac{\partial f}{\partial z} = 2.$$

Definition 2.1.1. The **gradient vector** for a function f is

$$\nabla f(p) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{bmatrix}.$$

Definition 2.1.2. The **Hessian Matrix** is

$$\mathbf{H}_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix},$$

Theorem 2.1.3 (Schwarz's Theorem)

For a function $f : \Omega \to \mathbb{R}$ defined on a set $\Omega \subset \mathbb{R}^n$, if $\mathbf{p} \in \mathbb{R}^n$ is a point such that some neighborhood of $\mathbf{p}$ is contained in Ω and f has continuous second partial derivatives at the point $\mathbf{p}$, then $\forall i, j \in \{1, 2, \ldots, n\}$

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{p}) = \frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{p}).$$

Proof. The proof is left as a search on wikipedia :p. □

Example 2.1.4

Find the gradient and hessian matrix of $f(x, y, z) = 2x^2 + 3y^2 + z^2$.

Solution. We find that

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix}$$

and

$$\mathbf{H}_f = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

□

Example 2.1.5

Find the gradient and hessian matrix of $f(x, y) = x^2 - 3xy + y^2$.

Solution. We find that

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ 2y - 3x \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}.$$

□

Example 2.1.6

Find the gradient and hessian matrix $f(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 - 6x_2x_3$.

Solution. We find that

$$\nabla f = \begin{bmatrix} 18x_1 - 2x_2 + 4x_3 \\ -2x_1 + 14x_2 - 6x_3 \\ 4x_1 - 6x_2 + 6x_3 \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 18 & -2 & 4 \\ -2 & 14 & -6 \\ 4 & -6 & 6 \end{bmatrix}.$$

□

Example 2.1.7

Let x be a function of t. Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial \dot{x}}, \dfrac{\partial f}{\partial \ddot{x}}$, and $\dfrac{df}{dt}$.

Theorem 2.1.8 (Generalized Chain Rule)

Let $w = f(x_1, x_2, \dots, x_m)$ be a differentialble function of m independent variables, and for each $i \in \{1, \dots, m\}$, let $x_i = x_i(t_1, t_2, \dots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any $j \in \{1, \dots, n\}$.

Proof. Deez nutz □

§2.2 Tangent Planes

We proceed by finding the equation of the tangent plane to $x^2 - y^2 - z^2 = 1$ at $(1, 0, 0)$. To begin, we find the gradient of $f(x, y, z) = x^2 - y^2 - z^2$ to be

$$\nabla f(1, 0, 0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Then, the "point-slope" form of a plane is

$$m_x(x - x_1) + m_y(y - y_1) + m_z(z - z_1) = 0.$$

Thus, we obtain the following tangent plane for our scenario: $2(x - 1) = 0$.

§2.3 Unconstrained Optimization

Definition 2.3.1. A **stationary point** is a critical point in higher dimensions. They can be found from the solution to the system of equations that results from letting the gradient equal zero.

Definition 2.3.2. A hessian is called **positive definite** if all the eigenvalues are positive.

Definition 2.3.3. A hessian is called **negative definite** if all the eigenvalues are negative.

Definition 2.3.4. A hessian is called **positive semidefinite** if all the eigenvalues are nonnegative and there exists at least one eigenvalue that is 0.

Definition 2.3.5. A hessian is called **negative semidefinite** if all the eignevalues are nonpositive and there exists at least one eigen value that is 0.

Definition 2.3.6. A point is a **saddle point** if the hessian has negative and positve eigenvalues.

As an alternative to the second derivative test in determining if a critical point is a max , min or an inflection point, the **hessian** will be used to determine if a stationary point is a max, min or inflection point.

Theorem 2.3.7 (Second Partial Derivative Test)

We can determine if the hessian is "positive" or "negative" by taking a look at its eigen values. Let $\mathbf{H}_f$ be the hessian for f, a differentiable function of n independent variables. Also, let $\Lambda = \{\lambda_i | 1 \leq i \leq n\}$ be the set of the eigenvalues of $\mathbf{H_f}$. If all elements in Λ are positive, then the hessian is called **positive definite**, giving a minimum. If all elements in Λ are negative, then the hessian is called **negative definite**, giving a maximum. If $\lambda_i \geq 0$, $\mathbf{H_f}$ is called **positive semidefinite**. If $\lambda_i \leq 0$, $\mathbf{H_f}$ is called **negative semidefinite**. If one eigenvalue is positive and one eigenvalue is negative, we have a **saddle point**.

Example 2.3.8

Find the critical points of $f(x_1, x_2) = x_1^2 - 2x_1x_2 + 4x_2^2$.

Proof. We begin by finding the gradient of f. This is

$$\nabla f = \begin{bmatrix} 2x_1 - 2x_2 \\ -2x_1 + 8x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$\begin{aligned} 2x_1 - 2x_2 &= 0 \\ -2x_1 + 8x_2 &= 0. \end{aligned}$$

This gives $(x_1, x_2) = (0, 0)$. They hessian, $\mathbf{H}_f$ of f is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -2 \\ -2 & 8 \end{bmatrix}.$$

Since the eigen values of this hessian are both positive, we have a minimum. □

Example 2.3.9

Find the critical points of $f(x_1, x_2) = -x_1^2 + 2x_1x_2 + 3x_2^2 + 8x_1$.

Proof. We begin by finding the gradient of f. This is

$$\nabla f = \begin{bmatrix} -2x_1 + 2x_2 + 8 \\ 2x_1 + 6x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$\begin{aligned} -2x_1 + 2x_2 &= -8 \\ 2x_1 + 6x_2 &= 0. \end{aligned}$$

Solving gives $(x_1, x_2) = (-3, 1)$. The hessian, $\mathbf{H}_f$, of f is

$$\mathbf{H}_f = \begin{bmatrix} -2 & 2 \\ 2 & 6 \end{bmatrix}.$$

Since one of the eignevalues of this hessian is positive and the other is negative, we have a saddle point. □

Example 2.3.10

Find the critical points of $f(x_1, x_2) = \left(x_1 - x_2^2\right)\left(x_1 - 3x_2^2\right)$.

Proof. To begin, we distribute to get $f(x_1, x_2) = x_1^2 - 4x_1x_2^2 + 3x_2^4$. The gradient of f is then

$$\nabla f = \begin{bmatrix} 2x_1 - 4x_2^2 \\ -8x_1x_2 + 12x_2^3 \end{bmatrix}.$$

Setting the gradient to zero and solving the resulting system of equations, we get the following critical point $(x_1, x_2) = (0, 0)$. The hessian of f is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -8x_2 \\ -8x_2 & -8x_1 + 36x_2^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since one eigenvalue is positive and the other is equal to 0, the hessian is postive semidefinite. □

Example 2.3.11

Kartik and Monica invested \$20,000 in the design and the development of a new product. They can manufacture it for \$2 per unit. They hired marketing consultants to determine the relation between selling price, the amount spent on advertising, and the number of units that would be sold as a result of the first two combined. The company determined that units sold would follow the equation

$$2000 + 4\sqrt{a} - 20p.$$

Determine the profit that Felicia and Megan will make as a function of the money spent on advertising, a, and the price of the product, p. Maximize that profit.

Proof. We first identify that the revenue gained from sales would be $p\left(2000 + 4\sqrt{a} - 20p\right)$. Then, the costs would be $20000 + 2\left(2000 + 4\sqrt{a} - 20p\right) + a$. Taking the difference, we get the profit P being

$$\begin{aligned} P(a, p) &= p\left(2000 + 4\sqrt{a} - 20p\right) - 20000 - 2\left(2000 + 4\sqrt{a} - 20p\right) - a \\ &= 2040p + 4p\sqrt{a} - 20p^2 - 24000 - 8\sqrt{a} - a. \end{aligned}$$

The gradient of P is

$$\nabla P = \begin{bmatrix} \dfrac{2p - 4}{\sqrt{a}} - 1 \\ -40p + 4\sqrt{a} + 2040 \end{bmatrix}.$$

Setting the gradient to 0 and solving the resulting system of equations, we get that $p = 63.25$ and $a = 15006.25$. The maximum profit is $\boxed{\$40025}$. □

§2.4 Constrained Optimization

Equality and Inequality constraints

Definition 2.4.1. The **Lagrangian** is

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)),$$

where $g_i(x)$ are constraints.

Example 2.4.2

Maximize $f(x_1, x_2) = 5 - (x_1 - 2)^2 - 2(x_2 - 1)^2$ subject to $x_1 + 4x_2 = 3$.

Proof. We get

$$\begin{aligned}\mathcal{L}(x, \lambda) &= f(x_1, x_2) + \lambda(x_1 + 4x_2 - 3) \\ &= 5 - x_1^2 + 4x_1 - 4 - 2x_2^2 + 4x_2 - 2 + \lambda(x_1 + 4x_2 - 3) \\ &= -x_1^2 - 2x_2^2 + 4x_1 + 4x_2 - 1 + \lambda(x_1 + 4x_2 - 3)\end{aligned}$$

Thus

$$\nabla \mathcal{L} = \begin{bmatrix} -2x_1 + \lambda + 4 \\ -4x_2 + 4\lambda + 4 \\ x_1 + 4x_2 - 3 \end{bmatrix} = 0$$

We get $(x_1, x_2, \lambda) = (5/3, 1/3, -2/3)$. Thus our critical point is $\boxed{(5/3, 1/3, 4)}$. □

Example 2.4.3

Let the sun be located at the origin of a coordinate plane. How close does Halley's comet come to the sun on its orbit?

$$171.725x^2 + 171.725y^2 + 297.37xy + 557.178x - 557.178y - 562.867 = 0.$$

Proof. We seek to optimize the distance from the comet to the sun. Thus, we seek to optimize $f(x, y) = x^2 + y^2$. Optimization with the lagrangian follows, where

$$\mathcal{L} = x^2 + y^2 - \lambda().$$

□

Example 2.4.4

Minimize $L(x) = x_1 e^{-(x_1^2 + x_2^2)} + \frac{x_1^2 + x_2^2}{20}$ subject to $f(x) = \frac{x_1 x_2}{2} + (x_1^2 + 2)^2 + (x_1^2 - 2)^2/2 - 2 \leq 0$.

Proof. We find that

$$\nabla L = \begin{bmatrix} e^{-(x_1^2 + x_2^2)} - 2x_1^2 e^{-(x_1^2 + x_2^2)} + x_1/10 \\ -2x_1 x_2 e^{-(x_1^2 + x_2^2)} + x_2/10 \end{bmatrix} = 0$$

We find that the critical points are $(-1/2, \sqrt{\ln(10) - 1/4})$ and $(-1/2, -\sqrt{\ln(10) - 1/4})$. □

3 Differential Equations

§3.1 Separable Differential Equations

A separable differential equation can be simplified to the form:

$$g(y)y' = f(x) \implies g(y)\,\mathrm{d}y = f(x)\,\mathrm{d}x,$$

where f and g are functions of x and y respectively. Letting $G'(y) = g(y)$ and $F'(x) = f(x)$ and taking the integral of both sides of the differential equation, we get

$$G(y) = F(x) + C$$

for some constant C. Using a separation of variables is most helpful as a first try when attempting first-order linear ordinary differential equations. It is also used as a very useful intermediate step in solving more complex differential equations. We go more in detail later in the chapter.

Example 3.1.1 (Separable Differential Eqiation)
Solve for y:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y - 2xy.$$

Solution. Separating the variables, we see that $\frac{1}{y}\,\mathrm{d}y = (x^2 - 2x)\,\mathrm{d}x$. Taking the integral of both sides yields

$$\ln y = \frac{1}{3}x^3 - x^2 + A \implies y = Ae^{\frac{1}{3}x^3 - x^2}.$$

$\square$

Question 3.1.2. What is y when given the initial condition $y(0) = e$?

§3.2 Homogeneous Ordinary Differential Equations

The general from of a linear ordinary differential equation(ODE) is

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x),$$

where $a_i \in \mathbb{R}[x]$ for $i \in \{0, 1, \ldots, n\}$ and $y^{(j)}$ denotes the jth derivative of y. It is then said that this ODE has **order** n since it is the highest derivative in the equation. An ODE is said to be **homogenous** if $f(x) = 0$. Otherwise, it is said to be a **nonhomogeneous** ODE. The most common method of approaching homogeneous ODEs is ansatz. This is an educated guess that is made to help solve a problem. This will become apparent not only in homogeneous ODEs but more complex ones as well.

Example 3.2.1 (Homogeneous ODE)

Solve for y:

$$y'' + 7y' + 12y = 0.$$

Solution. Functions of the form $y = e^{mx}$ are an educated guess for solving homoegeneous ODEs. So, letting $y = e^{mx}$ means $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging these in our differential equation,

$$\begin{aligned} m^2e^{mx} + 7me^{mx} + 12e^{mx} &= 0 \\ e^{mx}\left(m^2 + 7m + 12\right) &= 0 \\ e^{mx}(m+4)(m+3) &= 0. \end{aligned}$$

Therefore, $m = -4, -3$, implying that $y_1 = C_1e^{-4x}$ and $y_2 = C_2e^{-3x}$. This means that our solution of the differential equation is

$$y_h = y_1 + y_2 = C_1e^{-4x} + C_2e^{-3x}.$$

□

At first, it might be puzzling why we consider $y = e^{mx}$ to be an educated guess for solving homogeneous ODEs with constant coeficients. However, let us consider the following first-order differential equation

$$y' + ky = 0 \implies y' = -ky.$$

Here, it is natural to guess that $y = e^{mx}$ because e^{mx} is the only function where its derivative is equal to some scalar multiple of itsself. So, for other homogeneous ODEs with constant coefficient and of higher degree, we also use the guess as a means of finding the general solution.

Exercise 3.2.2 (Homogeneous ODE with initial values).

$$y'' + y' - 2y = 0,\ y(0) = 4,\ y'(0) = -5.$$

Now, we consider what happens if we get a polynomial with repeated roots or if we are already given a solution to the differential equation.

Example 3.2.3 (Homogeneous ODE with repeated roots)

Solve for y:

$$y'' + 4y' + 4y = 0.$$

Solution. Let $y = e^{mx}$. This means $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging in, we get

$$\begin{aligned} m^2e^{mx} + 4me^{mx} + 4e^{mx} &= 0 \\ e^{mx}(m+2)^2 &= 0. \end{aligned}$$

Therefore, the first fundamental solution is $y_1 = e^{-2x}$. However, it is not clear what the second fundamental solution to this differential equation is. In order to determine this, we use the method of **reduction of order**. This involves letting the second fundamental solution being the product of y_1 and some function of x, say $f(x)$. So, let $y_2 = f(x) \cdot y_1$.

Now, we seek to find what $f(x)$ is. Thus, we want to find the first and second derivatives of y_2 in order to plug it back into the original differential equation. This will aid us in finding out more information regarding $f(x)$. Therefore, we have

$$\begin{aligned}
y_2' &= f'(x)y_1 + f(x)y_1' \\
&= f'(x)e^{-2x} - 2f(x)e^{-2x} \\
y_2'' &= f''(x)y_1 + 2f'(x)y_1' + f(x)y_1'' \\
&= f''(x)e^{-2x} - 4f'(x)e^{-2x} + 4f(x)e^{-2x}.
\end{aligned}$$

Plugging this into the original differential equation and simplifying fully yields $f''(x) = 0$. Now, let $g(x) = f'(x)$. This means $g'(x) = f''(x)$. So, $g'(x) = 0$. Solving for $g(x)$ we get $g(x) = A$. Therefore, $f'(x) = A$. Solving for $f(x)$ then gives $f(x) = Ax + B$. Thus,

$$y_2 = (Ax + B)e^{-2x} \implies y_2 = xe^{-2x}$$

since y_2 must be linearly independent from y_1. Finally, we find the solution to the differential equation to be

$$y = c_1y_1 + c_2y_2 = c_1e^{-2x} + c_2xe^{-2x}.$$

□

Theorem 3.2.4 (Reduction of Order)
Consider a general homogeneous linear ODE of order two:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

and suppose $y_1(x)$ is one solution to the differential equation. Then the second solution to the differential equation is

$$y_2(x) = y_1(x) \int \frac{u(x)}{[y_1(x)]^2}\, \mathrm{d}x,$$

where $u(x) = e^{-\int p(x)\, \mathrm{d}x}$.

Proof. We assume the second solution is of the form $y_2(x) = f(x)y_1(x)$ for some function f. Thus,

$$y_2' = f'y_1 + fy_1' \qquad y_2'' = f''y_1 + 2f'y_1' + fy_1''.$$

Substituting these expressions back into the original differential equation, we get

$$f''y_1 + 2f'y_1' + fy_1'' + p(f'y_1 + fy_1') + qfy_1 = 0.$$

Rearranging these terms, we obtain

$$f(y_1'' + py_1' + qy_1) + f''y_1 + 2f'y_1' + pf'y_1 = 0.$$

Notice that the term in the parentheses becomes 0 since y_1 is a solution to the differential equation. Therefore, we are simply left with

$$f''y_1 + 2f'y_1' + pf'y_1 = 0.$$

Rearranging and letting $g(x) = f'(x)$,

$$g'(x) = -g(x)\left(2\frac{y_1'(x)}{y_1(x)} + p(x)\right).$$

Now, we simply have a first-order ODE in terms of $g(x)$. It can then be shown that $f(x) = \int \frac{u(x)}{[y_1(x)]^2}\,\mathrm{d}x$ where $u(x) = e^{-\int p(x)\,\mathrm{d}x}$. It then follows that

$$y_2(x) = y_1(x)\int \frac{u(x)}{[y_1(x)]^2}\,\mathrm{d}x.$$

□

Corollary 3.2.5 (Reduction of Order with Constant Coefficients)
Consider a homogeneous linear ODE with constant coeffiecients:

$$ay''(x) + by'(x) + cy(x) = 0$$

where $a, b, c \in \mathbb{R}\backslash\{0\}$ and suppose the discriminant, $b^2 - 4ac$, vanishes due to the presence of a repeated root in the characteristic equation. Additionally suppose $y_1(x)$ is one solution to the differential equation. Then the second solution to the differential equation is $y_2(x) = xy_1(x)$.

Proof. Letting $y = e^{mx}$ and solving for m from the resulting characteristic equation gives $m = -\frac{b}{2a}$. Therefore $y_1(x) = e^{-\frac{b}{2a}x}$. □

Example 3.2.6 (Reduction of Order)
Solve for y:

$$y'' + y' + (1/4)y = 0;\ y(0) = 3;\ y'(0) = -3.5.$$

Solution. Let $y = e^{mx}$. This means $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging in, we get

$$\begin{aligned} 4m^2e^{mx} + 4me^{mx} + e^{mx} &= 0 \\ e^{mx}(4m^2 + 4m + 1) &= 0 \\ e^{mx}(2m + 1)^2 &= 0. \end{aligned}$$

Therefore $y_1 = e^{-x/2}$. Then, by the general result of reduction of order on a homogeneous linear ODE with constant coefficients gives $y_2 = x \cdot y_1$. So, the solution to the differential equation is

$$y = c_1e^{-x/2} + c_2xe^{-x/2}$$

Using the inital values to solve for constants c_1 and c_2, we finally find

$$y = 3e^{-x/2} - 2xe^{-x/2}.$$

□

Example 3.2.7 (Reduction of Order)

$$y''' + 3y'' - 4y = 0.$$

Solution. Let $y = e^{mx}$. This mean $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging in, we get

$$\begin{aligned} m^3e^{mx} + 3m^2e^{mx} - 4e^{mx} &= 0 \\ e^{mx}\left(m^3 + 3m^2 - 4\right) &= 0 \\ e^{mx}(m-1)(m^2+4m+4) &= 0 \\ e^{mx}(m-1)(m+2)^2 &= 0. \end{aligned}$$

Therefore, $y_1 = e^x$, $y_2 = e^{-2x}$, and $y_3 = xe^{-2x}$. Thus,

$$y_h = c_1e^x + c_2e^{-2x} + c_3xe^{-2x}.$$

□

Thus far, we have examined ODEs which had characteristic equations with real roots. However, it may be the case that these characteristic equations obtain complex roots. The help of **euler's formula** helps to clear any complex numbers in the solution.

Theorem 3.2.8 (Euler's Formula)
For any complex number x,

$$e^{ix} = \cos x + i \sin x.$$

Proof. A taylor series expansion of both sides gives the result. □

Example 3.2.9 (Complex Roots)
Solve for y:

$$y'' + 4y = 0.$$

Solution. Let $y = e^{mx}$. This means that $y'' = m^2e^{mx}$. Plugging in gives $m = \pm 2i$. Thus,

$$y = c_1e^{2ix} + c_2e^{-2ix}.$$

By Euler's Formula,

$$\begin{aligned} y &= c_1(\cos 2x + i \sin 2x) + c_2(\cos 2x - i \sin 2x) \\ y &= c_1 \cos 2x + c_2 \sin 2x. \end{aligned}$$

□

Example 3.2.10 (Complex Roots)

$$y'' + 8y' + 25y = 0.$$

Solution. Let $y = e^{mx}$. This meas that $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging this in gives $-4 \pm 3i$. Thus,

$$\begin{aligned}
y_h &= C_1 e^{(-4+3i)x} + C_2 e^{(-4-3i)x} \\
y_h &= e^{-4x}\left(C_1 e^{3ix} + C_2 e^{-3ix}\right) \\
y_h &= e^{-4x}\left(C_1 \cos(3x) + C_2 \sin(3x)\right).
\end{aligned}$$

□

§3.3 Nonhomogeneous Ordinary Differential Equations

A **nonhomogeneous** differential equation is of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

where $a_i \in \mathbb{R}[x]$ for $i \in \{1, 2, \ldots, i\}$ and $f(x) \neq 0$. The general solution to a nonhomogeneous differential equation is $y = y_h + y_p$, where y_h is found from letting the left hand side equal 0 and the particular solution, y_p, is found by using one of two methods, namely **undetermined coefficients** or **variation of parameters**. Undetermined coefficients can only be used with differential equations with scalar coefficients while variation of parameters can be used with scalar or variable coefficients.

Theorem 3.3.1 (Undetermined Coefficients)
Consider the differential equation

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = f(x).$$

In order to solve for the particular solution, y_p, we guess y_p based on the follwing table

$f(x)$	$y_p(x)$
$ae^{\beta x}$	$Ae^{\beta x}$
$a\cos(\beta x)$	$A\cos(\beta x) + B\sin(\beta x)$
$b\sin(\beta x)$	$A\cos(\beta x) + B\sin(\beta x)$
$a\cos(\beta x) + b\sin(\beta x)$	$A\cos(\beta x) + B\sin(\beta x)$
nth deg polynomial	$A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0$

In addition, if $f(x)$ is a linear combination of any of the forms above, then $y_p(x)$ is also a linear combination of the guesses for each of the forms present in $f(x)$.

Example 3.3.2 (Undetermined Coefficients)

$$y'' - 4y' + 3y = e^{-x}; \; y(0) = 1; \; y'(0) = 0.$$

Solution. We first find y_h using $y'' - 4y' + 3y = 0$. Thus we obtain

$$y_h = C_1 e^{x} + C_2 e^{3x}.$$

Our guess for $y_p = Ce^{-x}$. Thus, $y_p' = -Ce^{-x}$ and $y_p'' = Ce^{-x}$. So, plugging this in, we get

$$\begin{aligned} Ce^{-x} + 4Ce^{-x} + 3Ce^{-x} = e^{-x} &\implies 8Ce^{-x} = e^{-x} \\ &\implies C = \frac{1}{8}. \end{aligned}$$

Therefore

$$y = y_h + y_p = C_1e^x + C_2e^{3x} + \frac{1}{8}e^{-x}.$$

Using the initial values, we find that

$$\boxed{y = \frac{5}{4}e^x - \frac{3}{8}e^{3x} + \frac{1}{8}e^{-x}}.$$

$\square$

Example 3.3.3 (Guessing Practice with Undetermined Coefficients)

$$y'' + 2y' + y = \cos(2x)$$

Solution. We first proceed to find y_h. We find that

$$y_h = C_1e^{-x} + C_2xe^{-x}.$$

We make the following guess: $y_p = a\cos(2x) + b\sin(2x)$. Solving for a and b, we find that $y_p = -\frac{3}{25}\cos(2x) + \frac{4}{25}\sin(2x)$. Therefore

$$\boxed{y = C_1e^{-x} + C_2xe^{-x} - \frac{3}{25}\cos(2x) + \frac{4}{25}\sin(2x)}.$$

$\square$

Example 3.3.4 (Bad Guessing)

$$y'' + 4y' + 3y = e^{-x}.$$

Solution. We notice $y_h = C_1e^{-x} + C_2e^{-3x}$. $y_p = Ce^{-x}$ is a bad guess since it is a term present in the homogeneous equation. So, we let $y_p = Cxe^{-x}$, which is a guess from the result of applying the reduction of order. Solving for C, we get that $C = \frac{1}{2}$. Therefore

$$\boxed{y = C_1e^{-x} + C_2e^{-3x} + \frac{1}{2}xe^{-x}}.$$

$\square$

Theorem 3.3.5 (Variation of Parameters)

The method of variation of parameters is a technique for finding a particular solution to a nonhomogeneous linear second order ODE:

$$y'' + P(x)y' + Q(x)y = R(x)$$

provided that the general solution of the corresponding homogeneous linear second order ODE:

$$y'' + P(x)y' + Q(x)y = 0$$

is already known. The particular solution is then

$$y_p = u(x)y_1(x) + v(x)y_2(x)$$

such that

$$u'(x) = -\frac{y_2R(x)}{W(y_1, y_2)} \text{ and } v'(x) = \frac{y_1R(x)}{W(y_1, y_2)},$$

where $W(y_1, y_2)$ denotes the Wronskian of y_1 and y_2.

Proof. First, we have

$$\begin{aligned} y_p' &= \left(u'y_1 + uy_1'\right) + \left(v'y_2 + vy_2'\right) \\ &= \left(uy_1' + vy_2'\right) + \left(u'y_1 + v'y_2\right) \end{aligned}$$

by the product rule. □

Example 3.3.6 (Variation of parameters)

$$y'' + 4y' + 4y = e^{-2x}/x^2.$$

Solution. We find that $y_h = C_1e^{-2x} + C_2xe^{-2x}$. Notice that the Wronskian is

$$w = \det\left(\begin{bmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{bmatrix}\right) = e^{-4x}.$$

Thus,

$$\begin{aligned} u'(x) &= -\frac{1}{x} \\ v'(x) &= \frac{1}{x^2}. \end{aligned}$$

Thus, $u(x) = -\ln|x|$ and $v(x) = -\frac{1}{x}$. Therefore,

$$y_p = \ln|x|e^{2x} - e^{-2x} \implies \boxed{y = C_1e^{-2x} + C_2xe^{-2x} + \ln|x|e^{2x}}.$$

□

Example 3.3.7 (Undetermined Coefficients)

$$y'' + y' + y = 3x^2 + 4.$$

Solution. We first find that

$$y_h = e^{(-1/2)x}\left(C_1\cos(\sqrt{3}x/2) + C_2\sin(\sqrt{3}x/2)\right).$$

Then we let $y_p = D_1x^2 + D_2x + D_3$. Therefore

$$\begin{aligned} 2D_1 + 2xD_1 + D_2 + D_1x^2 + D_2x + D_3 &= 3x^2 + 4 \\ x^2(D_1) + x(2D_1 + D_2) + (2D_1 + D_2 + D_3) &= 3x^2 + 4. \end{aligned}$$

Solving the resulting system yields $D_1 = 3, D_2 = -6, D_3 = 4$. Therefore

$$y_p = 3x^2 - 6x + 4.$$

So,

$$\boxed{y = e^{(-1/2)x}\left(C_1\cos(\sqrt{3}x/2) + C_2\sin(\sqrt{3}x/2)\right) + 3x^2 - 6x + 4}.$$

□

Example 3.3.8 (Variation of parameters)

$$y'' + 4y = \tan x.$$

Solution. We first find $y_h = C_1\cos(2x) + C_2\sin(2x)$. Now, we notice that wronskian is

$$w = \det\left(\begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}\right) = 2.$$

Thus,

$$u'(x) = -\frac{\tan x\sin(2x)}{2} = -\sin^2(x)$$

and

$$v'(x) = \frac{1}{2}\ln|\cos x| - \frac{\cos(2x)}{4} = \frac{\sin(2x)}{2} - \frac{1}{2}\tan x.$$

Solving, we find $u(x) = \frac{\sin(2x)}{4} - \frac{1}{2}x$. $v(x) = \frac{1}{2}\ln|\cos x| - \frac{\cos 2x}{4}$. Thus,

$$y_p = -\frac{1}{2}\cos(2x) + \frac{1}{2}\sin(2x)\ln|\cos x|.$$

So,

$$y = C_1\cos(2x) + C_2\sin(2x) - \frac{1}{2}\cos(2x) + \frac{1}{2}\sin(2x)\ln|\cos x|$$

□

Exercise 3.3.9. Find y for the differential equation $y'' + 3y' + 2y = \frac{1}{1+e^x}$.

§3.4 Ordinary Differential Equations with Variable Coefficients

Euler's Equation is the following

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = f(x).$$

The methods for solving these differential equations are the **transformation method**, which changes to an equation with constant coefficients, and the **direct method**.

Theorem 3.4.1 (Transformation Method)
Let $x = e^z$. Plugging in e^z into $y(x)$, we get $\hat{y}(z)$. Thus,

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}\hat{y}}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x} = \hat{y}'\frac{\mathrm{d}z}{\mathrm{d}x} = \hat{y}'\frac{1}{x}.$$

Then, y'' follows to be

$$y'' = \frac{\hat{y}'' - \hat{y}'}{x^2}.$$

Exercise 3.4.2. Find y'''.

Example 3.4.3 (Transformation Method)
We seek to solve the differential equation

$$x^2 y'' - x y' + y = 0.$$

Using the transformation method, we get

$$x^2\left(\frac{\hat{y}'' - \hat{y}'}{x^2}\right) - x\left(\frac{\hat{y}'}{x}\right) + \hat{y} = 0.$$

Simplifying gives

$$\hat{y}'' - 2\hat{y}' + \hat{y} = 0.$$

This is now a differential equation with constant coefficients, which we know how to solve easily. Thus, we let $\hat{y} = e^{mz}$. Then, $\hat{y_h} = Ae^z + Bze^z$. Putting everything in x, we get

$$y_h = Ax + Bx\ln x.$$

Theorem 3.4.4 (Direct Method)
Assume $y = x^m$, where $m \in \mathbb{C}$.

Example 3.4.5 (Direct Method)

We seek to solve the differential equation

$$x^2y'' - xy' + y = 0.$$

Letting $y = x^m$, we have $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Plugging in, we get $x^m\left(m^2 - 2m + 1\right) = 0$. Therefore, $m = 1$, meaning that $y_1 = C_1x$ We now proceed to use reduction of order. Thus, $y_2 = f(x)y_1$. So,

$$\begin{aligned} y_2' &= f'y_1 + fy_1' = f'x + f \\ y_2'' &= f''y_1 + 2f'y_1' + y_1'' = f''x + 2f'. \end{aligned}$$

Plugging this back into the original differential equation, we get

$$x^2\left(xf''(x) + 2f'(x)\right) - x\left(xf'(x) + f(x)\right) + xf(x) = 0.$$

Simplifying and rearranging, we get

$$x^3f''(x) + x^2f'(x) = 0.$$

Letting $g(x) = f'(x)$, we obtain the following separable differential equation

$$\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}x} = -\frac{1}{x} \implies g(x) = \frac{1}{x}.$$

Thus, $f(x) = \ln x$. Finally, $y_2 = x\ln x$ and

$$y_h = Ax + Bx\ln x.$$

Exercise 3.4.6. Solve the differential equation $x^3y''' + 2x^2y'' + xy' - y = 0$.

Example 3.4.7 (Nonhomogeneous Euler's)

We seek to solve the differential equation

$$x^2y'' - xy' + y = x^5.$$

Using the trasformation method, we get

$$\hat{y}'' - 2\hat{y}' + \hat{y} = e^{5z}.$$

Now, we can simply use undetermined coefficients. Solving for the homogeneous solution, we get $\hat{y_h} = Ae^z + Bze^z$. Then, we get $\hat{y_p} = \frac{1}{16}e^{5z}$ So, $y_h = Ax + Bx\ln x$ and $y_p = \frac{1}{16}x^5$. Finally,

$$y = Ax + Bx\ln x + \frac{1}{16}x^5.$$

Exercise 3.4.8. Solve the differentiual equation $x^2y'' + 3xy + y = \frac{1}{x(\ln x)^2}$. Hint: use variation of parameters.

§3.5 System of Linear ODEs

An example of a system of linear odes is a spring system in series. Our first goal is to change the system that can be written as

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \mathbf{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where $\mathbf{A}$ is a square matrix. Now, assume, $y = [C]e^{\lambda x}$ where $[C]$ and λ are an eigenvalue/vector pair of $\mathbf{A}$. Then, the eigenvalues and eigenvectors of $\mathbf{A}$ form the fundamential solutions

$$y_h = \alpha_1 c_1 e^{\lambda_1 x} + \alpha_2 c_2 e^{\lambda_2 x}.$$

Example 3.5.1

We seek to solve the following system of differential equations.

$$\begin{aligned} y_1' + 2y_1 + y_2' + 6y_2 &= 0 \\ 2y_1' + 3y_1 + 3y_2' + 8y_2 &= 0, \end{aligned}$$

where

$$y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

First, we isolate y_1' from the first equation to obtain

$$y_1' = -2y_1 - y_2' - 6y_2.$$

Then, plugging this into the second equation and simplifying, we find

$$y_2' = 4y_2 + y_1.$$

y_1' follows to be

$$y_1' = -10y_2 - 3y_1.$$

Thus,

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3 & -10 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = -1,\ \lambda_2 = 2,\ \mathbf{E}_1 = \begin{bmatrix} -5 \\ 1 \end{bmatrix},\ \mathbf{E}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Therefore,

$$y_h = \alpha_1 \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-x} + \alpha_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2x}.$$

Solving for α_1 and α_2, we get $\alpha_1 = -\frac{1}{3}$ and $\alpha_2 = \frac{1}{3}$. Finally,

$$y_h = -\frac{1}{3} \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-x} + \frac{1}{3} \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2x}.$$

Exercise 3.5.2. Solve the following system of linear odes:

$$y' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} y;\ y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Example 3.5.3 (Repeated Roots with a basis)

We seek to solve the following system of differential equations

$$y' = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} y;\ y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues for $\mathbf{A}$ are

$$\lambda_{1/2} = -1,\ \lambda_3 = 5.$$

The eigenvectors for $\mathbf{A}$ are

$$\mathbf{E}_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{E}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{E}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore,

$$y_h = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-x} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-x} + \alpha_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{5x}.$$

Solving for the constants, we find

$$y_h = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} e^{-x} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{5x}.$$

Example 3.5.4 (Repeated Roots with no Basis)
We seek to solve the following systems of differentials

$$y' = \begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} y;\ y(0) = \begin{bmatrix} 17 \\ 42 \end{bmatrix}.$$

The eigenvalue of $\mathbf{A}$ is $\lambda = 6$, where the eigenvector is $\mathbf{C} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Therefore, $y_1 = \alpha_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{6x}$. Now, we proceed to use a method that is similar to reduction of order. Thus, we let

$$y_2 = \begin{bmatrix} a + bx \\ c + dx \end{bmatrix} e^{6x} \implies y_2' = \begin{bmatrix} b \\ d \end{bmatrix} e^{6x} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} 6e^{6x}$$

Plugging into the original differentual equation, we have

$$\begin{bmatrix} b \\ d \end{bmatrix} e^{6x} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} 6e^{6x} = \begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} a + bx \\ c + dx \end{bmatrix} e^{6x}$$

Therefore,

$$\begin{bmatrix} b + 6a \\ d + 6c \end{bmatrix} e^{6x} + \begin{bmatrix} 6b \\ 6d \end{bmatrix} xe^{6x} = \begin{bmatrix} 8a + c \\ -4a + 4c \end{bmatrix} e^{6x} + \begin{bmatrix} 8b + d \\ -4b + 4d \end{bmatrix} xe^{6x}.$$

Solving the system, we have

$$a = -\frac{m + 2n}{4} \ b = -\frac{m}{2} \ c = n \ d = m$$

for $m, n \in \mathbb{R}$. Letting $n = 1$ and $m = -2$, we get $a = 0, b = 1, c = 1, d = -2$. Therefore,

$$y_2 = \alpha_2 \begin{bmatrix} x \\ 1 - 2x \end{bmatrix} e^{6x}.$$

Solving for α_1 and α_2, we find

$$y = -17 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{6x} + 76 \begin{bmatrix} x \\ 1 - 2x \end{bmatrix} e^{6x} \implies y = \begin{bmatrix} 17 \\ 42 \end{bmatrix} e^{6x} + 76 \begin{bmatrix} 1 \\ -2 \end{bmatrix} xe^{6x}.$$

Exercise 3.5.5 (Undetermined Coefficients)**.** Solve the following sytem of differential equations:

$$y' = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} y + \begin{bmatrix} 1 + x \\ e^x \end{bmatrix}.$$

Solution. The homogeneous solution is

$$y_h = \alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3x}.$$

The particular solution is

$$y_p = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^x + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} x + \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}.$$

Plugging into the original differential equation, we obtain

$$\begin{aligned}
\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^x + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \left(\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^x + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} x + \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \right) + \begin{bmatrix} 1+x \\ e^x \end{bmatrix} \\
\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^x + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} &= \begin{bmatrix} d_1 + d_2 \\ -d_1 + 4d_2 + 1 \end{bmatrix} e^x + \begin{bmatrix} r_1 + 2r_2 + 1 \\ -r_1 + 4r_2 \end{bmatrix} x + \begin{bmatrix} s_1 + 2s_2 + 1 \\ -s_1 + 4s_2 \end{bmatrix}.
\end{aligned}$$

Comparing coefficients, we find that

$$\begin{aligned}
d_1 &= 1 & d_2 &= 0 \\
r_1 &= -2/3 & r_2 &= -1/6 \\
s_1 &= 19/18 & s_2 &= -11/36
\end{aligned}$$

Thus

$$\boxed{y = \alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^x + \alpha_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^x + \begin{bmatrix} -2/3 \\ -1/6 \end{bmatrix} x + \begin{bmatrix} 19/18 \\ -11/36 \end{bmatrix}}.$$

□

We now proceed to examine the method of variation of parameters in solving systems of differentual equations. Suppose we have the following system:

$$y' = \mathbf{A}y + f.$$

Let the $y_1, y_2, \ldots, y_n$ be the fundamental solutions of the system and

$$\mathbf{Y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}.$$

Then we have

$$y_p = \mathbf{Y} \int \mathbf{Y}^{-1} f \,\mathrm{d}x.$$

Example 3.5.6 (Variation of Parameters)

Solve the following system of differential equations:

$$y' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} y + \begin{bmatrix} 3 \\ -3 \end{bmatrix} \sqrt{t} e^{-2t}.$$

Solving for the homogeneous portion, we have

$$y_h = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2x}.$$

So, we let

$$\mathbf{Y} = \begin{bmatrix} e^{4x} & e^{-2x} \\ e^{4x} & e^{-2x} \end{bmatrix}.$$

The inverse of $\mathbf{Y}$ is

$$\mathbf{Y}^{-1} = \frac{1}{-2e^{2x}} \begin{bmatrix} -e^{-2x} & -e^{-2x} \\ -e^{4x} & e^{4x} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-4x} & e^{-4x} \\ e^{2x} & -e^{2x} \end{bmatrix}$$

From this

$$\mathbf{Y}^{-1} f = \begin{bmatrix} 0 \\ 3\sqrt{t} \end{bmatrix}.$$

Thus,

$$\int \mathbf{Y}^{-1} f \,\mathrm{d}t = \int \begin{bmatrix} 0 \\ 3\sqrt{t} \end{bmatrix} \mathrm{d}t = \begin{bmatrix} 0 \\ 2t^{3/2} \end{bmatrix}.$$

So,

$$\mathbf{Y} \int \mathbf{Y}^{-1} f \,\mathrm{d}t = \begin{bmatrix} 2t^{3/2} e^{-2t} \\ -2t^{3/2} e^{-2t} \end{bmatrix}.$$

Our general solution is therefore,

$$y = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} 2t^{3/2} e^{-2t}.$$

§3.6 Power Series Approach

Sometimes an ordinary differential equation is not solvable using the standard methods introduced thus far. Therefore, using a power series as a guess for the general solution can be a powerful approach. To examine this, we take a look at an example that may also be solved using a separation of variables.

Example 3.6.1 (Power Series)
Solve for y:
$$(x-3)y' + 2y = 0.$$

Solution. The guess we make for the general solution of y is

$$y = a_0x^0 + a_1x^1 + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n.$$

Taking the derivative of this, we get

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} a_nnx^{n-1}.$$

Plugging in this into our differential equation, we obtain

$$\begin{aligned}(x-3)\sum_{n=1}^{\infty} a_nnx^{n-1} + 2\sum_{n=1}^{\infty} a_nx^n &= 0\\ x\sum_{n=1}^{\infty} a_nnx^{n-1} - 3\sum_{n=1}^{\infty} a_nnx^{n-1} + 2\sum_{n=0}^{\infty} a_nx^n &= 0\\ \sum_{n=1}^{\infty} a_nnx^{n} - 3\sum_{n=1}^{\infty} a_nnx^{n-1} + 2\sum_{n=0}^{\infty} a_nx^n &= 0\end{aligned}$$

Now, we seek to obtain the same exponent on x for each summation so that it becomes easier to combine each sum. Therefore, we perform a **change of indices** on the middle term, letting $n \to n+1$. This gives

$$\begin{aligned}\sum_{n=1}^{\infty} a_nnx^n - 3\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + 2\sum_{n=0}^{\infty} a_nx^n &= 0\\ \sum_{n=1}^{\infty} a_nnx^n + \sum_{n=0}^{\infty}\left(2a_n - 3a_{n+1}(n+1)\right)x^n &= 0\\ \sum_{n=1}^{\infty} a_nnx^n + \sum_{n=1}^{\infty}\left(2a_n - 3a_{n+1}(n+1)\right)x^n &= 3a_1 - 2a_0\\ \sum_{n=1}^{\infty}\left((n+2)a_n - 3a_{n+1}(n+1)\right)x^n &= 3a_1 - 2a_0.\end{aligned}$$

Comparing coefficients gives $2a_0 - 3a_1 = 0$ and $(n+2)a_n - 3a_{n+1}(n+1) = 0$ for all $n \geq 1$. Thus, we have the following recursion

$$a_{n+1} = \frac{(n+2)a_n}{3(n+1)},$$

where $a_1 = (2/3)a_0$. We try now to find a closed form for a_n. This is usually done by finding a pattern among the terms. Thus, listing out the first few terms, we have

n	a_n
0	$(1/3^0)a_0$
1	$(2/3^1)a_0$
2	$(3/3^2)a_0$
3	$(4/3^3)a_0$
4	$(5/3^4)a_0$
5	$(6/3^5)a_0$

Therefore, the closed form for (a_n) is

$$a_n = \left(\frac{n+1}{3^n}\right) a_0.$$

This result may be proved by induction for completeness but that is left as an exercise for the reader. So, we have

$$\begin{aligned} y = a_0 \sum_{n=0}^{\infty} \left(\frac{n+1}{3^n}\right) x^n &= a_0 \sum_{n=0}^{\infty} n \left(\frac{x}{3}\right)^n + a_0 \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n \\ &= a_0 \left(\frac{x/3}{(1-(x/3))^2} + \frac{1}{1-(x/3)}\right) = \frac{a_0}{(1-(x/3))^2} \\ &= \frac{9a_0}{(3-x)^2}. \end{aligned}$$

Combining the numerator into one constant, we finally obtain the general solution for y to be

$$y = \frac{c}{(3-x)^2},$$

which can be shown is equivalent to the one derived from a separation of variables. □

Theorem 3.6.2 (Frobenius Method)

The frobenius method is a way to find an infinite series solution to a second-order ordinary differential equation of the form

$$x^2 y'' + p(x)xy' + q(x)y = 0.$$

Dividing by x^2 gives a differential equation of the form

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0.$$

The method then finds a solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

where the value of r is determined by the maximal solution to the **indicial equation** $r(r-1) + p(0)r + q(0) = 0$. The second fundamental solution may be found by a reduction of order:

$$y_2 = y_1(x) \int \frac{u(x)}{[y_1(x)]^2}\,\mathrm{d}x \text{ where } u(x) = e^{-\int p(x)/x\,\mathrm{d}x}.$$

Example 3.6.3 (Frobenius)
Solve for y:

$$xy'' + 2y' + 4xy = 0.$$

Solution. Re-expressing the differentual equation, we have

$$y'' + \frac{2}{x}y' + \frac{4x^2}{x^2}y = 0.$$

Therefore, our indicial equation is $r(r-1)+2r = 0$, giving $r = 0$. So, our general solution is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Finding the first and second derivatives and plugging in to the given differential equation gives

$$\begin{aligned} x\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 2\sum_{n=1}^{\infty} a_n n x^{n-1} + 4x\sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} + \sum_{n=1}^{\infty} 2a_n n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^{n+1} &= 0. \end{aligned}$$

Performing a change of indices on the last term by letting $n \to n-2$, we now have

$$\begin{aligned} \sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} + \sum_{n=1}^{\infty} 2a_n n x^{n-1} + \sum_{n=2}^{\infty} 4a_{n-2} x^{n-1} &= 0 \\ 2a_1 + \sum_{n=2}^{\infty} \left(a_n n(n+1) + 4a_{n-2}\right) x^{n-1} &= 0. \end{aligned}$$

Comparing coefficients, we have $a_1 = 0$ and $a_n n(n+1) + 4a_{n-2} = 0$ for all $n \geq 2$. So, we have the recursion

$$a_n = -\frac{4a_{n-2}}{n(n+1)}.$$

Listing out the first few terms yields

n	a_n
0	$((-4)^0/1!)a_0$
1	0
2	$((-4)^1/3!)a_0$
3	0
4	$((-4)^2/5!)a_0$
5	0

Therefore, the closed form of the sequence (a_n) is

$$a_{2n} = \left(\frac{(-4)^n}{(2n+1)!}\right) a_0$$

Plugging this into our guess for the first fundamental solution gives us

$$y_1 = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n+1)!} = a_0 \left(\frac{\sin 2x}{2x}\right).$$

Then, a reduction of order allows us to obtain the second solution, namely $y_2 = \frac{\cos 2x}{2x}$. Therefore, the general solution to our differential equation is

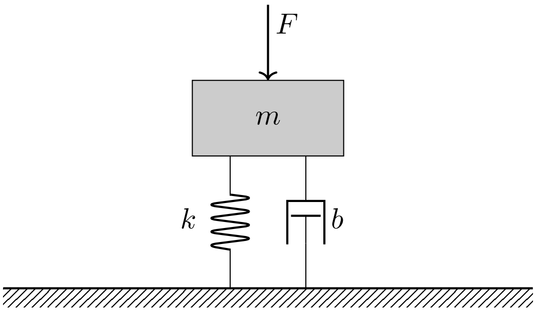
$$y = c_1 \left(\frac{\sin 2x}{2x} \right) + c_2 \left(\frac{\cos 2x}{2x} \right).$$

□

4 Applications of Mathematical Methods

§4.1 Fundamentals of Translational and Rotational Systems

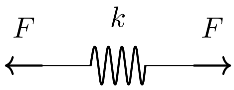
Damper systems do not have equilibrium but spring systems do. The number odes needed to be solved is based on the number of degrees of freedom in the system.



A **stiffness element** is a spring as shown below. A spring with an unstretched length of l_0 has a length of

$$l(t) = l_0 + x_2(t) - x_1(t)$$

with the application of the shown forces in the figure below. For a linear spring $F = k(x_1 - x_1)$, where k is the **stiffness coefficient**. Simplifying this gives $F = kx(t)$ or just $F = kx$.



5 Heat Transfer and Transport

We begin by introducting some fundamental equations required.

Theorem 5.0.1 (Fundamental Equation of accumulation)
$A = I - O + G - C$

$$\text{Accumulation} \;=\; \text{In} \;-\; \text{Out} \;+\; \text{Generative} \;-\; \text{Consumption}$$

Theorem 5.0.2 (Fourier's Law)
$\frac{Q}{A} = q = -\frac{k\,\mathrm{d}T}{\mathrm{d}x}$ or $-\frac{k\,\mathrm{d}T}{\mathrm{d}r}$, where Q is the total heat, A is the area, and q is the heat flux. $q = h(T_1 - T_2)$, where h is the heat transfer coefficient and $T_1 - T_2$ is the change in the temperature.

Example 5.0.3

Consider a slab going into the paper with a width of w, as shown below.
Heat moves away from $x = 0$, and the interior has .

Since we assume the system to be stead state, the accumulation is equal to 0. Thus, $0 = I - O + G - C$. $I = q(x)wL$, $O = q_i(x + \Delta x)wL$, and $G = +S_0 wL\Delta x$. Plugging in and simplifying gives

$$0 = q_i(x) - q_i(x + \Delta x) + S_0\Delta x.$$

Dividing by ΔX and taking a limit as Δx approaches 0, we can obtain

$$\lim_{\Delta x \to 0} \frac{q_i(x + \Delta x) - q_i(x)}{\Delta x} = S_0.$$

This is the limit definition of a derivative. Therefore we have

$$\begin{aligned} \frac{\mathrm{d}q_i}{\mathrm{d}x} = S_0 \implies \int \mathrm{d}q_i = \int S_0 \,\mathrm{d}x \\ \implies q_i(x) = S_0 x + c_1 \end{aligned}$$

Example 5.0.4

Consider a cylindrical slab going into the paper with a length L, an inner radius of R_1, an outer radius of R_2. Same initial conditions as prior example.

We notice first that $q = h(T_s - T_f)$, where T_s is the temperature on the surface and T_f is the temperature on the outside. Additionally, since we assume the system to be steady state, the accumulation is equal to 0. Firstly, we proceed to solve for the heat flux of the inner material. We have

$$\begin{aligned}
I &= q_i(x)2\pi rL \\
O &= q_i(x)(r+\Delta r)2\pi(r+\Delta r)L \\
G &= S_0\pi L((r+\Delta r)^2 - r^2) \\
C &= 0.
\end{aligned}$$

Thus, since the accumulation is equal to 0,

$$q_i(x)2\pi rL - q_i(x)(r+\Delta r)2\pi(r+\Delta r)L + S_0\pi L((r+\Delta r)^2 - r^2) = 0.$$

After simplification, rearrangment and taking a limit, we attain the following equation

$$\lim_{\Delta r\to 0}\frac{q_i(r+\Delta r)(r+\Delta r) - q_i(r)r}{\Delta r} = \lim_{\Delta r\to 0} S_0\left(r+\frac{\Delta r}{2}\right).$$

Using the limit definition of the derivative, this simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}r}(q_i(r)r) = S_0 r.$$

Therefore,

$$q_i(r)r = \frac{1}{2}S_0r^2 + c_1.$$

Since $q_i(0) = 0$, $c_1 = 0$, giving

$$q_i(r) = \frac{1}{2}S_0 r.$$

Now, we proceed to solve for the heat flux of the cladding. We have

$$\begin{aligned}
I &= q_c(x)2\pi rL \\
O &= q_c(x)(r+\Delta r)2\pi(r+\Delta r)L.
\end{aligned}$$

So, since the accumulation is equal to 0,

$$q_c(x)2\pi rL - q_c(x)(r+\Delta r)2\pi(r+\Delta r)L = 0.$$

Simplifying, rearranging, and taking a limit we obtain

$$\lim_{\Delta r\to 0}\frac{q_c(r+\Delta r)(r+\Delta r) - q_c(r)r}{\Delta r} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}r}q_c(r)r = 0.$$

Solving this differential equation, $q_c(r)r = c_2$. Since $q_i(R_1) = q_c(R_2)$ and $q_i(R_1) = \frac{1}{2}S_0R_1$, $c_2 = \frac{1}{2}S_0R_1^2$. Thus,

$$q_c(r) = \frac{S_0R_1^2}{2r}.$$

By Fourier's law,

$$\begin{aligned}
q_c = \frac{-K_I\,\mathrm{d}T_c}{\mathrm{d}r} &\implies \frac{S_0R_1^2}{2r} = \frac{-K_I\,\mathrm{d}T_c}{\mathrm{d}r} \\
&\implies \int \frac{S_0R_1^2}{2}\frac{1}{r}\,\mathrm{d}r = \int -K_I\,\mathrm{d}T_c \\
&\implies c_3 + \frac{S_0R_1^2}{2}\ln r = -K_IT_c(r) \\
&\implies T_c(r) = -\frac{S_0R_1^2}{2K_I}\ln r + c_3.
\end{aligned}$$

We then use the fact that $q_c = h(T_s - T_f) \implies q_c(R_2) = h(T_c(R_2) - T_f)$. This is equivalent to

$$\frac{S_0 R_1^2}{2R_2} = h\left(-\frac{S_0 R_1^2}{2K_I}\ln R_2 + c_3 - T_f\right) \implies c_3 = \frac{S_0 R_1^2}{2R_2 h} + \frac{S_0 R_1^2}{2K_I}\ln R_2 + T_f$$

6 Electrical Systems

A **resistor** is an electrical component that resists current. The units of resistance are ohms (Ω). An important fact involving resistors is **Ohm's Law**.

Theorem 6.0.1 (Ohm's Law)
The voltage drop across a resistor is equal to the product of the current passing through the resistor and the resistance of the resistor.

$$V_R = iR.$$

A **capacitor** is an electrical component that stores charge, usually in the form of two parallel plates. A capacitor has a capacitance which is measure in the units of Farads (F). The current passing through the capacitor is

$$i = C\frac{\mathrm{d}V_c}{\mathrm{d}t} = C\dot{V}_c.$$

An **inductor** is an electrical component that resists a change in current. The inductance of an inductor is measured in Henry's. The voltage drop across an inductor is

$$V_L = L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

Two important laws play a key role in electrical systems: **kirchoff's voltage law** and **kirchoff's current law**.

Theorem 6.0.2 (Kirchoff's Voltage Law)
The sum of all voltage drops in a closed loop is equal to zero.

$$\sum_k V_k = 0.$$

Theorem 6.0.3
The

7 Laplace Transforms

The **laplace transform** of a function, $f(t)$, is

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t,$$

where $s = a + bi$ for $a > 0$.

§7.1 Well Known Laplace Transforms

$\mathcal{L}\{t^n f(t)\} = \frac{\mathrm{d}^{(n)}}{\mathrm{d}s^{(n)}}\mathcal{L}\{f(t)\}(-1)^n$

$f(t)$	$\mathcal{L}\{f(t)\}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\frac{\mathrm{d}y}{\mathrm{d}t}$	$-y(0) + s\mathcal{Y}(s)$
$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$	$-y'(0) - sy(0) + s^2\mathcal{Y}(s)$
$e^{-at}f(t)$	$F(s+a)$

§7.2 Ordinary Differential Equations

Example 7.2.1
Solve for $\omega(t)$:

$$\dot{\omega} + 2\omega = 6,$$

given $\omega(0) = 0$.

Solution. Taking the laplacian of both sides, we obtain

$$\mathcal{L}\{\dot{\omega} + 2\omega\} = \mathcal{L}\{6\} \implies -\omega(0) + s\Omega(s) + 2\Omega(s) = \frac{6}{s}.$$

Proceeding to solve for $\Omega(s)$ results in

$$\Omega(s) = \frac{6}{s(s+2)} = \frac{3}{s} - \frac{3}{s+2}.$$

So, taking the inverse laplace transform gives

$$w(t) = 3 - 3e^{-2t}.$$

$\square$

Example 7.2.2
Solve for $x(t)$:
$$\ddot{x} + 3\dot{x} + 2x = 1,$$
given $x(0) = \dot{x}(0) = 0$.

Solution. Taking the laplace transform of both sides, we obtain
$$\mathcal{L}\{\ddot{x} + 3\dot{x} + 2x\} = \mathcal{L}\{1\}$$
Therefore,
$$-\dot{x}(0) - sx(0) + s^2X(s) + 3(-x(0) + sX(s)) + 2X(s) = \frac{1}{s}.$$
Solving for $X(s)$ results in
$$X(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} + \frac{-2}{s+1} + \frac{1/2}{s+2}$$
Taking the inverse laplace transform of both sides gives
$$x(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$
□

Example 7.2.3
Solve for $y(t)$:
$$y'' + 4y' - 5y = 0,$$
given $y(0) = 1$, and $y'(0) = 0$.

Solution. Taking the laplace transform of both sides we obtain
$$\left(-y'(0) - sy(0) + s^2Y(s)\right) + 4\left(-y(0) + sY(s)\right) - 5Y(s) = 0.$$
Solving for $\mathcal{Y}(s)$, we obtain
$$\mathcal{Y}(s) = \frac{s+4}{s^2+4s-5} = \frac{5/6}{s-1} + \frac{1/6}{s+5}$$
Taking the inverse laplace transform of both sides, we obtain
$$y(t) = \frac{5}{6}e^{t} + \frac{1}{6}e^{-5t}.$$
□

Theorem 7.2.4 (Shifting Theorem)
$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$

Example 7.2.5

Solve for $y(t)$:

$$y'' + 4y' + 4y = 0,$$

where $y(0) = 1$, and $y'(0) = 0$.

Solution. Taking the laplace transfrom of both sides, we obtain

$$\left(-y'(0) - sy(0) + s^2\mathcal{Y}(s)\right) + 4\left(-y(0) + s\mathcal{Y}(s)\right) + 4\mathcal{Y}(s).$$

Solving for $\mathcal{Y}(s)$,

$$\mathcal{Y}(s) = \frac{s+4}{s^2+4s+4} = \frac{1}{s+2} + \frac{2}{(s+2)^2}.$$

Taking the inverse laplace transform of both sides, we have

$$y(t) = e^{-2t} + 2te^{-2t}$$

□

Example 7.2.6

Solve for $y(t)$:

$$y'' + 4y = 0,$$

given $y(0) = 1$ and $y'(0) = 0$.

Solution. Taking the laplace transform of both sides, we obtain

$$\left(-y'(0) - sy(0) + s^2\mathcal{Y}(s)\right) + 4\mathcal{Y}(s) = 0.$$

Solving for $\mathcal{Y}(s)$,

$$\mathcal{Y}(s) = \frac{s}{s^2+4}$$

Therefore, taking the inverse laplace transform of both sides results in

$$y(t) = \cos 2t.$$

□

Example 7.2.7

Solve for $y(t)$:

$$y'' + 8y' + 25y = 0,$$

given $y(0) = 1$ and $y'(0) = 0$.

Solution. Taking the laplace transform of both sides, we obtain

$$\left(-y'(0) - sy(0) + s^2\mathcal{Y}(s)\right) + 8\left(-y(0) + \mathcal{Y}(s)\right) + 25\mathcal{Y}(s) = 0.$$

Solving for $\mathcal{Y}(s)$,

$$\mathcal{Y}(s) = \frac{s+8}{s^2+8s+25} = \frac{s+4}{(s+4)^2+9} + \frac{4}{3}\left(\frac{3}{(s+4)^2+9}\right).$$

Thus, by the **shifting theorem** and taking the inverse laplace transform of both sides results in

$$y(t) = e^{-4t}\cos(3t) + \frac{4}{3}e^{-4t}\sin(3t).$$

□

Example 7.2.8

Solve for $y(t)$:

$$y'' - 6y' + 15y = 2\sin(3t),$$

given $y(0) = -1$ and $y'(0) = -4$.

Solution. Taking the laplace transform of both sides results in

$$\left(-y'(0) - sy(0) + s^2\mathcal{Y}(s)\right) - 6\left(-y(0) + s\mathcal{Y}(s)\right) + 15\mathcal{Y}(s) = \frac{2}{s^2+9}.$$

Solving for $\mathcal{Y}(s)$,

$$\begin{aligned}
\mathcal{Y}(s) &= \frac{6}{(s^2+9)(s^2-6s+15)} - \frac{s-2}{s^2-6s+15} \\
&= \frac{-s^3+2s^2-9s+24}{(s^2+9)(s^2-6s+15)} \\
&= \frac{s+1}{10(s^2+9)} - \frac{11s-25}{10(s^2-6s+15)} \\
&= \frac{1}{10}\left(\frac{s+1}{s^2+9} - \frac{11s-25}{(s-3)^2+6}\right) \\
&= \frac{1}{10}\left(\frac{s}{s^2+9} + \frac{1}{3}\left(\frac{3}{s^2+9}\right) - \frac{11(s-3)}{(s-3)^2+6} - \frac{8}{\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}\right).
\end{aligned}$$

Then, taking the inverse laplace transform, we obtain

$$y(t) = \frac{1}{10}\left(\cos(3t) + \frac{1}{3}\sin(3t) - 11e^{3t}\cos(\sqrt{6}t) - \frac{8}{\sqrt{6}}e^{3t}\sin(\sqrt{6}t)\right).$$

□

§7.3 Systems of Differential Equations

Laplace transforms can also be used to solve systems of ordinary differential equations.

Example 7.3.1

Solve for $y_1(t)$ and $y_2(t)$:

$$\begin{aligned}
y_1' &= -3y_1 - 10y_2 \\
y_2' &= y_1 + 4y_2.
\end{aligned}$$

given $y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Solution. Taking the laplace transform of both equations, we find that

$$\begin{aligned}-y_1(0) + s\mathcal{Y}_1(s) &= -3\mathcal{Y}_1(s) - 10\mathcal{Y}_2(s)\\ -y_2(0) + s\mathcal{Y}_2(s) &= \mathcal{Y}_1(s) + 4\mathcal{Y}_2(s)\end{aligned}$$

Plugging in the initial values and solving for $\mathcal{Y}_1(s)$ and $\mathcal{Y}_2(s)$ gives

$$\begin{aligned}\mathcal{Y}_1(s) &= \frac{s-4}{s^2-s-2} = \frac{5/3}{s+1} - \frac{2/3}{s-2}\\ \mathcal{Y}_2(s) &= \frac{1}{s^2-s-2} = \frac{1/3}{s-2} - \frac{1/3}{s+1}.\end{aligned}$$

Taking the inverse laplace transfrom of both we obtain the solution to be

$$\begin{aligned}y_1(t) &= \frac{5}{3}e^{-t} - \frac{2}{3}e^{2t}\\ y_2(t) &= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}.\end{aligned}$$

□

Example 7.3.2

Solve for $y_1(t)$ and $y_2(t)$:

$$\begin{aligned}y_1' &= y_2\\ y_2' &= -4y_1,\end{aligned}$$

given $y(0) = \begin{pmatrix}0\\1\end{pmatrix}$.

Solution. Taking the laplace transform of both equations, we find that

$$\begin{aligned}-y_1(0) + s\mathcal{Y}_1(s) &= \mathcal{Y}_2(s)\\ -y_2(0) + s\mathcal{Y}_2(s) &= -4\mathcal{Y}_1(s).\end{aligned}$$

Plugging in the initial values ans solving for $\mathcal{Y}_1(s)$ and $\mathcal{Y}_2(s)$ gives

$$\begin{aligned}\mathcal{Y}_1(s) &= \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+4}\\ \mathcal{Y}_2(s) &= \frac{s}{s^2+4}.\end{aligned}$$

Taking the inverse laplace transform of both we obtain the solution to be

$$\begin{aligned}y_1(t) &= \frac{1}{2}\sin(2t)\\ y_2(t) &= \cos(2t).\end{aligned}$$

□

Example 7.3.3

Solve for $y_1(t)$ and $y_2(t)$:

$$\begin{aligned} y_1' &= 8y_1 + y_2 \\ y_2' &= -4y_1 + 4y_2, \end{aligned}$$

given $y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution. Taking the laplace transfrom of both equations, we find that

$$\begin{aligned} -y_1(0) + s\mathcal{Y}_1(s) &= 8\mathcal{Y}_1(s) + \mathcal{Y}_2(s) \\ -y_2(0) + s\mathcal{Y}_2(s) &= -4\mathcal{Y}_1(s) + 4\mathcal{Y}_2(s). \end{aligned}$$

Plugging in the initial values and solving for $\mathcal{Y}_1(s)$ and $\mathcal{Y}_2(s)$ gives

$$\begin{aligned} \mathcal{Y}_1(s) &= \\ \mathcal{Y}_2(s) &= \end{aligned}$$

□

Example 7.3.4

Solve for $y(t)$:

$$ty'' - (t+1)y' + y = 0,$$

given $y'(0) = 0$.

Solution. We first notice that

$$\begin{aligned} \mathcal{L}\{ty''\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(-y'(0) - sy(0) + s^2\mathcal{Y}(s)\right) \\ &= -2s\mathcal{Y}(s) - s^2\frac{\mathrm{d}\mathcal{Y}(s)}{\mathrm{d}s} \\ \mathcal{L}\{ty'\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(-y(0) + s\mathcal{Y}(s)\right) \\ &= -\mathcal{Y}(s) - s\frac{\mathrm{d}\mathcal{Y}(s)}{\mathrm{d}s}. \end{aligned}$$

Thus, taking the laplace transform of both sides and rearranging gives

$$\begin{aligned} (s-s^2)\mathcal{Y}'(s) + (2-3s)\mathcal{Y}(s) = 0 &\implies (s-s^2)\mathcal{Y}'(s) = (3s-2)\mathcal{Y}(s) \\ &\implies \frac{\mathcal{Y}'(s)}{\mathcal{Y}(s)} = \frac{2-3s}{s^2-s} \\ &\implies \ln\mathcal{Y}(s) = \int -\frac{2}{s} - \frac{1}{s-1} \\ &\implies \ln\mathcal{Y}(s) = -2\ln|s| - \ln|s-1| \\ &\implies \mathcal{Y}(s) = \frac{1}{s^2(s-1)} \\ &\implies \mathcal{Y}(s) = -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1} \\ &\implies y(t) = e^t - t - 1. \end{aligned}$$

□

Example 7.3.5

Solve for $y(t)$:

$$ty'' + 2y' + 4ty = 0,$$

given $y(0) = 1$.

Solution. We first notice that

$$\begin{aligned}
\mathcal{L}\{ty''\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(-y'(0) - sy(0) + s^2\mathcal{Y}(s)\right) \\
&= -s\mathcal{Y}(s) - s^2\frac{\mathrm{d}\mathcal{Y}(s)}{\mathrm{d}s} \\
\mathcal{Y}\{ty\} &=
\end{aligned}$$

□