Post Calculus

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Contents

1 Linear Algebra

§1.1 Eigenvectors and Eigenvalues

Definition 1.1.1. A **homogeneous linear system** is one where $\mathbf{A}\mathbf{x} = \mathbf{0}$.

The trivial solution is when $\mathbf{x} = \mathbf{0}$. Non-trivial solutions exist iff $\det(\mathbf{A}) = 0$.

Theorem 1.1.2

Let $\mathbf{A}$ be a square matrix.

$$\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$$

if and only if $\vec{\mathbf{v}}$ is an eigenvector and λ is an eigenvalue.

We notice that

$$\begin{aligned} \mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} &\implies \mathbf{A}\vec{\mathbf{v}} = \lambda\mathbf{I}\vec{\mathbf{v}} \\ &\implies \vec{\mathbf{v}}\left(\mathbf{A} - \lambda\mathbf{I}\right) = \mathbf{0} \\ &\implies \det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0. \end{aligned}$$

The result above is known as the **characteristeric equation**.

Example 1.1.3

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix},$$

find the eigenvalues and eigenvectors of $\mathbf{A}$.

We first notice that $\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$. Meaning that

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix}\right) = 0,$$

implying that $\lambda^2 + 4 = 0$. Thus, $\lambda = \pm 2i$. For $\lambda_1 = 2i$, we have

$$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix}\vec{\mathbf{v}} = \mathbf{0}.$$

By gaussian elimination we arrive at the first eigenvector

$$\vec{\mathbf{v}_1} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Similarly

$$\vec{\mathbf{v}_2} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

for $\lambda_2 = -2i$.

§1.2 Diagonalization

For $\mathbf{y} = \mathbf{A}\mathbf{x}$, assume $\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$$

be a transformation matix. Let

$$\mathbf{y} = \mathbf{P}\mathbf{y}' \text{ and } \mathbf{x} = \mathbf{P}\mathbf{x}'.$$

We can then write a new linear system relating $\mathbf{y}'$ and $\mathbf{x}'$, giving

$$\begin{aligned} \mathbf{P}\mathbf{y}' = \mathbf{A}\mathbf{P}\mathbf{x}' &\implies \mathbf{P}^{-1}\mathbf{P}\mathbf{y}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{P}\mathbf{x}' \\ &\implies \mathbf{y}' = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\mathbf{x}', \end{aligned}$$

where $\mathbf{D} = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)$ is a diagonal matrix such that the diagonals turn out to be the eigenvalues of $\mathbf{A}$.

> **Remark 1.2.1.** In order to find the diagonalization matrix, begin by finding the eigenvalues and eigenvectors of $\mathbf{A}$. Then find $\mathbf{P}$ from the eigenvetors. The diagonlization matrix follows.

Example 1.2.2

Diagonalize

$$\begin{bmatrix} 7/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 5/4 \end{bmatrix}.$$

§1.3 Rotating Conics

Let Q be a quadratic form such that

$$Q = ax_1^2 + (b+c)x_1x_2 + dx_2^2.$$

We can express this as

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We can convert the quadratic form to the canonical form by using the diagonalized matrix of $\mathbf{A}$:

$$C = \begin{bmatrix} x_1' & x_2' \end{bmatrix} \mathbf{D} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

Additionally, note that the transformation matrix is the rotation matrix, meaning that

$$\mathbf{P} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

Theorem 1.3.1

Let Q_n be a quadratic form with n dimensions and C_n the corresponding canonical form. Then, for an $n \times n$ matrix $\mathbf{A}$ and its diagonalized matrix $\mathbf{D}$,

$$Q_n = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$C_n = \begin{bmatrix} x'_1 & x'_2 & \cdots & x'_n \end{bmatrix} \mathbf{D} \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix}.$$

If any linear terms are present, they can be expressed as the product of the coefficient matrix and the matrix with each variable. So,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Exercise 1.3.2.

2 3D Optimization

§2.1 Partial Derivatives

Let $f(x, y, z) = 2x^2 + 3y^2 + z^2$. Partial derivatives treat the other variables as constants. Thus,

$$\frac{\partial f}{\partial x} = 4x, \ \frac{\partial f}{\partial y} = 6y, \ \frac{\partial f}{\partial z} = 2.$$

Definition 2.1.1. The **gradient vector** for a function f is

$$\nabla f(p) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{bmatrix}.$$

Definition 2.1.2. The **Hessian Matrix** is

$$\mathbf{H}_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix},$$

Theorem 2.1.3 (Schwarz's Theorem)
For a function $f : \Omega \to \mathbb{R}$ defined on a set $\Omega \subset \mathbb{R}^n$, if $\mathbf{p} \in \mathbb{R}^n$ is a point such that some neighborhood of $\mathbf{p}$ is contained in Ω and f has continuous second partial derivatives at the point $\mathbf{p}$, then $\forall i, j \in \{1, 2, \ldots, n\}$

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{p}) = \frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{p}).$$

Proof. The proof is left as a search on wikipedia :p. □

Example 2.1.4
Find the gradient and hessian matrix of $f(x, y, z) = 2x^2 + 3y^2 + z^2$.

Solution. We find that

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix}$$

and

$$\mathbf{H}_f = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

□

Example 2.1.5

Find the gradient and hessian matrix of $f(x, y) = x^2 - 3xy + y^2$.

Solution. We find that

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ 2y - 3x \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}.$$

□

Example 2.1.6

Find the gradient and hessian matrix $f(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 - 6x_2x_3$.

Solution. We find that

$$\nabla f = \begin{bmatrix} 18x_1 - 2x_2 + 4x_3 \\ -2x_1 + 14x_2 - 6x_3 \\ 4x_1 - 6x_2 + 6x_3 \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 18 & -2 & 4 \\ -2 & 14 & -6 \\ 4 & -6 & 6 \end{bmatrix}.$$

□

Example 2.1.7

Let x be a function of t. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial \dot{x}}, \frac{\partial f}{\partial \ddot{x}}$, and $\frac{df}{dt}$.

Theorem 2.1.8 (Generalized Chain Rule)
Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentialble function of m independent variables, and for each $i \in \{1, \ldots, m\}$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any $j \in \{1, \ldots, n\}$.

Proof. Deez nutz □

§2.2 Tangent Planes

We proceed by finding the equation of the tangent plane to $x^2 - y^2 - z^2 = 1$ at $(1, 0, 0)$. To begin, we find the gradient of $f(x, y, z) = x^2 - y^2 - z^2$ to be

$$\nabla f(1, 0, 0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Then, the "point-slope" form of a plane is

$$m_x(x - x_1) + m_y(y - y_1) + m_z(z - z_1) = 0.$$

Thus, we obtain the following tangent plane for our scenario: $2(x - 1) = 0$.

§2.3 Unconstrained Optimization

Definition 2.3.1. A **stationary point** is a critical point in higher dimensions. They can be found from the solution to the system of equations that results from letting the gradient equal zero.

Definition 2.3.2. A hessian is called **positive definite** if all the eigenvalues are positive.

Definition 2.3.3. A hessian is called **negative definite** if all the eigenvalues are negative.

Definition 2.3.4. A hessian is called **positive semidefinite** if all the eigenvalues are nonnegative and there exists at least one eigenvalue that is 0.

Definition 2.3.5. A hessian is called **negative semidefinite** if all the eignevalues are nonpositive and there exists at least one eigen value that is 0.

Definition 2.3.6. A point is a **saddle point** if the hessian has negative and positve eigenvalues.

As an alternative to the second derivative test in determining if a critical point is a max , min or an inflection point, the **hessian** will be used to determine if a stationary point is a max, min or inflection point.

Theorem 2.3.7 (Second Partial Derivative Test)

We can determine if the hessian is "positive" or "negative" by taking a look at its eigen values. Let $\mathbf{H}_f$ be the hessian for f, a differentiable function of n independent variables. Also, let $\Lambda = \{\lambda_i | 1 \leq i \leq n\}$ be the set of the eigenvalues of $\mathbf{H_f}$. If all elements in Λ are positive, then the hessian is called **positive definite**, giving a minimum. If all elements in Λ are negative, then the hessian is called **negative definite**, giving a maximum. If $\lambda_i \geq 0$, $\mathbf{H_f}$ is called **positive semidefinite**. If $\lambda_i \leq 0$, $\mathbf{H_f}$ is called **negative semidefinite**. If one eigenvalue is positive and one eigenvalue is negative, we have a **saddle point**.

Example 2.3.8

Find the critical points of $f(x_1, x_2) = x_1^2 - 2x_1x_2 + 4x_2^2$.

Proof. We begin by finding the gradient of f. This is

$$\nabla f = \begin{bmatrix} 2x_1 - 2x_2 \\ -2x_1 + 8x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$\begin{aligned} 2x_1 - 2x_2 &= 0 \\ -2x_1 + 8x_2 &= 0. \end{aligned}$$

This gives $(x_1, x_2) = (0, 0)$. They hessian, $\mathbf{H}_f$ of f is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -2 \\ -2 & 8 \end{bmatrix}.$$

Since the eigen values of this hessian are both positive, we have a minimum. □

Example 2.3.9

Find the critical points of $f(x_1, x_2) = -x_1^2 + 2x_1x_2 + 3x_2^2 + 8x_1$.

Proof. We begin by finding the gradient of f. This is

$$\nabla f = \begin{bmatrix} -2x_1 + 2x_2 + 8 \\ 2x_1 + 6x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$\begin{aligned} -2x_1 + 2x_2 &= -8 \\ 2x_1 + 6x_2 &= 0. \end{aligned}$$

Solving gives $(x_1, x_2) = (-3, 1)$. The hessian, $\mathbf{H}_f$, of f is

$$\mathbf{H}_f = \begin{bmatrix} -2 & 2 \\ 2 & 6 \end{bmatrix}.$$

Since one of the eignevalues of this hessian is positive and the other is negative, we have a saddle point. □

Example 2.3.10

Find the critical points of $f(x_1, x_2) = \left(x_1 - x_2^2\right)\left(x_1 - 3x_2^2\right)$.

Proof. To begin, we distribute to get $f(x_1, x_2) = x_1^2 - 4x_1x_2^2 + 3x_2^4$. The gradient of f is then

$$\nabla f = \begin{bmatrix} 2x_1 - 4x_2^2 \\ -8x_1x_2 + 12x_2^3 \end{bmatrix}.$$

Setting the gradient to zero and solving the resulting system of equations, we get the following critical point $(x_1, x_2) = (0, 0)$. The hessian of f is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -8x_2 \\ -8x_2 & -8x_1 + 36x_2^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since one eigenvalue is positive and the other is equal to 0, the hessian is postive semidefinite. □

Example 2.3.11

Kartik and Monica invested \$20,000 in the design and the development of a new product. They can manufacture it for \$2 per unit. They hired marketing consultants to determine the relation between selling price, the amount spent on advertising, and the number of units that would be sold as a result of the first two combined. The company determined that units sold would follow the equation

$$2000 + 4\sqrt{a} - 20p.$$

Determine the profit that Felicia and Megan will make as a function of the money spent on advertising, a, and the price of the product, p. Maximize that profit.

Proof. We first identify that the revenue gained from sales would be $p\left(2000 + 4\sqrt{a} - 20p\right)$. Then, the costs would be $20000 + 2\left(2000 + 4\sqrt{a} - 20p\right) + a$. Taking the difference, we get the profit P being

$$\begin{aligned} P(a, p) &= p\left(2000 + 4\sqrt{a} - 20p\right) - 20000 - 2\left(2000 + 4\sqrt{a} - 20p\right) - a \\ &= 2040p + 4p\sqrt{a} - 20p^2 - 24000 - 8\sqrt{a} - a. \end{aligned}$$

The gradient of P is

$$\nabla P = \begin{bmatrix} \dfrac{2p - 4}{\sqrt{a}} - 1 \\ -40p + 4\sqrt{a} + 2040 \end{bmatrix}.$$

Setting the gradient to 0 and solving the resulting system of equations, we get that $p = 63.25$ and $a = 15006.25$. The maximum profit is $\boxed{\$40025}$. □

§2.4 Constrained Optimization

Equality and Inequality constraints

Definition 2.4.1. The **Lagrangian** is

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)),$$

where $g_i(x)$ are constraints.

Example 2.4.2

Maximize $f(x_1, x_2) = 5 - (x_1 - 2)^2 - 2(x_2 - 1)^2$ subject to $x_1 + 4x_2 = 3$.

Proof. We get

$$\begin{aligned}\mathcal{L}(x, \lambda) &= f(x_1, x_2) + \lambda(x_1 + 4x_2 - 3) \\ &= 5 - x_1^2 + 4x_1 - 4 - 2x_2^2 + 4x_2 - 2 + \lambda(x_1 + 4x_2 - 3) \\ &= -x_1^2 - 2x_2^2 + 4x_1 + 4x_2 - 1 + \lambda(x_1 + 4x_2 - 3)\end{aligned}$$

Thus

$$\nabla \mathcal{L} = \begin{bmatrix} -2x_1 + \lambda + 4 \\ -4x_2 + 4\lambda + 4 \\ x_1 + 4x_2 - 3 \end{bmatrix} = 0$$

We get $(x_1, x_2, \lambda) = (5/3, 1/3, -2/3)$. Thus our critical point is $\boxed{(5/3, 1/3, 4)}$. □

Example 2.4.3

Let the sun be located at the origin of a coordinate plane. How close does Halley's comet come to the sun on its orbit?

$$171.725x^2 + 171.725y^2 + 297.37xy + 557.178x - 557.178y - 562.867 = 0.$$

Proof. We seek to optimize the distance from the comet to the sun. Thus, we seek to optimize $f(x, y) = x^2 + y^2$. Optimization with the lagrangian follows, where

$$\mathcal{L} = x^2 + y^2 - \lambda().$$

□

Example 2.4.4

Minimize $L(x) = x_1 e^{-(x_1^2 + x_2^2)} + \frac{x_1^2 + x_2^2}{20}$ subject to $f(x) = \frac{x_1 x_2}{2} + (x_1^2 + 2)^2 + (x_1^2 - 2)^2/2 - 2 \leq 0$.

Proof. We find that

$$\nabla L = \begin{bmatrix} e^{-(x_1^2 + x_2^2)} - 2x_1^2 e^{-(x_1^2 + x_2^2)} + x_1/10 \\ -2x_1 x_2 e^{-(x_1^2 + x_2^2)} + x_2/10 \end{bmatrix} = 0$$

We find that the critical points are $(-1/2, \sqrt{\ln(10) - 1/4})$ and $(-1/2, -\sqrt{\ln(10) - 1/4})$. □

3 Ordinary Differential Equations

§3.1 Separable Differential Equations

A separable differential equation can be simplified to the form:

$$g(y)y' = f(x) \implies g(y)\,\mathrm{d}y = f(x)\,\mathrm{d}x,$$

where f and g are functions of x and y respectively. Letting $G'(y) = g(y)$ and $F'(x) = f(x)$ and taking the integral of both sides of the differential equation, we get

$$G(y) = F(x) + C$$

for some constant C. Using a separation of variables is most helpful as a first try when attempting first-order linear ordinary differential equations. We go more in detail later in the chapter.

Example 3.1.1
Find y for the following differential equation:

$$\frac{dy}{dx} = x^2y - 2xy.$$

Solution. Separating the variables, we see that $\frac{1}{y}\,\mathrm{d}y = (x^2 - 2x)\,\mathrm{d}x$. Taking the integral of both sides we get that

$$\ln|y| = \frac{1}{3}x^3 - x^2 + C \implies y = Ce^{\frac{1}{3}x^3 - x^2}.$$

□

Question 3.1.2. What is y when given the initial condition $y(0) = e$?

§3.2 Homogeneous Ordinary Differential Equations

The general from of a linear ordinary differential equation(ODE) is

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x),$$

where $a_i \in \mathbb{R}[x]$ for $i \in \{0, 1, \ldots, n\}$ and $y^{(j)}$ denotes the jth derivative of y. It is then said that this ODE has **order** n since it is the highest derivative in the equation.

Example 3.2.1 (ODE)

$$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} + 5 = 0 \implies y'' + 3y' + 5 = 0.$$

Remark 3.2.2. Partial differential equations also exist.

Definition 3.2.3. A **homogeneous** ordinary differential equation is one with $f(x) = 0$ and has constant coefficients.

Example 3.2.4 (Homogeneous ODE)

Consider the differential equation

$$y'' + 7y' + 12y = 0.$$

Functions of the form $y = e^{mx}$ are an educated guess for solving homoegeneous ODEs. So, letting $y = e^{mx}$ means $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging these in our differential equation,

$$\begin{aligned} m^2e^{mx} + 7me^{mx} + 12e^{mx} &= 0 \\ e^{mx}\left(m^2 + 7m + 12\right) &= 0 \\ e^{mx}(m+4)(m+3) &= 0. \end{aligned}$$

Therefore, $m = -4, -3$, implying that $y_1 = C_1e^{-4x}$ and $y_2 = C_2e^{-3x}$. This means that our solution of the differential equation is

$$y_h = y_1 + y_2 = C_1e^{-4x} + C_2e^{-3x}.$$

At first, it might be puzzling why we consider $y = e^{mx}$ to be an educated guess for solving homogeneous ODEs with constant coeficients. However, let us consider the following first-order differential equation

$$y' + ky = 0 \implies y' = -ky.$$

Here, it is natural to guess that $y = e^{mx}$ because e^{mx} is the only function where its derivative is equal to some scalar multiple of itsself. So, for other homogeneous ODEs with constant coefficient and of higher degree, we also use the guess as a means of finding the general solution.

Exercise 3.2.5 (Homogeneous ODE with initial values)**.**

$$y'' + y' - 2y = 0, \; y(0) = 4, \; y'(0) = -5.$$

Now, we consider what happens if we get a polynomial with repeated roots or if we are already given a solution to the differential equation.

Example 3.2.6 (Homogeneous ODE with repeated roots)

$$y'' + 4y' + 4y = 0.$$

Let $y = e^{mx}$. This means $y' = me^{mx}$ and $y'' = m^2e^{mx}$. PLugging in, we get

$$\begin{aligned} m^2e^{mx} + 4me^{mx} + 4e^{mx} &= 0 \\ e^{mx}(m+2)^2 &= 0. \end{aligned}$$

Thus, $\boxed{y_1 = C_1e^{-2x}}$. We now proceed to use the **reduction of order** method. So, $y_2 = f(x) \cdot y_1$. First, we have

$$\begin{aligned} y_2' &= f'(x)y_1 + f(x)y_1' = f'(x)e^{-2x} - 2f(x)e^{-2x} \\ y_2'' &= f''(x)y_1 + 2f'(x)y_1' + f(x)y_1'' = f''(x)e^{-2x} - 4f'(x)e^{-2x} + 4f(x)e^{-2x}. \end{aligned}$$

Plugging this into the original differential equation, we get

$$f''(x)e^{-2x} - 4f'(x)e^{-2x} + 4f(x)e^{-2x} + 4\left(f'(x)e^{-2x} - 2f(x)e^{-2x}\right) + 4f(x)e^{-2x} = 0.$$

Simplifying yields $f''(x) = 0$. Now, let $g(x) = f'(x)$. This means $g'(x) = f''(x)$. So, $g'(x) = 0$. This is equivalent to $\frac{dg}{dx} = 0$. Using separation of variables, we get $g(x) = C$. So, $f'(x) = C$. This is the same as $\frac{df}{dx} = C \implies f(x) = C_1x + C_2$. Thus,

$$\boxed{y_2 = (C_1x + C_2)\, e^{-2x}}.$$

Adding the two solutions, we obtain

$$\begin{aligned} y_h &= y_1 + y_2 = C_0e^{-2x} + C_1xe^{-2x} + C_2e^{-2x} \\ y_h &= Ce^{-2x} + Dxe^{-2x}. \end{aligned}$$

Theorem 3.2.7 (General Reduction of Order)

Consider a general homogeneous linear ODE:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

and suppose $y_1(x)$ is one solution to the differential equation. Then the second solution to the differential equation is

$$y_2(x) = y_1(x) \int \frac{u(x)}{[y_1(x)]^2}\, \mathrm{d}x,$$

where $u(x) = e^{-\int p(x)\, \mathrm{d}x}$.

Proof. We assume the second solution is of the form $y_2(x) = f(x)y_1(x)$ for some function f. Thus,

$$y_2' = f'y_1 + fy_1' \qquad y_2'' = f''y_1 + 2f'y_1' + fy_1''.$$

Substituting these expressions back into the original differential equation, we get

$$f''y_1 + 2f'y_1' + fy_1'' + p(f'y_1 + fy_1') + qfy_1 = 0.$$

Rearranging these terms, we obtain

$$f(y_1'' + py_1' + qy_1) + f''y_1 + 2f'y_1' + pf'y_1 = 0.$$

Notice that the term in the parentheses becomes 0 since y_1 is a solution to the differential equation. Therefore, we are simply left with

$$f''y_1 + 2f'y_1' + pf'y_1 = 0.$$

Rearranging and letting $g(x) = f'(x)$,

$$g'(x) = -g(x)\left(2\frac{y_1'(x)}{y_1(x)} + p(x)\right).$$

Now, we simply have a first-order ODE in terms of $g(x)$. It can then be shown that $f(x) = \int \frac{u(x)}{[y_1(x)]^2}\,\mathrm{d}x$ where $u(x) = e^{-\int p(x)\,\mathrm{d}x}$. It then follows that

$$y_2(x) = y_1(x)\int \frac{u(x)}{[y_1(x)]^2}\,\mathrm{d}x.$$

□

Corollary 3.2.8 (Reduction of Order with Constant Coefficients)

Consider a homogeneous linear ODE with constant coeffiecients:

$$ay''(x) + by'(x) + cy(x) = 0$$

where $a, b, c \in \mathbb{R}\backslash\{0\}$ and suppose the discriminant, $b^2 - 4ac$, vanishes due to the presence of a repeated root in the characteristic equation. Additionally suppose $y_1(x)$ is one solution to the differential equation. Then the second solution to the differential equation is $y_2(x) = xy_1(x)$.

Proof. Letting $y = e^{mx}$ and solving for m from the resulting characteristic equation gives $m = -\frac{b}{2a}$. Therefore $y_1(x) = e^{-\frac{b}{2a}x}$. □

Example 3.2.9

$$y'' + y' + (1/4)y = 0;\ \ y(0) = 3;\ \ y'(0) = -3.5.$$

Solution. Let $y = e^{mx}$. This means $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging in, we get

$$\begin{aligned} 4m^2e^{mx} + 4me^{mx} + e^{mx} &= 0 \\ e^{mx}(4m^2 + 4m + 1) &= 0 \\ e^{mx}(2m+1)^2 &= 0. \end{aligned}$$

Thus, $\boxed{y_1 = C_0e^{-\frac{1}{2}x}}$. We now proceed to use reduction of order. So, $y_2 = f(x)\cdot y_1$. First, we have

$$\begin{aligned} y_2' &= f'(x)y_1 + f(x)y_1' = f'(x)e^{-x/2} - \frac{1}{2}f(x)e^{-x/2} \\ y_2'' &= f''(x)y_1 + 2f'(x)y_1' + f(x)y_1'' = f''(x)e^{-x/2} - f'(x)e^{-x/2} + \frac{1}{4}f(x)e^{-x/2}. \end{aligned}$$

Plugging this into the original differential equation, we get

$$f''(x)e^{-x/2} - f'(x)e^{-x/2} + \frac{1}{4}f(x)e^{-x/2} + f'(x)e^{-x/2} - \frac{1}{2}f(x)e^{-x/2} + \frac{1}{4}f(x)e^{-x/2} = 0$$
$$f''(x) = 0.$$

Now, let $g(x) = f'(x)$. This means that $g'(x) = f''(x)$. So, $g'(x) = 0$. This means $g(x) = C$. So, $f'(x) = C$. Thus, $f(x) = C_1x + C_2$. THerefore,

$$y_2 = (C_1x + C_2)e^{-x/2}.$$

Adding the two solutions, we obtain

$$\begin{aligned} y_h = y_1 + y_2 &= C_0e^{-x/2} + C_1xe^{-x/2} + C_2e^{-x/2} \\ &= Ce^{-x/2} + Dxe^{-x/2}. \end{aligned}$$

Using the inital values, we obtain

$$y_h = 3e^{-x/2} - 2xe^{-x/2}.$$

□

In general $y_2 = xy_1$.

Example 3.2.10

$$y''' + 3y'' - 4y = 0.$$

Solution. Let $y = e^{mx}$. This mean $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging in, we get

$$\begin{aligned} m^3e^{mx} + 3m^2e^{mx} - 4e^{mx} &= 0 \\ e^{mx}\left(m^3 + 3m^2 - 4\right) &= 0 \\ e^{mx}(m-1)(m^2 + 4m + 4) &= 0 \\ e^{mx}(m-1)(m+2)^2 &= 0. \end{aligned}$$

Therefore, $y_1 = C_1e^x$, $y_2 = C_2e^{-2x}$, and $y_3 = C_3xe^{-2x}$. Thus,

$$\boxed{y_h = c_1e^x + C_2e^{-2x} + C_3xe^{-2x}}.$$

□

Example 3.2.11 (Complex Roots)

$$y'' + 4y = 0.$$

Solution. Let $y = e^{mx}$. This means that $y'' = m^2e^{mx}$. Plugging in gives $m = \pm 2i$. Thus, $y_h = C_1e^{2ix} + C_2e^{-2ix}$. Recall **Euler's Theorem**

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Plugging this in gives

$$\begin{aligned} y_h &= C_1(\cos 2x + i\sin 2x) + C_2(\cos 2x - i\sin 2x) \\ y_h &= C_1\cos 2x + C_2\sin 2x. \end{aligned}$$

□

Example 3.2.12 (Complex Roots)

$$y'' + 8y' + 25y = 0.$$

Solution. Let $y = e^{mx}$. This meas that $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging this in gives $-4 \pm 3i$. Thus,

$$\begin{aligned} y_h &= C_1e^{(-4+3i)x} + C_2e^{(-4-3i)x} \\ y_h &= e^{-4x}\left(C_1e^{3ix} + C_2e^{-3ix}\right) \\ y_h &= e^{-4x}\left(C_1\cos(3x) + C_2\sin(3x)\right). \end{aligned}$$

□

§3.3 Non Homogeneous ODEs

This occurs when $f(x) \neq 0$. Our solution will still include the homogeneous potion $y = y_h(x) + y_p(x)$. Some methods for solving these are **undetermined coefficients**, which can only be used for scalar coefficients, and **variation of parameters**, which can be used with scalar or variable coefficients.

Example 3.3.1 (Undetermined Coefficients)

$$y'' - 4y' + 3y = e^{-x};\ y(0) = 1;\ y'(0) = 0.$$

Solution. We first find y_h using $y'' - 4y' + 3y = 0$. Thus we obtain

$$y_h = C_1 e^x + C_2 e^{3x}.$$

Our guess for $y_p = Ce^{-x}$. Thus, $y_p' = -Ce^{-x}$ and $y_p'' = Ce^{-x}$. So, plugging this in, we get

$$\begin{aligned} Ce^{-x} + 4Ce^{-x} + 3Ce^{-x} = e^{-x} &\implies 8Ce^{-x} = e^{-x} \\ &\implies C = \frac{1}{8}. \end{aligned}$$

Therefore

$$y = y_h + y_p = C_1 e^x + C_2 e^{3x} + \frac{1}{8} e^{-x}.$$

Using the initial values, we find that

$$\boxed{y = \frac{5}{4} e^x - \frac{3}{8} e^{3x} + \frac{1}{8} e^{-x}}.$$

□

The following is a useful guessing guide:

$$\begin{aligned} r(x) &\implies y_p(x) \\ ke^{rx} &\implies Ce^{rx} \\ kx^n &\implies P_n = a_n x^n + a_{n-1} x^{n-1} + \cdots a_0 \\ k\cos(wx) \text{ or } k\sin(wx) &\implies a\cos(wx) + b\sin(wx). \end{aligned}$$

Example 3.3.2 (Guessing Practice with Undetermined Coefficients)

$$y'' + 2y' + y = \cos(2x)$$

Solution. We first proceed to find y_h. We find that

$$y_h = C_1 e^{-x} + C_2 x e^{-x}.$$

We make the following guess: $y_p = a\cos(2x) + b\sin(2x)$. Solving for a and b, we find that $y_p = -\frac{3}{25}\cos(2x) + \frac{4}{25}\sin(2x)$. Therefore

$$\boxed{y = C_1 e^{-x} + C_2 x e^{-x} - \frac{3}{25}\cos(2x) + \frac{4}{25}\sin(2x)}.$$

□

Example 3.3.3 (Bad Guessing)

$$y'' + 4y' + 3y = e^{-x}.$$

Solution. We notice $y_h = C_1e^{-x} + C_2e^{-3x}$. $y_p = Ce^{-x}$ is a bad guess since it is a term present in the homogeneous equation. So, we let $y_p = Cxe^{-x}$, which is a guess from the result of applying the reduction of order. Solving for C, we get that $C = \frac{1}{2}$. Therefore

$$\boxed{y = C_1e^{-x} + C_2e^{-3x} + \frac{1}{2}xe^{-x}}.$$

□

Theorem 3.3.4 (Variation of Parameters)

The method of variation of parameters is a technique for finding a particular solution to a nonhomogeneous linear second order ODE:

$$y'' + P(x)y' + Q(x)y = R(x)$$

provided that the general solution of the corresponding homogeneous linear second order ODE:

$$y'' + P(x)y' + Q(x)y = 0$$

is already known. The particular solution is then

$$y_p = u(x)y_1(x) + v(x)y_2(x)$$

such that

$$u'(x) = -\frac{y_2R(x)}{W(y_1, y_2)} \text{ and } v'(x) = \frac{y_1R(x)}{W(y_1, y_2)},$$

where $W(y_1, y_2)$ denotes the Wronskian of y_1 and y_2.

Proof. First, we have

$$\begin{aligned} y_p' &= (u'y_1 + uy_1') + (v'y_2 + vy_2') \\ &= (uy_1' + vy_2') + (u'y_1 + v'y_2) \end{aligned}$$

by the product rule. □

Example 3.3.5 (Variation of parameters)

$$y'' + 4y' + 4y = e^{-2x}/x^2.$$

Solution. We find that $y_h = C_1e^{-2x} + C_2xe^{-2x}$. Notice that the Wronskian is

$$w = \det\left(\begin{bmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{bmatrix}\right) = e^{-4x}.$$

Thus,

$$u'(x) = -\frac{1}{x}$$
$$v'(x) = \frac{1}{x^2}.$$

Thus, $u(x) = -\ln|x|$ and $v(x) = -\frac{1}{x}$. Therefore,

$$y_p = \ln|x|e^{2x} - e^{-2x} \implies \boxed{y = C_1e^{-2x} + C_2xe^{-2x} + \ln|x|e^{2x}}.$$

□

Example 3.3.6 (Undetermined Coefficients)

$$y'' + y' + y = 3x^2 + 4.$$

Solution. We first find that

$$y_h = e^{(-1/2)x}\left(C_1\cos(\sqrt{3}x/2) + C_2\sin(\sqrt{3}x/2)\right).$$

Then we let $y_p = D_1x^2 + D_2x + D_3$. Therefore

$$2D_1 + 2xD_1 + D_2 + D_1x^2 + D_2x + D_3 = 3x^2 + 4$$
$$x^2(D_1) + x(2D_1 + D_2) + (2D_1 + D_2 + D_3) = 3x^2 + 4.$$

Solving the resulting system yields $D_1 = 3, D_2 = -6, D_3 = 4$. Therefore

$$y_p = 3x^2 - 6x + 4.$$

So,

$$\boxed{y = e^{(-1/2)x}\left(C_1\cos(\sqrt{3}x/2) + C_2\sin(\sqrt{3}x/2)\right) + 3x^2 - 6x + 4}.$$

□

Example 3.3.7 (Variation of parameters)

$$y'' + 4y = \tan x.$$

Solution. We first find $y_h = C_1\cos(2x) + C_2\sin(2x)$. Now, we notice that wronskian is

$$w = \det\left(\begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}\right) = 2.$$

Thus,

$$u'(x) = -\frac{\tan x\sin(2x)}{2} = -\sin^2(x)$$

and

$$v'(x) = \frac{1}{2}\ln|\cos x| - \frac{\cos(2x)}{4} = \frac{\sin(2x)}{2} - \frac{1}{2}\tan x.$$

Solving, we find $u(x) = \frac{\sin(2x)}{4} - \frac{1}{2}x$. □