Post Calculus

Lazo Attar

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Contents

1 Linear Algebra

§1.1 Eigenvectors and Eigenvalues

Definition 1.1.1. A **homogeneous linear system** is one where $\mathbf{A}\mathbf{x} = \mathbf{0}$.

The trivial solution is when $\mathbf{x} = \mathbf{0}$. Non-trivial solutions exist iff $\det(\mathbf{A}) = 0$.

Theorem 1.1.2

Let $\mathbf{A}$ be a square matrix.

$$\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$$

if and only if $\vec{\mathbf{v}}$ is an eigenvector and λ is an eigenvalue.

We notice that

$$\begin{aligned}
\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} &\implies \mathbf{A}\vec{\mathbf{v}} = \lambda\mathbf{I}\vec{\mathbf{v}} \\
&\implies \vec{\mathbf{v}}\left(\mathbf{A} - \lambda\mathbf{I}\right) = \mathbf{0} \\
&\implies \det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0.
\end{aligned}$$

The result above is known as the **characteristeric equation**.

Example 1.1.3

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix},$$

find the eigenvalues and eigenvectors of $\mathbf{A}$.

We first notice that $\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$. Meaning that

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix}\right) = 0,$$

implying that $\lambda^2 + 4 = 0$. Thus, $\lambda = \pm 2i$. For $\lambda_1 = 2i$, we have

$$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix}\vec{\mathbf{v}} = \mathbf{0}.$$

By gaussian elimination we arrive at the first eigenvector

$$\vec{\mathbf{v}_1} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Similarly

$$\vec{\mathbf{v}_2} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

for $\lambda_2 = -2i$.

§1.2 Diagonalization

For $\mathbf{y} = \mathbf{A}\mathbf{x}$, assume $\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$$

be a transformation matix. Let

$$\mathbf{y} = \mathbf{P}\mathbf{y}' \text{ and } \mathbf{x} = \mathbf{P}\mathbf{x}'.$$

We can then write a new linear system relating $\mathbf{y}'$ and $\mathbf{x}'$, giving

$$\begin{aligned} \mathbf{P}\mathbf{y}' = \mathbf{A}\mathbf{P}\mathbf{x}' &\implies \mathbf{P}^{-1}\mathbf{P}\mathbf{y}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{P}\mathbf{x}' \\ &\implies \mathbf{y}' = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\mathbf{x}', \end{aligned}$$

where $\mathbf{D} = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)$ is a diagonal matrix such that the diagonals turn out to be the eigenvalues of $\mathbf{A}$.

> **Remark 1.2.1.** In order to find the diagonalization matrix, begin by finding the eigenvalues and eigenvectors of $\mathbf{A}$. Then find $\mathbf{P}$ from the eigenvetors. The diagonlization matrix follows.

Example 1.2.2

Diagonalize

$$\begin{bmatrix} 7/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 5/4 \end{bmatrix}.$$

§1.3 Rotating Conics

Let Q be a quadratic form such that

$$Q = ax_1^2 + (b+c)x_1x_2 + dx_2^2.$$

We can express this as

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We can convert the quadratic form to the canonical form by using the diagonalized matrix of $\mathbf{A}$:

$$C = \begin{bmatrix} x_1' & x_2' \end{bmatrix} \mathbf{D} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

Additionally, note that the transformation matrix is the rotation matrix, meaning that

$$\mathbf{P} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

Theorem 1.3.1

Let Q_n be a quadratic form with n dimensions and C_n the corresponding canonical form. Then, for an $n \times n$ matrix $\mathbf{A}$ and its diagonalized matrix $\mathbf{D}$,

$$Q_n = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$C_n = \begin{bmatrix} x_1' & x_2' & \cdots & x_n' \end{bmatrix} \mathbf{D} \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}.$$

If any linear terms are present, they can be expressed as the product of the coefficient matrix and the matrix with each variable. So,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Exercise 1.3.2.

2 3D Optimization

§2.1 Partial Derivatives

Let $f(x,y,z) = 2x^2 + 3y^2 + z^2$. Partial derivatives treat the other variables as constants. Thus,

$$\frac{\partial f}{\partial x} = 4x,\ \frac{\partial f}{\partial y} = 6y,\ \frac{\partial f}{\partial z} = 2.$$

Definition 2.1.1. The **gradient vector** for a function f is

$$\nabla f(p) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{bmatrix}.$$

Definition 2.1.2. The **Hessian Matrix** is

$$\mathbf{H}_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix},$$

Theorem 2.1.3 (Schwarz's Theorem)

For a function $f : \Omega \to \mathbb{R}$ defined on a set $\Omega \subset \mathbb{R}^n$, if $\mathbf{p} \in \mathbb{R}^n$ is a point such that some neighborhood of $\mathbf{p}$ is contained in Ω and f has continuous second partial derivatives at the point $\mathbf{p}$, then $\forall i, j \in \{1, 2, \dots, n\}$

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{p}) = \frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{p}).$$

Proof. The proof is left as a search on wikipedia :p. □

Example 2.1.4

Find the gradient and hessian matrix of $f(x,y,z) = 2x^2 + 3y^2 + z^2$.

Solution. We find that

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix}$$

and

$$\mathbf{H}_f = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

□

Example 2.1.5

Find the gradient and hessian matrix of $f(x, y) = x^2 - 3xy + y^2$.

Solution. We find that

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ 2y - 3x \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}.$$

□

Example 2.1.6

Find the gradient and hessian matrix $f(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 - 6x_2x_3$.

Solution. We find that

$$\nabla f = \begin{bmatrix} 18x_1 - 2x_2 + 4x_3 \\ -2x_1 + 14x_2 - 6x_3 \\ 4x_1 - 6x_2 + 6x_3 \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 18 & -2 & 4 \\ -2 & 14 & -6 \\ 4 & -6 & 6 \end{bmatrix}.$$

□

Example 2.1.7

Let x be a function of t. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial \dot{x}}, \frac{\partial f}{\partial \ddot{x}}$, and $\frac{df}{dt}$.

Theorem 2.1.8 (Generalized Chain Rule)

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentialble function of m independent variables, and for each $i \in \{1, \ldots, m\}$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any $j \in \{1, \ldots, n\}$.

Proof. Deez nutz □

§2.2 Tangent Planes

We proceed by finding the equation of the tangent plane to $x^2 - y^2 - z^2 = 1$ at $(1, 0, 0)$. To begin, we find the gradient of $f(x, y, z) = x^2 - y^2 - z^2$ to be

$$\nabla f(1, 0, 0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Then, the "point-slope" form of a plane is

$$m_x(x - x_1) + m_y(y - y_1) + m_z(z - z_1) = 0.$$

Thus, we obtain the following tangent plane for our scenario: $2(x - 1) = 0$.

§2.3 Unconstrained Optimization

Definition 2.3.1. A **stationary point** is a critical point in higher dimensions. They can be found from the solution to the system of equations that results from letting the gradient equal zero.

Definition 2.3.2. A hessian is called **positive definite** if all the eigenvalues are positive.

Definition 2.3.3. A hessian is called **negative definite** if all the eigenvalues are negative.

Definition 2.3.4. A hessian is called **positive semidefinite** if all the eigenvalues are nonnegative and there exists at least one eigenvalue that is 0.

Definition 2.3.5. A hessian is called **negative semidefinite** if all the eignevalues are nonpositive and there exists at least one eigen value that is 0.

Definition 2.3.6. A point is a **saddle point** if the hessian has negative and positve eigenvalues.

As an alternative to the second derivative test in determining if a critical point is a max , min or an inflection point, the **hessian** will be used to determine if a stationary point is a max, min or inflection point.

Theorem 2.3.7 (Second Partial Derivative Test)

We can determine if the hessian is "positive" or "negative" by taking a look at its eigen values. Let $\mathbf{H}_f$ be the hessian for f, a differentiable function of n independent variables. Also, let $\Lambda = \{\lambda_i | 1 \leq i \leq n\}$ be the set of the eigenvalues of $\mathbf{H_f}$. If all elements in Λ are positive, then the hessian is called **positive definite**, giving a minimum. If all elements in Λ are negative, then the hessian is called **negative definite**, giving a maximum. If $\lambda_i \geq 0$, $\mathbf{H_f}$ is called **positive semidefinite**. If $\lambda_i \leq 0$, $\mathbf{H_f}$ is called **negative semidefinite**. If one eigenvalue is positive and one eigenvalue is negative, we have a **saddle point**.

11

Example 2.3.8

Find the critical points of $f(x_1, x_2) = x_1^2 - 2x_1x_2 + 4x_2^2$.

Proof. We begin by finding the gradient of f. This is

$$\nabla f = \begin{bmatrix} 2x_1 - 2x_2 \\ -2x_1 + 8x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$\begin{aligned} 2x_1 - 2x_2 &= 0 \\ -2x_1 + 8x_2 &= 0. \end{aligned}$$

This gives $(x_1, x_2) = (0, 0)$. They hessian, $\mathbf{H}_f$ of f is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -2 \\ -2 & 8 \end{bmatrix}.$$

Since the eigen values of this hessian are both positive, we have a minimum. □

Example 2.3.9

Find the critical points of $f(x_1, x_2) = -x_1^2 + 2x_1x_2 + 3x_2^2 + 8x_1$.

Proof. We begin by finding the gradient of f. This is

$$\nabla f = \begin{bmatrix} -2x_1 + 2x_2 + 8 \\ 2x_1 + 6x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$\begin{aligned} -2x_1 + 2x_2 &= -8 \\ 2x_1 + 6x_2 &= 0. \end{aligned}$$

Solving gives $(x_1, x_2) = (-3, 1)$. The hessian, $\mathbf{H}_f$, of f is

$$\mathbf{H}_f = \begin{bmatrix} -2 & 2 \\ 2 & 6 \end{bmatrix}.$$

Since one of the eignevalues of this hessian is positive and the other is negative, we have a saddle point. □

Example 2.3.10

Find the critical points of $f(x_1, x_2) = \left(x_1 - x_2^2\right)\left(x_1 - 3x_2^2\right)$.

Proof. To begin, we distribute to get $f(x_1, x_2) = x_1^2 - 4x_1x_2^2 + 3x_2^4$. The gradient of f is then

$$\nabla f = \begin{bmatrix} 2x_1 - 4x_2^2 \\ -8x_1x_2 + 12x_2^3 \end{bmatrix}.$$

Setting the gradient to zero and solving the resulting system of equations, we get the following critical point $(x_1, x_2) = (0, 0)$. The hessian of f is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -8x_2 \\ -8x_2 & -8x_1 + 36x_2^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since one eigenvalue is positive and the other is equal to 0, the hessian is postive semidefinite. □

Example 2.3.11

Kartik and Monica invested \$20,000 in the design and the development of a new product. They can manufacture it for \$2 per unit. They hired marketing consultants to determine the relation between selling price, the amount spent on advertising, and the number of units that would be sold as a result of the first two combined. The company determined that units sold would follow the equation

$$2000 + 4\sqrt{a} - 20p.$$

Determine the profit that Felicia and Megan will make as a function of the money spent on advertising, a, and the price of the product, p. Maximize that profit.

Proof. We first identify that the revenue gained from sales would be $p\,(2000 + 4\sqrt{a} - 20p)$. Then, the costs would be $20000 + 2\,(2000 + 4\sqrt{a} - 20p) + a$. Taking the difference, we get the profit P being

$$\begin{aligned} P(a,p) &= p\left(2000 + 4\sqrt{a} - 20p\right) - 20000 - 2\left(2000 + 4\sqrt{a} - 20p\right) - a \\ &= 2040p + 4p\sqrt{a} - 20p^2 - 24000 - 8\sqrt{a} - a. \end{aligned}$$

The gradient of P is

$$\nabla P = \begin{bmatrix} \dfrac{2p-4}{\sqrt{a}} - 1 \\ -40p + 4\sqrt{a} + 2040 \end{bmatrix}.$$

Setting the gradient to 0 and solving the resulting system of equations, we get that $p = 63.25$ and $a = 15006.25$. The maximum profit is $\boxed{\$40025}$. □

§2.4 Constrained Optimization

Equality and Inequality constraints

Definition 2.4.1. The **Lagrangian** is

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)),$$

where $g_i(x)$ are constraints.

Example 2.4.2

Maximize $f(x_1, x_2) = 5 - (x_1 - 2)^2 - 2(x_2 - 1)^2$ subject to $x_1 + 4x_2 = 3$.

Proof. We get

$$\begin{aligned}\mathcal{L}(x,\lambda) &= f(x_1,x_2)+\lambda(x_1+4x_2-3)\\ &= 5-x_1^2+4x_1-4-2x_2^2+4x_2-2+\lambda(x_1+4x_2-3)\\ &= -x_1^2-2x_2^2+4x_1+4x_2-1+\lambda(x_1+4x_2-3)\end{aligned}$$

Thus

$$\nabla\mathcal{L} = \begin{bmatrix} -2x_1+\lambda+4 \\ -4x_2+4\lambda+4 \\ x_1+4x_2-3 \end{bmatrix} = 0$$

We get $(x_1,x_2,\lambda)=(5/3,1/3,-2/3)$. Thus our critical point is $\boxed{(5/3,1/3,4)}$. □

Example 2.4.3

Let the sun be located at the origin of a coordinate plane. How close does Halley's comet come to the sun on its orbit?

$$171.725x^2+171.725y^2+297.37xy+557.178x-557.178y-562.867=0.$$

Proof. We seek to optimize the distance from the comet to the sun. Thus, we seek to optimize $f(x,y)=x^2+y^2$. Optimization with the lagrangian follows, where

$$\mathcal{L}=x^2+y^2-\lambda().$$

□

Example 2.4.4

Minimize $L(x)=x_1e^{-(x_1^2+x_2^2)}+\frac{x_1^2+x_2^2}{20}$ subject to $f(x)=\frac{x_1x_2}{2}+(x_1^2+2)^2+(x_1^2-2)^2/2-2\le 0$.

Proof. We find that

$$\nabla L = \begin{bmatrix} e^{-(x_1^2+x_2^2)}-2x_1^2e^{-(x_1^2+x_2^2)}+x_1/10 \\ -2x_1x_2e^{-(x_1^2+x_2^2)}+x_2/10 \end{bmatrix} = 0$$

We find that the critical points are $(-1/2,\sqrt{\ln(10)-1/4})$ and $(-1/2,-\sqrt{\ln(10)-1/4})$. □

3 Ordinary Differential Equations

§3.1 Separable Differential Equations

Example 3.1.1

Find y for the following differential equation:

$$\frac{dy}{dx} = x.$$

Solution. We see that $dy = x\ dx$. Taking the integral of both sides, we obtain

$$y = \frac{1}{2}x^2 + C.$$

□

Example 3.1.2

Find y for the following differential equation:

$$\frac{dy}{dx} = \frac{1}{x}.$$

Solution. We see that $dy = \frac{1}{x}\ dx$. Taking the integral of both sides, we obtain

$$y = \ln|x| + C.$$

□

Example 3.1.3

Find y for the following differential equation:

$$\frac{dy}{dx} = \cos x.$$

Solution. We see that $dy = \cos x\ dx$. Taking the integral of both sides, we obtain

$$\boxed{y = \sin x + C}.$$

□

Example 3.1.4

Find y for the following differential equation:

$$\frac{dy}{dx} = e^x.$$

Solution. We see that $dy = e^x \ dx$. Taking the integral of borth sides, we obtain

$$\boxed{y = e^x + C}.$$

□

Example 3.1.5

Find y for the following differential equation:

$$\frac{dy}{dx} = xy.$$

Solution. We see that $\frac{1}{y} \ dy = x \ dx$. Taking the integral of both sides, we get

$$\ln |y| = \frac{1}{2}x^2 + C \implies y = e^{\frac{1}{2}x^2 + C}.$$

□

Example 3.1.6

Find y for the following differential equation:

$$\frac{dy}{dx} = x^2y - 2xy.$$

Solution. We see that $dy = xy(x-2) \ dx \implies \frac{1}{y} \ dy = x^2 - 2x \ dx$. Taking the integral of both sides we get that

$$\ln |y| = \frac{1}{3}x^3 - x^2 + C \implies y = e^{\frac{1}{3}x^3 - x^2 + C}.$$

□

Example 3.1.7

Find y for the following differential equation:

$$(x^2 + x) \frac{dy}{dx} = y - 1.$$

Solution. We see that $\frac{1}{y-1} \ dy = \frac{1}{x^2+x} \ dx$. Taking the integral of both sides, we find that

$$\ln |y - 1| = \ln |x| - \ln |x + 1| + C \implies y = C \left| \frac{x}{x+1} \right| + 1.$$

□

Example 3.1.8

Find y for the following differential equation:

$$\frac{dy}{dx} = x^2y - 2xy,$$

where $y(0) = e$.

16

Solution. By a previous example we have

$$y = C\left(e^{\frac{1}{3}x^3 - x^2}\right).$$

By the initial value, we solve for C to get $C = e$. Therefore,

$$\boxed{y = e^{\frac{1}{3}x^3 - x^2 + 1}}.$$

□

§3.2 ODEs and PDEs

Example 3.2.1 (ODE)

$$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} + 5 = 0 \implies y'' + 3y' + 5 = 0.$$

The general from of a linear ODE is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x),$$

where $a_i \in \mathbb{R}$ such that $i \in \{0, 1, \ldots, n\}$.

Example 3.2.2 (PDE)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

The order of a differential equation is the highest derivative in the equation.

For first order ODEs, we try separation of variables first.

Definition 3.2.3. A **homogeneous** ordinary differential equation is one with $f(x) = 0$ with constant coefficients.

Example 3.2.4 (Homogeneous ODE)

$$y'' + 7y' + 12y = 0.$$

Functions of the form $y = e^{mx}$ are an "educated guess" for solving ODEs.

Solution. Let $y = e^{mx}$. This implies $y' = me^{mx}$ and $y'' = m^2 e^{mx}$. Plugging in, we get

$$\begin{aligned} m^2 e^{mx} + 7me^{mx} + 12e^{mx} &= 0 \\ e^{mx}\left(m^2 + 7m + 12\right) &= 0 \\ e^{mx}(m+4)(m+3) &= 0. \end{aligned}$$

Thus we have $y_h = C_1 e^{-4x} + C_2 e^{-3x}$. □

Example 3.2.5 (Homogeneous ODE with initial values)

$$y'' + y; -2y = 0, y(0) = 4, y'(0) = -5.$$

Solution. Let $y = e^{mx}$. This means $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging in, we get

$$\begin{aligned} m^2e^{mx} + me^{mx} - 2e^{mx} &= 0 \\ e^{mx}\left(m^2 + m - 2\right) &= 0 \\ e^{mx}(m-1)(m+2) &= 0. \end{aligned}$$

So, $m = -2, 1$. Therefore $y_h = C_1e^{-2x} + C_2e^x$. Then, we get the following system from the initial values:

$$\begin{aligned} C_1 + C_2 &= 4 \\ -2C_1 + C_2 &= -5. \end{aligned}$$

Solving, we get $C_1 = 3$ and $C_2 = 1$. So, $\boxed{y_h = 3e^{-2x} + e^x}$. □

Example 3.2.6 (Homogeneous ODE with repeated roots)

$$y'' + 4y' + 4y = 0.$$

Solution. Let $y = e^{mx}$. This means $y' = me^{mx}$ and $y'' = m^2e^{mx}$. PLugging in, we get

$$\begin{aligned} m^2e^{mx} + 4me^{mx} + 4e^{mx} &= 0 \\ e^{mx}(m+2)^2 &= 0. \end{aligned}$$

Thus, $\boxed{y_1 = C_1e^{-2x}}$. We now proceed to use the **reduction of order** method. So, $y_2 = f(x) \cdot y_1$. First, we have

$$\begin{aligned} y_2' &= f'(x)y_1 + f(x)y_1' = f'(x)e^{-2x} - 2f(x)e^{-2x} \\ y_2'' &= f''(x)y_1 + 2f'(x)y_1' + f(x)y_1'' = f''(x)e^{-2x} - 4f'(x)e^{-2x} + 4f(x)e^{-2x}. \end{aligned}$$

Plugging this into the original differential equation, we get

$$f''(x)e^{-2x} - 4f'(x)e^{-2x} + 4f(x)e^{-2x} + 4\left(f'(x)e^{-2x} - 2f(x)e^{-2x}\right) + 4f(x)e^{-2x} = 0.$$

Simplifying yields $f''(x) = 0$. Now, let $g(x) = f'(x)$. This means $g'(x) = f''(x)$. So, $g'(x) = 0$. This is equivalent to $\frac{dg}{dx} = 0$. Using separation of variables, we get $g(x) = C$. So, $f'(x) = C$. This is the same as $\frac{df}{dx} = C \implies f(x) = C_1x + C_2$. Thus,

$$\boxed{y_2 = \left(C_1x + C_2\right)e^{-2x}}.$$

Adding the two solutions, we obtain

$$\begin{aligned} y_h &= y_1 + y_2 = C_0e^{-2x} + C_1xe^{-2x} + C_2e^{-2x} \\ y_h &= Ce^{-2x} + Dxe^{-2x}. \end{aligned}$$

□

Example 3.2.7

$$y'' + y' + (1/4)y = 0;\ y(0) = 3;\ y'(0) = -3.5.$$

Solution. Let $y = e^{mx}$. This means $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging in, we get

$$\begin{aligned} 4m^2e^{mx} + 4me^{mx} + e^{mx} &= 0 \\ e^{mx}(4m^2 + 4m + 1) &= 0 \\ e^{mx}(2m+1)^2 &= 0. \end{aligned}$$

Thus, $\boxed{y_1 = C_0e^{-\frac{1}{2}x}}$. We now proceed to use reduction of order. So, $y_2 = f(x) \cdot y_1$. First, we have

$$\begin{aligned} y_2' &= f'(x)y_1 + f(x)y_1' = f'(x)e^{-x/2} - \frac{1}{2}f(x)e^{-x/2} \\ y_2'' &= f''(x)y_1 + 2f'(x)y_1' + f(x)y_1'' = f''(x)e^{-x/2} - f'(x)e^{-x/2} + \frac{1}{4}f(x)e^{-x/2}. \end{aligned}$$

Plugging this into the original differential equation, we get

$$\begin{aligned} f''(x)e^{-x/2} - f'(x)e^{-x/2} + \frac{1}{4}f(x)e^{-x/2} + f'(x)e^{-x/2} - \frac{1}{2}f(x)e^{-x/2} + \frac{1}{4}f(x)e^{-x/2} &= 0 \\ f''(x) &= 0. \end{aligned}$$

Now, let $g(x) = f'(x)$. This means that $g'(x) = f''(x)$. So, $g'(x) = 0$. This means $g(x) = C$. So, $f'(x) = C$. Thus, $f(x) = C_1x + C_2$. THerefore,

$$y_2 = (C_1x + C_2)e^{-x/2}.$$

Adding the two solutions, we obtain

$$\begin{aligned} y_h = y_1 + y_2 &= C_0e^{-x/2} + C_1xe^{-x/2} + C_2e^{-x/2} \\ &= Ce^{-x/2} + Dxe^{-x/2}. \end{aligned}$$

Using the inital values, we obtain

$$y_h = 3e^{-x/2} - 2xe^{-x/2}.$$

□

In general $y_2 = xy_1$.

Example 3.2.8

$$y''' + 3y'' - 4y = 0.$$

Solution. Let $y = e^{mx}$. This mean $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging in, we get

$$\begin{aligned} m^3e^{mx} + 3m^2e^{mx} - 4e^{mx} &= 0 \\ e^{mx}\left(m^3 + 3m^2 - 4\right) &= 0 \\ e^{mx}(m-1)(m^2 + 4m + 4) &= 0 \\ e^{mx}(m-1)(m+2)^2 &= 0. \end{aligned}$$

Therefore, $y_1 = C_1e^x$, $y_2 = C_2e^{-2x}$, and $y_3 = C_3xe^{-2x}$. Thus,

$$\boxed{y_h = c_1e^x + C_2e^{-2x} + C_3xe^{-2x}}.$$

□

Example 3.2.9 (Complex Roots)

$$y'' + 4y = 0.$$

Solution. Let $y = e^{mx}$. This means that $y'' = m^2e^{mx}$. Plugging in gives $m = \pm 2i$. Thus, $y_h = C_1e^{2ix} + C_2e^{-2ix}$. Recall **Euler's Theorem**

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Plugging this in gives

$$\begin{aligned} y_h &= C_1(\cos 2x + i\sin 2x) + C_2(\cos 2x - i\sin 2x) \\ y_h &= C_1\cos 2x + C_2\sin 2x. \end{aligned}$$

□

Example 3.2.10 (Complex Roots)

$$y'' + 8y' + 25y = 0.$$

Solution. Let $y = e^{mx}$. This meas that $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Plugging this in gives $-4 \pm 3i$. Thus,

$$\begin{aligned} y_h &= C_1e^{(-4+3i)x} + C_2e^{(-4-3i)x} \\ y_h &= e^{-4x}\left(C_1e^{3ix} + C_2e^{-3ix}\right) \\ y_h &= e^{-4x}\left(C_1\cos(3x) + C_2\sin(3x)\right). \end{aligned}$$

□

§3.3 Non Homogeneous ODEs

This occurs when $f(x) \neq 0$. Our solution will still include the homogeneous potion $y = y_h(x) + y_p(x)$. Some methods for solving these are **undetermined coefficients**, which can only be used for scalar coefficients, and **variation of parameters**, which can be used with scalar or variable coefficients.

Example 3.3.1 (Undetermined Coefficients)

$$y'' - 4y' + 3y = e^{-x};\ y(0) = 1;\ y'(0) = 0.$$

Solution. We first find y_h using $y'' - 4y' + 3y = 0$. Thus we obtain

$$y_h = C_1 e^x + C_2 e^{3x}.$$

Our guess for $y_p = Ce^{-x}$. Thus, $y_p' = -Ce^{-x}$ and $y_p'' = Ce^{-x}$. So, plugging this in, we get

$$\begin{aligned} Ce^{-x} + 4Ce^{-x} + 3Ce^{-x} = e^{-x} &\implies 8Ce^{-x} = e^{-x} \\ &\implies C = \frac{1}{8}. \end{aligned}$$

Therefore

$$y = y_h + y_p = C_1 e^x + C_2 e^{3x} + \frac{1}{8} e^{-x}.$$

Using the initial values, we find that

$$\boxed{y = \frac{5}{4} e^x - \frac{3}{8} e^{3x} + \frac{1}{8} e^{-x}}.$$

□

The following is a useful guessing guide:

$$\begin{aligned} r(x) &\implies y_p(x) \\ ke^{rx} &\implies Ce^{rx} \\ kx^n &\implies P_n = a_n x^n + a_{n-1} x^{n-1} + \cdots a_0 \\ k\cos(wx) \text{ or } k\sin(wx) &\implies a\cos(wx) + b\sin(wx). \end{aligned}$$

Example 3.3.2 (Guessing Practice with Undetermined Coefficients)

$$y'' + 2y' + y = \cos(2x)$$

Solution. We first proceed to find y_h. We find that

$$y_h = C_1 e^{-x} + C_2 x e^{-x}.$$

We make the following guess: $y_p = a\cos(2x) + b\sin(2x)$. Solving for a and b, we find that $y_p = -\frac{3}{25}\cos(2x) + \frac{4}{25}\sin(2x)$. Therefore

$$\boxed{y = C_1 e^{-x} + C_2 x e^{-x} - \frac{3}{25}\cos(2x) + \frac{4}{25}\sin(2x)}.$$

□

21

Example 3.3.3 (Bad Guessing)

$$y'' + 4y' + 3y = e^{-x}.$$

Solution. We notice $y_h = C_1e^{-x} + C_2e^{-3x}$. $y_p = Ce^{-x}$ is a bad guess since it is a term present in the homogeneous equation. So, we let $y_p = Cxe^{-x}$, which is a guess from the result of applying the reduction of order. Solving for C, we get that $C = \frac{1}{2}$. Therefore

$$\boxed{y = C_1e^{-x} + C_2e^{-3x} + \frac{1}{2}xe^{-x}}.$$

□

Definition 3.3.4. **Variation of Parameter** involve differential equations of variable coefficients. For $a_2y'' + a_1y' + a_0y = f(x)$. First, we make the leading coefficient 1 by dividing by a_2. Thus,

$$y'' + p(x)y' + q(x)y = r(x).$$

We still do the homogeneous portion first: $y_h = C_1y_1 + C_2y_2$. Then $y_p = u(x)y_1(x) + v(x)y_2(x)$, where $u' = -ry_2/w$ and $v' = ry_1/w$, where

$$w = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right).$$

This matrix is called the **Wronskian**.

Example 3.3.5 (Variation of parameters)

$$y'' + 4y' + 4y = e^{-2x}/x^2.$$

Solution. We find that $y_h = C_1e^{-2x} + C_2xe^{-2x}$. Notice that the Wronskian is

$$w = \det\left(\begin{bmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{bmatrix}\right) = e^{-4x}.$$

Thus,

$$u'(x) = -\frac{1}{x}$$
$$v'(x) = \frac{1}{x^2}.$$

Thus, $u(x) = -\ln|x| + C$ and $v(x) = -\frac{1}{x} + C$. □