Post Calculus

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Contents

§1 Week 2

§1.1 Eigenvectors and Eigenvalues

Definition 1.1. A **homogeneous linear system** is one where $\mathbf{Ax} = \mathbf{0}$.

The trivial solution is when $\mathbf{x} = \mathbf{0}$. Non-trivial solutions exist iff $\det(\mathbf{A}) = 0$.

Theorem 1.2

Let $\mathbf{A}$ be a square matrix.
$$\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$$
if and only if $\vec{\mathbf{v}}$ is an eigenvector and λ is an eigenvalue.

We notice that

$$\begin{aligned}
\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} &\implies \mathbf{A}\vec{\mathbf{v}} = \lambda\mathbf{I}\vec{\mathbf{v}} \\
&\implies \vec{\mathbf{v}}\left(\mathbf{A} - \lambda\mathbf{I}\right) = \mathbf{0} \\
&\implies \det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0.
\end{aligned}$$

The result above is known as the **characteristeric equation**.

Example 1.3

Let
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix},$$
find the eigenvalues and eigenvectors of $\mathbf{A}$.

We first notice that $\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$. Meaning that

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix}\right) = 0,$$

implying that $\lambda^2 + 4 = 0$. Thus, $\lambda = \pm 2i$. For $\lambda_1 = 2i$, we have

$$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \vec{\mathbf{v}} = \mathbf{0}.$$

By gaussian elimination we arrive at the first eigenvector

$$\vec{\mathbf{v_1}} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Similarly

$$\vec{\mathbf{v_2}} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

for $\lambda_2 = -2i$.

§1.2 Diagonalization

For $\mathbf{y} = \mathbf{A}\mathbf{x}$, assume $\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$$

be a transformation matix. Let

$$\mathbf{y} = \mathbf{P}\mathbf{y}' \text{ and } \mathbf{x} = \mathbf{P}\mathbf{x}'.$$

We can then write a new linear system relating $\mathbf{y}'$ and $\mathbf{x}'$, giving

$$\mathbf{P}\mathbf{y}' = \mathbf{A}\mathbf{P}\mathbf{x}' \implies \mathbf{P}^{-1}\mathbf{P}\mathbf{y}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{P}\mathbf{x}'$$
$$\implies \mathbf{y}' = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\mathbf{x}',$$

where $\mathbf{D} = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)$ is a diagonal matrix such that the diagonals turn out to be the eigenvalues of $\mathbf{A}$.

> **Remark 1.4.** In order to find the diagonalization matrix, begin by finding the eigenvalues and eigenvectors of $\mathbf{A}$. Then find $\mathbf{P}$ from the eigenvetors. The diagonlization matrix follows.

Example 1.5

Diagonalize

$$\begin{bmatrix} 7/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 5/4 \end{bmatrix}.$$

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§1.3 Rotating Conics

Let Q be a quadratic form such that

$$Q = ax_1^2 + (b+c)x_1 x_2 + dx_2^2.$$

We can express this as

$$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We can convert the quadratic form to the canonical form by using the diagonalized matrix of $\mathbf{A}$:

$$C = \begin{bmatrix} x_1' & x_2' \end{bmatrix} \mathbf{D} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

Additionally, note that the transformation matrix is the rotation matrix, meaning that

$$\mathbf{P} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

> **Theorem 1.6**
>
> Let Q_n be a quadratic form with n dimensions and C_n the corresponding canonical form. Then, for an $n \times n$ matrix $\mathbf{A}$ and its diagonalized matrix $\mathbf{D}$,
>
> $$Q_n = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$
>
> $$C_n = \begin{bmatrix} x'_1 & x'_2 & \cdots & x'_n \end{bmatrix} \mathbf{D} \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix}.$$

If any linear terms are present, they can be expressed as the product of the coefficient matrix and the matrix with each variable. So,

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Exercise 1.7.

§2 Week 3

§2.1 Partial Derivatives

Let $f(x, y, z) = 2x^2 + 3y^2 + z^2$. Partial derivatives treat the other variables as constants. Thus,

$$\frac{\partial f}{\partial x} = 4x, \ \frac{\partial f}{\partial y} = 6y, \ \frac{\partial f}{\partial z} = 2.$$

Definition 2.1. The **gradient vector** for a function f is

$$\nabla f(p) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{bmatrix}.$$

Definition 2.2. The **Hessian Matrix** is

$$\mathbf{H}_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \, \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \, \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \, \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \, \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \, \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \, \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix},$$

Theorem 2.3 (Schwarz's Theorem)

For a function $f : \Omega \to \mathbb{R}$ defined on a set $\Omega \subset \mathbb{R}^n$, if $\mathbf{p} \in \mathbb{R}^n$ is a point such that some neighborhood of $\mathbf{p}$ is contained in Ω and f has continuous second partial derivatives at the point $\mathbf{p}$, then $\forall i, j \in \{1, 2, \ldots, n\}$

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{p}) = \frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{p}).$$

Proof. The proof is left as a search on wikipedia :p. $\square$

Example 2.4

Find the gradient and hessian matrix of $f(x, y, z) = 2x^2 + 3y^2 + z^2$.

Proof. We find that

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix}$$

and

$$\mathbf{H}_f = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$\square$

Example 2.5

Find the gradient and hessian matrix of $f(x, y) = x^2 - 3xy + y^2$.

Proof. We find that

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ 2y - 3x \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}.$$

$\square$

Example 2.6

Find the gradient and hessian matrix $f(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 - 6x_2x_3$.

Proof. We find that

$$\nabla f = \begin{bmatrix} 18x_1 - 2x_2 + 4x_3 \\ -2x_1 + 14x_2 - 6x_3 \\ 4x_1 - 6x_2 + 6x_3 \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 18 & -2 & 4 \\ -2 & 14 & -6 \\ 4 & -6 & 6 \end{bmatrix}.$$

$\square$

Example 2.7

Let x be a function of t. Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial \dot{x}}, \dfrac{\partial f}{\partial \ddot{x}}$, and $\dfrac{df}{dt}$.

Theorem 2.8 (Generalized Chain Rule)

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentialble function of m independent variables, and for each $i \in \{1, \ldots, m\}$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any $j \in \{1, \ldots, n\}$.

Proof. Deez nutz

$\square$

§2.2 Tangent Planes

We proceed by finding the equation of the tangent plane to $x^2 - y^2 - z^2 = 1$ at $(1, 0, 0)$. To begin, we find the gradient of $f(x, y, z) = x^2 - y^2 - z^2$ to be

$$\nabla f(1, 0, 0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Then, the "point-slope" form of a plane is

$$m_x(x - x_1) + m_y(y - y_1) + m_z(z - z_1) = 0.$$

Thus, we obtain the following tangent plane for our scenario: $2(x - 1) = 0$.

§2.3 Unconstrained Optimization

Definition 2.9. A **stationary point** is a critical point in higher dimensions. They can be found from the solution to the system of equations that results from letting the gradient equal zero.

Definition 2.10. A hessian is called **positive definite** if all the eigenvalues are positive.

Definition 2.11. A hessian is called **negative definite** if all the eigenvalues are negative.

Definition 2.12. A hessian is called **positive semidefinite** if all the eigenvalues are nonnegative and there exists at least one eigenvalue that is 0.

Definition 2.13. A hessian is called **negative semidefinite** if all the eignevalues are nonpositive and there exists at least one eigen value that is 0.

Definition 2.14. A point is a **saddle point** if the hessian has negative and positve eigenvalues.

As an alternative to the second derivative test in determining if a critical point is a max , min or an inflection point, the **hessian** will be used to determine if a stationary point is a max, min or inflection point.

> **Theorem 2.15** (Second Partial Derivative Test)

We can determine if the hessian is "positive" or "negative" by taking a look at its eigen values. Let $\mathbf{H}_f$ be the hessian for f, a differentiable function of n independent variables. Also, let $\Lambda = \{\lambda_i | 1 \leq i \leq n\}$ be the set of the eigenvalues of $\mathbf{H_f}$. If all elements in Λ are positive, then the hessian is called **positive definite**, giving a minimum. If all elements in Λ are negative, then the hessian is called **negative definite**, giving a maximum. If $\lambda_i \geq 0$, $\mathbf{H_f}$ is called **positive semidefinite**. If $\lambda_i \leq 0$, $\mathbf{H_f}$ is called **negative semidefinite**. If one eigenvalue is positive and one eigenvalue is negative, we have a **saddle point**.

> **Example 2.16**
> Find the critical points of $f(x_1, x_2) = x_1^2 - 2x_1x_2 + 4x_2^2$.

Proof. We begin by finding the gradient of f. This is

$$\nabla f = \begin{bmatrix} 2x_1 - 2x_2 \\ -2x_1 + 8x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$2x_1 - 2x_2 = 0$$
$$-2x_1 + 8x_2 = 0.$$

This gives $(x_1, x_2) = (0, 0)$. They hessian, $\mathbf{H}_f$ of f is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -2 \\ -2 & 8 \end{bmatrix}.$$

Since the eigen values of this hessian are both positive, we have a minimum. $\quad\square$

Example 2.17

Find the critical points of $f(x_1, x_2) = -x_1^2 + 2x_1 x_2 + 3x_2^2 + 8x_1$.

Proof. We begin by finding the gradient of f. This is

$$\nabla f = \begin{bmatrix} -2x_1 + 2x_2 + 8 \\ 2x_1 + 6x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$-2x_1 + 2x_2 = -8$$
$$2x_1 + 6x_2 = 0.$$

Solving gives $(x_1, x_2) = (-3, 1)$. The hessian, $\mathbf{H}_f$, of f is

$$\mathbf{H}_f = \begin{bmatrix} -2 & 2 \\ 2 & 6 \end{bmatrix}.$$

Since one of the eignevalues of this hessian is positive and the other is negative, we have a saddle point. $\quad\square$

Example 2.18

Find the critical points of $f(x_1, x_2) = \left(x_1 - x_2^2\right)\left(x_1 - 3x_2^2\right)$.

Proof. To begin, we distribute to get $f(x_1, x_2) = x_1^2 - 4x_1 x_2^2 + 3x_2^4$. The gradient of f is then

$$\nabla f = \begin{bmatrix} 2x_1 - 4x_2^2 \\ -8x_1 x_2 + 12x_2^3 \end{bmatrix}.$$

Setting the gradient to zero and solving the resulting system of equations, we get the following critical point $(x_1, x_2) = (0, 0)$. The hessian of f is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -8x_2 \\ -8x_2 & -8x_1 + 36x_2^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since one eigenvalue is positive and the other is equal to 0, the hessian is postive semidefinite. $\quad\square$

> **Example 2.19**
>
> Kartik and Monica invested $20,000 in the design and the development of a new product. They can manufacture it for $2 per unit. They hired marketing consultants to determine the relation between selling price, the amount spent on advertising, and the number of units that would be sold as a result of the first two combined. The company determined that units sold would follow the equation
>
> $$2000 + 4\sqrt{a} - 20p.$$
>
> Determine the profit that Felicia and Megan will make as a function of the money spent on advertising, a, and the price of the product, p. Maximize that profit.

Proof. We first identify that the revenue gained from sales would be $p\left(2000 + 4\sqrt{a} - 20p\right)$. Then, the costs would be $20000 + 2\left(2000 + 4\sqrt{a} - 20p\right) + a$. Taking the difference, we get the profit P being

$$P(a,p) = p\left(2000 + 4\sqrt{a} - 20p\right) - 20000 - 2\left(2000 + 4\sqrt{a} - 20p\right) - a$$
$$= 2040p + 4p\sqrt{a} - 20p^2 - 24000 - 8\sqrt{a} - a.$$

The gradient of P is

$$\nabla P = \begin{bmatrix} \dfrac{2p-4}{\sqrt{a}} - 1 \\ -40p + 4\sqrt{a} + 2040 \end{bmatrix}.$$

Setting the gradient to 0 and solving the resulting system of equations, we get that $p = 63.25$ and $a = 15006.25$. The maximum profit is $\boxed{\$40025}$. $\hspace{1cm}\square$

§2.4 Constrained Optimization

Equality and Inequality constraints

Definition 2.20. The **Lagrangian** is

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)),$$

where $g_i(x)$ are constraints.

> **Example 2.21**
>
> Maximize $f(x_1, x_2) = 5 - (x_1 - 2)^2 - 2(x_2 - 1)^2$ subject to $x_1 + 4x_2 = 3$.

Proof. We get

$$\mathcal{L}(x, \lambda) = f(x_1, x_2) + \lambda(x_1 + 4x_2 - 3)$$
$$= 5 - x_1^2 + 4x_1 - 4 - 2x_2^2 + 4x_2 - 2 + \lambda(x_1 + 4x_2 - 3)$$
$$= -x_1^2 - 2x_2^2 + 4x_1 + 4x_2 - 1 + \lambda(x_1 + 4x_2 - 3)$$

Thus

$$\nabla\mathcal{L} = \begin{bmatrix} -2x_1 + \lambda + 4 \\ -4x_2 + 4\lambda + 4 \\ x_1 + 4x_2 - 3 \end{bmatrix} = 0$$

We get $(x_1, x_2, \lambda) = (5/3, 1/3, -2/3)$. Thus our critical point is $\boxed{(5/3, 1/3, 4)}$. $\hspace{0.5cm}\square$

> **Example 2.22**
>
> Let the sun be located at the origin of a coordinate plane. How close does Halley's comet come to the sun on its orbit?
>
> $$171.725x^2 + 171.725y^2 + 297.37xy + 557.178x - 557.178y - 562.867 = 0.$$

Proof. We seek to optimize the distance from the comet to the sun. Thus, we seek to optimize $f(x, y) = x^2 + y^2$. Optimization with the lagrangian follows, where

$$\mathcal{L} = x^2 + y^2 - \lambda().$$

$\square$

> **Example 2.23**
>
> Minimize $L(x) = x_1 e^{-(x_1^2 + x_2^2)} + \frac{x_1^2 + x_2^2}{20}$ subject to $f(x) = \frac{x_1 x_2}{2} + (x_1^2 + 2)^2 + (x_1^2 - 2)^2/2 - 2 \leq 0$.

Proof. We find that

$$\nabla L = \begin{bmatrix} e^{-(x_1^2 + x_2^2)} - 2x_1^2 e^{-(x_1^2 + x_2^2)} + x_1/10 \\ -2x_1 x_2 e^{-(x_1^2 + x_2^2)} + x_2/10 \end{bmatrix} = 0$$

We find that the critical points are $(-1/2, \sqrt{\ln(10) - 1/4})$ and $(-1/2, -\sqrt{\ln(10) - 1/4})$.
$\square$