

# Post Calculus

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# 1 Linear Algebra

## §1.1 Eigenvectors and Eigenvalues

**Definition 1.1.1.** A homogeneous linear system is one where  $\mathbf{Ax} = \mathbf{0}$ .

The trivial solution is when  $\mathbf{x} = \mathbf{0}$ . Non-trivial solutions exist iff  $\det(\mathbf{A}) = 0$ .

### Theorem 1.1.2

Let  $\mathbf{A}$  be a square matrix.

$$\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$$

if and only if  $\vec{\mathbf{v}}$  is an eigenvector and  $\lambda$  is an eigenvalue.

We notice that

$$\begin{aligned}\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} &\implies \mathbf{A}\vec{\mathbf{v}} = \lambda\mathbf{I}\vec{\mathbf{v}} \\ &\implies \vec{\mathbf{v}}(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0} \\ &\implies \det(\mathbf{A} - \lambda\mathbf{I}) = 0.\end{aligned}$$

The result above is known as the **characteristic equation**.

### Example 1.1.3

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix},$$

find the eigenvalues and eigenvectors of  $\mathbf{A}$ .

We first notice that  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ . Meaning that

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix}\right) = 0,$$

implying that  $\lambda^2 + 4 = 0$ . Thus,  $\lambda = \pm 2i$ . For  $\lambda_1 = 2i$ , we have

$$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \vec{\mathbf{v}} = \mathbf{0}.$$

By gaussian elimination we arrive at the first eigenvector

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Similarly

$$\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

for  $\lambda_2 = -2i$ .

## §1.2 Diagonalization

For  $\mathbf{y} = \mathbf{A}\mathbf{x}$ , assume  $\mathbf{A}$  has a basis of eigenvectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ , with corresponding eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , and let the matrix

$$\mathbf{P} = (\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n)$$

be a transformation matrix. Let

$$\mathbf{y} = \mathbf{P}\mathbf{y}' \text{ and } \mathbf{x} = \mathbf{P}\mathbf{x}'.$$

We can then write a new linear system relating  $\mathbf{y}'$  and  $\mathbf{x}'$ , giving

$$\begin{aligned} \mathbf{P}\mathbf{y}' &= \mathbf{A}\mathbf{P}\mathbf{x}' \implies \mathbf{P}^{-1}\mathbf{P}\mathbf{y}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{P}\mathbf{x}' \\ &\implies \mathbf{y}' = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\mathbf{x}', \end{aligned}$$

where  $\mathbf{D} = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})$  is a diagonal matrix such that the diagonals turn out to be the eigenvalues of  $\mathbf{A}$ .

**Remark 1.2.1.** In order to find the diagonalization matrix, begin by finding the eigenvalues and eigenvectors of  $\mathbf{A}$ . Then find  $\mathbf{P}$  from the eigenvectors. The diagonalization matrix follows.

### Example 1.2.2

Diagonalize

$$\begin{bmatrix} 7/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 5/4 \end{bmatrix}.$$

## §1.3 Rotating Conics

Let  $Q$  be a quadratic form such that

$$Q = ax_1^2 + (b+c)x_1x_2 + dx_2^2.$$

We can express this as

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We can convert the quadratic form to the canonical form by using the diagonalized matrix of  $\mathbf{A}$ :

$$C = \begin{bmatrix} x'_1 & x'_2 \end{bmatrix} \mathbf{D} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}.$$

Additionally, note that the transformation matrix is the rotation matrix, meaning that

$$\mathbf{P} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

**Theorem 1.3.1**

Let  $Q_n$  be a quadratic form with  $n$  dimensions and  $C_n$  the corresponding canonical form. Then, for an  $n \times n$  matrix  $\mathbf{A}$  and its diagonalized matrix  $\mathbf{D}$ ,

$$Q_n = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$C_n = \begin{bmatrix} x'_1 & x'_2 & \cdots & x'_n \end{bmatrix} \mathbf{D} \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix}.$$

If any linear terms are present, they can be expressed as the product of the coefficient matrix and the matrix with each variable. So,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

**Exercise 1.3.2.**





# 2 3D Optimization

## §2.1 Partial Derivatives

Let  $f(x, y, z) = 2x^2 + 3y^2 + z^2$ . Partial derivatives treat the other variables as constants. Thus,

$$\frac{\partial f}{\partial x} = 4x, \quad \frac{\partial f}{\partial y} = 6y, \quad \frac{\partial f}{\partial z} = 2.$$

**Definition 2.1.1.** The **gradient vector** for a function  $f$  is

$$\nabla f(p) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{bmatrix}.$$

**Definition 2.1.2.** The **Hessian Matrix** is

$$\mathbf{H}_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix},$$

### Theorem 2.1.3 (Schwarz's Theorem)

For a function  $f : \Omega \rightarrow \mathbb{R}$  defined on a set  $\Omega \subset \mathbb{R}^n$ , if  $\mathbf{p} \in \mathbb{R}^n$  is a point such that some neighborhood of  $\mathbf{p}$  is contained in  $\Omega$  and  $f$  has continuous second partial derivatives at the point  $\mathbf{p}$ , then  $\forall i, j \in \{1, 2, \dots, n\}$

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{p}) = \frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{p}).$$

*Proof.* The proof is left as a search on wikipedia :p. □

### Example 2.1.4

Find the gradient and hessian matrix of  $f(x, y, z) = 2x^2 + 3y^2 + z^2$ .

*Solution.* We find that

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix}$$

and

$$\mathbf{H}_f = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

□

### Example 2.1.5

Find the gradient and hessian matrix of  $f(x, y) = x^2 - 3xy + y^2$ .

*Solution.* We find that

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ 2y - 3x \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}.$$

□

### Example 2.1.6

Find the gradient and hessian matrix  $f(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 - 6x_2x_3$ .

*Solution.* We find that

$$\nabla f = \begin{bmatrix} 18x_1 - 2x_2 + 4x_3 \\ -2x_1 + 14x_2 - 6x_3 \\ 4x_1 - 6x_2 + 6x_3 \end{bmatrix},$$

$$\mathbf{H}_f = \begin{bmatrix} 18 & -2 & 4 \\ -2 & 14 & -6 \\ 4 & -6 & 6 \end{bmatrix}.$$

□

### Example 2.1.7

Let  $x$  be a function of  $t$ . Find  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial \dot{x}}$ ,  $\frac{\partial f}{\partial \ddot{x}}$ , and  $\frac{df}{dt}$ .

### Theorem 2.1.8 (Generalized Chain Rule)

Let  $w = f(x_1, x_2, \dots, x_m)$  be a differentiable function of  $m$  independent variables, and for each  $i \in \{1, \dots, m\}$ , let  $x_i = x_i(t_1, t_2, \dots, t_n)$  be a differentiable function of  $n$  independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_j} + \dots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial t_j}$$

for any  $j \in \{1, \dots, n\}$ .

*Proof.* Deez nutz

□

## §2.2 Tangent Planes

We proceed by finding the equation of the tangent plane to  $x^2 - y^2 - z^2 = 1$  at  $(1, 0, 0)$ . To begin, we find the gradient of  $f(x, y, z) = x^2 - y^2 - z^2$  to be

$$\nabla f(1, 0, 0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Then, the "point-slope" form of a plane is

$$m_x(x - x_1) + m_y(y - y_1) + m_z(z - z_1) = 0.$$

Thus, we obtain the following tangent plane for our scenario:  $2(x - 1) = 0$ .

## §2.3 Unconstrained Optimization

**Definition 2.3.1.** A **stationary point** is a critical point in higher dimensions. They can be found from the solution to the system of equations that results from letting the gradient equal zero.

**Definition 2.3.2.** A hessian is called **positive definite** if all the eigenvalues are positive.

**Definition 2.3.3.** A hessian is called **negative definite** if all the eigenvalues are negative.

**Definition 2.3.4.** A hessian is called **positive semidefinite** if all the eigenvalues are nonnegative and there exists at least one eigenvalue that is 0.

**Definition 2.3.5.** A hessian is called **negative semidefinite** if all the eigenvalues are nonpositive and there exists at least one eigenvalue that is 0.

**Definition 2.3.6.** A point is a **saddle point** if the hessian has negative and positive eigenvalues.

As an alternative to the second derivative test in determining if a critical point is a max, min or an inflection point, the **hessian** will be used to determine if a stationary point is a max, min or inflection point.

**Theorem 2.3.7 (Second Partial Derivative Test)**

We can determine if the hessian is "positive" or "negative" by taking a look at its eigen values. Let  $\mathbf{H}_f$  be the hessian for  $f$ , a differentiable function of  $n$  independent variables. Also, let  $\Lambda = \{\lambda_i | 1 \leq i \leq n\}$  be the set of the eigenvalues of  $\mathbf{H}_f$ . If all elements in  $\Lambda$  are positive, then the hessian is called **positive definite**, giving a minimum. If all elements in  $\Lambda$  are negative, then the hessian is called **negative definite**, giving a maximum. If  $\lambda_i \geq 0$ ,  $\mathbf{H}_f$  is called **positive semidefinite**. If  $\lambda_i \leq 0$ ,  $\mathbf{H}_f$  is called **negative semidefinite**. If one eigenvalue is positive and one eigenvalue is negative, we have a **saddle point**.

**Example 2.3.8**

Find the critical points of  $f(x_1, x_2) = x_1^2 - 2x_1x_2 + 4x_2^2$ .

*Proof.* We begin by finding the gradient of  $f$ . This is

$$\nabla f = \begin{bmatrix} 2x_1 - 2x_2 \\ -2x_1 + 8x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$\begin{aligned} 2x_1 - 2x_2 &= 0 \\ -2x_1 + 8x_2 &= 0. \end{aligned}$$

This gives  $(x_1, x_2) = (0, 0)$ . The hessian,  $\mathbf{H}_f$  of  $f$  is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -2 \\ -2 & 8 \end{bmatrix}.$$

Since the eigen values of this hessian are both positive, we have a minimum.  $\square$

**Example 2.3.9**

Find the critical points of  $f(x_1, x_2) = -x_1^2 + 2x_1x_2 + 3x_2^2 + 8x_1$ .

*Proof.* We begin by finding the gradient of  $f$ . This is

$$\nabla f = \begin{bmatrix} -2x_1 + 2x_2 + 8 \\ 2x_1 + 6x_2 \end{bmatrix}.$$

Setting the gradient to zero, we get the following system of equations:

$$\begin{aligned} -2x_1 + 2x_2 &= -8 \\ 2x_1 + 6x_2 &= 0. \end{aligned}$$

Solving gives  $(x_1, x_2) = (-3, 1)$ . The hessian,  $\mathbf{H}_f$ , of  $f$  is

$$\mathbf{H}_f = \begin{bmatrix} -2 & 2 \\ 2 & 6 \end{bmatrix}.$$

Since one of the eigenvalues of this hessian is positive and the other is negative, we have a saddle point.  $\square$

**Example 2.3.10**

Find the critical points of  $f(x_1, x_2) = (x_1 - x_2^2)(x_1 - 3x_2^2)$ .

*Proof.* To begin, we distribute to get  $f(x_1, x_2) = x_1^2 - 4x_1x_2^2 + 3x_2^4$ . The gradient of  $f$  is then

$$\nabla f = \begin{bmatrix} 2x_1 - 4x_2^2 \\ -8x_1x_2 + 12x_2^3 \end{bmatrix}.$$

Setting the gradient to zero and solving the resulting system of equations, we get the following critical point  $(x_1, x_2) = (0, 0)$ . The hessian of  $f$  is

$$\mathbf{H}_f = \begin{bmatrix} 2 & -8x_2 \\ -8x_2 & -8x_1 + 36x_2^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since one eigenvalue is positive and the other is equal to 0, the hessian is positive semidefinite.  $\square$

### Example 2.3.11

Kartik and Monica invested \$20,000 in the design and the development of a new product. They can manufacture it for \$2 per unit. They hired marketing consultants to determine the relation between selling price, the amount spent on advertising, and the number of units that would be sold as a result of the first two combined. The company determined that units sold would follow the equation

$$2000 + 4\sqrt{a} - 20p.$$

Determine the profit that Felicia and Megan will make as a function of the money spent on advertising,  $a$ , and the price of the product,  $p$ . Maximize that profit.

*Proof.* We first identify that the revenue gained from sales would be  $p(2000 + 4\sqrt{a} - 20p)$ . Then, the costs would be  $20000 + 2(2000 + 4\sqrt{a} - 20p) + a$ . Taking the difference, we get the profit  $P$  being

$$\begin{aligned} P(a, p) &= p(2000 + 4\sqrt{a} - 20p) - 20000 - 2(2000 + 4\sqrt{a} - 20p) - a \\ &= 2040p + 4p\sqrt{a} - 20p^2 - 24000 - 8\sqrt{a} - a. \end{aligned}$$

The gradient of  $P$  is

$$\nabla P = \begin{bmatrix} \frac{2p-4}{\sqrt{a}} - 1 \\ -40p + 4\sqrt{a} + 2040 \end{bmatrix}.$$

Setting the gradient to 0 and solving the resulting system of equations, we get that  $p = 63.25$  and  $a = 15006.25$ . The maximum profit is  $\boxed{\$40025}$ .  $\square$

## §2.4 Constrained Optimization

Equality and Inequality constraints

**Definition 2.4.1.** The **Lagrangian** is

$$L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i (b_i - g_i(x)),$$

where  $g_i(x)$  are constraints.

### Example 2.4.2

Maximize  $f(x_1, x_2) = 5 - (x_1 - 2)^2 - 2(x_2 - 1)^2$  subject to  $x_1 + 4x_2 = 3$ .

*Proof.* We get

$$\begin{aligned}\mathcal{L}(x, \lambda) &= f(x_1, x_2) + \lambda(x_1 + 4x_2 - 3) \\ &= 5 - x_1^2 + 4x_1 - 4 - 2x_2^2 + 4x_2 - 2 + \lambda(x_1 + 4x_2 - 3) \\ &= -x_1^2 - 2x_2^2 + 4x_1 + 4x_2 - 1 + \lambda(x_1 + 4x_2 - 3)\end{aligned}$$

Thus

$$\nabla \mathcal{L} = \begin{bmatrix} -2x_1 + \lambda + 4 \\ -4x_2 + 4\lambda + 4 \\ x_1 + 4x_2 - 3 \end{bmatrix} = 0$$

We get  $(x_1, x_2, \lambda) = (5/3, 1/3, -2/3)$ . Thus our critical point is  $\boxed{(5/3, 1/3, 4)}$ .  $\square$

### Example 2.4.3

Let the sun be located at the origin of a coordinate plane. How close does Halley's comet come to the sun on its orbit?

$$171.725x^2 + 171.725y^2 + 297.37xy + 557.178x - 557.178y - 562.867 = 0.$$

*Proof.* We seek to optimize the distance from the comet to the sun. Thus, we seek to optimize  $f(x, y) = x^2 + y^2$ . Optimization with the lagrangian follows, where

$$\mathcal{L} = x^2 + y^2 - \lambda().$$

$\square$

### Example 2.4.4

Minimize  $L(x) = x_1 e^{-(x_1^2 + x_2^2)} + \frac{x_1^2 + x_2^2}{20}$  subject to  $f(x) = \frac{x_1 x_2}{2} + (x_1^2 + 2)^2 + (x_1^2 - 2)^2/2 - 2 \leq 0$ .

*Proof.* We find that

$$\nabla L = \begin{bmatrix} e^{-(x_1^2 + x_2^2)} - 2x_1^2 e^{-(x_1^2 + x_2^2)} + x_1/10 \\ -2x_1 x_2 e^{-(x_1^2 + x_2^2)} + x_2/10 \end{bmatrix} = 0$$

We find that the critical points are  $(-1/2, \sqrt{\ln(10) - 1/4})$  and  $(-1/2, -\sqrt{\ln(10) - 1/4})$ .  $\square$

# 3 Ordinary Differential Equations

## §3.1 Separable Differential Equations

### Example 3.1.1

Find  $y$  for the following differential equation:

$$\frac{dy}{dx} = x.$$

*Solution.* We see that  $dy = x \, dx$ . Taking the integral of both sides, we obtain

$$y = \frac{1}{2}x^2 + C.$$

□

### Example 3.1.2

Find  $y$  for the following differential equation:

$$\frac{dy}{dx} = \frac{1}{x}.$$

*Solution.* We see that  $dy = \frac{1}{x} \, dx$ . Taking the integral of both sides, we obtain

$$y = \ln |x| + C.$$

□

### Example 3.1.3

Find  $y$  for the following differential equation:

$$\frac{dy}{dx} = \cos x.$$

*Solution.* We see that  $dy = \cos x \, dx$ . Taking the integral of both sides, we obtain

$$y = \sin x + C.$$

□

### Example 3.1.4

Find  $y$  for the following differential equation:

$$\frac{dy}{dx} = e^x.$$

*Solution.* We see that  $dy = e^x dx$ . Taking the integral of both sides, we obtain

$$y = e^x + C.$$

□

### Example 3.1.5

Find  $y$  for the following differential equation:

$$\frac{dy}{dx} = xy.$$

*Solution.* We see that  $\frac{1}{y} dy = x dx$ . Taking the integral of both sides, we get

$$\ln |y| = \frac{1}{2}x^2 + C \implies y = e^{\frac{1}{2}x^2 + C}.$$

□

### Example 3.1.6

Find  $y$  for the following differential equation:

$$\frac{dy}{dx} = x^2y - 2xy.$$

*Solution.* We see that  $dy = xy(x - 2) dx \implies \frac{1}{y} dy = x^2 - 2x dx$ . Taking the integral of both sides we get that

$$\ln |y| = \frac{1}{3}x^3 - x^2 + C \implies y = e^{\frac{1}{3}x^3 - x^2 + C}.$$

□

### Example 3.1.7

Find  $y$  for the following differential equation:

$$(x^2 + x) \frac{dy}{dx} = y - 1.$$

*Solution.* We see that  $\frac{1}{y-1} dy = \frac{1}{x^2+x} dx$ . Taking the integral of both sides, we find that

$$\ln |y - 1| = \ln |x| - \ln |x + 1| + C \implies y = C \left| \frac{x}{x + 1} \right| + 1.$$

□

### Example 3.1.8

Find  $y$  for the following differential equation:

$$\frac{dy}{dx} = x^2y - 2xy,$$

where  $y(0) = e$ .



*Solution.* By a previous example we have

$$y = C \left( e^{\frac{1}{3}x^3 - x^2} \right).$$

By the initial value, we solve for  $C$  to get  $C = e$ . Therefore,

$$y = e^{\frac{1}{3}x^3 - x^2 + 1}.$$

□

## §3.2 ODEs and PDEs

### Example 3.2.1 (ODE)

$$\frac{d^2 f}{dx^2} + 3 \frac{df}{dx} + 5 = 0 \implies y'' + 3y' + 5 = 0.$$

The general form of a linear ODE is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x),$$

where  $a_i \in \mathbb{R}$  such that  $i \in \{0, 1, \dots, n\}$ .

### Example 3.2.2 (PDE)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

The order of a differential equation is the highest derivative in the equation.

For first order ODEs, we try separation of variables first.

**Definition 3.2.3.** A **homogeneous** ordinary differential equation is one with  $f(x) = 0$  with constant coefficients.

### Example 3.2.4 (Homogeneous ODE)

$$y'' + 7y' + 12y = 0.$$

Functions of the form  $y = e^{mx}$  are an "educated guess" for solving ODEs.

*Solution.* Let  $y = e^{mx}$ . This implies  $y' = me^{mx}$  and  $y'' = m^2 e^{mx}$ . Plugging in, we get

$$m^2 e^{mx} + 7m e^{mx} + 12e^{mx} = 0$$

$$e^{mx} (m^2 + 7m + 12) = 0$$

$$e^{mx} (m + 4)(m + 3) = 0.$$

Thus we have  $y_h = C_1 e^{-4x} + C_2 e^{-3x}$ .

□

**Example 3.2.5** (Homogeneous ODE with initial values)

$$y'' + y; -2y = 0, y(0) = 4, y'(0) = -5.$$

*Solution.* Let  $y = e^{mx}$ . This means  $y' = me^{mx}$  and  $y'' = m^2e^{mx}$ . Plugging in, we get

$$\begin{aligned} m^2e^{mx} + me^{mx} - 2e^{mx} &= 0 \\ e^{mx}(m^2 + m - 2) &= 0 \\ e^{mx}(m - 1)(m + 2) &= 0. \end{aligned}$$

So,  $m = -2, 1$ . Therefore  $y_h = C_1e^{-2x} + C_2e^x$ . Then, we get the following system from the initial values:

$$\begin{aligned} C_1 + C_2 &= 4 \\ -2C_1 + C_2 &= -5. \end{aligned}$$

Solving, we get  $C_1 = 3$  and  $C_2 = 1$ . So,  $y_h = 3e^{-2x} + e^x$ . □

**Example 3.2.6** (Homogeneous ODE with repeated roots)

$$y'' + 4y' + 4y = 0.$$

*Solution.* Let  $y = e^{mx}$ . This means  $y' = me^{mx}$  and  $y'' = m^2e^{mx}$ . Plugging in, we get

$$\begin{aligned} m^2e^{mx} + 4me^{mx} + 4e^{mx} &= 0 \\ e^{mx}(m + 2)^2 &= 0. \end{aligned}$$

Thus,  $y_1 = C_1e^{-2x}$ . We now proceed to use the **reduction of order** method. So,  $y_2 = f(x) \cdot y_1$ . First, we have

$$\begin{aligned} y_2' &= f'(x)y_1 + f(x)y_1' = f'(x)e^{-2x} - 2f(x)e^{-2x} \\ y_2'' &= f''(x)y_1 + 2f'(x)y_1' + f(x)y_1'' = f''(x)e^{-2x} - 4f'(x)e^{-2x} + 4f(x)e^{-2x}. \end{aligned}$$

Plugging this into the original differential equation, we get

$$f''(x)e^{-2x} - 4f'(x)e^{-2x} + 4f(x)e^{-2x} + 4(f'(x)e^{-2x} - 2f(x)e^{-2x}) + 4f(x)e^{-2x} = 0.$$

Simplifying yields  $f''(x) = 0$ . Now, let  $g(x) = f'(x)$ . This means  $g'(x) = f''(x)$ . So,  $g'(x) = 0$ . This is equivalent to  $\frac{dg}{dx} = 0$ . Using separation of variables, we get  $g(x) = C$ . So,  $f'(x) = C$ . This is the same as  $\frac{df}{dx} = C \implies f(x) = C_1x + C_2$ . Thus,

$$y_2 = (C_1x + C_2)e^{-2x}.$$

Adding the two solutions, we obtain

$$\begin{aligned} y_h &= y_1 + y_2 = C_0e^{-2x} + C_1xe^{-2x} + C_2e^{-2x} \\ y_h &= Ce^{-2x} + Dxe^{-2x}. \end{aligned}$$

□

**Example 3.2.7**

$$y'' + y' + (1/4)y = 0; \quad y(0) = 3; \quad y'(0) = -3.5.$$

*Solution.* Let  $y = e^{mx}$ . This means  $y' = me^{mx}$  and  $y'' = m^2e^{mx}$ . Plugging in, we get

$$\begin{aligned} 4m^2e^{mx} + 4me^{mx} + e^{mx} &= 0 \\ e^{mx}(4m^2 + 4m + 1) &= 0 \\ e^{mx}(2m + 1)^2 &= 0. \end{aligned}$$

Thus,  $y_1 = C_0e^{-\frac{1}{2}x}$ . We now proceed to use reduction of order. So,  $y_2 = f(x) \cdot y_1$ . First, we have

$$\begin{aligned} y_2' &= f'(x)y_1 + f(x)y_1' = f'(x)e^{-x/2} - \frac{1}{2}f(x)e^{-x/2} \\ y_2'' &= f''(x)y_1 + 2f'(x)y_1' + f(x)y_1'' = f''(x)e^{-x/2} - f'(x)e^{-x/2} + \frac{1}{4}f(x)e^{-x/2}. \end{aligned}$$

Plugging this into the original differential equation, we get

$$\begin{aligned} f''(x)e^{-x/2} - f'(x)e^{-x/2} + \frac{1}{4}f(x)e^{-x/2} + f'(x)e^{-x/2} - \frac{1}{2}f(x)e^{-x/2} + \frac{1}{4}f(x)e^{-x/2} &= 0 \\ f''(x) &= 0. \end{aligned}$$

Now, let  $g(x) = f'(x)$ . This means that  $g'(x) = f''(x)$ . So,  $g'(x) = 0$ . This means  $g(x) = C$ . So,  $f'(x) = C$ . Thus,  $f(x) = C_1x + C_2$ . Therefore,

$$y_2 = (C_1x + C_2)e^{-x/2}.$$

Adding the two solutions, we obtain

$$\begin{aligned} y_h &= y_1 + y_2 = C_0e^{-x/2} + C_1xe^{-x/2} + C_2e^{-x/2} \\ &= Ce^{-x/2} + Dxe^{-x/2}. \end{aligned}$$

Using the initial values, we obtain

$$y_h = 3e^{-x/2} - 2xe^{-x/2}.$$

□

In general  $y_2 = xy_1$ .

**Example 3.2.8**

$$y''' + 3y'' - 4y = 0.$$

*Solution.* Let  $y = e^{mx}$ . This mean  $y' = me^{mx}$  and  $y'' = m^2e^{mx}$ . Plugging in, we get

$$\begin{aligned} m^3e^{mx} + 3m^2e^{mx} - 4e^{mx} &= 0 \\ e^{mx}(m^3 + 3m^2 - 4) &= 0 \\ e^{mx}(m - 1)(m^2 + 4m + 4) &= 0 \\ e^{mx}(m - 1)(m + 2)^2 &= 0. \end{aligned}$$

Therefore,  $y_1 = C_1 e^x$ ,  $y_2 = C_2 e^{-2x}$ , and  $y_3 = C_3 x e^{-2x}$ . Thus,

$$y_h = c_1 e^x + C_2 e^{-2x} + C_3 x e^{-2x}.$$

□

### Example 3.2.9 (Complex Roots)

$$y'' + 4y = 0.$$

*Solution.* Let  $y = e^{mx}$ . This means that  $y'' = m^2 e^{mx}$ . Plugging in gives  $m = \pm 2i$ . Thus,  $y_h = C_1 e^{2ix} + C_2 e^{-2ix}$ . Recall **Euler's Theorem**

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Plugging this in gives

$$\begin{aligned} y_h &= C_1 (\cos 2x + i \sin 2x) + C_2 (\cos 2x - i \sin 2x) \\ y_h &= C_1 \cos 2x + C_2 \sin 2x. \end{aligned}$$

□

### Example 3.2.10 (Complex Roots)

$$y'' + 8y' + 25y = 0.$$

*Solution.* Let  $y = e^{mx}$ . This means that  $y' = m e^{mx}$  and  $y'' = m^2 e^{mx}$ . Plugging this in gives  $-4 \pm 3i$ . Thus,

$$\begin{aligned} y_h &= C_1 e^{(-4+3i)x} + C_2 e^{(-4-3i)x} \\ y_h &= e^{-4x} (C_1 e^{3ix} + C_2 e^{-3ix}) \\ y_h &= e^{-4x} (C_1 \cos(3x) + C_2 \sin(3x)). \end{aligned}$$

□

## §3.3 Non Homogeneous ODEs

This occurs when  $f(x) \neq 0$ . Our solution will still include the homogeneous portion  $y = y_h(x) + y_p(x)$ . Some methods for solving these are **undetermined coefficients**, which can only be used for scalar coefficients, and **variation of parameters**, which can be used with scalar or variable coefficients.

**Example 3.3.1** (Undetermined Coefficients)

$$y'' - 4y' + 3y = e^{-x}; \quad y(0) = 1; \quad y'(0) = 0.$$

*Solution.* We first find  $y_h$  using  $y'' - 4y' + 3y = 0$ . Thus we obtain

$$y_h = C_1 e^x + C_2 e^{3x}.$$

Our guess for  $y_p = C e^{-x}$ . Thus,  $y'_p = -C e^{-x}$  and  $y''_p = C e^{-x}$ . So, plugging this in, we get

$$\begin{aligned} C e^{-x} + 4C e^{-x} + 3C e^{-x} &= e^{-x} \implies 8C e^{-x} = e^{-x} \\ &\implies C = \frac{1}{8}. \end{aligned}$$

Therefore

$$y = y_h + y_p = C_1 e^x + C_2 e^{3x} + \frac{1}{8} e^{-x}.$$

Using the initial values, we find that

$$y = \frac{5}{4} e^x - \frac{3}{8} e^{3x} + \frac{1}{8} e^{-x}.$$

□

The following is a useful guessing guide:

$$\begin{aligned} r(x) &\implies y_p(x) \\ k e^{rx} &\implies C e^{rx} \\ k x^n &\implies P_n = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \\ k \cos(wx) \text{ or } k \sin(wx) &\implies a \cos(wx) + b \sin(wx). \end{aligned}$$

**Example 3.3.2** (Guessing Practice with Undetermined Coefficients)

$$y'' + 2y' + y = \cos(2x)$$

*Solution.* We first proceed to find  $y_h$ . We find that

$$y_h = C_1 e^{-x} + C_2 x e^{-x}.$$

We make the following guess:  $y_p = a \cos(2x) + b \sin(2x)$ . Solving for  $a$  and  $b$ , we find that  $y_p = -\frac{3}{25} \cos(2x) + \frac{4}{25} \sin(2x)$ . Therefore

$$y = C_1 e^{-x} + C_2 x e^{-x} - \frac{3}{25} \cos(2x) + \frac{4}{25} \sin(2x).$$

□

**Example 3.3.3** (Bad Guessing)

$$y'' + 4y' + 3y = e^{-x}.$$

*Solution.* We notice  $y_h = C_1e^{-x} + C_2e^{-3x}$ .  $y_p = Ce^{-x}$  is a bad guess since it is a term present in the homogeneous equation. So, we let  $y_p = Cxe^{-x}$ , which is a guess from the result of applying the reduction of order. Solving for  $C$ , we get that  $C = \frac{1}{2}$ . Therefore

$$y = C_1e^{-x} + C_2e^{-3x} + \frac{1}{2}xe^{-x}.$$

□

**Definition 3.3.4. Variation of Parameter** involve differential equations of variable coefficients. For  $a_2y'' + a_1y' + a_0y = f(x)$ . First, we make the leading coefficient 1 by dividing by  $a_2$ . Thus,

$$y'' + p(x)y' + q(x)y = r(x).$$

We still do the homogeneous portion first:  $y_h = C_1y_1 + C_2y_2$ . Then  $y_p = u(x)y_1(x) + v(x)y_2(x)$ , where  $u' = -ry_2/w$  and  $v' = ry_1/w$ , where

$$w = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

This determinant is called the **Wronskian**.

**Example 3.3.5** (Variation of parameters)

$$y'' + 4y' + 4y = e^{-2x}/x^2.$$

*Solution.* We find that  $y_h = C_1e^{-2x} + C_2xe^{-2x}$ . Notice that the Wronskian is

$$w = \det \begin{pmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{pmatrix} = e^{-4x}.$$

Thus,

$$\begin{aligned} u'(x) &= -\frac{1}{x} \\ v'(x) &= \frac{1}{x^2}. \end{aligned}$$

Thus,  $u(x) = -\ln|x|$  and  $v(x) = -\frac{1}{x}$ . Therefore,

$$y_p = \ln|x|e^{2x} - e^{-2x} \implies y = C_1e^{-2x} + C_2xe^{-2x} + \ln|x|e^{2x}.$$

□

**Example 3.3.6** (Undetermined Coefficients)

$$y'' + y' + y = 3x^2 + 4.$$

*Solution.* We first find that

$$y_h = e^{(-1/2)x} \left( C_1 \cos(\sqrt{3}x/2) + C_2 \sin(\sqrt{3}x/2) \right).$$

Then we let  $y_p = D_1x^2 + D_2x + D_3$ . Therefore

$$\begin{aligned} 2D_1 + 2xD_1 + D_2 + D_1x^2 + D_2x + D_3 &= 3x^2 + 4 \\ x^2(D_1) + x(2D_1 + D_2) + (2D_1 + D_2 + D_3) &= 3x^2 + 4. \end{aligned}$$

Solving the resulting system yields  $D_1 = 3, D_2 = -6, D_3 = 4$ . Therefore

$$y_p = 3x^2 - 6x + 4.$$

So,

$$y = e^{(-1/2)x} \left( C_1 \cos(\sqrt{3}x/2) + C_2 \sin(\sqrt{3}x/2) \right) + 3x^2 - 6x + 4.$$

□