Parabolas

February 28, 2019

1 Parabolas and Where to Find Them

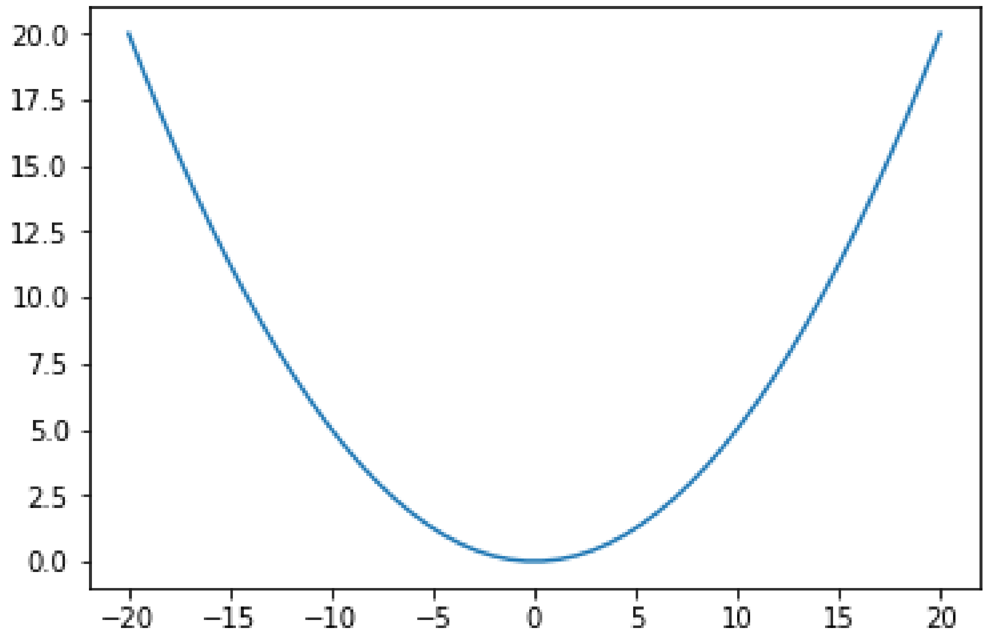
A regression back to High School math, with the goal of understanding how to fit a 2D rotated parabola.

```
In [1]: # %matplotlib notebook
        import matplotlib.pylab as plt
        from mpl_toolkits.mplot3d import axes3d, Axes3D
        import numpy as np
```

1.1 1D Parabola

```
In [2]: # create some x axis data
        xs = np.linspace(-20, 20)

In [3]: xs

Out[3]: array([-20.        , -19.18367347, -18.36734694, -17.55102041,
               -16.73469388, -15.91836735, -15.10204082, -14.28571429,
               -13.46938776, -12.65306122, -11.83673469, -11.02040816,
               -10.20408163,  -9.3877551 ,  -8.57142857,  -7.75510204,
                -6.93877551,  -6.12244898,  -5.30612245,  -4.48979592,
                -3.67346939,  -2.85714286,  -2.04081633,  -1.2244898 ,
                -0.40816327,   0.40816327,   1.2244898 ,   2.04081633,
                 2.85714286,   3.67346939,   4.48979592,   5.30612245,
                 6.12244898,   6.93877551,   7.75510204,   8.57142857,
                 9.3877551 ,  10.20408163,  11.02040816,  11.83673469,
                12.65306122,  13.46938776,  14.28571429,  15.10204082,
                15.91836735,  16.73469388,  17.55102041,  18.36734694,
                19.18367347,  20.        ])

In [4]: # and our y axis data uses the equation for a parabola
        f = 5.
        zs = (1 / (4.*f))*(xs)**2

In [5]: # make sure it looks right!
        fig = plt.figure()
        ax = fig.gca()
        ax.plot(xs, zs)
```

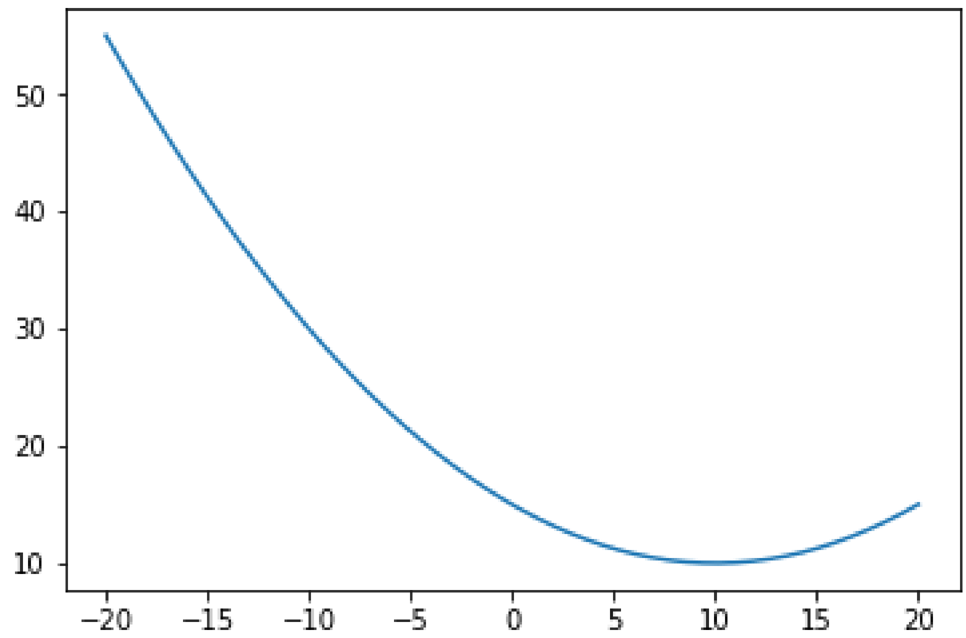
Out[5]: [<matplotlib.lines.Line2D at 0x7f1b80a2bb90>]



In [6]:
```
# now apply some vertices and watch the displacement happen
v1 = 10.
v2 = 10.
zs2 = ((1/(4.*f))*(xs - v1)**2) + v2
```

In [7]:
```
fig = plt.figure()
ax = fig.gca()
ax.plot(xs, zs2)
```

Out[7]: [<matplotlib.lines.Line2D at 0x7f1b7e9136d0>]

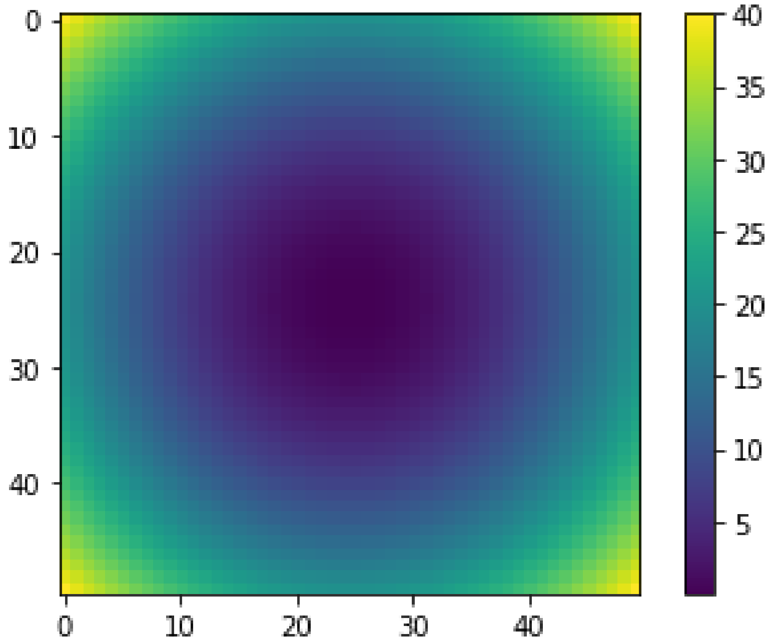


1.2 2D Parabola

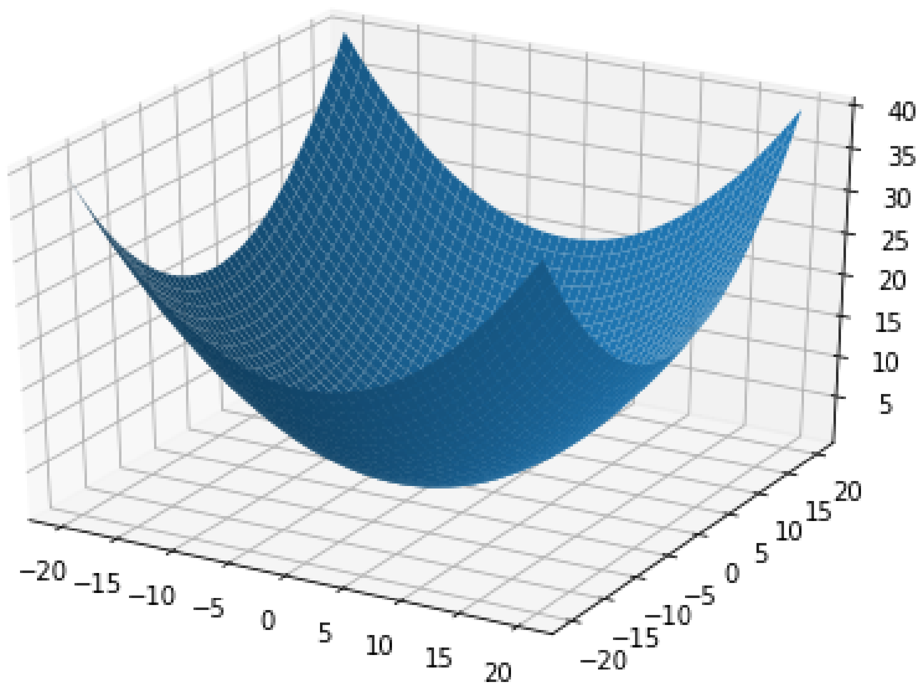
```
In [8]: # now we need to create x and y axis data
        ys = np.linspace(-20., 20.)
        xs2d, ys2d = np.meshgrid(xs, ys)
        # and here we apply a 2D parabola equation
        zs2d = (1 / (4.*f))*(xs2d)**2 +  (1 / (4.*f))*(ys2d)**2
        zs2d.shape
```

```
Out[8]: (50, 50)
```

```
In [9]: # make sure this 2D parabola makes sense
        from mpl_toolkits.mplot3d import axes3d, Axes3D
        fig = plt.figure()
        ax = fig.gca()
        cax = ax.imshow(zs2d)
        cbar = fig.colorbar(cax)
```



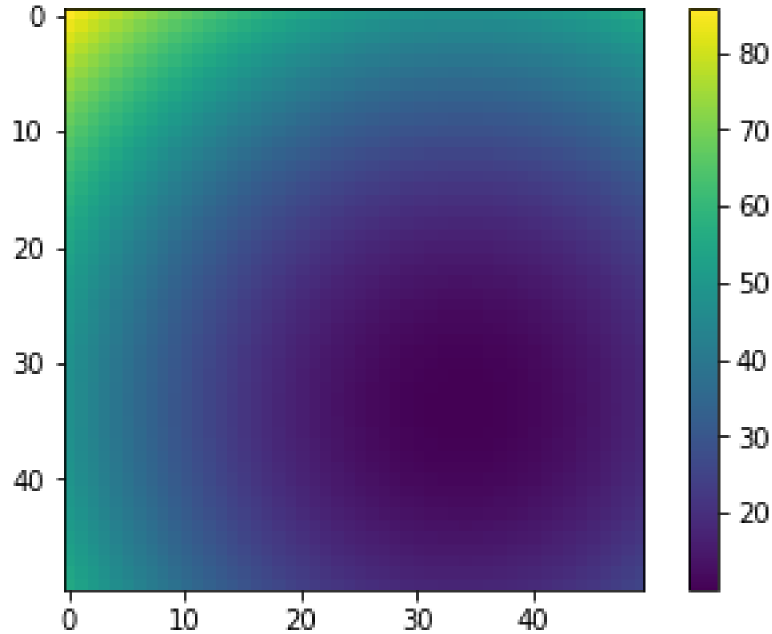

```
In [10]: # surface plot is even more informative
         fig = plt.figure()
         ax = Axes3D(fig)
         ax.plot_surface(xs2d, ys2d, zs2d)

Out[10]: <mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x7f1b7e5be090>
```




```
In [11]: # now let's apply the equation for a 2D Parabola but with vertices (displacement)
         v2 = 10.
         v1x = v1y = 7.5
         zs2d2 = (1 / (4.*f))*(xs2d - v1x)**2 + (1 / (4.*f))*(ys2d - v1y)**2 + v2;
```

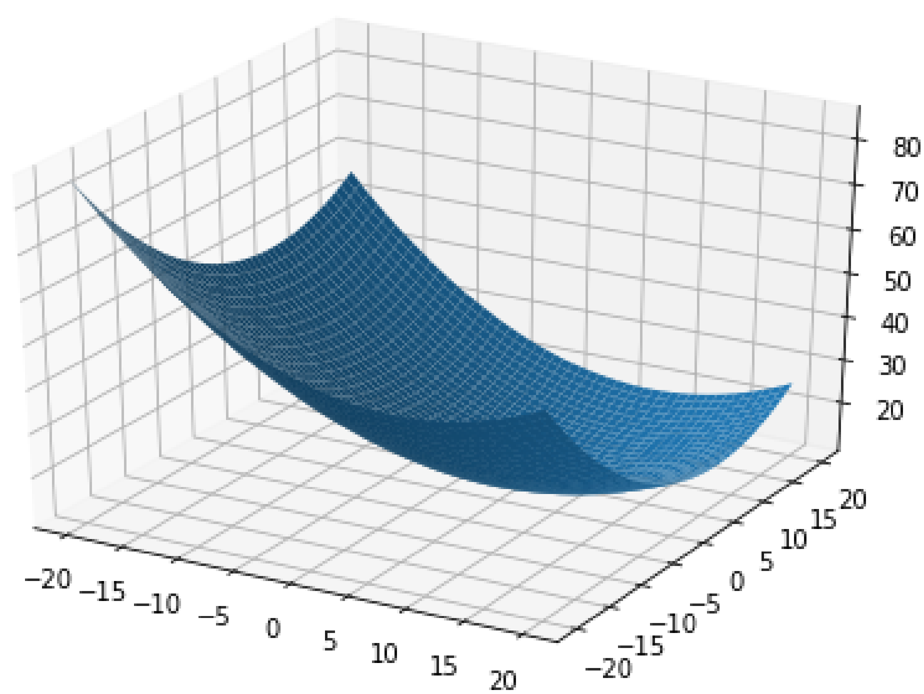
```
In [12]: fig = plt.figure()
         ax = fig.gca()
         cax = ax.imshow(zs2d2)
         fig.colorbar(cax)
```

```
Out[12]: <matplotlib.colorbar.Colorbar at 0x7f1b7e0b6710>
```




```
In [13]: fig = plt.figure()
         ax = Axes3D(fig)
         ax.plot_surface(xs2d, ys2d, zs2d2)
```

```
Out[13]: <mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x7f1b7df88a50>
```



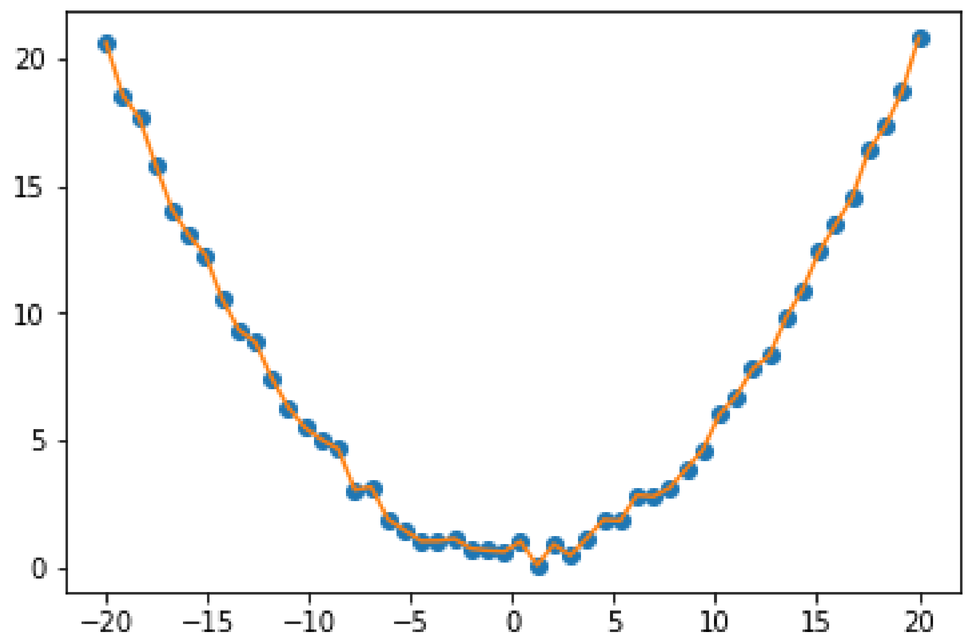
1.3 Simple 1D Parabola Fit

```
In [14]: # now create a simple 1 D parabola but with added noise
         focal = 5.
         data1D = (1 / (4.*focal))*(xs)**2 + (np.random.rand(50))
         data1D.shape
```

```
Out[14]: (50,)
```

```
In [15]: # make sure it looks right
         fig = plt.figure()
         ax = fig.gca()
         ax.plot(xs, data1D, 'o', xs, data1D)
```

```
Out[15]: [<matplotlib.lines.Line2D at 0x7f1b7c514dd0>,
          <matplotlib.lines.Line2D at 0x7f1b7e500610>]
```



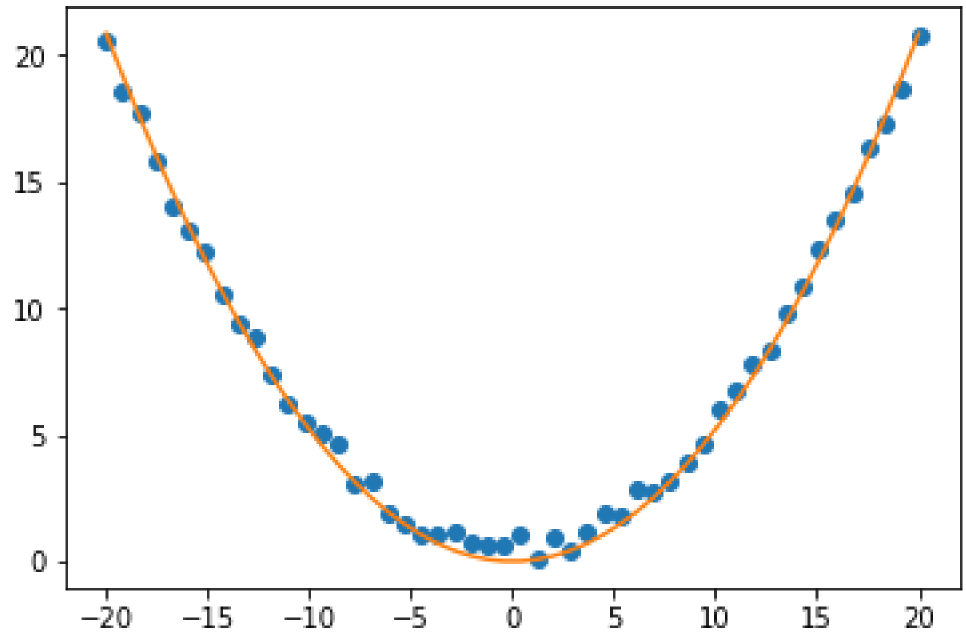

```
In [16]: # we want to fit this data to the equation of a 1D parabola (NO vertices)
         def fun(coeffs, xdata):
             return (1 / (4.*coeffs[0]))*(xdata)**2
```

```
In [17]: # and what we want to minimize is the difference between the function above and our mad
         def errfun(coeffs, xdata, ydata):
             return fun(coeffs, xdata) - ydata
```

```
In [18]: # here we use scipy to fit our data
         from scipy.optimize import least_squares
         #r = least_squares(errfun, [3.5], args=(xs, data1D), ftol=1e-15)
         # we set our inital guess close to the original value of the focus of 5
         guess = [3.5]
         r = least_squares(errfun, guess, args=(xs, data1D))
         #print r
         # what's the answer?  Was it able to find an answer?
         print r.success
         print r.message
         print r.nfev
         print r.x

         # let's see what the fitted data looks like: that is, the application of the fit
         # we found from above to the parabola function
         fittedFocus = r.x
         fittedData = fun([fittedFocus], xs)

         # graph the two together to see how close the fit is
         fig = plt.figure()
         ax = fig.gca()
         ax.plot(xs, data1D, 'o', xs, fittedData)
```

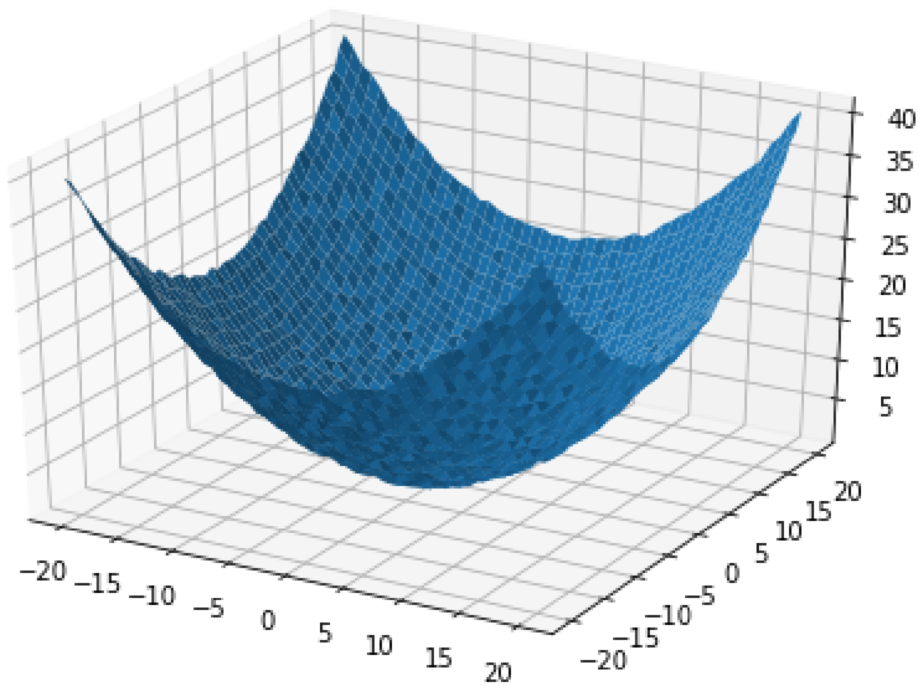
```
True
`xtol` termination condition is satisfied.
6
[4.79274099]
```

```
Out[18]: [<matplotlib.lines.Line2D at 0x7f1b36f65150>,
          <matplotlib.lines.Line2D at 0x7f1b36f74190>]
```



1.4 Simple 2D Parabola Fitting

```
In [19]: # same thing as above, but in 2D - make some noisy data
         xs2d, ys2d = np.meshgrid(xs, ys)
         data2D = (1 / (4.*f))*(xs2d)**2 +  (1 / (4.*f))*(ys2d)**2 + np.random.rand(50, 50)
         fig = plt.figure()
         ax = Axes3D(fig)
         ax.plot_surface(xs2d, ys2d, data2D)

Out[19]: <mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x7f1b34f18d50>
```

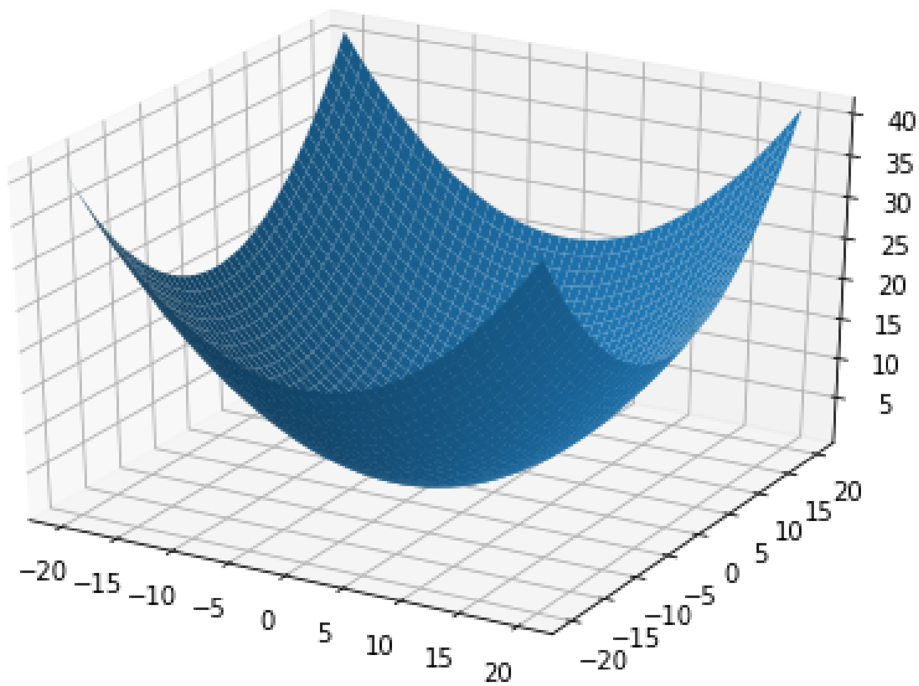



```
In [20]: def fun(coeffs, xdata, ydata):
             return (1 / (4.*coeffs[0]))*(xdata)**2 + (1 / (4.*coeffs[0]))*(ydata)**2

In [21]: def errfun(coeffs, xdata, ydata, zdata):
             return fun(coeffs, xdata, ydata) - zdata

In [22]: # find a fit for the 2D data
         r = least_squares(errfun, [3.5], args=(xs2d.flatten(), ys2d.flatten(), data2D.flatten()
         #print r
         print r.success
         print r.message
         print r.nfev
         print r.x

         # take a look at the fitted data
         fittedFocus = r.x
         fittedData = fun([fittedFocus], xs2d, ys2d)
         fig = plt.figure()
         ax = Axes3D(fig)
         ax.plot_surface(xs2d, ys2d, fittedData)
```

```
True
Both `ftol` and `xtol` termination conditions are satisfied.
```

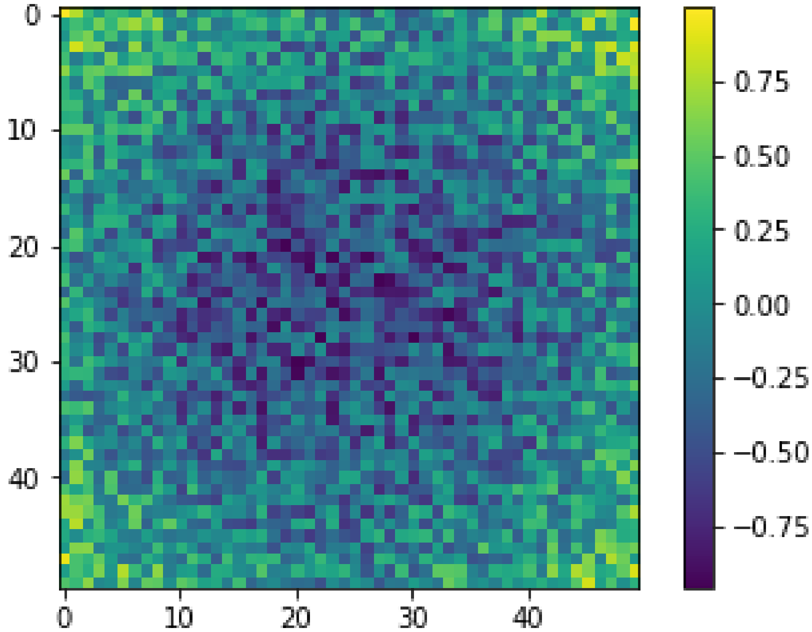
```
6
[4.87389014]
```

Out[22]: <mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x7f1b34f1f1d0>



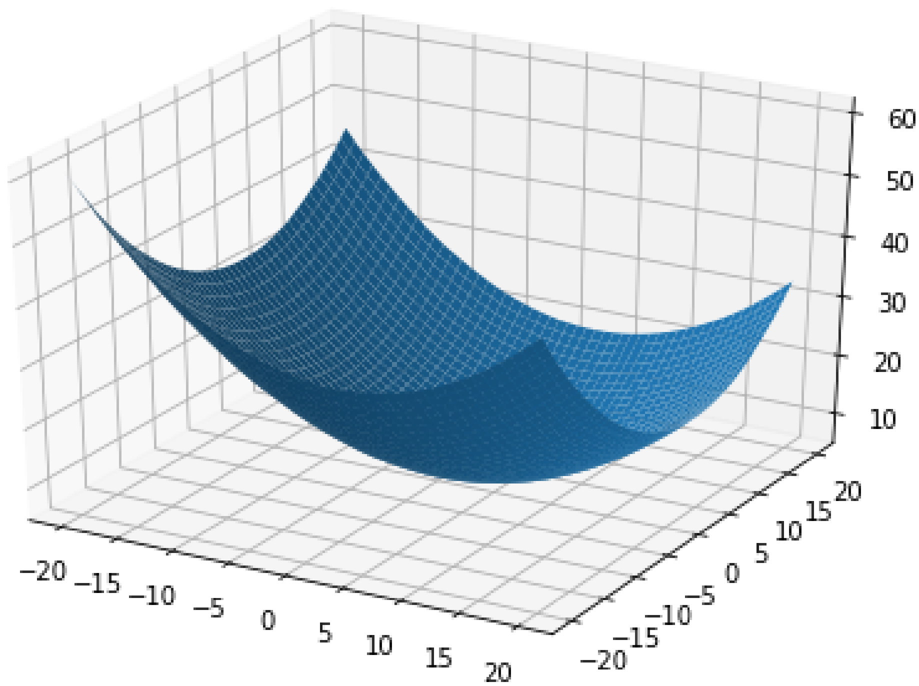
In [23]:
```python
# from the graphs above, the fit looks good, but lets find the difference
# between the fit and the original data anyways
residuals = fittedData - data2D
fig = plt.figure()
ax = fig.gca()
cax = ax.imshow(residuals)
fig.colorbar(cax)
print "focus", f
```

```
focus 5.0
```



1.5 2D Parabola Fitting with vertices

```
In [24]: # Same thing as above, just our parabola function for creating the noisy data,
         # and the function we are fitting to, just get a little more complicated
         xs2d, ys2d = np.meshgrid(xs, ys)
         v1x = 3.0
         v1y = 4.0
         v2 = 6.0
         data2D = (1 / (4.*f))*(xs2d  - v1x)**2 +  (1 / (4.*f))*(ys2d - v1y)**2 + v2
         #np.random.rand(50, 50)
         fig = plt.figure()
         ax = Axes3D(fig)
         ax.plot_surface(xs2d, ys2d, data2D)

Out[24]: <mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x7f1b346e3990>
```




```
In [25]: def fun(coeffs, xdata, ydata):
             f = coeffs[0]
             v1x = coeffs[1]
             v1y = coeffs[2]
             v2 = coeffs[3]
             return (1 / (4.*f))*(xdata - v1x)**2 + (1 / (4.*f))*(ydata - v1y)**2 + v2    #
             #r2 = (1 / (4.*f))*(xdata - v1x)**2 + (1 / (4.*f))*(ydata - v1y)**2
             #hr = 100.
             #return (r2 * (hr**2 > r2)) + v2
```

```
In [26]: def errfun(coeffs, xdata, ydata, zdata):
             return fun(coeffs, xdata, ydata) - zdata
```

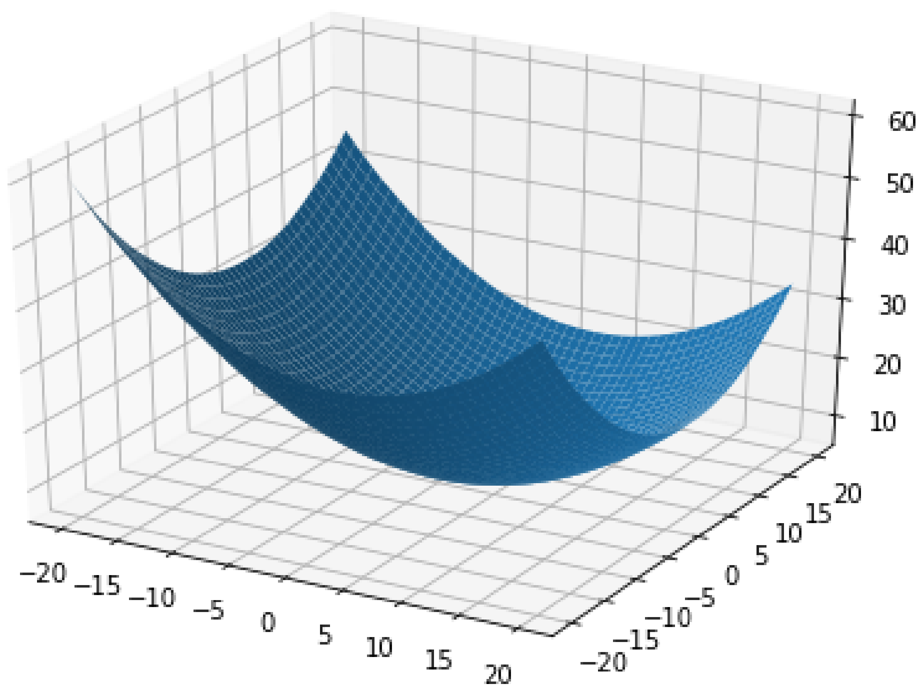
```
In [27]: # see if we come up with a fit, using a guess pretty close the original params that
         # were used to create the data
         #guess = [3.5, 2.0, 2.0, 2.0]
         guess = [f, v1x, v1y, v2]
         r = least_squares(errfun, guess, args=(xs2d.flatten(), ys2d.flatten(), data2D.flatten()
                           method='trf',
                           ftol=1e-15,
                           xtol=1e-15)
         #print r
         print r.success
```

```
print r.message
print r.nfev
print r.x
```

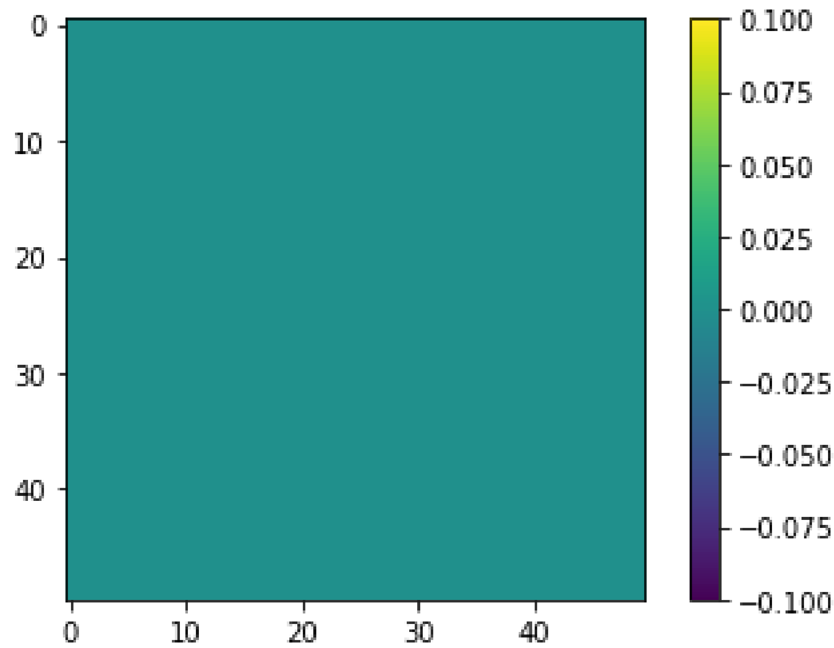
```
True
`gtol` termination condition is satisfied.
1
[5. 3. 4. 6.]
```

In [28]:
```
# take a look at the fitted data
fittedFocus = r.x
fittedData = fun(r.x, xs2d, ys2d)
fig = plt.figure()
ax = Axes3D(fig)
ax.plot_surface(xs2d, ys2d, fittedData)
```

Out[28]:
```
<mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x7f1b3430fdd0>
```



In [29]:
```
# and again, make sure the difference between the fit and the original data is small
residuals = fittedData - data2D
fig = plt.figure()
ax = fig.gca()
cax = ax.imshow(residuals)
cbar = fig.colorbar(cax)
```




```
In [30]: np.max(residuals)
```

```
Out[30]: 0.0
```

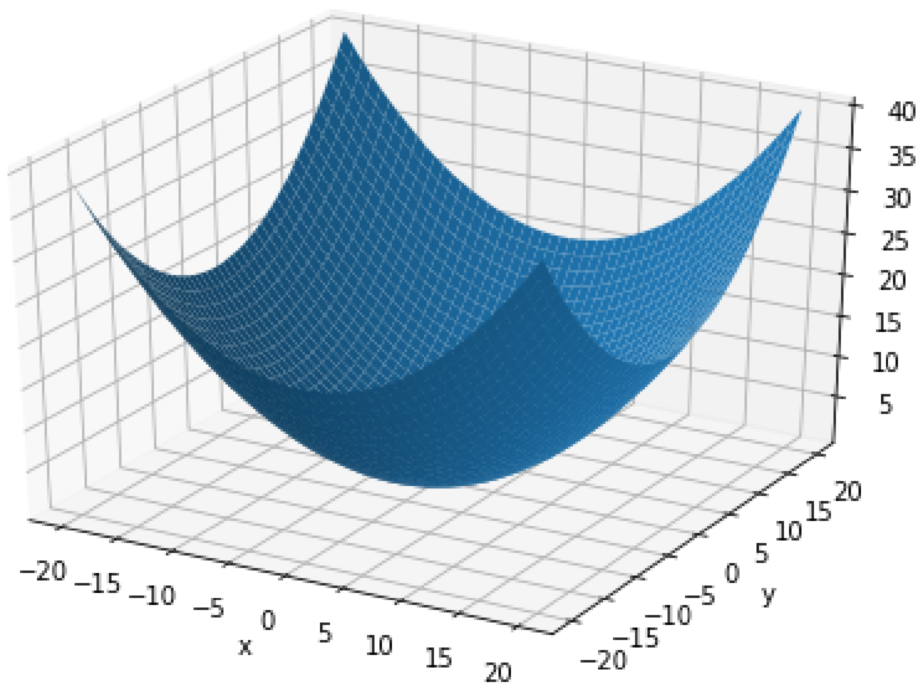
1.6 2D Rotated Parabola

Now things get complicated. Let's take our 2D parabola and rotate it.

```
In [31]: xs2d, ys2d = np.meshgrid(xs, ys)
         # first, don't use an displacement
         v1x = 0.0
         v1y = 0.0
         v2 = 0.0
         # equation for a 2D parabola
         data2d = (1 / (4.*f))*(xs2d  - v1x)**2 +  (1 / (4.*f))*(ys2d - v1y)**2 + v2
```

```
In [32]: # make sure the original data looks sensible
         fig = plt.figure()
         ax = Axes3D(fig)
         ax.plot_surface(xs2d, ys2d, data2d)
         plt.xlabel("x")
         plt.ylabel("y")
```

```
Out[32]: Text(0.5,0,'y')
```

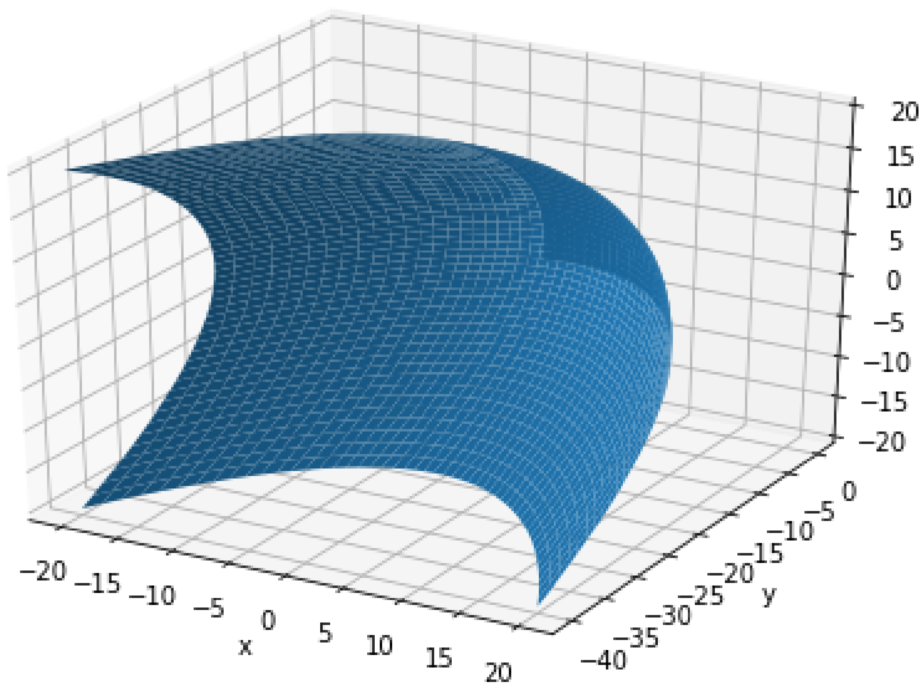


In [33]:
```
# now import the rotation functions we'll need
from rotate import *
```

In [34]:
```
# actually rotate the data!

qtrTurn = np.pi / 2
xs2dRot, ys2dRot, zs2dRot = rotateXY(xs2d, ys2d, data2d, qtrTurn, qtrTurn)
```

In [35]:
```
# now view this to make sure the rotation looks right!
fig = plt.figure()
ax = Axes3D(fig)
ax.plot_surface(xs2dRot, ys2dRot, zs2dRot)
plt.xlabel("x")
plt.ylabel("y")
```

Out[35]:
```
Text(0.5,0,'y')
```



1.7 Fit 2D Rotated Parabola

Now we'll try to fit a 2D rotated parabola. However, things get really complicated, so now we'll switch to using a python module for our parabola methods. This module comes with a test suite, TestParabolas.py, as well.

```
In [36]: from parabolas import *
```

1.8 Try limits of fitting routine

First let's see what works, and what doesn't. We'll use a function that simulates parabolas based off the inputs, and tries to find fits for them. This is like what you'll see in the unit tests

```
In [37]: # first, try no rotation
         f = 5.0
         v1x = 0.0
         v1y = 0.0
         v2 = 10.0
         xRot = 0.
         yRot = 0.
         data = [f, v1x, v1y, v2, xRot, yRot]
         _, _, _, diff = tryFit(data)
         # with no rotation, we easily get the right answer!
```

```
# here's the diff between the right answer and our fit
print diff
```

```
[0. 0. 0. 0. 0. 0.]
```

```
In [38]: from copy import copy
         # even when our inital guess is off
         guess = [1.0, 0., 0., 0., 0., 0.]
         _, _, _, diff = tryFit(data, guess)
         # with no rotation, we easily get the right answer!
         print diff
```

```
[0.00000000e+00 3.18571680e-15 2.47811901e-16 0.00000000e+00
 2.59557157e-17 8.79718302e-17]
```

```
In [39]: # we can still have success, when just one axis is rotated
         import numpy as np
         xRot = np.pi/2
         data = [f, v1x, v1y, v2, xRot, yRot]
         _, _, _, diff = tryFit(data)
         # here's the diff between the right answer and our fit
         print diff
```

```
[0. 0. 0. 0. 0. 0.]
```

```
In [40]: # but trying to rotate at all in the other axis as well brings problems
         yRot = 0.1
         data = [f, v1x, v1y, v2, xRot, yRot]
         _, _, _, diff = tryFit(data)
         # here's the diff between the right answer and our fit.
         # some parts of the fit are way off
         print diff
```

```
[8.88178420e-16 1.38968162e-15 8.73292814e-16 0.00000000e+00
 0.00000000e+00 1.00000000e-01]
```

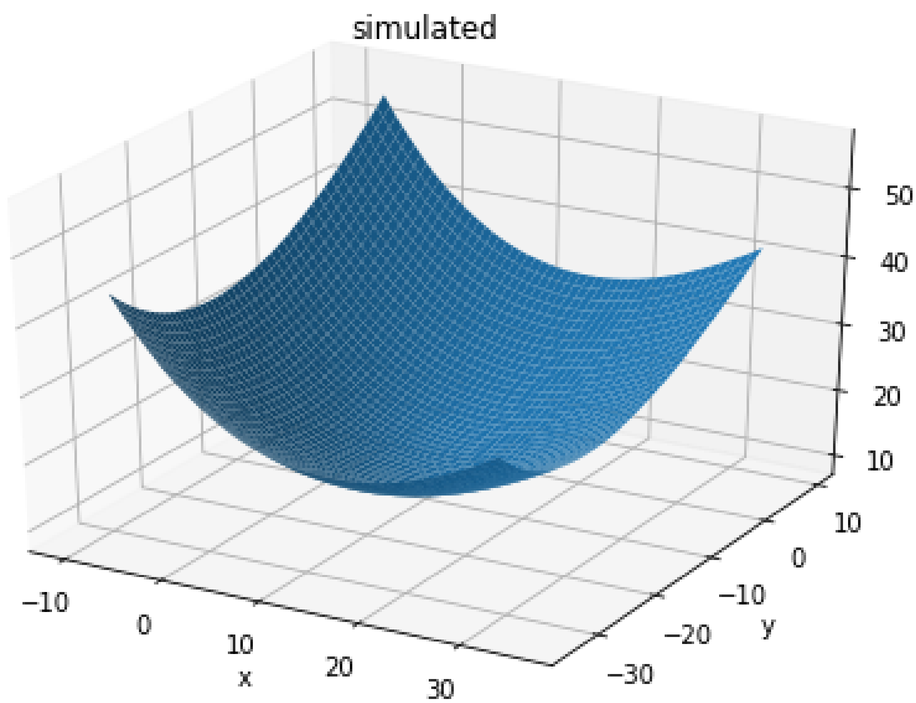
```
In [41]: # if you want to rotate both axis, keep the angles small
         xRot = np.pi/10
         yRot = np.pi/10
         data = [f, v1x, v1y, v2, xRot, yRot]
         _, _, _, diff = tryFit(data)
         # here's the diff between the right answer and our fit
         print diff
```

```
[0.00000000e+00 3.37948543e-15 1.40797674e-15 1.77635684e-15
 1.50492712e-02 1.58624900e-02]
```

1.9 Visualize fits

Here we'll repeat the last example, execpt visualize the simulated vs. fitted data

In [42]:
```
# create the simulated data
xs = np.linspace(-20, 20)
ys = np.linspace(-20, 20)

xs2d, ys2d = np.meshgrid(xs, ys)

#answer = [f, v1x, v1y, v2, -xRot, -yRot]

xdata, ydata, zdata = simData(xs2d, ys2d, f, v1x, v1y, v2, xRot, yRot)

# and plot it
surface3dPlot(xdata, ydata, zdata, "simulated")
```



In [43]:
```
# now fit this data
# make a really good guess
guess = data
r = lsqFit(xdata, ydata, zdata, guess)
print "fit: ", r.x
```

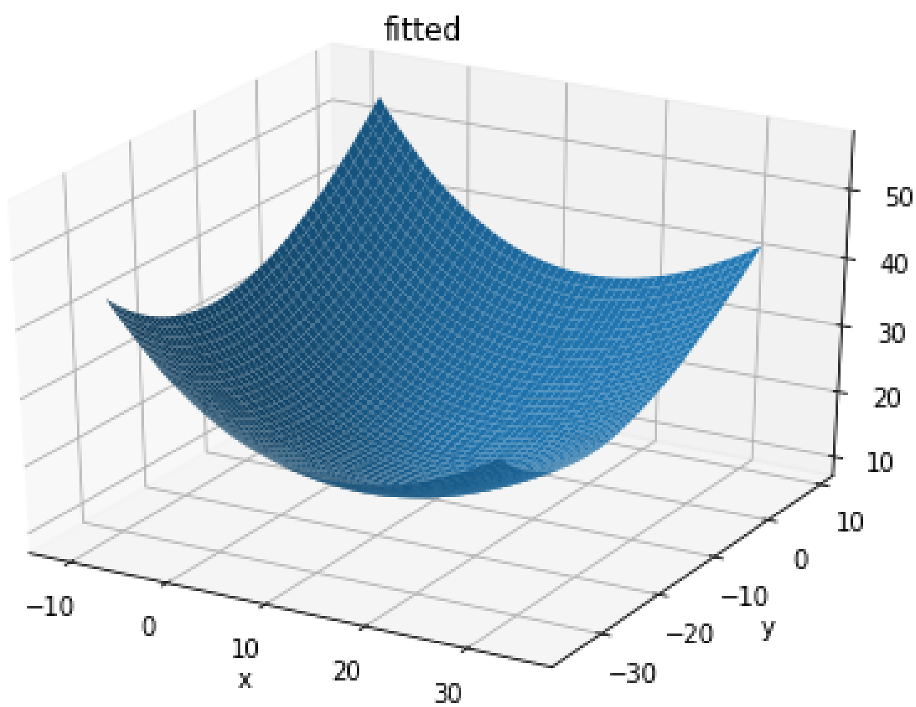
```
# now visualize what this fitted data looks like:
#fFit, v1xFit, v1yFit, v2Fit, xRotFit, yRotFit = r.x
# note how we invert the rotations
#xfit, yfit, zfit = simData(xdata, zdata, fFit, v1xFit, v1yFit, v2Fit, -xRotFit, -yRotF

# and plot it
#surface3dPlot(xfit, yfit, zfit, "fitted")
```

```
fit:  [ 5.00000000e+00 -2.79216248e-15  8.73543312e-16  1.00000000e+01
 -3.29208537e-01 -2.98296775e-01]
```

In [44]:
```
print xRot, yRot
```

```
0.314159265359 0.314159265359
```

In [45]:
```
# now visualize what this fitted data looks like:
fFit, v1xFit, v1yFit, v2Fit, xRotFit, yRotFit = r.x
# note how we invert the rotations
xfit, yfit, zfit = simData(xs2d, ys2d, fFit, v1xFit, v1yFit, v2Fit, -xRotFit, -yRotFit)

# and plot it
surface3dPlot(xfit, yfit, zfit, "fitted")
```



```
In [ ]:
```