第一章　多项式习题解答

P44.1　1) $f(x)=g(x)(\frac{1}{3}x-\frac{7}{9})+(-\frac{26}{9}x-\frac{2}{9})$

2) $f(x)=g(x)(x^2+x-1)+(-5x+7)$

P44.2　1) $x^2+mx-1\,|\,x^3+9x+q \Rightarrow$ 余式 $(p+1+m^2)x+(q-m)=0$

$\therefore \begin{cases} m=q \\ p=q^2-1 \end{cases}$

方法二，

设 $x^3+px+q=(x^2+m-1)(x+q) \Rightarrow \begin{cases} m-q=0 \\ -mq-1=p \end{cases}$ 同样。

2) $x^2+mx+1\,|\,x^4+px^2+q \Rightarrow$ 余式 $m(p+2-m^2)x-(q-p+1+m^2)=0$

$\therefore m(m^2+p-2)=0. \qquad m^2+p=1+q,(x^2=1-p+q)$

P44.3.1 用 $g(x)=x+3$ 除 $f(x)=2x^5-5x^3-8x$

解：

$\therefore f(x)=2(x+3)^5-30(x+3)^4+175(x+3)^3-495(x+3)^2+667(x+3)-327$

P44.3 .2)

$$\begin{aligned}&\therefore (x^3-x^2-x)\\&=(x-1+2i)^3+(2-8i)(x-1+2i)^2\\&-(12+8i)(x-1+2i)-(9-8i)\end{aligned}$$

即余式 $-9+8i$

商 $x^2-2ix-(5+2i)$

P44 4.1).　$f(x)=x^5,x_0=1$；即

$\therefore f(x)=(x-1)^5+5(x-1)^4+10(x-1)^3+10(x-1)^2+5(x-1)+1$

当然也可以 $f(x)=x^5=[(x-1)+1]^5$

$=(x-1)^5+5(x-1)^4+\cdots+1$

P44.4　2) 结果

$$f(x)=x^4-2x^2+3=(x+2)^4-8(x+2)^3+22(x+2)^2-24(x+2)+11$$

3)

$$\begin{aligned}f(x)&=x^4+2ix^3-(1+i)x^2+3x+7+i\\&=(x+i-i)^4+2i(x+i-i)^3-(1+i)(x+i-i)^2-3(x+i-i)+7+i\\&=(x+i)^4-2i(x+i)^3+(1+i)(x+i)^2-5(x+i)+7+5i\end{aligned}$$

P45.5

(1) $g(x)=(x-1)(x^2+2x+1)=(x-1)(x+1)^2$

$f(x)=(x+1)(x^3-3x-1)$

$\therefore (f(x),g(x))=x+1$

(2) $g(x)=x^3-3x^2+1$不可约

$f(x)=x^4-4x^3+1$不可约

$\therefore \left(f(x),g(x)\right)=1$

(3) $f(x)=x^4-10x^2+1=(x^2+2\sqrt{2}x-1)(x^2-2\sqrt{2}x-1)$

$g(x)=x^4-4\sqrt{2}x^3+6x^2+6\sqrt{2}x+1, f(x)=4\sqrt{2}(-x^3+2\sqrt{2}x^2+x)=(x^2-2\sqrt{2}x-1)^2$ $\therefore$

$\left(f(x),g(x)\right)=x^2-2\sqrt{2}x-1$

P45.6

(1) $f(x)=(x+1)^2(x^2-2)$ $\quad g(x)=(x^2-2)(x^2+x+1)$

$\because (x+1)^2\left[-(x+1)\right]+(x^2+x+1)(x+2)=1$

$\therefore (x^2-2)=-(x+1)f(x)+(x+2)g(x)$

(2) $f(x)=(x-1)(4x^3+2x^2-14x-y)$， $g(x)=(x-1)(2x^2+x-4)$

$=(x-1)f_1(x)$ $\quad =(x-1)g_1(x)$

而

$f_1(x)=g_1(x)\cdot 2x\text{-}3(2x+3)$

$g_1(x)=(2x+3)\cdot(x-1)$

$\therefore 1=(2x+3)(x-1)-g_1=(\frac{2x}{3}y_1-\frac{1}{3}f_1)(x-1)-g_1$

$\because x-1=-\frac{1}{3}(x-1)f(x)+(\frac{2}{3}x^2-\frac{2}{3}x-1)g(x)$

(3) $f(x)=x^4-x^3-4x^2+4x+1$， $g(x)=x^2-x-1$

$\therefore f(x)=g(x)(x^2-3)+(x-2)$， $g(x)=(x-2)(x+1)+1$

$\because 1=-(f-g(x^2-3))(x+1)+g$

$=-(x+1)f(x)+(x^3+x^2-3x-2)g(x)$.

P45.7

$f(x)=g(x)1+(1+t)x^2+(2-t)x+u=r(x)$

$g(x)=r(x)(\frac{1}{1+t}x+\frac{t-2}{(1+t)^2})+\frac{(t^2+t+u)+(t-2)^2}{(1+t)^2}x+(1-\frac{t-2}{(t+1)^2})u$

由题意$r(x)$与g(x)的公因式应为二次所以$r(x)\mid g(x)$

$$\therefore \begin{cases}\dfrac{t^3+3t^2-(u+3)t+(4-u)}{(1+t)^2}=0\\ \dfrac{t^2+t+3}{(1+t)^2}u=0\end{cases}$$

$t\neq -1$否则$r(x)$为一次的

$$\therefore \Rightarrow\begin{cases}t^3+3t^2-(u+3)t+(4-u)=0\\ (t^2+t+3)u=0\end{cases}$$

解出(ⅰ)当$u=0$时$t^3+3t^2-3t+4=0(t+4)(t^2-t+1)$

∴ $t=-4$或$t=\dfrac{1\pm\sqrt{3}\mathrm{i}}{2}=e^{\pm\frac{\pi}{3}\mathrm{i}}$

（ⅱ）当$u\neq 0$时,只有$t^2+t+3=0,\dfrac{1}{t+1}=-\dfrac{t}{3}$

$$t^3+3t^2-(u+3)t+(4-u)\Rightarrow u=\frac{t^3+3t^2-3t+4}{t+1}=-\frac{t}{3}(t^3+3t^2-3t+4)$$

∴ $u=-\dfrac{1}{3}[(t^2+t+3)(t^2+2t-8)+6t+24]=-2(t+4)$

即 $\begin{cases} u=-2(t+4) \\ t^2+t+3=0 \end{cases}$

$$t=\frac{-1\pm\sqrt{-11}}{2}$$

P45、8 $d(x)\mid f(x),d(x)\mid g(x)$ 表明 $d(x)$ 是公因式

又已知：$d(x)$是$f(x)$与$g(x)$的组合　　表明任何公因式整除 $d(x)$

所以 $d(x)$ 是一个最大的公因式。

P45，9. 证明$(f(x)h(x),g(x)h(x))=(f(x),g(x)h(x))$（$h(x)$ 的首系=1）

证：设$(f(x)h(x),g(x)h(x))=m(x)$ 由

$(f(x),g(x))h(x)\mid f(x)h(x)$　　　$(f(x),g(x))h(x)\mid g(x)h(x).$

$\therefore (f(x),g(x))h(x)\mid m(x)$　　$\therefore (f(x),g(x))h(x)$ 是一个公因式。

设 $d(x)=(f(x),g(x))=u(x)f(x)+v(x)g(x).$

$\therefore d(x)h(x)=(f(x),g(x))h(x)=u(x)f(x)h(x)+v(x)g(x)h(x).$

而首项系数=1，又是公因式得（由 P45、8），它是最大公因式，且

$(f(x),g(x))h(x)=(f(x)h(x),g(x)h(x)).$

P45、10 已知 $f(x),g(x)$ 不全为 0。证明 $\left(\dfrac{f(x)}{(f(x),g(x))},\dfrac{g(x)}{(f(x),g(x))}\right)=1.$

证：设 $d(x)=(f(x),g(x)).$ 则 $d(x)\neq 0.$

设 $\dfrac{f(x)}{d(x)}=f_1(x),$　　$\dfrac{g(x)}{d(x)}=g_1(x),$ 及 $d(x)=u(x)f(x)+v(x)g(x).$

所以 $d(x)=u(x)f_1(x)d(x)+v(x)g_1(x)d(x).$

消去 $d(x)\neq 0$ 得 $1=u(x)f_1(x)+v(x)g_1(x)$

P45.11　证：设$(f(x),g(x))=d(x)\neq 0,\ f(x)=f_1(x)d(x),g(x)=g_1(x)d(x)$

∴ $u(x)f_1(x)d(x)+u(x)g_1(x)d(x)=d(x),u(x)f_1(x)+u(x)g_1(x)=1$

P45.12

设$uf+vg=1,\ u_1f+v_1h=1\ \Rightarrow uu_1f^2+ufv_1h+vgu_1f+vu_1gh=1$

$\therefore (uu_1f+uv_1h+vgu_1)f+(v_1u)gh=1\Rightarrow (f,gh)=1$.

P45.13

$\because\ (f_i,\mathrm{g}_i)=1$,

固定$i:(f_i,g_1g_2)=1$

$(f_i,g_1.g_2.g_n)=1$

P45.14

$(f,g)=1\Rightarrow uf+vg=1\Rightarrow (u-v)f+v(g+f)=1\Rightarrow (f,g+f)=1$

同理$(g,g+f)=1$.

由 12 题$(fg,f+g)=1$

令$g=g_1g_2\cdots g_n$

$\therefore$每个$i,(f_i,g)=1$

$\Rightarrow (f_1f_1,g)=1,$

$\Rightarrow\ (f_1f_2f_3,g)=1$,

$\Rightarrow\ (f_1f_2\cdots f_m,g_1g_2\cdots g_n)=1$（注反复归纳用 12 题）。

推广

若$(f(x),g(x))=1,$则$\forall$ m，n有$(f(x)^m,g(x)^n)=1$

P45,15

f(x)=x^3+2x^2+2x+1, g(x)=$x^4+x^3+2x^2+x+1$

解：g(x)=f(x)(x-1)+2(x^2+x+1),

f(x)=(x^2+x+1)(x+1)

即(f(x)，g(x)) = x^2+x+1.

令(x^2+x+1)=0 得$\varepsilon_1=\dfrac{-1+\sqrt{3}i}{2},\varepsilon_2=\dfrac{-1-\sqrt{3}i}{2}$

$\therefore$f(x)与g(x)的公共根为$\varepsilon_1,\varepsilon_2$.

P45.16 判断有无重因式

① $f(x)=x^5-5\ x^4+7x^3+2x^2+4x-8$　　② $f(x)=x^4+4x^2-4x-3$

解① $f'(x)=5x^4-20x^3+21x^2-4x+4$

$5f(x)=f'(x)(x-1)-3(2x^3-5x^2 4x+12)$

$2\ f'(x)=(2x^3-5x^2-4x+12)(5x-\dfrac{15}{2})+\dfrac{49}{2}(x^2-4x+4)$

$(2x^3-5x^2-4x+12)=(x^2-4x+4)(2x+3)$

故 $f(x)$ 有重因式 $(x-2)^3$

② $f'(x)=4x^3+8x-4$

$f(x)=(x^3+2x-1)x+(2x^2-3x+3)$

$f'(x)=(2x^2-3x+3)(2x+3)+(11x-13)$

$11(2x^2-3x+3)=(11x-13)(2x-\frac{6}{11})+(33+\frac{6\times 13}{11})$

$\therefore (f(x).f'(x))=1$

P45.17 $t=?$时$f(x)=x^3-3x^2+tx-1$ 有重因式（有重根）

解. $f'(x)=3x^2-6x+t \quad 3f(x)=f'(x)(x-1)+(2t-6)x+(t-3)$

如 $t=3$ 则 有重因式：3 重因式 $(x-1)^3=f(x)$

如 $t\neq 3.$ 则 $2f'(x)=(2x+2)(3\times -\frac{15}{2})+(2t+\frac{15}{2})$

此时必须 $t=-\frac{15}{4}$ 有重因式 $f(x)=(x+\frac{1}{2})^2(x-4)$

P45.18 求多项式 $f(x)=x^3+px+q$ 有重根因式的条件

证 $f(x)=3x^2+p$

$3f(x)=(3x^2+p)x+2px+3q \qquad (p\neq 0)$

$(3x^2+p)=(2px+3q)(\frac{3}{2p}x-\frac{3a}{4p^2})+(p+\frac{27q^2}{4p^2})$

$\therefore 4p^3+27q^2=0$

p45.19 令 $f(x)=Ax^4+Bx^2+1,$因为$(x-1)^2\mid f(x),$所以$(x-1)\Big|\ f'(x))$

即 $f'(x)=4Ax^3+2Bx=(x-1)(ax^2+bx+c)$

$$\begin{cases} a=4A \\ b-a=0 \\ c-b=2B \\ -c=0 \end{cases} \qquad \begin{aligned} &\therefore 4A=a=b=-2B \\ &\therefore 2A+B=0 \end{aligned}$$

又 $(x-1)f(x)\Rightarrow f(x)=(x-1)(a'x^3+b'x^2+c'x+d')$

$$\begin{cases} a'=A \\ b'-a'=0 \\ c'-b'=0 \\ -d'=1 \end{cases} \qquad \begin{aligned} &\therefore a'=A \\ &b'=a' \\ &-1-b'=B \qquad \therefore A+B+1=0 \end{aligned}$$

$\therefore A=1,B=-2$

P46，20 证 $f(x)=1+x+\frac{x^2}{2!}+\cdots\frac{x^n}{n!}$ 无重因式（重根）

证： $f'(x)=f(x)-\frac{x^n}{n!}$

$\therefore (f',f)=(f,\frac{x^y}{n!})$ $\because (f,x)=1)$ $(f,x^n)=1 \Rightarrow f(x)$无重因式.

P46，21

g′(x)=$\frac{1}{2}$[f′(x)+ f′(a)]+$\frac{x-a}{2}f''(x)-f'(x)\Rightarrow$ g′(a)=0

又 g(a)=0

$$g''(x)=\frac{1}{2}f'''(x)+\frac{1}{2}f''(x)+\frac{x-a}{2}f'''(x)-f''(x)=\frac{1}{2}(x-a)f''(x)\Rightarrow g''(a)=0$$

$$g'''(x)=\frac{1}{2}f'''(x)+\frac{x-a}{2}f^{(4)}x$$

$\therefore a$是$g(x),g'(x),g''(x),g'''(x)$根，且使$g''(x)$的k+1重根

$\therefore$ a是g(x)的k+3重根.

P46，22

“$\Leftarrow$”必要性显然（见定理 6 推论 1）

“$\Rightarrow$”若x_0是f(x)的t重根，t＞k，

由定理$\Rightarrow f^{(k)}(x_0)=0$

若t＜k$\Rightarrow f^{(k-1)}(x_0)\neq 0$，所以矛盾.

P46.23

例如 $f(x)=x^{m+1}$,则$x=0$是$f'(x)=(m+1)x^m$的m重根

但$x=0$不是$f(x)$的根.

P46.24 若$(x-1)f(x)^n$则$(x^n-1)|f(x^n)$

证若 $f(x)=(x-1)g(x)+r$(由上节课命题2)

$$f(x^n)=(x^n-1)g(x^n)+r=\overline{g}(x)+r\Rightarrow r=0$$

所以 $x^n-1|f(x^n)$

P46,25

证明 设x^2+x+1 的两个根 $\varepsilon_1,\varepsilon_2,\varepsilon_i^3=1$

$$x^2+x+1=(x-\varepsilon_1)(x-\varepsilon_2)$$

$$\therefore\begin{cases}f_1(\varepsilon_1^3)+\varepsilon_1 f_2(\varepsilon_1^3)=0\\ f_2(\varepsilon_2^3)+\varepsilon_2 f_2(\varepsilon_2^3)=0\end{cases}$$

即$\begin{cases} f_1(1)+\varepsilon_1 f_2(1)=0 \\ f_1(1)+\varepsilon_2 f_2(1)=0 \end{cases}$

$\Rightarrow f_2(1)=0 \quad f_1(1)=0$

$\Rightarrow (x-1)\Big| \ f_1(x), f_2(x)$.

P46、26 分解 x^n-1.

$$设\varepsilon_0=1 \quad \varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}, k=0,1,2\cdots,n-1$$

$$(i)在\mathbb{C}中, x^n-1=\prod_{i=0}^{n-1}(x-\varepsilon_i)=(x-1)\prod_{k=1}^{n-1}(x-(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}))$$

$$(ii)在\mathbb{R}中, n为奇 \quad x^n-1=(x-1)\prod_{k=1}^{2m-2}(x-\varepsilon_k)=(x-1)\prod_{k=1}^{m-1}(x-\varepsilon_k)(x-\varepsilon_{n-k})$$

$$n=2m-1 \qquad =(x-1)\prod_{k=1}^{m-1}(x^2-2\cos\frac{2k\pi}{n}x+1)$$

$$当n=2m时: x^n-1=(x-1)(x+1)\prod_{k=1}^{m-1}(x^2-2\cos\frac{2k\pi}{n}x+1)$$

p46,27 求有理根：

（1）$x^3-6x^2+15x-14=f(x)$.

解：有理根可能为±1、±2、±7、±14。

∵当 a<0 时 f（a）<0，所以 f(x)的有理根是可能 1,2,7,14

f(1)=-4≠0,f(2)=0,f(7)=140≠0,f(14)=1764≠0,只有一个x=2

（2）$4x^4-7x^2-5x-1=f(x)$.

解：有理根可能为±1、±$\frac{1}{2}$、±$\frac{1}{4}$，∵f(1)=-9≠0，f(-1)=1≠0,

f($\frac{1}{2}$)=-5,f(-$\frac{1}{2}$)=0,f($\frac{1}{4}$)=-2$\frac{43}{64}$,f(-$\frac{1}{4}$)=-$\frac{11}{64}$

所以f(x)只有一个有理根x=-$\frac{1}{2}$

（3）$f(x)=x^5+x^4-6x^3-14x^2-11x-3=f(x)$.

解：可能有有理根为±1、±3、f(1)=-32,f(-1)=0,f(3)=0　f(-3)=-96

故 f(x)有两个有理想-1,3

P46,28

①x^2+1：解 y=y+1, $x^2+1=y^2+2y+2$ 不可约

② $x^4-8x^3+12x^2+2$ 解取P=2，由Eisenstein判别法，不可约。

③x^6+x^3+1, 解令x=y+1 则

$x^6+x^3+1=y^6+6y^5+15y^4+21y^3+15y^2+9y+3$ 取P=3 即可。

④x^p+px+1 为奇素数

解：取y=x+1, $x^3+px+1=y^p+\sum_{i=1}^{p}(c_p^i y^{p-i}(-1)^i+p(y-1)+1$

$=y^p+\sum_{i=1}^{p-2}(c_p^i(-1)^i y^{p-i}+2py-p$

取 p 素数，即可

⑤$x^4+4kx+1$ k为整数

解：令x=y+1,则$f(x)=x^4+4kx+1=y^4+4y^3+6y^2+(4+4k)y+(4k+2)$

取p=2，则 $p2|4k+2$,

即可由Eisenstein判别法，f(x)于 $\mathbb{Z}(\mathbb{Q})$ 上不可约。

P47.1:证$\because f_1,g_1$都是f,g的组合,所以若$c(x)$是f,g的公因式,则必有$c(x)\mid f_1,c(x)\mid g_1$, 为$f_1,g_1$的公因式,即

$CD\{f(x),g(x)\}\subseteq CD\{f_1(x),g_1(x)\}$

反过来，得 $f(x)=\frac{1}{ad-bc}(df_1(x)-bg_1(x)),g(x)\frac{1}{ad-bc}(-cf_1(x)+ag_1(x))$

$\therefore f,g$也是f_1,g_1的组合,同上理,有

$CD\{f_1(x),g_1(x)\}\subseteq CD\{f(x),g(x)\}$

即，f与g和f_1与g_1的公因式一致,最大公因式也一致,那

$(f(x), g(x))=(f_1(x),g_1(x))$

注：不可约多项式也称既约定多项式

$f(x)\neq 0,a$,则$f(x)$不是既约,则称f(x)可约

P47,2

证：$\because d_1(x)f(x)\neq v_1(x)g(x)\Rightarrow$（f(x)，g(x)）=d(x).

$f(x)=f_i(x)d(x)$, $g(x)=g_i(x)d(x)$

$\therefore u_1(x)f_i(x)+v_1(x)g_1(x)=1$ 带余除法

令$u_1(x)=q_1(x)g_1(x)+u(x)$ $\partial(u)<\partial(g_1(x))=\partial(\frac{g(x)}{g(x)+y(x)})$

$v_1(x)=q_2(x)f_1(x)+v(x)$ $\quad\partial(v)<\partial(f_1(x))=\partial(f(x)/(f(x),g(x))$

则$fu+gv+fg_1q_1+f_1gq_2$

$=f(x)u(x)+g(x)v(x)+f_1(x)g_1(x)d(x)(q_1+q_2)=d(x)$

∴$d-(uf+vg)=f_1g_1d(q_1+q_2)$,左边次数$<\partial(f_1)+\partial(y_1)+\partial(d)\le$右边次数

故左、右两侧只有为 0，$d-(uf+vg)=0$

$u(x)f(x)+v(x)y(x)=d(x)$

且$\partial(u(x))<\partial(g_1(x)),\partial(v(x)<\partial(f_1(x))$

P47.3 若$f(x)$与$g(x)$互素,则$\forall m\ge 1, f(x^m$与$g(x^m)$也互素

证：$\because f(x)$与$g(x)$互素,$\because \exists u(x),v(x),u(x)f(x)+v(x)g(x)=1$

由推广令$\varphi(x)=x^m,u(x^m)f(x^m)+v(x^m)g(x^m)=1$

$\therefore (f(x^m),g(x^m))=1$,即$f(x^m)$与$g(x^m)$互素

P47 补 4

由定义有$(f_{1,}f_2\cdots f_s)=((f_1,\cdots,f_{s-1}),f_s)$,

证明$\exists d_i(x)$使得$u_1f_1+u_2f_2+\cdots+u_sf_s=(f_1,f_2\cdots f_s)$.

证：设$d=(f_1,f_2\cdots f_s)$，$d_1=(f_1\cdots f_{s-1})$ $\quad d'=(d_1,f_s)$

显然$d|f_s$及$d|d_1\Rightarrow d|d'$，

反之，$d'\Rightarrow d|d_1'$，$d'|f_s\Rightarrow d|f_i$（$\forall i$）$\Rightarrow d'|d$。

又d,d'首项系数$=1\Rightarrow d=d'$.

证：由归纳方式$\exists u_i'$，使$u_1'f_1+\cdots+u_{s-1}'f_{s-1}=d_1$,又$\exists v,u$使得$vd_1+u_sf_s=d'$，

∴$d=d'=vd_1+u_sf_s=v(\sum_{i=1}^{s-1}u_i'f_i)+u_sf_s$ $\quad$ 令$u_i=vu_i'$ $\quad i=1,2\cdots,s-1$

$=u_1f_1+\cdots+u_{s-1}f_{s-1}+u_sf_s$.

P48，补 5

证明 若：$f(x)g(x)$首项系数都$=1$ $\quad$ 则$[f,g]=\dfrac{fg}{(f,g)}$

证：令$(f,g)=d,f=f_1d,g=g_1d$,则$(f_1,g_1)=1$,设$m(x)=f_1g_1d$

显然① $f|m, g|m$, 故 m 是一个公倍式

再设② $f\,|\,l, g\,|\,l. \therefore d\,|\,l$，令$l=dl_1$，$\Rightarrow f_1|l_1, g_1|l_1$

$\because (f_1g_1)=1$，$\therefore f_1g_1|l_1 \Rightarrow f_1g_1d_1|l$　即$m|l$

m 是 f、g 的一个最小 公倍　式

即证得：$[f(x)、g(x)]=f_1(x)g_1(x)d(x)=\dfrac{f(x)\cdot g(x)}{(f(x)\cdot g(x))}$

$p48.7: f(x)$首项系数$=1. \partial(f(x))>0$,则 $f(x)$为某不可约多项式$p(x)$的方幂的充要条件是 $\forall g(x)$ 或者 $(f,g)=1$ 或者 $\exists m: f(x)\,|\,g^m(x)$

证明"$\Leftarrow$"反设不是则$f(x)=p_1^r(x)h(x)$,而$\partial(h(x))>0, p_1(x)+h(x)\Rightarrow$

$(p_1,h)=1$,即$h\nmid p_1$,取$g(x)=p_1(x)$,则$(f,g)\neq 1$,且$\forall\ \ m, f\,|\,g^m$,否则$h=p_1^s(x)$,矛盾.

"$\Rightarrow$" $f=p^r, \forall g(x)$,若$(p,g)=1 \Rightarrow (p^r,g)=(f,g)=1$,若$(p,g)\neq 1 \Rightarrow p\,|\,g \Rightarrow f\,|\,g^r(x)$

$p48.8: f(x)$首项系数$=1, \partial(f(x))>0$,则$f(x)$为某不可约多项式的方幂$\Leftrightarrow$

$\forall g(x)\,|\,h(x)$,由$f\,|\,gh \Rightarrow f\,|\,g$或者$\exists m, f(x)/h^m(x)$

证明"$\Rightarrow$"设$f=p^r$,若$f\,|\,gh, (p,h)=1 \Rightarrow (p^r,h)=1 \Rightarrow (f,h)=1 \Rightarrow f\,|\,g$

$$(p,h)\neq 1 \Rightarrow p\ \ |\,h \Rightarrow p^r\,|\ \ h^r \Rightarrow f\,|\,h^r(x)$$

"$\Leftarrow$"反设不是,则$f=p_1^r h$,而$\partial(h)>0, p_1+h$,令$g=p_1^r, h=h(x)$,则

$f\,|\,gh$却$f+g, f+h^m, \forall m(\because (p,h)=1 \Rightarrow (p^r,h^m)=1)$

P48，补 9 证：$x^n+ax^{n-m}+b$没有重数>2 的非零根

证：反设 $f(x)=x^n+ax^{n-m}+b$有k重根α,　(k＞2，$\alpha\neq 0$)

$g(x)=f'(x)=nx^{n-1}+a(n-m)x^{n-m-1}$有$k$重根$\alpha\neq 0$

$$\Rightarrow nx^{n-m-1}(x^m+\frac{a(n-m)}{n})\text{有重根}\alpha\neq 0$$

$\therefore h(x)=x^m+\dfrac{a(n-m)}{n}$　有重根$\alpha\neq 0$

但$h'(x)=mx^{m-1}$　　$\therefore (h(x)\ h'(x))=\begin{cases}1\\ h'(x)\end{cases}$　$\begin{cases}\text{无重数根}\\ \text{重根}0\end{cases}$

P48、补 10

$0\neq f(x)\in C[x]$,且$f(x)\,|\ \ f(x^n), n>1$,

证明$f(x)$的根只能为0或单位根(即满足某x^m=1的根).

证:设α为f(x)的根,由f(x^n)==f(x)g(x)

$\therefore$ f(α^n)=0, α^n为$f(x)$的根,

$\therefore f(\alpha^{n^2})=0, \alpha^{n^2}$为$f(x)$的根,

$\Rightarrow \alpha, \alpha^{n}, \alpha^{n^2}, \alpha^{n^3}, \cdots$都为f(x)的根.

$\because f(x) \neq 0, \therefore f(x)$不可能有无限个根,其中必有相等者:

$\alpha^{n^i} = \alpha^{n^j}$ (不妨设$i > j$),

$\therefore \alpha^{n^2}(\alpha^{n^i - n^j} - 1) = 0$,令$n^i - n^j = m$.

则或$\alpha = 0$,或α是$x^m = 1$的根.

P48、补 11 题:

$\because f'(x) \mid f(x) \Rightarrow f(x) = a(x-b)^n \quad \therefore f(x)$有n重根b.

补充P48 12题: $a_1, a_2 \cdots a_n$的两两不同. F(x)=(x−a_1)(x−a_2)···(x−a_n)

证:(1)

$$\sum_{i=1}^{n} \frac{F(x)}{(f-a_1)F'(a_i)} = 1 \quad \because l_i = \frac{F(x)}{(x-a_i)F'(a_i)} = \frac{(x-a_1)(x-a_{i-1})(x-a_{i+1})\cdots(x-a_n)}{(a_i-a_1)(a_i-a_{i-1})(a_2-a_{i+1})(a_i-a_n)}$$

$l_i(a_j) = 0, l_i\ (a_i) = 1, \therefore \sum_{i=1}^{n} l_j(a_j) = 1, \forall j = 1 \cdots n. \forall i, \sum_{i=1}^{n} l_i(x_i), i = 1 \cdots n$

为$n-1$次多项式,$\therefore \sum_{i=1}^{n} l_i(x) = 1$, (2)设$f(x) = F(x)q(x) + r(x)$,则$f(a_i) = r(a_i)$,

而$n-1$形式多项式$\sum_{r=1}^{n} f(a_i) l_i(x) = h(x) \ : h(a_j) = f(a_j)$

$= r(a_j) \quad \therefore h(x) = r(x)$

p49、补 13 题:

(1)求 $f(x) \quad \partial(f(x)) < 4$且$f(2) = 3 \quad f(3) = -1 \quad f(4) = 0 \quad f(5) = 2$

$$l_1(x) = \frac{(x-3)(x-4)(x-5)}{(2-3)(2-4)(2-5)} = -\frac{1}{6}(x^3 - 12x^2 + 47x - 60)$$

$$l_2(x) = \frac{(x-2)(x-4)(x-5)}{(3-2)(3-4)(3-5)} = \frac{1}{2}x^3 - \frac{11}{2}x^2 + 19x - 20$$

$$l_3(x) = \frac{(x-2)(x-3)(x-5)}{(4-2)(4-3)(4-5)} = \frac{1}{2}x^3 + 5x^2 + \frac{31}{2}x + 15$$

$$l_4(x) = \frac{(x-2)(x-3)(x-4)}{(5-2)(5-3)(5-4)} = \frac{1}{6}x^3 + \frac{3}{2}x^2 + \frac{13}{3}x - 4$$

$$\therefore f(x) = \sum_{i=1}^{4} (l_i(x)) f(a_i) = 3l_1 - l_2 + 0 \cdot l_3 + 2l_4 = -\frac{2}{3}x^3 + \frac{17}{2}x^2 - \frac{203}{6}x + 42$$

②求一个二次多项式f(x)，$f(0)=\sin 0, f(\frac{\pi}{2})=\sin\frac{\pi}{2}, f(\pi)=\sin\pi=0$.

$l_1(x)=\cdots,$

$l_2=\dfrac{(x-0)(x-\pi)}{(\frac{\pi}{2}-0)(\frac{\pi}{2}-\pi)},$

$l_3=\cdots,$

$\therefore f(x)=f(\frac{n}{2})l_2=\dfrac{x(x-\pi)}{-\frac{\pi^2}{4}}$

③ $f(x)$可能低次项：f(0)=1　$f(1)=2$　　$f(2)=5$　　$f(3)=10$

$$l_1(x)=\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}=-\frac{1}{6}x^3+x^2-\frac{11}{6}x+1$$

$$l_2(x)=\frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}=\frac{1}{2}x^3-\frac{5}{2}x+3x$$

$$l_3(x)=\frac{(x-0)(x-2)(x-3)}{(2-0)(2-1)(2-3)}=-\frac{x^3}{2}+2x^2+3x$$

$$l_4(x)=\frac{(x-0)(x-2)(x-3)}{(3-0)(3-1)(3-2)}=\frac{1}{6}x^3-\frac{1}{2}x^2+\frac{1}{3}x$$

$$\therefore f(x)=l_1(x)+2l_2(x)+5l_3(x)+10l_4(x)=x^2+1$$

P49.补 14)，$f(x)\in\mathbb{Z}[x], f(0), f(1)$奇，则f(x)无整数根.

证：反设f(x)有整数数根m，则x-m|f(x)，

f(0)奇$\Rightarrow$-m奇　　f(1)奇$\Rightarrow$1-m奇　　矛盾！

第二章　　行列式习题解答

P96.1 ① $\tau(134782695)=0+1+1+3+3+0+1+1=10\therefore 13478695$偶排列

② $\tau(217986354)=1+0+4+5+4+3+0+1=18\therefore 21798354$偶排列

③ $\tau(98765432)=8+7+6+\cdots 1+2+1=36\therefore 987654321$偶排列

P96.2①若 1274i56k9 偶则 i,k=3.8 或 8.3

τ (127435689)=5，τ (127485639)=10　　∴i=8，k=3

②若 1i25j4897 奇则 i，k=3，6 或 6.3

τ (132564897)=4 τ(162534897)=7 ∴i=6 k=3

P96.3 3 1 2 4 3 5 →2 1 4 3 5 →2 5 4 3 1 →2 5 3 4 1 即得

P96.4 ∵τ(n(n-1),… 3 2 1)=$C_n^2=\dfrac{n(n-1)}{2}$

∴当 4|n或 4|n-1 即n=4k或n=4k+1 时，C_n^2为偶数，偶排列。

当n=4n+2，n=4n+3，则C_n^2为奇数，是奇排列

P96.5 排列π_1：$x_1x_2\cdots x_n$与π_2：$x_nx_{n-1}\cdots x_2x_1$中，任取两个数x_i, x_j

若x_i, x_j在π_1中有逆序，则在π_2中没有，反之在π_1中没有逆序，则π_2中有逆序，∴ $\tau(\pi_1)+\tau(\pi_2)=C_n^2$

即$\tau(x_nx_{n-1}\cdots x_2x_1)=C_n^2-\tau(x_1x_2\cdots x_n)$.

P97.6. 由于$\tau(234516)+\tau(312645)=8.a_{23}a_{31}a_{42}+a_{56}a_{14}a_{65}$带正号

由$\tau(341562)+\tau(234165)=10\ \therefore a_{32}a_{43}a_{14}a_{51}a_{66}a_{23}$带正号

P96.7 $j_1j_2j_3j_4$由于$j_2=3$，∴$j_1j_3j_4$取 1、2、4 的排列

$j_1j_3j_4=124$

$\Rightarrow\tau(1324)=1, j_1j_3j_4\Rightarrow\tau(1342)=2; j_1j_3j_4=214\Rightarrow\tau(2314)=2$

$j_1j_3j_4=241\Rightarrow\tau(2341)=3. j_1j_3j_4=142\Rightarrow\tau(1342)=2; j_1j_3j_4=214\Rightarrow\tau(2314)=2$

$j_1j_3j_4=241\Rightarrow\tau(2341)=3, j_1j_3j_4=412\Rightarrow\tau(4312)=5; j_1j_3j_4=421\Rightarrow\tau(4321)=6$

$\therefore$取负号只有$-a_{11}a_{23}a_{32}a_{44}, -a_{12}a_{23}a_{34}a_{41}, -a_{14}a_{23}a_{31}a_{42}$

$P97.8(1)D=(-1)^{\tau(\text{n}\cdots 321)}123\cdots(n-1)n=(-1)\dfrac{n(n-1)}{2}\bullet n$

$P97.8(2)D=(-1)^{\tau(23\cdots \text{n}1)}123\cdots n=(-1)^{n-1}.n1$

$P97.8(3)D=(-1)^{\tau((\text{n}-1)\ (\text{n}-2)\cdots 21\text{n})}123\cdots n=(-1)\dfrac{(n-1)(n-2)}{2}.n$

$$P97.9D=\sum_{j_1j_2j_3j_4j_5}(-1)^{(j_1j_2j_3j_4j_5)}aj_1bj_2cj_3dj_4ej_5$$

因后三行后三列为 0，所以非零的项只有，$j_3\le 2, j_4\le 2, j_5\le 2$，而$j_3, j_4, j_5$是互不相

同的数,这是不可能的,所以没有非 0 的项,D=0

P97.10　f(x)= $\begin{vmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$ 求 x^4，x^3 的系数

解：∵行列式中每项由每行出一元相乘，故 x^4 必须将 2.3.4 行的x都取，这时第i行取第i列，这是行列式的一项，ax^4，系数为a=2。

x^3 项必有一元 a_{ij} 在对角线外，于是i行，j列的x不能再取了，故当i=1, j>2 时，至少去掉3 个x,不含 x^3 项了，对于i>2, j=2 同理

其它情形，至少去掉两个x且第一行（或第二列）的两个x只能取一个，故不含 x^3 项，只剩下i=1, j=2 时，a_{12} 本身是x项为

$(-1)^{(2134)}a_{12}a_{21}a_{33}a_{44}=-x1xx=-x^3$, 系数为-1

P97.11　$d=\sum\limits_{j_1j_2\cdots j_n}(-1)^{\tau(j_1,j_2\cdots j_n)}\bullet 1=0$　故 $\sum$ 中 $+1$ 与 -1 一样多,即 $+$ 号, $-$ 号一样多,

也即奇偶排列一样多, $\therefore$ n ≥ 2 时,奇偶排列各占一半.

P98.12①

$$p(x)=\begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \end{vmatrix}=V\,x^{n-1}+\cdots(\text{按第一行展开})$$

$\because a_1a_2,\cdots a_{n-1}$ 两两不同 $\therefore V_{n-1}\neq 0$ 即 $\partial(p(x))=n-1$

$\because p(a_1)=p(a_2)=\cdots=p(a_{n-1})=0$ (总有两行相同)最多 $n-1$ 个根,

②即p(x)的所有根为 $a_1,a_2,\cdots,a_{n-1}$

$$P98.13\ ②\ \xrightleftharpoons[]{\substack{1\times(1)+3\\2\times(1)+3}}\begin{vmatrix} x & x & x+y \\ y & x+y & x \\ 2(x+y) & 2(x+y) & 2(x+y) \end{vmatrix}\xrightleftharpoons[]{3(2(x+y))^{-1}}2(x+y)$$

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ 1 & 1 & 1 \end{vmatrix}\xrightleftharpoons[]{\substack{3(-x-y)+1\\3(-x)+2}}2(x+y)\begin{vmatrix} -y & -x & 0 \\ y-x & y & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$=2(x+y)[-y^2+xy-x^2]=-2(x^3+y^3)$$

$$P48.13③\begin{vmatrix}3&1&1&1\\1&3&1&1\\1&1&3&1\\1&1&1&3\end{vmatrix}\xlongequal{P2.23.3.4(例)}(3+(4-1)\cdot1)(3-1)^{4-1}=6.2^3=48$$

P98.13④（法一）：

$$\begin{vmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{vmatrix}=\begin{vmatrix}1&2&3&4\\0&-1&-2&-7\\0&-1&-8&-10\\0&-7&-10&-13\end{vmatrix}=-\begin{vmatrix}1&2&3&4\\0&1&2&7\\0&0&-4&4\\0&0&4&36\end{vmatrix}=\begin{vmatrix}1&2&3&4\\0&1&2&7\\0&0&4&-4\\0&0&0&40\end{vmatrix}=160$$

法二：

$$=\begin{vmatrix}10&2&3&4\\10&3&4&7\\10&4&1&2\\10&1&2&3\end{vmatrix}=10\begin{vmatrix}1&2&3&4\\0&1&1&-3\\0&2&-2&-2\\0&-1&-1&-1\end{vmatrix}=20\begin{vmatrix}1&2&3&4\\0&1&1&-3\\0&2&0&0\\0&-1&-1&-1\end{vmatrix}=20\begin{Vmatrix}1&2&3&4\\0&2&0&0\\0&0&1&-3\\0&0&-1&-1\end{Vmatrix}$$

$$=(-20)\begin{vmatrix}1&2&3&4\\0&2&0&0\\0&0&1&-3\\0&0&0&-4\end{vmatrix}=160$$

法三：

$f(x)=1+2x+3x^2+4x^3$ $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$为4次单位根$\pm1,\pm i$

令$\varepsilon_1=1,\varepsilon_2=-1,\varepsilon_3=i,\varepsilon_4=-i$,则

行列式$d=(-1)^{c_{4-1}^2}f(1)f(-1)f(i)f(-i)$

$=(-1)^3\bullet10\bullet(-2)\bullet(-2-2i)\bullet(-2+2i)=20[(-2)^2-(2i)^2]=160$

令

$$\begin{vmatrix}1+x&1&1&1\\1&1-x&1&1\\1&1&1+y&1\\1&1&1&1-y\end{vmatrix}=\begin{vmatrix}1&1&1&1&1\\0&1+x&1&1&1\\0&1&1-x&1&1\\0&1&1&1+y&1\\0&1&1&1&1-y\end{vmatrix}=\begin{vmatrix}1&1&1&1&1\\-1&x&0&0&0\\-1&0&-x&0&0\\-1&0&0&y&0\\-1&0&0&0&-y\end{vmatrix}=x^2y^2$$

P98. 13⑤解：

解法二, 设f(x, y)= $\begin{vmatrix} 1+x & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$ 则第一行减第二行 $\begin{vmatrix} x & x & 0 & 0 \\ & & * & \\ & & & \end{vmatrix}$

$\therefore x \mid f(x,y)$

又因为 $f(-x,y)=\begin{vmatrix} 1-x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} \xrightleftharpoons[]{\substack{交换1,2行 \\ 再交换1,2列}} \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = f(x,y).$

∴ f(x, y)关于x是偶函数，即 $x^2 \mid f(x,y)$.

同理，y|f(x, y)，且也是偶函数，所以 $y^2 \mid f(x,y)$

$\because \partial(f(x,y)) \le 4 \qquad \therefore x^2y^2 \mid f(x,y) \Rightarrow f(x,y)=kx^2y^2$

而 $f(x,y)$ 中 x^2y^2 的系数为1.　故有 $f(x,y)=x^2y^2$.

P98　13⑥

$$\xrightleftharpoons[\substack{1\times(-1)+2 \\ 1\times(-1)+3 \\ 1x\times(-1)+4}]{} \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix} \xrightleftharpoons[\substack{2\times(-2)+3 \\ 2\times(-3)+4}]{} \begin{vmatrix} a^2 & 2a+1 & 2 & 6 \\ b^2 & 2b+1 & 2 & 6 \\ c^2 & 2c+1 & 2 & 6 \\ d^2 & 2d+1 & 2 & 6 \end{vmatrix} \xrightleftharpoons[]{性质5} 0$$

$$\xrightleftharpoons[2\times(-1)+3]{} \begin{vmatrix} 246 & 427 & -100 \\ 1014 & 543 & -100 \\ -342 & 721 & -100 \end{vmatrix} \xrightleftharpoons[\substack{3\times(246)+1 \\ 3\times(427)+2}]{} \begin{vmatrix} 0 & 0 & 1 \\ 768 & 116 & 1 \\ -588 & 294 & 1 \end{vmatrix} \times(-100) \xrightleftharpoons[2\times(2)+1]{}$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1000 & 116 & 1 \\ 0 & 294 & 1 \end{vmatrix} \times(-100) \xrightleftharpoons[性质7]{} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 116 & 1000 \\ 1 & 294 & 0 \end{vmatrix} \times(100) = -100\times \begin{vmatrix} 1 & 0 & 0 \\ 1 & 294 & 0 \\ 1 & 116 & 1000 \end{vmatrix} = -29400000$$

$P98.14$　左 $\xrightleftharpoons[\substack{2\times(1)+1 \\ 3\times(1)+1 \\ 1\times(\frac{1}{2})}]{} 2\begin{vmatrix} a+b+c & c+a & a+b \\ a_1+b_1+c_1 & c_1+a_1 & a_1+b_1 \\ a_2+b_2+c_2 & c_2+a_2 & a_2+b_2 \end{vmatrix} \xrightleftharpoons[\substack{1\times(-1)+2 \\ 1\times(-1)+3}]{} 2\begin{vmatrix} a+b+c & -b & -c \\ a_1+b_1+c_1 & -b_1 & -c_1 \\ a_2+b_2+c_2 & -b_2 & -c_2 \end{vmatrix}$

$$\xrightleftharpoons[\substack{2\times(1)+1 \\ 2\times(1)+1}]{} 2\begin{vmatrix} a & -b & -c \\ a_1 & -b_1 & -c_1 \\ a_2 & -b_2 & -c_2 \end{vmatrix} \xrightleftharpoons[\substack{2(-1) \\ 3(-1)}]{} 右(-1)^2 = 右$$

P98. 15 求出所有代数余子式

① $\begin{vmatrix}1&2&1&4\\0&-1&2&1\\0&0&2&1\\0&0&0&3\end{vmatrix}$直接计算有$\begin{pmatrix}A_{11}&A_{12}&A_{13}&A_{14}\\A_{21}&A_{22}&A_{23}&A_{24}\\A_{31}&A_{32}&A_{33}&A_{34}\\A_{41}&A_{42}&A_{43}&A_{44}\end{pmatrix}=\begin{pmatrix}-6&0&0&0\\-12&6&0&0\\15&-6&-3&0\\7&0&1&-2\end{pmatrix}$

② $\begin{vmatrix}1&-1&2\\3&2&1\\0&1&4\end{vmatrix}$直接计算有$\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix}=\begin{pmatrix}7&-12&3\\6&4&-1\\-5&5&5\end{pmatrix}$

P99.16①

$$=-\begin{vmatrix}-1&+3&+5&1&2\\2&0&1&2&1\\0&1&2&-1&4\\3&3&1&2&1\\2&1&0&3&5\end{vmatrix}\xlongequal{\substack{1\times(2)+2\\1\times(3)=4\\1\times(2)+5}}\begin{vmatrix}-1&3&5&1&2\\0&6&11&4&5\\0&1&2&-1&4\\0&12&16&5&7\\0&7&10&5&9\end{vmatrix}\xlongequal{\substack{2(-)3\\2(-1)}}\begin{vmatrix}-1&3&5&1&2\\0&-1&-2&1&4\\0&6&11&4&5\\0&12&16&5&7\\0&7&10&5&9\end{vmatrix}$$

$$\xlongequal{\substack{2\times(6)+3\\2\times(2)+4\\2\times(7)+5}}\begin{vmatrix}-1&3&5&1&2\\0&-1&-2&1&-4\\0&0&-1&-1&9\\0&0&-8&17&-41\\0&0&-4&12&-19\end{vmatrix}\xlongequal{\substack{3(-1)\\3\times(8)+4\\3\times(4)+5}}\begin{vmatrix}-1&3&5&1&2\\0&-1&-2&1&-4\\0&0&-1&-1&9\\0&0&-8&63&111\\0&0&0&-28&57\end{vmatrix}\xlongequal{4\times(-2)+3}$$

$$\begin{vmatrix}-1&3&5&1&2\\0&-1&-2&1&-4\\0&0&-1&-10&19\\0&0&0&-7&-3\\0&0&0&-28&57\end{vmatrix}$$

$$\xlongequal{\substack{1\times(-1)\\2\times(-1)\\4\times(-4)+5}}\begin{vmatrix}1&-&-5&-1&-2\\0&1&2&-1&4\\0&0&1&-10&19\\0&0&0&-10&-3\\0&0&0&0&69\end{vmatrix}=(-7)69=-483$$

P99.16②

$$\underline{\underline{\begin{matrix}1\times(2)\\5\times(2)\\4\times(2)\end{matrix}}}\frac{1}{8}\begin{vmatrix}2&1&0&4&-2\\2&0&-1&2&2\\3&2&1&1&0\\1&-1&0&2&2\\4&2&6&0&1\end{vmatrix}\underline{\underline{\begin{matrix}1\to 4\\1\times(2)+2\\1\times(-3)+3\\1\times(2)+4\\1\times(-4)+5\end{matrix}}}\frac{1}{8}\begin{vmatrix}1&-1&0&2&2\\0&2&-1&-2&-2\\0&5&1&-5&-6\\0&3&0&0&-6\\0&6&6&-8&-7\end{vmatrix}$$

$$\underline{\underline{\begin{matrix}2-4\\4\times(-1)+2\end{matrix}}}+\frac{1}{8}\begin{vmatrix}1&-1&0&2&2\\0&2&1&2&-4\\0&5&1&-5&-6\\0&2&-1&-2&-2\\0&6&6&-8&-7\end{vmatrix}$$

$$\underline{\underline{\begin{matrix}2\times(-5)+3\\2\times(-2)+4\\2\times(-8)+5\end{matrix}}}\frac{1}{8}\begin{vmatrix}1&-1&0&2&2\\0&2&1&2&-4\\0&0&-4&-15&-14\\0&0&-3&-6&6\\0&0&0&-20&17\end{vmatrix}\underline{\underline{\begin{matrix}4-3\\2\times\frac{1}{3}\\3\times(-4)+4\end{matrix}}}-\frac{3}{8}\begin{vmatrix}1&-1&0&2&2\\0&1&1&2&4\\0&0&-1&-2&2\\0&0&0&-7&6\\0&0&0&0&-\frac{1}{7}\end{vmatrix}$$

$$=-\frac{3}{8}\cdot 1\cdot 1\cdot(-1)\cdot 7\cdot(\frac{1}{7})=\frac{3}{8}$$

P99.17①若

$j_1j_2\cdots j_n$中$j_n=n$,则j_{n-1},取j_{n-1}取$n-1$, j_{n-2}取$n-2\cdots$, $j_2=2$, $j_1=1$或若$j_n=1$,则

$\Rightarrow j_1=2, j_2=3,\cdots j_{n-1}=n$.故只有两项. $\tau(123\cdots \mathrm{n})=0$, $\tau(2,3,\cdots n1)=\mathrm{n}-1$

$\therefore \mathrm{d}=\sum=\mathrm{x}^{\mathrm{n}}+(-1)^{\mathrm{n}-1}\mathrm{y}^{\mathrm{n}}$

P99.17②

$n=1$则$d=a_1-b_1$

$$\begin{aligned}n=2,\text{则}d&=(a_1-b_1)(a_2-b_2)-(a_1-b_2)(a_2-b_1)\\&=a_1b_1-a_2b_2+a_1b_1-b_2a_2\\&=(a_2-a_1)(b_2-b_1)\end{aligned}$$

当$n\geq 3$时 $\begin{vmatrix}a_1-b_1&b_1-b_2&\vdots&a_1-b_1\\a_2-b_2&b_1-b_2&\vdots&b_1-b_n\\\vdots&\vdots&\vdots&b_1-b_n\\a_n-b_1&b_1-b_n&\vdots&b_1-b_n\end{vmatrix}=0$第2列与第$n$列成比例

$$P99.17③ \xrightleftharpoons[\substack{2\times(1)+1\\3\times(1)+1\\n\times(1)+1}]{} \begin{vmatrix} \sum_{i=1}^{m} x_i - m & x_2 & \vdots & x_n \\ \vdots & x_2 - m & \vdots & \vdots \\ \vdots & \vdots & \vdots & x_n \\ \sum_{i=1}^{m} x_i - m & x_2 & \vdots & x_{n-m} \end{vmatrix} \xrightleftharpoons[\substack{1(\sum_{i=1}^{n}(x_{i-m})^{-1})\\1\times(-x_2)+2\\1\times(-x_3)+3\\1\times(-x_n)+n}]{} \begin{vmatrix} 1 & 0 & \vdots & 0 \\ 1 & -m & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ 1 & 0 & \vdots & -m \end{vmatrix}$$

$$(\sum_{i=1}^{h} X_i - m) = (\sum_{i=1}^{h} X_i - m)(-m)^{n-1} = (-1)^n (m - \sum_{i=1}^{h} X_i) m^{n-1}.$$

$$P99.17\ ④ \xrightleftharpoons[]{\substack{2\times(-1)+3\\2\times(i-4)+4}} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & \cdots & \cdots & n-2 \end{vmatrix} \xrightleftharpoons[]{\substack{\text{性质}\\1\leftrightarrow 2}} \begin{vmatrix} 2 & 2 & & \cdots & 2 \\ -1 & 0 & & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ & & & 2 & \\ & & & & n-2 \end{vmatrix}$$

$$\xrightleftharpoons[1\leftrightarrow 2]{1(\frac{1}{2})} 2\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & \\ & 0 & 1 & & 0 \\ & & 0 & & \\ & 0 & 0 & & n-2 \end{vmatrix}$$

=−2·(n−2)!　　（n≥2），且 n=1 时。D=1（左上角 1）

$P99.17.5$从最后一列开始，第n列加到第n−1列，再第n−1列加到第n−2列…，第2列加

$$= \begin{vmatrix} \frac{n(n+1)}{2} & \frac{n(n+1)}{2} & \frac{n(n-1)}{2}-3 & \cdots & 2n-1-n \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1-n \end{vmatrix}$$

$$= \frac{n(n+1)}{2} \bullet (-1)(-2)\cdots(1-n) = (-1)^{n-1} \bullet \frac{1}{2} \bullet (n+1)!$$

P100.18①:从第二列起：有列（第三列）

乘以 $-\frac{1}{a_{i-1}}$ 加到第一列，则有

$$D=\begin{vmatrix} a_o\sum_{i=2}^{n+1}\frac{1}{a_{i-1}} & 1 & 1 & \cdots & 1 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_n \end{vmatrix}$$

$$=a_1a_2\cdots a_n(a_o-\sum_{i=2}^{n+1}\frac{1}{a_{i-1}})=a_1a_2\cdots a_n(a_o-\sum_{i=1}^{n}\frac{1}{a_i})$$

$$=a_oa_1\cdots a_n+\sum_{i=1}^{n}a_1\cdots a_{i-1}a_{i+1}\cdots a_n,\quad （\alpha_{\mathrm{i}}\neq 0）$$

②

P100.18④

$$D_n=\begin{vmatrix} \cos a & 1 & & & \\ 1 & 2\cos a & 1 & & \\ & 1 & 2\cos a & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2\cos a \end{vmatrix}=\cos n\alpha$$

证法一，用归纳法，D_1成立，

设$k<n$时$D_k=\cos k\alpha$,当$k=n$时,因为

$Dn=\cos 2\alpha D_{n-1}-1\cdot D\ _{n-2}=(2\cos\alpha\cdot\cos(n-1)\alpha-\cos(n-2)\alpha)=(\cos na+\cos(n-2a-\cos(n-2)a))$

$=\cos na$.证毕

证法二：

$\because D_n=2\cos cxD_{n-1}-D_{n-2}$　找不出适当倍数左移.($i^2=-1$),

$D_n-(\cos\alpha+i\sin\alpha)D_{n-1}=(\cos\alpha-i\sin\alpha)[D_{n-1}-(\cos\alpha+i\sin\alpha)D_{n-2}]$

同理$D_n=(\cos a-i\sin a)D_{n-1}=(\cos(n-1)a+i\sin(n-1)a)(i\sin a)$

相减:$2i\sin a\mathrm{D}_{\mathrm{n}}=i\sin a[\cos na+\sin na+\cos na-\sin na]$

即$D_n=\frac{1}{2}(2\cos na)=\cos na$

.

P100.18⑤以第一行×(-1)加到后面各行

$$=\begin{vmatrix}1+a_1 & 1 & 1 & \cdots & 1\\ -a_1 & a_2 & 0 & \cdots & 0\\ -a_1 & 0 & a_3 & 0 & 0\\ -a_n & 0 & \cdots & 0 & a_n\end{vmatrix}$$

$$(\text{属于交行列式})=a_1\cdots a_n(1+a_1-\sum_{i=2}^{n}\frac{-a_1}{a_2})$$

$$=a_1a_2\cdots a_n(\frac{1}{a_1}+1+\sum_{i=2}^{n}\frac{1}{a_i})$$

要求 $(a_i\neq 0)$ $\quad=a_1a_2\cdots a_n(1+\sum_{i=2}^{n}\frac{1}{a_i})$

p101.19①

$$D=\begin{vmatrix}2 & -2 & 3 & 2\\ 3 & -3 & 3 & 2\\ 3 & -3 & -1 & 2\\ 3 & -1 & 3 & -1\end{vmatrix}=\begin{vmatrix}2 & -1 & 3 & 2\\ -3 & 0 & -6 & -4\\ 1 & 0 & -4 & 0\\ 1 & 0 & 0 & -3\end{vmatrix}=\begin{vmatrix}-3 & -6 & -4\\ 0 & -4 & 0\\ 1 & 0 & -3\end{vmatrix}=\left|\begin{vmatrix}0 & -6 & -13\\ 0 & -4 & 3\\ 1 & 0 & -3\end{vmatrix}\right|=-18-52=-70$$

且 $D_1=D_2=D_3=D_4=-70 \quad \therefore \quad x_i=\frac{D_i}{D}=1 \quad i=1,2,3,4$

P101，19②

$$D=\begin{vmatrix}1 & 2 & 3 & -2\\ 2 & -1 & -2 & -3\\ 3 & 2 & -1 & 2\\ 2 & -3 & 2 & 1\end{vmatrix}=\begin{vmatrix}1 & 2 & 3 & -2\\ 0 & -5 & -8 & 1\\ 0 & -4 & -10 & 8\\ 0 & -7 & -4 & 5\end{vmatrix}=\begin{vmatrix}5 & 8 & 1\\ 4 & 10 & 8\\ 7 & 4 & 5\end{vmatrix}=\begin{vmatrix}0 & 0 & 1\\ -36 & -54 & 8\\ -18 & -36 & 5\end{vmatrix}=\begin{vmatrix}36 & 54\\ 18 & 36\end{vmatrix}$$

$$=18^2\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}=18^2=324$$

且 $D_1=324$，$D_2=648$，$D_3=-324$，$D_4=-648$，

$$\therefore x_1=D_1/D=1, x_2=D_2/D=2, x_3=-1, x_4=-2$$

P101 19 (3), 即 $D=\begin{vmatrix}1 & 2 & -2 & 4 & -1\\ 2 & -1 & 3 & -4 & 2\\ 3 & 1 & -1 & 2 & -1\\ 4 & 3 & 4 & 2 & 2\\ 1 & -1 & -1 & 2 & -3\end{vmatrix}=\begin{vmatrix}0 & 3 & -1 & 2 & 2\\ 0 & 1 & 5 & -8 & 8\\ 0 & 4 & 2 & -4 & 8\\ 0 & 7 & 8 & -6 & -14\\ 1 & -1 & -1 & 2 & -3\end{vmatrix}=\begin{vmatrix}0 & -16 & 26 & -22\\ 1 & 5 & -8 & 8\\ 0 & -18 & 38 & -24\\ 0 & -27 & 50 & -42\end{vmatrix}$

$$=-\begin{vmatrix}2 & -2 & 2\\ -18 & 28 & -24\\ -9 & 22 & -18\end{vmatrix}=-\begin{vmatrix}0 & -2 & 0\\ 10 & 28 & 4\\ 13 & 22 & 4\end{vmatrix}=-(-2)\cdot(x-1)^{1+2}\begin{vmatrix}10 & 4\\ 13 & 4\end{vmatrix}=-8\begin{vmatrix}10 & 1\\ 13 & 1\end{vmatrix}=24$$

同理算出$d_1=96, d_2=-336, d_3=-96, d_4=169, d_5=312$

即得 $x_1=-4, x_2=-14, x_3=-4, x_4=7, x_5=13$

(消元法解)

$$\overline{A}=\begin{pmatrix}1 & 2 & -2 & 4 & -1 & -1\\ 2 & -1 & 3 & -4 & 2 & 8\\ 3 & 1 & -1 & 2 & -1 & 3\\ 4 & 3 & 4 & 2 & 2 & -2\\ 1 & -1 & -1 & 2 & -3 & -3\end{pmatrix}\xrightarrow[\text{再相减}]{\text{5行移最上}}\begin{pmatrix}1 & -1 & -1 & 2 & -3 & -3\\ 0 & 1 & -1 & 2 & 2 & 2\\ 0 & 1 & 5 & -8 & 8 & 14\\ 0 & 4 & 2 & -4 & 8 & 12\\ 1 & 7 & 8 & -6 & 14 & 10\end{pmatrix}\xrightarrow{\text{3行移到第2行再相减}}$$

$$\begin{pmatrix}1 & 0 & 4 & -6 & 5 & 11\\ 0 & 1 & 5 & -8 & 8 & 14\\ 0 & 0 & -16 & 26 & -22 & -40\\ 0 & 0 & -18 & 28 & -24 & -44\\ 0 & 0 & -27 & 50 & 42 & -88\end{pmatrix}\xrightarrow[(5)\times(4)\text{的2倍}]{(3)-(4)\text{后再乘}\frac{1}{2}}$$

$$\begin{pmatrix}1 & 0 & 4 & -6 & 5 & 11\\ 0 & 1 & 5 & -8 & 8 & 14\\ 0 & 0 & 1 & -1 & 1 & 2\\ 0 & 0 & -18 & 28 & -24 & -44\\ 0 & 0 & 9 & -6 & 6 & 0\end{pmatrix}\xrightarrow{(4)+(5)\text{的2倍后乘以}\frac{1}{4}\text{再用3行的相反倍加到各行}}$$

$$\begin{pmatrix}1 & 0 & 0 & -2 & 1 & 3\\ 0 & 1 & 0 & -3 & 3 & 4\\ 0 & 0 & 1 & -1 & 1 & 2\\ 0 & 0 & 0 & 4 & -3 & -11\end{pmatrix}\xrightarrow{(5)\times\frac{1}{3}\text{后再乘相应倍加到各行}}\begin{pmatrix}1 & 0 & 0 & 0 & -1 & -9\\ 0 & 1 & 0 & 0 & 0 & -14\\ 0 & 0 & 1 & 0 & 0 & -4\\ 0 & 0 & 0 & 0 & 1 & 13\\ 0 & 0 & 0 & 0 & -1 & -6\end{pmatrix}$$

$$\xrightarrow[(5)+(4)\text{后再交换}(4),(5)]{(1)+(4)}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 4\\ 0 & 1 & 0 & 0 & 0 & -14\\ 0 & 0 & 1 & 0 & 0 & -4\\ 0 & 0 & 0 & 1 & 0 & 7\\ 0 & 0 & 0 & 0 & 1 & 13\end{pmatrix}\text{即得.}$$

P101，19④

$$D=\begin{vmatrix}5&6&0&0&0\\1&5&6&0&0\\0&1&5&6&0\\0&0&1&5&6\\0&0&0&1&15\end{vmatrix}=\frac{2^6-3^6}{2-3}=3^6-2^6=665$$

见例2,

$$D_n=\begin{vmatrix}\alpha+\beta&\alpha\beta&\\1&\alpha+\beta&\ddots\\&\ddots&\ddots\end{vmatrix}=\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}$$

取$\alpha=2,\beta=3$

且, $D_1=1507\ \ D_2=-1145\ \ D_3=703\ \ D_4=-395\ \ D_5=212$

$\therefore x_1=\dfrac{1507}{665},x_2=\dfrac{-229}{133},x_3=\dfrac{37}{35},x_4=-\dfrac{79}{133},x_5=\dfrac{212}{665}$

P101.20 解：

代入各a于f(x) $\begin{cases}c_0a_1{}^{n-1}+c_1a_1{}^{n-2}+\cdots c_{n-1}=b_1\\c_0a_2{}^{n-1}+c_1a_2{}^{n-2}+\cdots c_{n-1}=b_2\\c_0a_n{}^{n-1}+c_1a_n{}^{n-2}+\cdots c_{n-1}=b_n\end{cases}$

系数行列式：

$$d=\begin{vmatrix}a_1{}^{n-1}&a_1{}^{n-2}\cdots a_1&1\\a_2{}^{n-1}&a_2{}^{n-2}\cdots a_2&1\\a_n{}^{n-1}&a_n{}^{n-2}\cdots a_n&1\end{vmatrix}=\begin{vmatrix}1&a_1&a_1{}^2\cdots a_1{}^{n-1}\\1&a_2&a_2{}^2\cdots a_n{}^{n-1}\\\cdots&\cdots&\cdots\\1&a_n&a_n^2\cdots a_n^{n-1}\end{vmatrix}=(-1)C_n^2V_n'$$

由于$a_1,a_2\cdots a_n$两两不同, 故$V_n'\neq 0,d\neq 0$由Cramer法则, 存在

唯一解$C_0,C_1,C_2\cdots c_{n-1}$, 即有$f(x)=\sum\limits_{i=0}^{n-1}C_ix^{n-1-i}$(唯一地)使$f(a_i)=b_i$

例P101.21 解：$\begin{cases}a_0&=13.60\\a_0+10a_1+100a_2+1000a_3=13.57\\a_0+20a_1+400a_2+8000a_3=13.55\\a_0+30a_1+900a_2+7000a_3=13.52\end{cases}$

$$d=\begin{vmatrix}1 & 0 & 0 & 0\\ 1 & 10 & 10^2 & 10^3\\ 1 & 20 & 20^2 & 20^3\\ 1 & 30 & 30^2 & 30^3\end{vmatrix}\begin{matrix}=1.2\times10^7\\ =(1a,20,30\cdot(20-10)(30-10)\cdot(30-20))\end{matrix}$$

$$d_0=1.632\times10^8,d_1=-50000,d_2=1800,d_3=-40$$

$$a_0={do}/{d}=13.6,a_1=-\frac{25}{6}\times10^{-3},a_2=1.5\times10^{-4},a_3=-\frac{1}{3}\times10^{-5}$$

用消元法:$\begin{pmatrix}1 & 0 & 0 & 0 & 13.6\\ 1 & 10 & 100 & 1000 & 13.57\\ 1 & 20 & 400 & 8000 & 13.55\\ 1 & 30 & 900 & 27000 & 13.52\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0 & 0 & 13.6\\ 0 & 10^3 & 10^4 & 10^5 & -3\\ 0 & 2\times10^3 & 4\times10^4 & 5\times50^5 & -5\\ 0 & 3\times30^3 & 9\times10^4 & 27\times10^5 & -8\end{pmatrix}$

$$\to\begin{pmatrix}1 & 0 & 0 & 0 & 13.6\\ 0 & 10^3 & 10^4 & 10^5 & -3\\ 0 & 0 & 2\times10^4 & 6\times10^5 & 1\\ 0 & 0 & 6\times10^4 & 24\times10^5 & 1\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0 & 0 & 13.6\\ 0 & 10^3 & 10^4 & 10^5 & -3\\ 0 & 0 & 2\times10^4 & 6\times10^5 & 1\\ 0 & 0 & 0 & 6\times10^5 & -2\end{pmatrix}\to(\text{略})$$

$$h=13.6-\frac{25}{6}\times10^{-3}\times t+\frac{3}{2}\times10^{-4}\times t^2-\frac{1}{3}\times10.5\times t^3(t=°c,h={g}/{cm_3})$$

故当t=15°c时,h=13.56(精确)-(${g}/{cm^3}$)

当t=40°c时,h=13.46(书上答案13.48是错的)

P102 补 1

$$D_1=\sum_{j_1j_2\cdots j_n}\begin{vmatrix}a_{1j_1} & a_{1j_2} & \cdots & a_{1j_n}\\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{nj_1} & a_{nj_2} & \cdots & a_{nj_n}\end{vmatrix}\quad \text{设}D=|a_{ij}|$$

$$\therefore D_1=\sum_{j_1j_2\cdots j_n}(-1)\tau^{(j_1j_2\cdots j_n)}D=\left(\sum_{j_1j_2\cdots j_n\text{取遍}}(-1)^{\tau(j_1j_2\ldots j_n)}\right)D=0$$

(n≥2 奇偶排列各半)

当 n=1 时,

$$P102\quad 补2 \because D=\sum_{j_1j_2\cdots j_n}(-1)^{\tau(j_1j_2\cdots j_n)}a_{1j_1}(t)a_{2j_2}(t)\cdots a_{nj_n}(t))$$

$$\therefore\frac{d}{dt}D=\sum_{j_1j_2\cdots j_n}(-1)^{\tau(j_1j_2\cdots j_n)}(\sum_{k=1}^{n}a_{1j_1}(t)\cdots a_{k-1j_{k-1}}(t)\frac{da_{kjk}(t)}{dt}a_{k+1j_{k+1}}\cdots a_{nj_n}(t))$$

$$=\sum_{k=1}^{n}(\sum_{j_1,j_2\cdots j_n}(-1)\tau^{(j_1,j_2\cdots j_n)})a_{1j_1}(t)a_{2j_2}(t)\cdots a_{k-1j_{k-1}}(t)\frac{da_{kj_k}(t)}{dt}a_{k+1j_{k+1}}\cdots a_{nj_n}(t))$$

$$=\sum_{k=1}^{n}\begin{vmatrix}a_{11}(t) & a_{12}(t) & a_{1n}(t)\\ \frac{d}{dt}a_{k1}(t) & \frac{d}{dt}a_{k2}(t) & \frac{d}{dt}a_{kn}(t)\\ a_{n1}(t) & a_{n2}(t) & a_{nn}(t)\end{vmatrix}$$

同理也有 $\frac{d}{dt}$ D= $\frac{d}{dt}$ D^T= $\sum_{n=1}^{n}\begin{vmatrix}a_{11}(t) & a_{21}(t) & a_{n1}(t)\\ \frac{d}{dt}(a_{1k}(t)) & \frac{d}{dt}a_{2k}(t) & \frac{d}{dt}a_{nk}(t)\\ \cdots & \cdots & \cdots\end{vmatrix}$

$$(转置)=\sum_{k=1}^{n}\begin{vmatrix}a_{11} & \frac{d}{dt}a_{1k}(t) & a_{1n}\\ a_{21} & \frac{d}{dt}a_{2k}(t) & a_{2n}\\ a_{n1} & \frac{d}{dt}a_{nk}(t) & a_{nn}\end{vmatrix}$$

$$左=\begin{vmatrix}a_{11}+x & a_{12}+x & \cdots & a_{1n}+x\\ a_{21}+x & a_{22}+x & \cdots & a_{2n}+x\\ \\ a_{n1}+x & a_{u2}+x & \cdots & a_{u3}+x\end{vmatrix}=\left\|\begin{matrix}1 & 0 & 0 & \cdots & 0\\ 1 & a_{11}+x & a_{12}+x & & a_{1n}+x\\ 1 & a_{21}+x & a_{22}+x & & a_{2n}+x\\ \\ 1 & au1+x & a_{u2}+x & & a_{2u}+x\end{matrix}\right|$$

P102 补 3 ①

$$=\begin{vmatrix}1 & -x & -x\cdots-x\\ 1 & a_{11} & a_{12}\cdots a_{12}\\ 1 & a_{21} & a_{22}\cdots a_{2n}\\ 1 & a_{n1} & a_{n2}\cdots a_{nn}\end{vmatrix}=\begin{vmatrix}a_{11}\cdots a_{1n}\\ a_{n1}\cdots a_{nn}\end{vmatrix}+(-x)\sum_{i=1}^{n}(-1)^{1+(j+1)}\begin{vmatrix}1 & a_{11}\cdots a_{1j-1} & a_{1j+1}\cdots a_{1n}\\ \vdots & & \\ 1 & a_{n1}\cdots a_{nj-1} & a_{nj+1}\cdots a_{nn}\end{vmatrix}$$

$$=D+X\sum_{i=1}^{n}(-1)^{j+1}\left(\begin{vmatrix} a_{11}\cdots a_{1j-1} & a_{1j+1}\cdots a_{1n} \\ a_{21}\cdots a_{2j-1} & a_{2j+1}\cdots a_{2n} \\ a_{n1}\cdots a_{nj-1} & a_{nj+1}\cdots a_{nn} \end{vmatrix}\right)(-1)^{j-1}$$

$$=D+X\sum_{j=1}^{n}\left(\sum_{i=1}^{n}1\bullet Aij\right)=D+X\sum_{i=1}^{n}\sum_{j=1}^{n}Aij$$

补 3 ②在①中令 X=1

$$\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}=\begin{vmatrix} a_{11}+1 & a_{12}+1 & \vdots & a_{1n}+1 \\ a_{21}+1 & a_{22}+1 & \vdots & a_{2n}+1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}+1 & a_{n2}+1 & & a_{nn}+1 \end{vmatrix}-D=$$

$$\begin{vmatrix} a_{11}-a_{12} & a_{12}-a_{13} & \cdots & a_{1n-1}-a_{1n} & a_{1n}+1 \\ a_{21}-a_{22} & a_{22}-a_{23} & \cdots & a_{2n-1}-a_{2n} & a_{2n}+1 \\ \cdots & & \cdots & & \cdots \\ a_{n1}-a_{n2} & a_{n2}-a_{n3} & \cdots & a_{nn-1}-a_{nn} & a_{nn}+1 \end{vmatrix}-\begin{vmatrix} a_{11}-a_{12} & a_{12}-a_{13} & \cdots & a_{1n-1}-a_{1n} & a_{1n} \\ a_{21}-a_{22} & a_{22}-a_{23} & \cdots & a_{2n-1}-a_{2n} & a_{2n} \\ \cdots & & \cdots & & \cdots \\ a_{n1}-a_{n2} & a_{n2}-a_{n3} & \cdots & a_{nn-1}-a_{nn} & a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{11}-a_{12} & a_{12}-a_{13} & \cdots & a_{1n-1}-a_{1n} & 1 \\ a_{21}-a_{22} & a_{22}-a_{23} & \cdots & a_{2n-1}-a_{2n} & 1 \\ \cdots & & \cdots & & \cdots \\ a_{n1}-a_{n2} & a_{n2}-a_{n3} & \cdots & a_{nn-1}-a_{nn} & 1 \end{vmatrix}$$

P103 补 4②：以第一行×(- $\frac{\beta}{\alpha}$)加到后面各行

$$\begin{vmatrix} \lambda & \alpha & \alpha & \cdots & \alpha \\ b-\frac{\beta\lambda}{\alpha} & \alpha-\beta & 0 & 0 & 0 \\ b-\frac{\beta\lambda}{\alpha} & 0 & \alpha-\beta & 0 & 0 \\ b-\frac{\beta\lambda}{\alpha} & 0 & 0 & 0 & \alpha-\beta \end{vmatrix}$$

$$=(\alpha-\beta)^{n-1}(\lambda-\sum_{i=2}^{n}\frac{\alpha b-\beta\lambda}{\alpha-\beta})$$

$$=\lambda(\alpha-\beta)^{n-1}-(n-1)(\alpha b-\beta\lambda)(\alpha-\beta)^{n-2}$$

P103 补 4、③ 见上面 4④得

$$D^n=[a(x+a)^n+a(x-a)^n]/2a=\frac{1}{2}\left[(x+a)^n+(x-a)^n\right]$$

P103 补 4④

$$D_n=\begin{vmatrix} & & & y \\ & & & y \\ * & & & \vdots \\ z & z\cdots z & & x-y+y \end{vmatrix}=\begin{vmatrix} & & & y \\ & & & y \\ * & & & \vdots \\ oo & & \cdots o & x-y \end{vmatrix}+\begin{vmatrix} x & y & \cdots\cdots & y \\ * & & & \\ z\ z & \cdots & \cdots\ x & y \\ z\ z\cdots & & \cdots\cdots z & y \end{vmatrix}$$

$$=(x-y)D_{n-1}+\begin{vmatrix} x-z & y-x & o & \cdots\cdots & o \\ o & x-z & y-x & & \cdots o \\ \cdots & \cdots & & & \cdots \\ o\cdots & & x-z & & o \\ z\cdots & \cdots & z & & y \end{vmatrix}=(x-y)D_{n-1}+y(x-z)^{n-1} \qquad (i)$$

y与z的对称位置有$Dn=(x-z)D_{n-1}+z(x-y)^{n-1} \qquad (ii)$

$(1)\times(x-z)-(ii)\times(x-y)$：得$(y-z)Dn=y(x-z)^n-z(x-y)^n$

$\therefore Dn=\left[y(x-z)^n-z(x-y)^n\right]/(y-z)$

由令, y=a, z=−a, 便得

P103，补 5，f(x)是一个 n+1 级行列式

$$f(x)=\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots 0 & x \\ 1 & 2 & 0 & 0 & \cdots 0 & x^3 \\ 1 & 3 & 3 & 0 & \cdots 0 & x^3 \\ \vdots & \vdots & \vdots & \vdots & \cdots\vdots & \vdots \\ 1 & n & c_n^2 & c_n^3 & \cdots c_n^{n-1} & x_n \\ 1 & n+1 & c_{n+1}^2 & c_{n+1}^3 & \cdots c_{n+1}^{n-1} & x^{n-1} \end{vmatrix}$$

计算 f（x+1），由于前 n 列完全一样，故以下只标出第 n+1 列

$$f(x+1)=\begin{vmatrix} * & x+1 \\ * & (x+1)^2 \\ * & (x+1)_3 \\ \cdots & \cdots \\ * & (x+1)^n \\ * & (x+1)^{n+1} \end{vmatrix}=\begin{vmatrix} * & x+1 \\ * & x^2+2x+1 \\ * & x^3+3x^2+3x+1 \\ \cdots & \cdots\quad\cdots \\ * & x^n+C_n^{n-1}x^{n-1}+\cdots+C_n^1x+1 \\ * & x^{n+1}+(n+1)x^n+\cdots+C_n^2x^2+(n+1)x\ \ +1) \end{vmatrix}$$

$$=\begin{vmatrix} & x \\ & x^2 \\ * & x^3 \\ & \vdots \\ & x^n \\ & x^{n+1}\end{vmatrix}+\begin{vmatrix} & 0 \\ & 0 \\ * & 0 \\ & \vdots \\ & 0 \\ & 0 \\ & (n+1)x^n\end{vmatrix}+\begin{vmatrix} & 0 \\ & 0 \\ & 0 \\ & \vdots \\ * & 0 \\ & c_n^2x^{n-1} \\ & c_{n+1}^2x^{n-2}\end{vmatrix}+\cdots+\begin{vmatrix} & 0 \\ & 2x \\ * & 3x \\ & \vdots \\ & (n-1)x \\ & nx \\ & (n+1)x\end{vmatrix}+\begin{vmatrix} & 1 \\ & 1 \\ & 1 \\ * & \vdots \\ & 1 \\ & 1 \\ & 1\end{vmatrix}$$

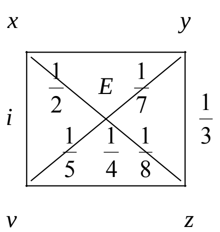
其 余 首 项 （ 可 算 出 后 面 每 个 行 列 式 的 最 后 一 列 都 与 前 面 某 列 比 例 =0 ）

$=f(x)+(\text{第2个行列式})D_2$

$$\text{而}D_2=\begin{vmatrix}1 & & & & \\ & 2 & & & \\ & & 3 & & \\ * & & & \ddots\, c_n^{n-1} & (n+1)xn\end{vmatrix}=(n+1)!x^n$$

$\therefore f(x+1)-f(x)=(n+1)!x^n$

P104 补 6　 分别用 U、X、Y、Z 表子该些点的电位



$$\begin{cases}(x-y)\big/\tfrac{1}{2}+x\Big/\tfrac{1}{6}+(x-0)\big/1=100\\ (y-x)\big/\tfrac{1}{2}+x\Big/\tfrac{1}{7}+(y-z)\Big/\tfrac{1}{3}=100\\ (x-y)\big/\tfrac{1}{2}+x\Big/\tfrac{1}{3}+(x-0)\big/1=100\\ (o-z)\Big/\tfrac{1}{4}+u\Big/\tfrac{1}{5}+(v-x)\big/1=100\end{cases}\qquad\begin{cases}ax-2y\quad -\upsilon=100\\ -2\times+12y-3z=100\\ -3y+15z-4\upsilon=100\\ -x\ -4z+10\upsilon=100\end{cases}$$

$$D=\begin{vmatrix}a & -2 & 0 & -1\\ -2 & 12 & -3 & 0\\ 0 & -3 & 15 & 14\\ -1 & 0 & -4 & 10\end{vmatrix}=\begin{vmatrix}-1 & 0 & -4 & 10\\ 0 & 12 & 5 & -20\\ 0 & -3 & 15 & -4\\ 0 & -2 & -36 & 89\end{vmatrix}=\begin{vmatrix}0 & 65 & -36\\ -1 & +51 & -93\\ 0 & -138 & 275\end{vmatrix}=\begin{vmatrix}65 & -36\\ 138 & 275\\ 17875-4968 & \end{vmatrix}$$

$dx=210100,dy=188400,dz=183300,do=223400.\quad =12907$

$\therefore x=\dfrac{210100}{12907},y=\dfrac{188400}{12907},z=\dfrac{183300}{12907},\upsilon=\dfrac{223400}{12907}$

第三章　　线性方程组习题参考答案

P154,T1

1)解:

$$\begin{pmatrix}1&3&5&-4&0&1\\1&3&2&-2&1&-1\\1&-2&1&-1&-1&3\\1&-4&1&1&-1&3\\1&2&1&-1&1&-1\end{pmatrix}\to\begin{pmatrix}1&3&5&-4&0&-1\\0&0&-3&2&1&-2\\0&-5&-4&3&-1&2\\0&-7&-4&5&-1&2\\0&-1&-4&3&1&-2\end{pmatrix}\to\begin{pmatrix}1&3&5&-4&0&1\\0&-1&-4&3&1&-2\\0&0&-3&2&1&-2\\0&0&16&-12&-6&12\\0&0&24&-16&-8&16\end{pmatrix}$$

$$\to\begin{pmatrix}1&3&5&-4&0&1\\0&-1&-4&3&1&-2\\0&0&1&-2&-1&2\\0&0&0&-4&-2&4\\0&0&0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&3&5&0&0&1\\0&1&4&3&1&-2\\0&0&1&-2&-1&2\\0&0&0&-4&-2&4\\0&0&0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&0&0&\frac{1}{2}&0\\0&1&0&0&\frac{1}{2}&-1\\0&0&1&0&0&0\\0&0&0&1&\frac{1}{2}&1\\0&0&0&0&0&0\end{pmatrix}$$

∴方程组的解是 $\begin{cases}x_1=-\frac{1}{2}k\\x_2=-1-\frac{1}{2}k\\x_3=0\\x_4=1-\frac{1}{2}k\\x_5=k\end{cases}$ k为任意数

2）解：

$$\begin{pmatrix}1&2&0&-3&2&1\\1&-1&-3&1&-3&2\\2&-3&4&-5&2&7\\9&-9&6&-16&2&25\end{pmatrix}\to\begin{pmatrix}1&-1&-3&1&-3&2\\0&3&3&-4&5&-1\\0&-1&10&-7&8&3\\0&0&33&-25&29&7\end{pmatrix}\to\begin{pmatrix}1&-1&-3&1&-3&2\\0&-1&10&-7&8&3\\0&0&22&-25&29&8\\0&0&33&-25&29&7\end{pmatrix}$$

$$\to\begin{pmatrix}1&-1&-3&1&-3&2\\0&1&-10&7&-8&-3\\0&0&33&-25&29&8\\0&0&0&0&0&-1\end{pmatrix}$$ 出现了（0,0,0,0,0,-1），无解

3）解：$\begin{pmatrix}1&-2&3&-4&4\\0&1&-1&1&-3\\&3&0&1&1\\0&-7&3&1&-3\end{pmatrix}\to\begin{pmatrix}1&-2&3&-4&4\\0&1&-1&1&-3\\0&5&-3&5&-3\\0&-7&3&1&-3\end{pmatrix}\to\begin{pmatrix}1&0&1&-2&-2\\0&1&-1&1&-3\\0&0&2&0&12\\0&0&-4&8&-24\end{pmatrix}$

$$\rightarrow\begin{pmatrix}1&0&0&-2&-8\\0&1&0&1&3\\0&0&1&0&6\\0&0&0&8&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&0&0&-8\\0&1&0&0&3\\0&0&1&0&6\\0&0&0&1&0\end{pmatrix}$$

有唯一解: x_1=-8, x_2=3, x_3=6, x_4=0

4）解：
$$\begin{pmatrix}3&4&-5&7&0\\2&-3&3&-2&0\\4&11&-13&16&0\\7&-2&1&3&0\end{pmatrix}\rightarrow\begin{pmatrix}1&7&-8&9\\2&-3&3&-2\\0&17&-19&20\\-1&-24&27&-29\end{pmatrix}\rightarrow\begin{pmatrix}1&7&-8&9\\0&-17&19&-20\\0&17&-19&20\\0&-17&19&-20\end{pmatrix}$$

$$\rightarrow\begin{pmatrix}1&7&-8&9\\0&-1&19/17&-20/17\\0&0&0&0\\0&0&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&7&-3/17&13/17\\0&-1&19/17&-20/17\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

得解：
$$\begin{cases}x_1=\dfrac{3}{17}k-\dfrac{13}{17}l\\x_2=\dfrac{19}{17}k-\dfrac{20}{17}l\\x_3=k\\x_4=l\end{cases}$$

5）解：

$$\begin{pmatrix}2&1&-1&1&1\\3&-2&2&-3&2\\5&1&-1&2&-1\\2&-1&1&-3&4\end{pmatrix}\rightarrow\begin{pmatrix}2&1&-1&1&1\\1&-3&3&-4&1\\1&-1&1&0&-3\\0&-2&2&4&3\end{pmatrix}\rightarrow\begin{pmatrix}1&1&1&0&-3\\0&-3&-3&1&7\\0&-1&2&-4&4\\0&-2&2&-4&3\end{pmatrix}\rightarrow\begin{pmatrix}1&-1&1&0&-3\\0&-3&-3&1&7\\0&-2&2&-4&4\\0&0&0&0&-1\end{pmatrix}$$

出现了（0,0,0,0,-1），无解

6）解：

$$\begin{pmatrix}1&2&3&-1&1\\3&2&1&-1&1\\2&3&1&-1&1\\2&2&2&-1&1\\5&5&2&0&2\end{pmatrix}\rightarrow\begin{pmatrix}1&2&3&-1&1\\0&-4&-8&3&-2\\0&-1&-5&2&-1\\0&-2&-4&4&-1\\0&0&0&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&-1&0&0\\0&1&5&-3&1\\0&0&12&-10&2\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&0&-\frac{5}{6}&\frac{1}{6}\\0&1&0&\frac{7}{6}&\frac{1}{6}\\0&0&1&-\frac{5}{6}&\frac{1}{6}\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

一般解为 $\begin{cases} x_1=\dfrac{1}{6}+\dfrac{5}{6}k \\ x_2=\dfrac{1}{6}-\dfrac{7}{6}k \\ x_3=\dfrac{1}{6}+\dfrac{5}{6}k \\ x_4=k\cdots \end{cases}$ 或 $\begin{cases} x_1=(1+5x_4)/6 \\ x_2=(1-7x)/6 \\ x_3=(1+5x_4)/6 \\ x_4\text{任意} \end{cases}$

P154，T2

1）解：设β=$x_1\alpha_1+x_2\alpha_2+x_3\alpha_3+x_4\alpha_4$，则

$$\begin{cases} x_1+x_2+x_3+x_4=1 \\ x_1+x_2-x_3-x_4=2 \\ x_1-x_2+x_3-x_4=1 \\ x_1-x_2-x_3+x_4=1 \end{cases} \Rightarrow \begin{cases} x_1=5/4 \\ x_2=1/4 \\ x_3=-1/4 \\ x_4=-1/4 \end{cases}$$

$$\therefore \alpha=\frac{5}{4}\alpha_1+\frac{1}{4}\alpha_2-\frac{1}{4}\alpha_3-\frac{1}{4}\alpha_4$$

2）解：设β=$x_1\alpha_1+x_2\alpha_2+x_3\alpha_3+x_4\alpha_4$，则

$$\begin{cases} x_1+2x_2+x_3=0 \\ x_1+x_2+x_3+x_4=0 \\ 0+3x_2-x_4=0 \\ x_1+x_2-x_4=0 \end{cases} \Rightarrow \begin{cases} x_1=1 \\ x_2=0 \\ x_3=-1 \\ x_4=0 \end{cases}$$

即β=$\alpha_1-\alpha_3$

P155.T3

证明：设$k_1, k_2, \cdots, k_r$, l不全为 0，使$k_1\alpha_1+k_2\alpha_2+\cdots k_r\alpha_r+l\beta=0$

若$\ell=0$，则$k_1, \cdots, k_r$也不全为 0，而$k_1\alpha_1+\cdots k_r\alpha_r=0$矛盾.

$\therefore \ell\neq 0$，即$\beta=(-\frac{k_1}{l})\alpha_1+(-\frac{k_2}{l})\alpha_2+\cdots(-\frac{k_s}{l})\alpha_r$线性表出

P155.T4

证明：设$x_1+\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n=0$，则

$$\begin{cases} a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n = 0 \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n = 0 \end{cases}$$

因为系数行列式$|(a_{ij})'| = |a_{ij}| \neq 0$，故上面方程组只有零解，于是$\alpha_1,\alpha_2,\cdots \alpha_n$线性无关。

P155.T5

证明：添加$t_{r+1},\cdots,t_n$, 使$t_1, t_2,\cdots,t_r, t_{r+1},\cdots,t_n$两两不同

得向量组

$$\begin{aligned} &\alpha_1 = (1, t_1, t_1^2, \cdots t_1^{n-1}) \\ &\alpha_r = (1, t_r, t_r^2, \cdots t_r^{n-1}) \\ &\alpha_{r+1} = (1, t_{r+1}, t_{r+1}^2, \cdots t_{r+1}^{n-1}) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\alpha_n = (1, t_n, t_n^2, \cdots t_n^{n-1}) \end{aligned}$$

由于$\alpha_1,\alpha_2,\cdots,\alpha_n$的分量作成一个Vandermonder行列式（公比两两不同）且不等于 0，由 上一题，$\alpha_1,\alpha_2,\cdots,\alpha_r,\cdots,\alpha_n$线性无关，于是它的任一部分线性无关

P155.T6

证：设$\beta_1=\alpha_2+\alpha_3$，$\beta_2=\alpha_3+\alpha_1$，$\beta_3=\alpha_1+\alpha_2$，

若$x_1\beta_1+s_2\beta_2+x_3\beta_3=0$，则即

$(x_2+x_3)\alpha_1+(x_3+x_1)\alpha_2+(x_1+x_2)\alpha_3=0$

即$\alpha_1, \alpha_2, \alpha_3$线性无关

$$\therefore \begin{cases} x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

∴x_1, x_2, x_3全为 0，即$\beta_1, \beta_2, \beta_3$线性无关。

而若$\alpha_1,\alpha_2,\alpha_3,\alpha_4$线性无关，则$\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_3+\alpha_4, \alpha_4+\alpha_1$,线性相关。

P155.T7

证明：设$\alpha_1,\alpha_2,\cdots,\alpha_s$（I)的一个极大无关组$\alpha_{i1}, \alpha_{i2},\cdots,\alpha_{ir}$(I)′ 及任一线性无关向量组$\alpha_{j1}, \alpha_{j2},\cdots,$ α_{jr}(I)″

任取（I）中的一个向量β有$\alpha_{j1},\cdots,\alpha_{jr},\beta\leftarrow(I)\rightleftarrows(I)'$ 而$(I)'$ 中只有r个向量，由定理 2，α_{j1},α_{jr}，β线性相关，而本来$(II)''$ 线性无关，故（临界定理）$\beta\leftarrow(I)''$，所以$(I)\leftarrow(I)''$所以$(I)''$是极大无关组。

P155.T8

证明：

设$\alpha_1,\alpha_2,\cdots,\alpha_s$（I），$\alpha_{i1},\alpha_{i2},\cdots,\alpha_{ir}$ $(I)'$ ，及(I)的一个极大无关组$\alpha_{i1},\alpha_{i2},\cdots,\alpha_{ir}$ $(I)''$ ，已知$(I)\leftarrow(I)'$，故有

$$(I)'\rightleftarrows(I)\rightleftarrows(I)''$$

所以取 $(I)'$ 的极大无关组$\alpha_{k1},\alpha_{k2},\ldots,\alpha_{kt}(I)''$ ，则t ≤r且 $(I)''\rightleftarrows(I)'\rightleftarrows(I)''$ ，那么$(I)'''\rightarrow(I)''$由于$(I)''$有r个向量且线性无关，所以（由定理 2 推论 1）r ≤t，即r=t，故$(I)'''=(I)'$，$(I)'$线性无关。

$(I)'$是(I)的一个极大无关组。

P155.T9

证明：设(I)的一个线性无关组$(I)'$

1° 逐个检查(I)中的向量α_i

2° a、若$\alpha_i\leftarrow(I)'$，则去掉α_i，检查下一个α

b、若存在$\alpha_i\not\leftarrow(I)'$，则添加$\alpha_i$到$(I)'$ 中将$(I)'$ 扩充为$(I)''$，回到检查第 1 个向量，重复 1°、2°

若干步后（∵有限步后，任意n+1 个n维向量也相关，必含停止），得到$(I)',(I)'',\ldots(I)^{(k-1)}$，$(I)^{(k)}$

而$(I)^{(k)}$不得再扩大，于是$(I)^{(k)}$是一个极大无关组，是$(I)'\subseteq(I)^{(k)}$。

P155.T10

1）解：∵α_1与α_2的分量不成比例，故α_1与α_2线性无关

2）解：考虑$\alpha_1,\alpha_2,\alpha_3$ ∵$3\alpha_1+\alpha_2=\alpha_3$ 去掉α_3

考虑$\alpha_1,\alpha_2,\alpha_4$,取它们的后三个分量$\begin{vmatrix}-1 & 2 & 4\\ 3 & 1 & 2\\ 1 & 2 & 0\end{vmatrix}=28\neq0$，∴增加一个分量后仍然线性无关。

即$\alpha_1, \alpha_2, \alpha_4$线性无关

再考虑$\alpha_1, \alpha_2, \alpha_4, \alpha_5$,因为分量行列式

$$\begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 3 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 5 & 6 \end{vmatrix} = 0$$

即$\alpha_5=\alpha_1+\alpha_2+\alpha_4$　所以它的极大线性无关组是$\alpha_1, \alpha_2, \alpha_4$

P155.T11

1）解：

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 6 & 4 & 1 & -1 & 2 \\ 1 & 0 & 2 & 3 & -4 \\ 1 & 4 & -9 & -16 & 22 \\ 7 & 1 & 0 & -1 & 3 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \to \begin{pmatrix} 1 & 0 & 2 & 3 & -4 \\ 0 & 4 & -11 & -19 & 26 \\ 0 & 4 & -11 & -19 & 26 \\ 0 & 1 & -14 & -22 & 31 \end{pmatrix} \begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array} \to \begin{pmatrix} 1 & 0 & 2 & 3 & -4 \\ 0 & 1 & -14 & -22 & 31 \\ 0 & 0 & 45 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} 2 \\ 4 \\ 1 \\ 3 \end{array}$$

∴秩$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$=3，且$\alpha_2, \alpha_3, \alpha_4$为一个极大无关组。

∴秩(A)=5

2）解：略

P156.T12

证：　设$(I)'$为$(II)''$分别为(I)、(II)的极大无关组，则有

$$(I)' \rightleftarrows (I) \leftarrow (II) \rightleftarrows (II)'$$

设$(I)'$含r个向量，$(II)'$含七个向量，因为$(I)'$线性无关，且$(I)' \leftarrow (II)'$，所以r≤t，即秩(I)≤秩(II)

P156.T13

证明：设$\alpha_{i1},\alpha_{i2},\cdots,\alpha_{ir}$为$\alpha_1,\alpha_2,\cdots,\alpha_n$的极大线性无关组则得下面表示序列

$$\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ir} \rightleftarrows \alpha_1, \alpha_2, \cdots, \alpha_n \to \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$$

因为单位向量组$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$线性无关，由（定理 2 推论），得n ≤ r故$\alpha_{i1},\alpha_{i2},\ldots,\alpha_{ir}$为$\alpha_1,\alpha_2,\cdots,\alpha_n$本身，即证得$\alpha_1,\alpha_2,\cdots \alpha_n$,线性无关。

P156.T14

证明：略

P156.T15

证明："$\Leftarrow$"若系数行列式$|a_{ij}|\neq 0$，则由Cramer法则，对任何常数$b_1,b_2,\ldots,b_n$有唯一解。

"$\Rightarrow$" $A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{12} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ 则原方程组为 $x_1\beta_1+x_2\beta_2+\cdots+x_n\beta_n=b$

其中$\beta_2,\beta_3,\ldots,\beta_n$为A的列向量组， ∵对任何 $b=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ 都有解。

依次定 $b=\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$，则得

$$\beta_1,\beta_2,\cdots,\beta_n\to\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$$

从而$\beta_2,\beta_3,\ldots,\beta_n$线性无关，列秩 (A)=n，即秩(A)=n，由定理 5，$|A|\neq 0$

P156.T16

证明： 设$\alpha_1,\alpha_2,\cdots,\alpha_r$(I)及$\alpha_1,\cdots\alpha_r,\alpha_{r+1},\cdots,\alpha_s$(II)，且秩(I)=秩(II)=t，设$\alpha_{i1},\alpha_{i2},\cdots,\alpha_{it}$(III)为(I)的极大无在组$\alpha_{i1},\alpha_{i2},\cdots,\alpha_{it}$(IV)为(II)的极大无关组，那么，$(III)\rightleftarrows(I)\leftarrow(II)\rightleftarrows(IV)$

任取(II)中一个向量β，组成α_{i1}，$\alpha_{i2},\cdots,\alpha_{it}$，β—(V)，则$(V)\leftarrow(II)\leftarrow(IV)$，因为（IV）只有七个向量，所以（V）线性相关，而（III）线性无关。

所以$\beta\leftarrow(III)$即$(II)\leftarrow(III)$ $\therefore(II)\rightleftarrows(III)$

$\therefore(I)\rightleftarrows(III)\rightleftarrows(II)$ （I）与（II）等价

P156.T17

证明： ∵$\beta_1=\alpha_2+\ldots+\alpha_r$，$\beta_2=\alpha_1+\alpha_3+\ldots+\alpha_r,\cdots,\beta_r=\alpha_1+\ldots+\alpha_{r-1}$

$\therefore\beta_1,\beta_2,\cdots,\beta_r\to\alpha_1,\alpha_2,\cdots,\alpha_r$

令 $r=\beta_1+\beta_2+\cdots+\beta_r=(r-1)(\alpha_1+\alpha_2+\cdots+\alpha_r)$

$\therefore\alpha_i=\dfrac{1}{r-1}r-\beta_i$ 即α_i可以$\beta_1,\beta_2,\ldots,\beta_r$线性表示

$\therefore\beta_1,\beta_2,\cdots,\beta_r\rightleftarrows\alpha_1,\alpha_2,\cdots,\alpha_r$ 秩必相同

P156.T18

① $A \to \begin{pmatrix} 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & -4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$ ∴秩(A)=4

② $A \to \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 3 & 0 & -4 & 1 \\ 0 & 3 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ ∴ 秩(A)=3

③ $A \to \begin{pmatrix} 6 & 104 & 21 & 9 & 17 \\ 7 & 6 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B$ 显然B中有2阶子式不等于0 且所有3阶子式等于0 ∴秩(B)=2 秩$(A)=2$

④ $A \to \begin{pmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 2 & 3 & 13 & 28 \\ 0 & 5 & 6 & 28 & 61 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 9 & 18 \\ 0 & 0 & 6 & 18 & 36 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ ∴秩(A)=3

⑤

$$A \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

∴秩(A)=5

P157. T19

①∵ $\begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda+2)(\lambda-1)^2$；

∴当$\lambda \neq 1$时，有唯一解。$x_1 = \dfrac{-(\lambda+1)}{\lambda+2}, x_2 = \dfrac{1}{\lambda+2}, x_3 = \dfrac{(\lambda+1)^2}{\lambda+2}$。

∴当λ=-2 时，三个方程解相加，得 0=3（无解）。

∴当λ=1 时，变为一个方程$x_1+x_2+x_3=1$ 即$x_1=1-x_2-x_3$. x_2 x_3任取。

②∵系数解列式$\begin{vmatrix} \lambda+3 & 1 & 2 \\ \lambda & \lambda-1 & 1 \\ 3(\lambda+1) & \lambda & a+3 \end{vmatrix} = \lambda^3-\lambda^2=\lambda^2(\lambda-1)$。

而$\lambda\neq0$ 且$\lambda\neq1$ 时，有唯一解：(用Craner法则)

$$x_1=\frac{\lambda^3+3^{\lambda 2}-15\lambda+9}{a^2(\lambda-1)},x_2=\frac{\lambda^3+12\lambda-9}{\lambda^2(\lambda-1)},x_3=\frac{-4\lambda^2+3\lambda^2+12\lambda-9}{\lambda^2(\lambda-1)}$$。

而当λ=0 时为$\begin{cases} 3x_1+x_2+2x_3=0 \\ -x_2+x_3=0 \\ 3x_1+3x_3=3 \end{cases}$ ①式+②式-③式得 0=-3，矛盾。

当λ=1 时为$\begin{cases} 4x_1+x_2+2x_3=1 \\ x_1+\cdots+x_3=2 \\ 6x_1+x_2+4x_3=3 \end{cases}$①式+②式两倍-③式得 0=2，矛盾。

③系数行列式$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 2b & 1 \end{vmatrix}=b(1-a)$。

当$\lambda\neq0$ 且a≠1 时，有唯一解。$x_1=\frac{1-2b}{b(1-a)},x_2=\frac{1}{b},x_3=\frac{4b-2ab-1}{b(1-a)}$。

若 b=0,则②式-③式得 0=-1。矛盾。

若b≠0 而a=1。化为$\begin{cases} x_1+x_2+x_3=4 \\ x_1+bx_2+x_3=3 \\ x_1+2bx_2+x_3=4 \end{cases}$

①-③得（1-2b）x_2=0

①-②得（1-b）x_2=1

∴ $x_2\neq0$ 义与$(1-2b)=0$ 即$b=\frac{1}{2}$.$\left(b\neq\frac{1}{2}\text{则矛盾无解}\right)$

此时化为$\begin{cases} x_1+x_2+x_3=4 \\ x_1+\frac{1}{2}x_2+x_3=3 \\ x_1+x_2+x_3=4 \end{cases}$即$\begin{cases} x_1+x_2+x_3=4 \\ x_2=2 \end{cases}$即$\begin{cases} x_1=2-x_3 \\ x_2=2 \end{cases}$ x_3任意值

P157.T20、

1) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 & -3 \\ 0 & 1 & 2 & 2 & 6 \\ 5 & 4 & 3 & 3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -6 \\ 0 & 1 & 2 & -6 \\ 0 & -1 & -2 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -1 & -5 \\ 0 & 1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

即x_3,x_4,x_5为自由未知量.

$$(1,0,0) \qquad \eta_1=(1,-2,1,0,0)$$

$$(0,1,0) \qquad \eta_2=(1,-2,0,1,0)$$

令 $(x_3,x_4,x_5)=(0,0,1)$ 得：$\eta_3=(1,-2,0,0,1)$

2）

$$\begin{pmatrix}1&1&0&-3&-1\\1&-1&2&-1&0\\4&-2&6&3&-4\\2&4&-2&4&-7\end{pmatrix}\to\begin{pmatrix}1&1&0&-3&-1\\0&-2&2&2&1\\0&-6&6&15&0\\0&2&-2&10&-5\end{pmatrix}\to\begin{pmatrix}1&1&0&-3&-1\\0&-2&2&2&1\\0&0&0&9&-3\\0&0&0&12&-4\end{pmatrix}$$

$$\to\begin{pmatrix}1&0&1&-2&-\frac{1}{2}\\0&1&-1&-1&-\frac{1}{2}\\0&0&0&1&-\frac{1}{3}\\0&0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&1&0&-\frac{7}{6}\\0&1&-1&0&-\frac{5}{6}\\0&0&0&1&-\frac{1}{3}\\0&0&0&0&0\end{pmatrix}$$

即 x_1,x_2,x_5 为基本，x_3,x_4 为自由未知量。

$$\eta_1=(-1,1,1,0,0)$$

$(x_3,x_4)=(1,0)$

令 $(x_3,x_4)=(0,1)$，得 $\eta_2=(\frac{7}{6},\frac{5}{6},0\frac{1}{3},1)$。

即基础解系为（$-1,1,1,0,-2$）和（$7,5,0,2,6$）

$$\begin{pmatrix}1&-2&1&-1&1\\2&1&-1&2&-3\\3&-2&-1&1&-2\\2&-5&1&-2&2\end{pmatrix}\to\begin{pmatrix}1&-2&1&-1&1\\0&5&-3&4&-5\\0&4&-4&4&-5\\0&-1&-1&0&0\end{pmatrix}\to\begin{pmatrix}1&0&3&-1&1\\0&1&1&0&0\\0&0&-8&4&-5\\0&0&-8&4&-5\end{pmatrix}$$

4）

$$\to\begin{pmatrix}1&0&0&+1/2&-7/8\\0&1&0&1/2&-5/8\\0&0&1&-1/2&5/8\\0&0&0&0&0\end{pmatrix}\text{即}\begin{cases}x_1=-\frac{1}{2}x_4+\frac{7}{8}x_5\\x_2=-\frac{1}{2}x_4+\frac{7}{8}x_5\\x_3=-\frac{1}{2}x_4+\frac{7}{8}x_5\end{cases}$$

$(x_4,x_5)=(1.0)(0.1)$ 得基础解系 $\begin{cases}y_1=\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},1,0\right)\\y_2=\left(\frac{7}{8},\frac{5}{8},-\frac{5}{8},0,1\right)\end{cases}$

令

P157.T22

$$\begin{pmatrix}1&1&1&1&1\\3&2&1&-3&a\\0&1&2&6&3\\5&4&3&-1&b\end{pmatrix}\to$$

$$\begin{pmatrix}1&1&1&1&1&1\\0&-1&-2&-2&-6&a-3\\0&1&2&2&6&3\\0&-1&-2&-2&-6&b-5\end{pmatrix}\to\begin{pmatrix}1&0&-1&-1&-5&-2\\0&1&2&2&6&3\\0&0&0&0&0&a\\0&0&0&0&0&b-2\end{pmatrix}$$

由于有解$\Leftrightarrow$秩（系）=秩（增）故有解$\Leftrightarrow$ a=0, b=2.

此时，x_1, x_2为基础未知量。特解为r_0=(-2, 3, 0, 0, 0)

x_3, x_4, x_r为自由未知量，依次取

（$x_3, x_4. x_r$）=(1, 0, 0)　　　　　η_1=(1, -2, 1, 0, 0)

（$x_3, x_4. x_r$）=(0, 1, 0)　　　得　η_2=(1, -2, 0, 1, 0)

（$x_3, x_4. x_r$）=(0, 0, 1)　　　　　η_3=(5, -6, 0, 0, 1)

通解为 $r_0+k_1\eta_1+k_2\eta_2+k_3\eta_3$（$k_1, k_2, k_3$为任意常数。）

P158.T23

增广矩阵$\begin{pmatrix}1&-1&0&0&0&a_1\\&1&-1&0&0&a_2\\&&&-1&0&a_3\\&&&1&-1&a_4\\-1&0&0&0&1&a_5\end{pmatrix}\to\begin{pmatrix}1&-1&0&0&0&a_1\\0&1&-1&0&0&a_2\\&&1&-1&0&a_3\\&&&1&-1&a_4\\0&0&0&0&0&a_1+a_2+a_3+a_4+a_5\end{pmatrix}$

因为系数矩阵秩为 4。增广矩阵秩为 5 $\Leftrightarrow\sum_{i=1}^{5}ai\neq0$

秩为 4 $\Leftrightarrow\sum_{i=1}^{5}a_i=0$

故由有解判别定理，方程组有解$\Leftrightarrow$秩(系)=秩(增) $\Leftrightarrow\sum_{i=1}^{5}a_i=0$

有解时，即 $\sum_{i=1}^{5} ai=0$。矩阵化为最简阶梯→$\begin{pmatrix} 1 & 0 & 0 & -1 & a_1+a_2+a_3+a_4 \\ 0 & 1 & 0 & -1 & a_2+a_3+a_4 \\ 0 & 0 & 1 & -1 & a_3+a_4 \\ 0 & 0 & 0 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

特解r_0=($a_1+a_2+a_3+a_4$, $a_2+a_3+a_4$, a_3+a_4, 0)。

导出组基础系（只有一个自由求知数x_5=1）为 η=（1，1，1，1，1）。

所以方程组的通解为$r_0+k\eta$。k为任意的数。

P158.T24

设η_1，$\eta_2,\cdots\eta_s$是某齐次方程组的基础解系，而ξ_1，$\xi_2\cdots\xi_t$是方程组的线形无关解组。则η_1，$\eta_2,\cdots\eta_s\Leftrightarrow\xi_1$，$\xi_2\cdots\xi_t$。由于等价且都线形无关，必有s = t. 由传递性，方程的任一解可由η_1，$\eta_2,\cdots\eta_s$线形表示，也可由ξ_1，$\xi_2\cdots\xi_t$线形表示。ξ_1，$\xi_2\cdots\xi_t$也是基础解系。

P158.T25

由于秩（系）=r. 故基础解系会$n-r$个向量 η_1，$\eta_2\cdots\eta_{n-r}$（$r<n$）

而它们任意$n-r$解 ζ_1，$\zeta_2\cdots\zeta_{n-r}$. 如果线性无关。则由（补 6，P157）的证明方法：（见 3.41.6.*）。秩（η_1，$\eta_2\cdots\cdots\eta_{n-r}$）=秩（$\zeta_1$，$\zeta_2\cdots\cdots\zeta_{n-r}$）=$n-r$。且 ζ_1，$\zeta_2\cdots\cdots\zeta_{n-r}\leftarrow\eta_1$，$\eta_2\cdots\cdots\eta_{n-r}$。故，$\zeta_1$，$\zeta_2\cdots\zeta_{n-r}\Leftrightarrow\eta_1\ \eta_2\cdots\cdots\eta_{n-r}$（再由上题（24 题）知向量 ζ_1，$\zeta_2\cdots\cdots\zeta_{n-r}$也为方程组的基础解系。

P158.T26

证明：设η_v=（k_{v1}，$k_{v2},\cdots k_{vn}$）（v=1，2,…t）为方程组$\sum_{j=1}^{n}a_{ij}x_j=b_j(j=1,2,\cdots s)$的解，

即必有 $\sum_{j=1}^{n}a_{ij}k_{vj}=b_j\quad\begin{pmatrix} v=1\cdots t \\ i=1\cdots s\end{pmatrix}$，那么 $\eta=\sum_{v=1}^{n}u_v\eta_v$ 代入方程组得 $\sum_{j=1}^{n}a_{ij}\sum_{v=1}^{t}u_vk_{vj}=\sum_{v=1}^{t}(\sum_{j=1}^{n}a_{ij}k_{vj})\ u_v=\sum_{v=1}^{t}b_iu_v=b_i\sum_{v=1}^{t}u_v=b_i$（由已知条件 $\sum_{v=1}^{t}u_v=1$）

$\therefore\ \eta=\sum_{v=1}^{n}u_v\eta_v$是方程的解。

此题反过来也成立，即

若$\sum_{v=1}^{t}u_v\eta_v$为非齐次方程的解，且η_v也是解，则必有$\sum_{v=1}^{t}u_v=1$。

P158.T27

$$\begin{vmatrix} a_0,a_1\cdots a_n \\ a_0\cdots a_n \\ a_0\cdots a_n \\ b_0\cdots b_m \\ b_0\cdots b_m \\ b_0\cdots b_m \end{vmatrix} = (-1)^m \begin{vmatrix} b_0,b_1\cdots b_n \\ a_0,a_1\cdots a_n \\ a_0\cdots a_n \\ a_0\cdots a_n \\ b_0\cdots b_m \\ b_0\cdots b_m \end{vmatrix} = (-1)^m \cdots = (-1)^{mn} \begin{vmatrix} b_0,b_r\cdots b_m \\ b_0\cdots b_m \\ a_0\cdots a_n \\ a_0\cdots a_m \end{vmatrix} = (-1)^{mn} R(g.f)$$

R（f·g）= 其中$n=\partial\left(f\left(x\right)\right)$ $m=\partial\left(g\left(x\right)\right)$

158.T28

①

$$R.\left(f.g\right)=\begin{vmatrix} 5 & -6x & 5x^2-16 & 0 \\ 0 & 5 & -6x & 5x^2-16 \\ 1 & -x-1 & 2x^2-x-4 & 0 \\ 0 & 1 & -x-1 & 2x^2-x-4 \end{vmatrix} = \begin{vmatrix} 0 & -x+5 & 0 \\ 0 & 0 & -5x^2+5x+4 \\ 1 & -x-1 & \\ 0 & 1 & 2x^2-x-4 \end{vmatrix}$$

$$=(-1)^{3+1}\begin{vmatrix} 0 & -6x^2+9x+9 & (2x^2-x-4) \\ 0 & -x+5 & -5x^2+5x+4 \\ 1 & -x-1 & 2x^2-x-4 \end{vmatrix}$$ 直接展开方程相加

$=32x^4-96x^3+96x-64$。$=32(x^4-3x^3+x^2+3x-2)=32(x^2-1)(x^2-3x+2)$

$=32(x-1)^2(x+1)(x-2)$

有 4 个解是 $x_1=x_2=1$，$x_3=2$，$x_4=-1$。

用 $x=1$ 代入在方程组得 $\begin{cases}5y^2-6y-11=0\\ y^2-2y-3=0\end{cases}$ 有公共解 $y=-1$, 即 $\begin{cases}x=1\\ y=-1\end{cases}$

用 $\gamma=2$ 代入在方程组得 $\begin{cases}5y^2-12y+4=0\\ y^2-3y-3=0\end{cases}$ 有公共解 $y=2$, 即 $\begin{cases}\gamma=2\\ y=-1\end{cases}$

用 $x=-1$ 代入在方程组得 $\begin{cases}5y^2+6y-11=0\\ y^2-1=0\end{cases}$ 有公共解 $y=1$, 即 $\begin{cases}x=-1\\ y=1\end{cases}$

即得到三组解 $\begin{cases}x=1\\ y=-1\end{cases}$ $\begin{cases}\gamma=2\\ y=-1\end{cases}$ $\begin{cases}x=-1\\ y=1\end{cases}$

第四章　矩阵练习题参考答案

P197.T1

①解:
$$AB=\begin{pmatrix}6&2&-2\\6&1&0\\8&-1&2\end{pmatrix}\qquad BA=\begin{pmatrix}1&1&-1\\2&-1&0\\1&0&1\end{pmatrix}\begin{pmatrix}3&1&1\\2&1&2\\1&2&3\end{pmatrix}=\begin{pmatrix}4&0&0\\4&1&0\\4&3&4\end{pmatrix}$$

∴
$$AB-BA=\begin{pmatrix}2&2&-2\\2&0&0\\4&-4&-2\end{pmatrix}$$

②解:
$$AB=\begin{pmatrix}a+b+c&a^2+b^2+c^2&ac+b^2+ac\\a+b+c&ac+b^2+ac&a^2+b^2+c^2\\3&a+b+c&a+b+c\end{pmatrix}$$

$$BA=\begin{pmatrix}1&a&c\\1&b&b\\1&c&a\end{pmatrix}\begin{pmatrix}a&b&c\\c&b&a\\1&1&1\end{pmatrix}=\begin{pmatrix}a+ac+c&b+ab+c&c+a^2+c\\a+bc+b&b+b^2+b&c+ab+b\\a+c^2+a&b+bc+a&c+ac+a\end{pmatrix}$$

$$\therefore AB-BA=\begin{pmatrix}b-ac&a^2+b^2+c^2-b-ab-c&b^2-a^2+2ac-2c\\c-bc&2(ac-b)&a^2+b^2+c^2-b-ab-c\\3-2a-c^2&c-bc&b-ac\end{pmatrix}$$

P198.T 2

①解:
$$\begin{pmatrix}3&1&1\\3&1&0\\0&1&2\end{pmatrix}^2=\begin{pmatrix}3&1&1\\3&1&0\\0&1&2\end{pmatrix}\begin{pmatrix}3&1&1\\3&1&0\\0&1&2\end{pmatrix}=\begin{pmatrix}12&5&5\\12&4&3\\3&3&4\end{pmatrix}$$

②解:
$$\begin{pmatrix}3&2\\-4&-2\end{pmatrix}^5=\begin{pmatrix}3&2\\-4&-2\end{pmatrix}^3\begin{pmatrix}1&2\\-4&-4\end{pmatrix}=\begin{pmatrix}-5&-2\\4&0\end{pmatrix}\begin{pmatrix}1&2\\-4&-4\end{pmatrix}=\begin{pmatrix}3&-2\\4&8\end{pmatrix}$$

④解: ∵
$$\begin{pmatrix}\cos\varphi&-\sin\varphi\\\sin\varphi&\cos\varphi\end{pmatrix}\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}=\begin{pmatrix}\cos(\varphi+\theta)&-\sin(\varphi+\theta)\\\sin(\varphi+\theta)&\cos(\varphi+\theta)\end{pmatrix}$$

∴
$$\begin{pmatrix}\cos\varphi&-\sin\varphi\\\sin\varphi&\cos\varphi\end{pmatrix}^n=\begin{pmatrix}\cos n\varphi&-\sin n\varphi\\\sin n\varphi&\cos n\varphi\end{pmatrix}$$

5 解:

$$(2,3,-1)\begin{pmatrix}1\\-1\\-1\end{pmatrix}=2-3+1=0$$

$$\begin{pmatrix}1\\-1\\-1\end{pmatrix}(2,3,-1)=\begin{pmatrix}2&3&-1\\-2&-3&1\\-2&-3&1\end{pmatrix}$$

6 解:

原式= $$(x,y,1)\begin{pmatrix}a_{11}x+a_{12}y+b_1\\a_{12}x+a_{22}y+b_2\\b_1x+b_2y+c\end{pmatrix}=a_{11}x^2+2a_{12}xy+a_{22}y^2+2b_1x+2b_1y+c$$

7 解: $$A=\begin{pmatrix}1&-1&-1&-1\\-1&1&-1&-1\\-1&-1&1&-1\\-1&-1&-1&1\end{pmatrix}A^2=\begin{pmatrix}4&0&0&0\\0&4&0&0\\0&0&4&0\\0&0&0&4\end{pmatrix}\therefore A^n=\begin{cases}2^nE & n=2k\text{时}\\2^{n-1}A & n=2k+1\text{时}\end{cases}$$

8 解: $$\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}^2=\begin{pmatrix}\lambda^2&2\lambda&1\\0&\lambda^2&2\lambda\\0&0&\lambda\end{pmatrix},$$ 而

$$\begin{pmatrix}\lambda^k&c_k^1\lambda^{k-1}&c_k^2\lambda^{k-2}\\0&\lambda^k&c_k^1\lambda^{k-1}\\0&0&\lambda^k\end{pmatrix}\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}\lambda^{k+1}&c_{k+1}^1\lambda^k&c_{k+1}^2\lambda^{k-1}\\0&\lambda^{k+1}&c_{k+1}^1\lambda^k\\0&0&\lambda^{k+1}\end{pmatrix}$$

$$\therefore\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}^n=\begin{pmatrix}\lambda^n&c_n^1\lambda^{n-1}&c_n^2\lambda^{n-2}\\0&\lambda^n&c_n^1\lambda^{n-1}\\0&0&\lambda^n\end{pmatrix}=\begin{pmatrix}\lambda^n&n\lambda^{n-1}&\frac{1}{2}n(n-1)\lambda^{n-2}\\0&\lambda^n&n\lambda^{n-1}\\0&0&\lambda^n\end{pmatrix}$$

P198.T3

① $$A^2=\begin{pmatrix}2&1&1\\3&1&2\\1&-1&0\end{pmatrix}^2=\begin{pmatrix}8&2&4\\11&2&5\\-1&0&-1\end{pmatrix}$$

$$\therefore f(A)=A^2-A-E=\begin{pmatrix}6&1&3\\8&0&3\\-2&2&-1\end{pmatrix}-E=\begin{pmatrix}5&1&3\\8&-1&3\\-2&2&-2\end{pmatrix}$$

② $$A^2=\begin{pmatrix}2&-1\\-3&3\end{pmatrix}^2=\begin{pmatrix}7&-5\\-15&12\end{pmatrix}$$

$\therefore f(A)=A^2-5A+3E=A^2-\begin{pmatrix}7 & -5\\ -15 & 12\end{pmatrix}=0$

P199.T4.

①设 $X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}, AX=\begin{pmatrix}a+c & b+d\\ c & d\end{pmatrix}, XA=\begin{pmatrix}a & a+b\\ c & c+d\end{pmatrix},$

由 $AX=XA\Rightarrow c=0$，$a+b=b+d\Rightarrow a=d$

$\therefore X=\begin{pmatrix}a & b\\ 0 & a\end{pmatrix}$ a,b 任取。

②$\because XA=AX\Leftrightarrow X(A-E)=(A-E)X.$ $\overline{A}=A-E=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 2\\ 3 & 1 & 1\end{pmatrix}$

设$X=\begin{pmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix}$, $X\overline{A}=\begin{pmatrix}3x_{13} & x_{13} & 2x_{12}+x_{13}\\ 3x_{23} & x_{23} & 2x_{22}+x_{23}\\ 3x_{33} & x_{33} & 2x_{32}+x_{33}\end{pmatrix}$,

$\overline{A}X=\begin{pmatrix}0 & 0 & 0\\ 2x_{31} & 2x_{32} & 2x_{33}\\ 3x_{11}+x_{21}+x_{31} & 3x_{12}+x_{22}+x_{32} & 3x_{13}+x_{23}+x_{33}\end{pmatrix}$

$\therefore x_{12}=x_{13}=0$

令 $\begin{matrix}x_{21}=3a\\ x_{22}=b\\ x_{23}=2c\end{matrix}\Rightarrow\begin{matrix}x_{31}=3c\\ x_{32}=c\\ x_{33}=b+c\end{matrix}\Rightarrow\begin{matrix}x_{11}=b+c-a-c\\ x_{22}=0\\ x_{13}=0\end{matrix}$ $\therefore x=\begin{pmatrix}b-a & 0 & 0\\ 3a & b & 2c\\ 3c & c & b+c\end{pmatrix}$

③同样设 $X=\begin{pmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix}$, $AX=\begin{pmatrix}x_{25} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\\ 0 & 0 & 0\end{pmatrix}$, $XA=\begin{pmatrix}0 & x_{11} & x_{12}\\ 0 & x_{21} & x_{22}\\ 0 & x_{31} & x_{32}\end{pmatrix}$

$\therefore x_{21}=x_{31}=x_{32}=0, x_1=x_{22}=x_{33}, x_{23}=x_{12}\therefore x=\begin{pmatrix}x_{11} & x_{12} & x_{13}\\ 0 & x_{11} & x_{12}\\ 0 & 0 & x_{11}\end{pmatrix}$

P199.T5

设 $X=\begin{pmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ & \cdots & \cdots & \\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{pmatrix}$ $\because AX=XA$

左边i行j列的元为 $a_i x_{ij}$

右边i行j列的元素 $x_{ij}a_j$

$\therefore i \neq j,\ \ a_i x_{ij} = a_j x_{ij} \Rightarrow \left(a_i - a_j\right) x_{ij} \Rightarrow x_{ij} = 0\ (\because a_i \neq a_j)$

$\therefore X = \begin{pmatrix} x_{11} & & & \\ & x_{22} & & \\ & & \cdots & \\ & & & x_{nn} \end{pmatrix}$ 只是对角矩阵

P199.T6

$A = \begin{pmatrix} a_1 E_{n1} & & & \\ & a_2 E_{n2} & & \\ & & \cdots & \\ & & & a_r E_{nr} \end{pmatrix}$ $(n_1 + n_2 + \cdots + n_r = n)$

令$X = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1r} \\ X_{21} & X_{22} & \cdots & X_{2r} \\ & \cdots & \cdots & \\ X_{r1} & X_{r2} & \cdots & X_{rr} \end{pmatrix}$

且 X_{ij}为$n_i \times n_j$型才能$AX = XA$分块相乘,应有

$\left.\begin{array}{l} \text{左边}AX\text{第}i\text{块行}j\text{块列为}a_i E_{ni} \cdot X_{ij} = a_i X_{ij} \\ \text{右边}XA\text{第}i\text{块行}j\text{块列为}X_{ij} \cdot a_j E_{nj} = a_j X_{ij} \end{array}\right\}$ $\because i \neq j.a_i \neq a_j$

$\therefore i \neq j$时$, x_{ij} = 0. \therefore$ $X = \begin{pmatrix} X_{11} & & \cdots & \\ & X_{22} & \cdots & \\ & \cdots & \cdots & \\ & & \cdots & X_{rr} \end{pmatrix}$为与$A$同类型的准对角矩阵

P199.T7

①设 $A = \left(a_{ij}\right)_{nxn}.$ $AE_{12} = \begin{pmatrix} 0 & a_{11} & \cdots 0 \\ 0 & a_{21} & \cdots 0 \\ 0 & a_{31} & \cdots 0 \\ 0 & \vdots & \cdots 0 \\ 0 & a_n & \cdots 0 \end{pmatrix}, E_{12}A = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$

$\therefore$A的第一列 $a_{11} = a_{22}$,其余$a_{k1} = 0\left(k > 1\right)$

A的第二行 $a_{22} = a_{11}$,其余$a_{2S} = 0\left(s \neq 2\right)$ $A = \begin{pmatrix} a_{11} & & & * & & \\ 0 & a_{11} & 0 & \cdots & \cdots & 0 \\ 0 & & & & & \\ \vdots & & & * & & \\ 0 & & & & & \end{pmatrix}$

$$AE_{ij}=\begin{pmatrix} & a_{1i} & \\ & a_{2i} & \\ 0 & \vdots & 0 \\ & a_{ni} & \end{pmatrix},E_{ij}A=\begin{pmatrix} 0 & & 0 & 0 \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ 0 & & 0 & 0 \end{pmatrix}i\text{行}$$

②

∴A的第i列：$a_{ii}=a_{jj}$，且$a_{ki}=0$，(k≠i)

A的第j行，$a_{jj}=a_{ii}$且$a_{js}=0$，(s≠j)

③由于A与所有n级矩阵可换，故A与$E_{11},E_{12},E_{13}\cdots E_{1n}$可换

∴A的第一列全为0，$AE_{11}=E_{11}A\Rightarrow A$的第一行只留下$a_{11}$可解非0

$AE_{12}=E_{12}A\Rightarrow A$的第二行只留下$a_{22}=a_{11}$其余全为0

$AE_{13}=E_{13}A\Rightarrow A$的第三行只留下$a_{33}=a_{11}$，其余全为0

$AE_{1n}=E_{1n}A\Rightarrow A$的第n行只留下$a_{nn}=a_{11}$. 其余全为0

所以 $$A=\begin{pmatrix} a_{11} & & & 0 \\ & a_{11} & & \\ & & a_{11} & \\ 0 & & & a_{11} \end{pmatrix}=aE \qquad (a=a_{11})$$

P200. T8

$$A(B+C)=AB+AC=BA+CA=(B+C)A$$

$$A(BC)=(AB)C=(BA)C=B(AC)=BC(A)=BC(A)=(BC)A$$

P200. T9

$$"\Rightarrow"\text{若}A^2=A,\text{则}\frac{1}{4}\left(B^2+2B+E\right)=\frac{1}{2}\left(B+E\right)\Rightarrow\frac{1}{4}B^2-\frac{1}{4}E=0\text{得 } B^2=E$$

$$"\Leftarrow"\text{若}B^2=E,\text{则}A^2=\frac{1}{4}\left(B^2+2B+E\right)=\frac{1}{4}\left(E+2B+E\right)=\frac{1}{2}\left(B+E\right)=A$$

P200. T10

反设$A\neq 0$,不防设$a_{st}\neq 0$,那么$a_{ts}\neq 0$,那么A^2中第s行s列的元素为

$$\sum_{k=1}^{n}a_{sk}a_{ks}=\sum_{k=1}^{n}a_{sk}a_{sk}=a_{s1}^2+a_{s2}^2+a_{st}^2+\cdots+a_{sn}^2>0.$$

$\therefore A^2\neq 0$，矛盾，即$A=0$。

P200. T11

$$"\Rightarrow"(AB)'=AB\Rightarrow AB=(AB)'=B'A'=BA(\because B'=B,A'=A)$$

$$"\Leftarrow"\text{如果}AB=BA,\text{那么}(AB)'=B'A'=BA=AB,\text{为对称矩阵}$$ 。

P200.T12

设A=B+C, $\left(B'=B,C'=-C\right)$

$\therefore A'=B'+C'=B-C$ $\therefore B=\frac{1}{2}\left(A+A'\right),C=\frac{1}{2}\left(A-A'\right)$

恰如 $B'=B,C'=-C,$即为所求.

P200.T13

令 $D=\begin{pmatrix} 1 & 1 & & 1 \\ x_1 & x_2 & \vdots & x_n \\ x_1^2 & x_2^2 & \vdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & & x_n^{n-1} \end{pmatrix},D'=\begin{pmatrix} 1 & x_1 & x_1^2 & & x_1^{n-1} \\ 1 & x_2 & x_2^2 & & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & & x_n^{n-1} \end{pmatrix}$

$\therefore DD'=\left(a_{ij}\right)_{nxn}=A,a_{ij}=\sum_{k=1}^{n}x_k^{i-1+j-1}=s_{i+j-2}$

$\therefore \left|A\right|=\left|\left(a_{ij}\right)\right|=\left|DD'\right|=\left|D\right|^2=\prod_{1\le i\le j\le n}\left(x_j-x_i\right)^2$

P200.T14

$"\Rightarrow"$取B_i为B的一个非0列 $\therefore AB_i=0,$而$AX=0$有非零解.故$\left|A\right|=0$.

$"\Leftarrow"\because \left|A\right|=0$ $\therefore AX=0$有非零解 $x_0\neq 0,$令$B_1=x_0,B_2=2x_0,\cdots B_n=nx_0$

而B的列由$B_1,B_2\cdots B_n$组成,所以$B\neq 0$

P200.T15

考虑AE.∵E的每一列 E_i 去乘A的各行为 0,∴AE=0

又 AE=A∴A=0

P200.T16.

①考虑齐线方程组, $C'x=0(\because C'_{nxr})$只含r个未知量,

而秩$\left(C'\right)=$秩$\left(C\right)=r=$未知量个数$\therefore C'X=0$ 只有零解

$\because BC=0\Rightarrow C'B'=0'=0$ $\therefore B'$的各列$\left(\text{都是适合}C'X=0\right)$都为0

$\therefore B'=0,B=0$

②若 $BC=C\Rightarrow\left(B-E\right)C=0\Rightarrow B-E=0\Rightarrow B=E$

P200.T17

设 A的行向量为$\alpha_1,\alpha_2\cdots\alpha_s$,（I）,B的行向量为 $\beta_1,\beta_2\cdots\beta_s$（II），而 $C=A+B$的行向量为$\gamma_1,\gamma_2\cdots\gamma_s$,（III）。那么

$r_1=\alpha_1+\beta_1, r_2=\alpha_2+\beta_2\cdots, r_m=\alpha_m+\beta_m$。

∴设 $\alpha_{i1},\cdots\alpha_{ir}$ (I)′ 为（Ⅰ）的极大无关组，那么秩（A）=秩（Ⅰ）=r

设 $\beta_{j1},\cdots\beta_{jp}$（Ⅱ′）为（Ⅱ）的极大无关组，那么秩（A）=秩（Ⅱ）= p

∴(Ⅲ)←（Ⅰ)U(Ⅱ) ←（Ⅰ）′U(Ⅱ)′= $\left\{\alpha_{i1},\cdots\alpha_{ir},\beta_{i1},\cdots\beta_{jp}\right\}\cdots$ (Ⅳ)

∴秩（A+B）=秩（C）=秩（Ⅲ）≤秩（Ⅳ）≤r+p=秩（A）+秩（B)。

P200.T18

设秩（A）=r,那么，线性方程组AX=0 的基础解系可设为 $\eta_1,\eta_2\cdots\eta_{n-r}$。

设B的各列为 $B_1,B_2\cdots\cdots B_m$. ∵AB=0.说明B的每列 B_j 乘以A的每行都为 0，即时 B_j 是AX=0 的解。∴

$B_j\leftarrow\eta_1,\eta_2\cdots\eta_{n-r}$

$\therefore B_1,B_2\cdots B_n,\leftarrow\eta_1,\eta_2\cdots\eta_{n-r}$

∴秩（B）=秩 $\left(B_1,B_2\cdots B_n\right)\le$ 秩 $\left(\eta_1,\eta_2\cdots\eta_{n-r}\right)=n-r$

$\therefore$ 秩 $(A)+$ 秩 $(B)\le r+n-r=n$

P200.T19

若 $A^k=0$

$\therefore\left(E-A\right)\left(E+A+A^2+\cdots+A^{k-1}\right)=E+A+A^2+\cdots+A^{k-1}-A-A^2-\cdots-A^{K-1}-A^k$

$=E-A^k=E-0=E$

$\therefore\left(E-A\right)^{-1}=E+A+A^2+\cdots+A^{k-1}$

P201.T20

① $A^*=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix},|A|=1$ ∴ $A^{-1}=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$

② $A=\left(\begin{array}{cc|c} 1 & 1 & -1\\ \hline 2 & 1 & 0\\ 1 & -1 & 0\end{array}\right)=\begin{pmatrix} A_1 & A_2\\ A_3 & 0\end{pmatrix}$, 令 $A^{-1}=X=\begin{pmatrix} X_1 & X_2\\ X_3 & X_4\end{pmatrix}$

∴ $AX=\begin{pmatrix} A_1X_1+A_2X_3 & A_1X_2+A_2X_4\\ A_3X_1 & A_3X_2\end{pmatrix}$

$A_3X_1=0\Rightarrow\because A_3^{-1}$ 存在 $\therefore X_1=\begin{pmatrix}0\\0\end{pmatrix}$

$A_3X_2=E_2\Rightarrow X_2=A_3^{-1}=-\frac{1}{3}\begin{pmatrix}-1 & -1\\ -1 & 2\end{pmatrix}$

$A_1X_1+A_2X_3=E_1\Rightarrow A_2X_3=E_1\Rightarrow X_3=-1$

而$A_1X_2+A_2X_4=0\Rightarrow X_4=-A_2^{-1}A_1X_2$

$$X_4=-(-1)(1,1)\left(-\frac{2}{3}\right)\begin{pmatrix}-1&-1\\-1&2\end{pmatrix}=\left(\frac{2}{3},-\frac{1}{3}\right)\quad\therefore\quad A^{-1}=\begin{pmatrix}0&\frac{1}{3}&\frac{1}{3}\\0&\frac{1}{3}&-\frac{2}{3}\\1&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$$

③ $$|A|=\begin{vmatrix}2&2&3\\1&-1&0\\-1&2&1\end{vmatrix}=-2+0+6-3-0-2=-1$$即 $$\begin{matrix}A_{11}=-1&A_{21}=4&A_{31}=3\\A_{12}=-1&A_{22}=5&A_{32}=3\\A_{13}=1&A_{23}=-6&A_{33}=-4\end{matrix}$$

$$\therefore A^*=\begin{pmatrix}-1&4&3\\-1&5&3\\1&-6&-4\end{pmatrix}\qquad\therefore A^{-1}=|A|A^{-1}=\begin{pmatrix}1&-4&-3\\1&-5&-3\\-1&6&4\end{pmatrix}$$

④ A= $$\left(\begin{array}{cc|cc}1&2&3&4\\2&3&1&2\\\hline 3&1&1&-1\\1&0&-2&-6\end{array}\right)\quad A_1=\begin{pmatrix}1&2\\2&3\end{pmatrix}A_2=\begin{pmatrix}3&4\\1&2\end{pmatrix}\quad A_2=\begin{pmatrix}1&1\\1&0\end{pmatrix}A_4=\begin{pmatrix}1&-1\\-2&-6\end{pmatrix}$$ A_1^{-1}= $\begin{pmatrix}-3&2\\2&-1\end{pmatrix}$

$$\begin{pmatrix}E&O\\-A_3A_1^{-1}&E\end{pmatrix}\begin{pmatrix}A_1&A_2\\A_3&A_4\end{pmatrix}=\begin{pmatrix}A_1&A_2\\0&A_4-A_3A^{-1}{}_1A_2\end{pmatrix}$$ $$A_3A_1^{-1}=\begin{pmatrix}-1&1\\-3&2\end{pmatrix}\quad A_4-A_3A_1^{-1}A_2=\begin{pmatrix}3&1\\5&2\end{pmatrix}=B_4$$

$$\therefore\begin{pmatrix}A_1&A_2\\A_3&A_4\end{pmatrix}^{-1}=\begin{pmatrix}A_1&A_2\\o&B_4\end{pmatrix}^{-1}\begin{pmatrix}E&o\\-A_3A^{-1}{}_1&E\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&3&1&2\\0&0&3&1\\0&0&5&2\end{pmatrix}^{-1}\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&-1&1&0\\3&-2&0&1\end{pmatrix}$$

A^{-1}= $$\begin{pmatrix}-3&2&-26&17\\2&-1&20&-13\\0&0&2&-1\\0&0&-5&3\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&-1&1&0\\3&-2&0&1\end{pmatrix}=\begin{pmatrix}22&-6&-26&17\\-17&5&20&-13\\-1&0&2&-1\\4&-1&-5&3\end{pmatrix}$$

5 法 1：$A^2=4E$ $\therefore A^{-1}=\frac{1}{4}A$

法 2：$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 00 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$

$$\to \left(\begin{array}{cccc|cccc} 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & -1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 2 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & -2 & 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 2 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -4 & -1 & 1 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 4 & 0 & 0 & 4 & 2 & 0 & 0 & 2 \\ 0 & 4 & 0 & 4 & 2 & 0 & -2 & 0 \\ 0 & 0 & 4 & -4 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 4 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 4 & 1 & -1 & -1 & 1 \end{array}\right) \Rightarrow \left(E \mid \frac{1}{4}(\ \ )\right)$$

$\therefore A^{-1} = \frac{1}{4}(\ \ ) = \frac{1}{4}A$

⑥A= $A = \begin{pmatrix} 3 & 3 & -4 & -3 \\ 0 & 6 & 1 & 1 \\ 5 & 4 & 2 & 1 \\ 2 & 3 & 3 & 2 \end{pmatrix} (A,E) \to \left(\begin{array}{cccc|cccc} 1 & 0 & -7 & -5 & 1 & 0 & 0 & -1 \\ 0 & 6 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 37 & 26 & -5 & 0 & 1 & 5 \\ 0 & 3 & 17 & 12 & -2 & 0 & 0 & 3 \end{array}\right)$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & -7 & -5 & 1 & 0 & 0 & -1 \\ 0 & 1 & 20 & 14 & -3 & 0 & 1 & 2 \\ 0 & 0 & -33 & -23 & 4 & 1 & 0 & -6 \\ 0 & 0 & -43 & -30 & 7 & 0 & -3 & -3 \end{array}\right) \to \cdots\cdots \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -7 & 5 & 12 & -14 \\ 0 & 1 & 0 & 0 & 3 & -2 & -5 & -8 \\ 0 & 0 & 1 & 0 & 41 & -30 & -69 & 111 \\ 0 & 0 & 0 & 1 & -59 & 43 & 99 & 159 \end{array}\right)$$

$A^{+}= \begin{pmatrix} -1 & 5 & 12 & -19 \\ 3 & -2 & -5 & 8 \\ 41 & -30 & -69 & 111 \\ -59 & 43 & 99 & -159 \end{pmatrix}$

7
$$A=\left(\begin{array}{cc|cc}1&3&-5&7\\0&1&2&-3\\\hline 0&0&1&2\\0&0&0&1\end{array}\right)\quad A^{-1}=\begin{pmatrix}\begin{pmatrix}1&3\\0&1\end{pmatrix}^{-1} & -\begin{pmatrix}1&3\\0&1\end{pmatrix}\begin{pmatrix}-5&7\\2&-3\end{pmatrix}\begin{pmatrix}1&-2\\0&1\end{pmatrix}\\ 0 & \begin{pmatrix}1&2\\0&1\end{pmatrix}^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}1&-3&11&-38\\0&1&-2&7\\0&0&1&-2\\0&0&0&1\end{pmatrix}$$

8
$$A=\left(\begin{array}{cc|cc}2&1&0&0\\3&2&0&0\\\hline 5&7&1&8\\-1&-3&-1&-6\end{array}\right)\Rightarrow\quad A^{-1}=\begin{pmatrix}\begin{pmatrix}2&1\\3&2\end{pmatrix}^{-1} & 0\\ -\begin{pmatrix}1&8\\-1&-6\end{pmatrix}^{-1}\begin{pmatrix}5&7\\-1&-3\end{pmatrix}\begin{pmatrix}2&1\\3&2\end{pmatrix}^{-1} & \begin{pmatrix}1&8\\-1&-6\end{pmatrix}^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}\begin{matrix}2&-1\\-3&2\end{matrix} & 0\\ \begin{pmatrix}-5&7\\2&-2\end{pmatrix} & \frac{1}{2}\begin{pmatrix}-6&-8\\1&1\end{pmatrix}\end{pmatrix}=\begin{pmatrix}2&-1&0&0\\-3&2&0&0\\-5&-7&-3&-4\\2&-2&\frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

⑨

$$\left(\begin{array}{cccc|cccc}0&0&1&-1&1&0&0&0\\0&3&1&4&0&1&0&0\\2&7&6&-1&0&0&1&0\\1&2&2&-1&0&0&0&1\end{array}\right)\to\left(\begin{array}{cccc|cccc}1&2&2&-1&0&0&0&1\\0&3&1&4&0&1&0&0\\0&0&1&-1&1&0&0&0\\0&3&2&1&0&0&1&-2\end{array}\right)$$

$$\to\left(\begin{array}{cccc|cccc}1&2&2&-1&0&0&0&1\\0&3&1&4&0&1&0&0\\0&0&1&-1&1&0&0&0\\0&0&1&3&0&-1&1&-2\end{array}\right)\to\left(\begin{array}{cccc|cccc}1&2&0&1&-2&0&0&1\\0&3&0&5&-1&1&0&0\\0&0&1&-1&1&0&0&0\\0&0&0&-2&-1&-1&1&-2\end{array}\right)$$

$$\to\left(\begin{array}{cccc|cccc}1&2&0&0&-5/2&-1/2&1/2&0\\0&3&0&0&-7/2&-3/2&5/2&-5\\0&0&1&0&3/2&1/2&-1/2&1\\0&0&0&1&1/2&1/2&-1/2&1\end{array}\right)$$

$$\rightarrow\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1/6 & 1/2 & -7/6 & 10/3 \\ 0 & 1 & 0 & 0 & -7/6 & -1/2 & 5/6 & 5/3 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 & -3/2 & 1 \\ 0 & 0 & 0 & 1 & 1/2 & 1/2 & -1/2 & 1 \end{array}\right)$$

∴$A^{-1}=\frac{1}{6}\begin{pmatrix} -1 & 3 & -7 & 20 \\ -7 & -3 & 5 & -10 \\ 9 & 3 & -3 & 6 \\ 3 & 3 & -3 & 6 \end{pmatrix}$

⑩ 求A^{-1}　　A=

方法 1:令B=$\frac{1}{2}\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$

∵$B^{-1}=(E-C)^{-1}=E+C+C^2+\ C^3+\ C^4$

$A^{-1}=(2B)^{-1}=\frac{1}{2}B^{-1}=\frac{1}{2}C+\frac{1}{2}C^2+\frac{1}{2}C^3+\frac{1}{2}C^4=\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{32} \\ & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} \\ & & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ & & & \frac{1}{2} & -\frac{1}{4} \\ & & & & \frac{1}{2} \end{pmatrix}$

方法 2:A=$\left(\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ \hline & & 2 & 1 & 0 \\ & & 0 & 2 & 1 \\ & & 0 & 0 & 2 \end{array}\right)=\begin{pmatrix} B & D \\ O & C \end{pmatrix}$

$$A^{0}=\begin{pmatrix} B^{-1} & -B^{-1}OC^{-1} \\ O & C^{-1} \end{pmatrix}=\left(\begin{array}{cc|ccc} \frac{1}{2} & -\frac{1}{4} & -B^{-1} & D & C \\ 0 & \frac{1}{2} & & & \\ \hline & & \frac{1}{2} & -\frac{1}{4} & 8 \\ 0 & & & \frac{1}{2} & -\frac{1}{4} \\ & & & & \end{array}\right)$$

$$=\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{32} \\ 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

P201.T21

设 A_{kxk}，C_{rr} 则 $X=\begin{pmatrix} 0 & A \\ C & 0 \end{pmatrix}$ 令 $Y=\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$ 才能乘

XY 与 YX，而 $XY=\begin{pmatrix} AY_3 & AY_4 \\ CY_1 & CY_2 \end{pmatrix}$ $YX=\begin{pmatrix} Y_2C & Y_1A \\ Y_4C & Y_3A \end{pmatrix}$

若 $Y=X^{-1}$，则 $XY=YX=E\Rightarrow \begin{matrix} CY_1=0, Y_1A=0\Rightarrow Y_1=0 \\ AY_4=0, Y_4C=0\Rightarrow Y_4=0 \end{matrix}$

$\therefore \begin{matrix} AY_3=Y_3A=E_k & Y_3=A^{-1} \\ CY_2=Y_2C=E_r & Y_2=C^{-1} \end{matrix}$ $\therefore$ $X^{-1}=\begin{pmatrix} 0 & C^{-1} \\ A^{-1} & 0 \end{pmatrix}$

P201.T22

将 $X=\begin{pmatrix} 0 & A \\ a_n & 0 \end{pmatrix}$ $A=\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{n-1} \end{pmatrix}$ 由 21 题，（见上面）

$$X^{-1}=\begin{pmatrix}0 & a^{n-1}\\ A^{-1} & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & \cdots & 0 & a_n^{-1}\\ a_1^{1} & 0 & & & 0\\ 0 & a_2^{-1} & & & 0\\ & & \cdots & & \\ 0 & & 0 & a_{n-1}^{-1} & 0\end{pmatrix}$$

P202. T23.

①∵ $\begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}^{-1}=\begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}$

∴ $X=\begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}^{-1}\begin{pmatrix}4 & -6\\ 2 & 1\end{pmatrix}=\begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}\begin{pmatrix}4 & -6\\ 2 & 1\end{pmatrix}=\begin{pmatrix}2 & -23\\ 0 & 8\end{pmatrix}$

②解

$$\left(\begin{array}{ccc|ccc}1 & 1 & -1 & 1 & -1 & 1\\ 0 & 2 & 2 & 1 & 1 & 0\\ 1 & -1 & 0 & 2 & 1 & 1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1 & 1 & -1 & 1 & -1 & 1\\ 0 & 2 & 2 & 1 & 1 & 0\\ 0 & -2 & 1 & 1 & 2 & 0\end{array}\right)$$

$$\to\left(\begin{array}{ccc|ccc}1 & 0 & -2 & \frac{1}{2} & -\frac{3}{2} & 1\\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 3 & 2 & 3 & 0\end{array}\right)\to\left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{11}{6} & \frac{1}{2} & 1\\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{2} & 0\\ 0 & 0 & 1 & \frac{2}{3} & 1 & 0\end{array}\right)$$

∴ $x=A^{-1}B=\begin{pmatrix}11/6 & 1/2 & 1\\ -1/6 & -1/2 & 0\\ 2/3 & 1 & 0\end{pmatrix}$

③∵AX=B，则X=A^{-1}B，故 $(A,B)=\left(\begin{array}{cccccc|cccccc}1 & 1 & 1 & \cdots & 1 & 1 & 2 & 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 2 & 1 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 1 & 1 & 0 & 1 & 2 & \cdots & 0 & 0\\ & & & & & \cdots & \cdots & & & & & \\ 0 & 0 & \cdots & & 0 & 1 & 0 & 0 & \cdots & & 1 & 2\end{array}\right)$

所以 $X=A^{-1}B=\begin{pmatrix}1 & -1 & -1 & & 0\\ 1 & 1 & -1 & \ddots & \\ & \ddots & \ddots & \ddots & -1\\ & & \ddots & 1 & -1\\ & 0 & & 1 & 2\end{pmatrix}$

4 $A=\begin{pmatrix}1&1&-1\\0&2&2\\1&-1&0\end{pmatrix}$, $|A|=6.$ $\begin{matrix}A_{11}=2 & A_{21}=1 & A_{31}=4\\ A_{12}=2 & A_{22}=1 & A_{32}=-2\\ A_{13}=-2 & A_{23}=2 & A_{33}=2\end{matrix}$

∴ $X=\begin{pmatrix}1&-1&1\\1&1&0\\2&1&1\end{pmatrix}.\frac{1}{6}\begin{pmatrix}2&1&4\\2&1&-2\\-2&-2&2\end{pmatrix}=\frac{1}{6}\begin{pmatrix}-2&2&8\\4&2&2\\4&5&8\end{pmatrix}$

P202.T24

①∵ $AA^{-1}=A^{-1}A=E$ ∴ $A'(A^{-1})'=(A^{-1})'A'=E$ ∴ $(A')^{-1}=(A^{-1})'$

若 $A'=A$,则$(A^{-1})'=(A')^{-1}=A^{-1}$,即$A^{-1}$对称

若 $A'=-A$,则$(A^{-1})'=(A')^{-1}=(-A)^{-1}=-(A^{-1})$即$A^{-1}$反对称

②若 $A'=-A$,那么,由$|kA|=k^n|A|$

∴ $|A|=|A'|=|A|=(-1)^n|A|=-|A|$ ∴ $|A|=0$

于是 A 不可逆。

P202.T25

①若A,B上三角形，则 $A=(a_{ij})$, $B=(b_{ij})$,当$i>j$时,$a_{ij}=0,b_{ij}=0$

∴ 当$i>j$ 时, $c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=\sum_{k=1}^{i-1}a_{ik}b_{kj}+\sum_{k=i}^{n}a_{ik}b_{kj}=\sum_{k=1}^{i-1}0\cdot b_{kj}+\sum_{k=i}^{n}a_{ik}0=0$

∴C=AB 为上三角

若A,B为下三角形,则$A=(a_{ij}),B=(b_{ij})$, 当$i<j$时,$a_{ij}=0,b_{ij}=0.$ $C=AB.C=(c_{ij})_{n\times n}$

∴ 当$i<j$时,$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=\sum_{k=1}^{i}a_{ik}b_{kj}+\sum_{k=i+1}^{n}a_{ik}b_{kj}=\sum_{k=1}^{i}a_{ik}0+\sum_{k=i+1}^{n}0b_{kj}=0+0=0$

∴C=AB 为下三角

② (i)若A为上三角,考虑$|A|$中$A_{ij},(i<j)$

∴

$$A_{ij}=(-1)^{i+j}M_{ij}=(-1)^{i+j}\begin{vmatrix} a_{11} & \cdots & a_{1i-1} & a_{1i} & \cdots & a_{1j-1} & a_{ij+1} & \cdots & a_{1n} \\ & \ddots & a_{i-1,i-1} & a_{i-1i} & \cdots & a_{i-1j-1} & a_{i-1j+1} & \cdots & a_{i-1n} \\ & & 0 & & a_{i+1i+1} & a_{i+1j-1} & a_{i+1j+1} & \cdots & a_{i+1n} \\ & & & 0 & \cdots & \cdots & & & \\ & & & & \ddots & a_{ij-1} & a_{i+1j+1} & \cdots & a_{j-1n} \\ & & & & & 0 & a_{ji+1} & \cdots & a_{jn} \\ & & & & & & a_{j+1j+1} & \cdots & a_{j+1n} \\ & & & & & & \cdots & & \\ & & & & & & \cdots & & \\ & & & & & & & & a_{nn} \end{vmatrix}=0$$

而$i<j$, A_{ij}位于A^*的对角线下方, $\therefore A^*$上三角，故A^{-1}上三角

$\because$当A为下三角时，A^T上三角$\therefore$（A^T）$^{-1}$为上三角，即（A^{-1}）^{T}为上三角，故A^{-1}为下三角。

P202.T26

$\because AA^*=A^*A=|A|E.\ \therefore |A^*||A|=|A|^n$

若$|A|\neq 0,\ \therefore |A^*|=|A|^{n-1}$

若$|A|=0$,由$P205.18$题$(P4.51.3.2).AA^*=0$ $\therefore$秩$(A)+$秩$(A^*)\leq 1$

(i)若秩$(A)=0\Rightarrow A=0\Rightarrow A^*=0\Rightarrow |A^*|=0$ $\therefore |A^*|=|A|^{n-1}(\because n\geq 2)$

(ii)若秩$(A)\neq 0\Rightarrow$秩$(A^*)<n\Rightarrow |A^*|=0\Rightarrow |A^*|=|A|^{n-1}=0$

总之，各种情形均有$|A^*|=|A|^{n-1}$

P202.T28

① (AE)=$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)\to\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -2 & -1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -2 & -1/2 & 1/2 & 1/2 & -1/2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 & 1/4 & -1/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & 0 & 1/4 & 1/4 & -1/4 & -1/4 \\ 0 & 0 & 0 & 1 & 1/4 & -1/4 & -1/4 & 1/4 \end{array}\right) \quad \therefore A^{-1}= 1/4\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

②A=$\begin{pmatrix} B & B \\ B & -B \end{pmatrix}$ B=$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $B^{-1}=-\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}=\frac{1}{2}B$

$$\begin{pmatrix} E & O \\ -E & E \end{pmatrix}\begin{pmatrix} B & B \\ B & -B \end{pmatrix}=\begin{pmatrix} B & B \\ O & -2B \end{pmatrix}$$ 而 $$\begin{pmatrix} B & B \\ O & -2B \end{pmatrix}^{-1}=\begin{pmatrix} B^{-1} & \frac{1}{2}B \\ 0 & (-2B)^{-1} \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} \frac{1}{2}B & \frac{1}{4}\mathbf{B} \\ 0 & -\frac{1}{4}B \end{pmatrix}\begin{pmatrix} E & 0 \\ -E & E \end{pmatrix}=\begin{pmatrix} \frac{1}{4}B & \frac{1}{4}B \\ \frac{1}{4}B & -\frac{1}{4}B \end{pmatrix}=\frac{1}{4}A$$

方法③：∵$A^2=4A$ ∴$A^{-1}=\frac{1}{4}A$

P203. T29∵

$$\begin{pmatrix} E_m & 0 \\ -A & E \end{pmatrix}\begin{pmatrix} E_m & B \\ A & E_n \end{pmatrix}=\begin{pmatrix} E_m & B \\ 0 & E_n-AB \end{pmatrix}, \text{即} \begin{pmatrix} E_m & 0 \\ A & E_n \end{pmatrix}\begin{pmatrix} E_m & 0 \\ -A & E_n \end{pmatrix}=\begin{pmatrix} E_m-BA & B \\ 0 & P \end{pmatrix}$$

$$\therefore \begin{vmatrix} E_m & B \\ A & E_n \end{vmatrix}=|E_m||E_n-AB|=|E_n-AB|=|E_m-BA||E_n|=|E_m-BA|$$

P203. T30 $\begin{pmatrix} E_m & B \\ A & \lambda E_n \end{pmatrix}$则$\begin{pmatrix} \lambda E_m & 0 \\ -A & E_n \end{pmatrix}\begin{pmatrix} E_m & B \\ A & \lambda E_n \end{pmatrix}=\begin{pmatrix} \lambda E_n & \lambda B \\ 0 & \lambda E_n-\lambda B \end{pmatrix}$

又 $\begin{pmatrix} E_m & B \\ A & \lambda E_n \end{pmatrix}\begin{pmatrix} \lambda E_m & 0 \\ -A & E_n \end{pmatrix}=\begin{pmatrix} \lambda E_n-\lambda B & B \\ 0 & \lambda E_n \end{pmatrix}$

∴ $\lambda_m\left|\lambda E_n-AB\right|=\left|\lambda E_n\right|\left|\lambda E_n-AB\right|=\begin{vmatrix} \lambda E_m & \lambda B \\ 0 & \lambda E_m-\lambda B \end{vmatrix}=\begin{vmatrix} \lambda E_m & 0 \\ -A & E_n \end{vmatrix}\begin{vmatrix} E_m & B \\ A & \lambda E_n \end{vmatrix}$

$=\begin{vmatrix} \lambda E_m-BA & B \\ 0 & \lambda E_n \end{vmatrix}=\left|\lambda E_m-BA\right|\left|\lambda E_n\right|=\left|\lambda E_m-BA\right|$

∴ $\left|\lambda E_m-AB\right|=\lambda^{n-m}\left|\lambda E_m-BA\right|$

第五章 二次型习题解答

P232.T1

（Ⅰ）②）化标准形，$f=x_1^2+2x_1x_2+2x_2^2+4x_2x_3+4x_3^2$

解：$f=(x_1+x_2)^2+x_2^2+4x_2x_3+4x_2^3$

$=(x_1+x_2)^2+(x_2+2x_3)^2+0$

令 $\begin{cases} y_1=x_1+x_2 \\ y_2=x_2+2x_3 \\ y_3=x_3 \end{cases}$ 即 $\begin{cases} x_1=y_1-y_2+2y_3 \\ x_2=y_2-2y_3 \\ x_3=y_3 \end{cases}$

则$f=y_1^2+y_2^2$

用矩阵验算 $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}'\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

（Ⅰ）③化标准形$f=x_1^2-3x_2^2-2x_1x_2+2x_1x_3-6x_2x_3$

解：$f=(x_1-x_2+x_3)^2-(x_2-x_3)^2-3x_2^2-6x_2x_3$

$=(x_1-x_2+x_3)^2-4x_2^2-4x_2x_3-x_3^2$

$=(x_1-x_2+x_3)^2-(2x_2+x_3)^2$

令 $\begin{cases} y_1=x_1-x_2+x_3 \\ y_2=2x_2+x_3 \\ y_3=x_3 \end{cases}$ 即 $\begin{cases} x_1=y_1+\frac{1}{2}y_2-\frac{3}{2}y_3 \\ x_2=\frac{1}{2}y_2-\frac{1}{2}y_3 \\ x_3=y_3 \end{cases}$

则$f=y_1^2-y_2^2$

$$\begin{pmatrix} 1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -1 & -3 & -3 \\ 1 & -3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

验算有：

（Ⅰ）④化标准形 $f=8x_1x_4+2x_3x_4+2x_2x_3+8x_2x_4$

解 $\begin{cases} x_1 = y_1 + y_4 \\ x_2 = y_2 \\ x_3 = y_3 \\ x_4 = y_1 - y_4 \end{cases}$ 则 $x=c_1yq$

$f=8(y^2_1-y^2_4)+2y_3(y_1-y_4)+2y_2y_3+8y_2(y_1-y_2)$

$=8y^2_1-8y^2_4+8y_1y_2+2y_1y_3+2y_2y_3-8y_2y_4-2y_3y_4$

$$\therefore f=8(y_1+\frac{1}{2}y+\frac{1}{8}y_3)^2-8(\frac{1}{2}y_2+\frac{1}{8}y_3)^2-8y_4^2+2y_2y_3-8y_2y_4-2y_3y_4$$

$$=8(y_1+\frac{1}{2}y_2+\frac{1}{8}y_3)^2-2(y_2-\frac{1}{4}y_3+2y_4)+2(-\frac{1}{4}y_3+2y_4)-\frac{1}{8}y_3^2-8y_4^2-2y_3y_4$$

$$=8(y_1+\frac{1}{2}y_2+\frac{1}{8}y_3)^2-2(y_2-\frac{1}{4}y_3+2y_4)^2-4y_3y_4$$

$$=8(y_1+\frac{1}{2}y_2+\frac{1}{8}y_3)^2-2(y_2-\frac{1}{4}y_3+2y_4)^2+(y_3-y_4)^2-(y_3+y_4)^2$$

令

$$\begin{cases} z_1 = y_1+\frac{1}{2}y_2+\frac{1}{8}y_3 \\ z_2 = y_2-\frac{1}{4}y_3+2y_4 \\ z_3 = y_3 - y_4 \\ z_4 = y_3 + y_4 \end{cases}$$

即 $$\begin{cases} y_1 = z_1-\frac{1}{2}z_2-\frac{5}{8}z_3+\frac{3}{8}z_4 \\ y_2 = z_2+\frac{9}{8}z_3-\frac{7}{8}z_4 \\ y_3 = \frac{1}{2}z_3+\frac{1}{2}z_4 \\ y_4 = -\frac{1}{2}z_3+\frac{1}{2}z_4 \end{cases}$$

则 $f=8z_1^2-2z_2^2+z_3^2-z_4^2$

矩阵验算略

（Ⅰ）⑤化标准形 $f=x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4$

解：

$$A=\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\left(\frac{A}{E}\right)\xrightarrow{Pi(2)}\begin{pmatrix}0&2&2&2\\2&0&2&2\\2&2&0&2\\2&2&2&0\\2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}\to\begin{pmatrix}4&2&4&4\\2&0&2&2\\4&2&0&2\\4&2&2&0\\2&0&0&0\\2&2&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}\to\begin{pmatrix}4&0&0&0\\0&-1&0&0\\0&0&-4&-2\\0&0&-2&-4\\2&-1&-2&-2\\2&1&-2&-2\\0&0&2&0\\0&0&0&2\end{pmatrix}$$

$$\to\begin{pmatrix}4&0&0&0\\0&-1&0&0\\0&0&-4&0\\0&0&0&-3\\2&-1&-2&-1\\2&1&-2&-1\\0&0&2&-1\\0&0&0&2\end{pmatrix}\quad\therefore\quad X=\begin{pmatrix}2&-1&-2&-1\\2&1&-2&-1\\0&0&2&-1\\0&0&0&2\end{pmatrix}y$$

则 $f=4y_1^2-y_2^2-4y_3^2-3y_4^2$

（Ⅰ）⑦化标准形 $f=x_1^2+x_2^2+x_3^2+x_4^2+2x_1x_2+2x_2x_3+2x_3x_4$

解：$A=\begin{pmatrix}1&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\end{pmatrix}$ 则 $\left(\frac{A}{E}\right)\to\begin{pmatrix}1&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&1&1\\0&0&1&1\\1&-1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

$$\xrightarrow{P(3,(-1))}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\\1&-1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&2&2&0\\0&2&0&0\\0&0&0&1\\0&-1&0&0\\0&1&0&0\\0&1&2&0\\0&-1&-2&1\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&-2&0\\0&0&0&1\\1&-1&1&0\\0&1&-1&0\\0&1&1&0\\0&-1&-1&1\end{pmatrix}$$

即令X=$\begin{pmatrix}1&-1&1&0\\0&1&-1&0\\0&1&1&0\\0&-1&-1&1\end{pmatrix}Y$

则 $f=y_1^2+2y_2^2-2y_3^2+y_4^2$

P233,T2

设秩（*A*）=*r*，则存在 *C* 满秩

$$C'AC=D=\begin{pmatrix}d_1&&&&&&\\&d_2&&&&&\\&&\ddots&&&&\\&&&d_r&&&\\&&&&0&&\\&&&&&\ddots&\\&&&&&&0\end{pmatrix}=\sum_{i=1}^{r}d_iE_{ii}$$

那么，$d_1E_{11},d_2E_{22},\dots d_rE_{rr}$的秩都等于 1，且为对称的。

$$\therefore A=(c')^{-1}\begin{pmatrix}\frac{r}{z}d_iE_{ii}\\ i=1\end{pmatrix}C^{-1}$$

$$A=(C')^{-1}(\sum_{i=1}^{r}d_iE_{ii})C^{-1}$$

$$=(C^{-1})'(\sum_{i=1}^{r}d_iE_{ii})C^{-1}$$

$$=\sum_{i=1}^{r}(C^{-1})'(d_iE_{ii})C^{-1}=\sum_{i=1}^{r}B_i$$

其中 $B_i=(c^{-1})'(d_iE_{ii})c^{-1}$

秩$(B_i)=$秩$(d_iE_{ii})=1$， $B_i'=(c^{-1)})'(d_iE_{ii})'(c^{-1})'=B_i$

$\therefore$ A为r个秩为 1 的，对称阵之和。

P233.T3

证明一:设 $A=\begin{pmatrix}\lambda_1&&&\\&\lambda_2&&\\&&\ddots&\\&&&\lambda_n\end{pmatrix}$ $B=\begin{pmatrix}\lambda_{i1}&&&\\&\lambda_{i2}&&\\&&\ddots&\\&&&\lambda_{in}\end{pmatrix}$

则 $A=\sum_{i=1}^{n}\lambda_iE_{ii}$ $B=\sum_{j=1}^{n}\lambda_{ij}E_{ij}$

又 $i_1,i_2,\cdots i_n$ 为 $1,2,\cdots n$ 的一个排列，所 $A=\sum_{j=1}^{n}\lambda_{i_j}E_{i_ji_j}$

考虑标准单位向量 $\varepsilon_1,\varepsilon_2,\cdots\varepsilon_n$ 作 $C=(\varepsilon_{i1},\varepsilon_{i2},\cdots\varepsilon_{in})$ 则C的n列线性无关，C可逆，且

$$C=\sum_{j=1}^{n}E_{i_j j}\ ,C^{'}=\sum_{j=1}^{n}E_{ji_j}$$

$$C^{'}AC=(\sum_{j=1}^{n}E_{ji_j})(\sum_{k=1}^{n}\lambda_{i_k}E_{i_k i_k})(\sum_{l=1}^{n}E_{i_l l})$$

$$=(\sum_{j=1}^{n}\lambda_{ij}E_{ji_j})(\sum_{l=1}^{n}E_{i_l l})$$

$$=\sum_{j=1}^{n}\lambda_{ij}E_{jj}$$

故 A 与 B 合同

△证法二（归纳法） n=1,显然，设 n-1 时命题成立。

考虑n情形，设$i_k=n$

1.若k=n,则i_1,…，i_{n-1}为 1，2…n-1 的一个排列，所以

$$\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\simeq\begin{pmatrix}\lambda_{i1} & & \\ & \ddots & \\ & & \lambda_{i1}\end{pmatrix}$$

$$p'(i_k,n)\begin{pmatrix}\lambda_i & & \\ & \ddots & \\ & & \lambda_{i+1}\end{pmatrix}P(i_k,n)=\begin{pmatrix}\lambda'_{i-1} & & & \\ & \ddots & & \\ & & \lambda'_{in-1} & \\ & & & \lambda'_{ik}\end{pmatrix}=B_1$$

2．若 $k<n,$

而i_1…i'_{n-1}为 1.2…n-1 的一个排列，所以

$$C_1'\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_{n-1}\end{pmatrix}C_1=\begin{pmatrix}\lambda'_{i1} & & \\ & \ddots & \\ & & \lambda'_{n-i1}\end{pmatrix}$$

$$\therefore\begin{pmatrix}C_1' & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \lambda_{n-1} & \\ & & & \lambda_n\end{pmatrix}\begin{pmatrix}C_1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\lambda'_i & & & \\ & \ddots & & \\ & & \lambda' i_{n-1} & \\ & & & \lambda_n\end{pmatrix}=B_1(\because\lambda_n=\lambda_{ik})$$

$\therefore A\simeq B_1$， $B\simeq B_1$， $\therefore A\simeq B$

由归纳原理，证明完毕。

$P_{233}4(1)\Rightarrow$ 若$A'=A$,要$X:a=X'AX=X'AX''=X'(-A)X=-a$

$\therefore a=0$,即$X,X'AX=0$

4(2) 令$X=\varepsilon_i$,则$f(\varepsilon_i)=0=\varepsilon_i'A\varepsilon_i=a_{ii}$,即$a_{ii}=0$

又令$X=\varepsilon_i+\varepsilon_j$则$f(\varepsilon_i+\varepsilon_j)=a_{ii}+a_{ij}+a_{ji}+a_{jj}=a_{ij}+a_{ji}=0$

$\therefore a_{ij}=-a_{ji}$,故$A'=-A$,证毕.

若$A'=A$,则$a_{ij}=a_{ji}$

$\therefore$ 作$f(x)=X'AX$, $f(\varepsilon_i)=0\Rightarrow a_{ii}=0(i=1,2,...n)$

又$f(\varepsilon_i+\varepsilon_j)=0=a_{ii}+a_{ij}+a_{ji}+a_{jj}=2a_{ij}$即$a_{jj}=0$

4② $\therefore A=0$

P233. T5

设实对称矩阵A,B秩为r_A,r_B,正惯性指数为P_A,P_B

$\therefore A \simeq B \Leftrightarrow r_A = r_B$且$p_A = p_B$

$\because 0 \le p \le r$,当$r = 0$时 只有$p = 0$此1类

当$r = 1$时 有$p = 0,1$,此2类

当$r = 2$时 有$p = 0,2$,此3类

当$r = n$时 有$p = 0,1,2,...n$,此$n+1$类

共有$1 + 2 + ... + (n+1) = C_{n+2}^2 = \frac{1}{2}(n+1)(n+2)$类

$P_{233}.6$ "$\Leftarrow$" $f = X'AX$, ①若f的秩$=1$,则$X = C_1Y$,C_1可逆. 使

$f = d_1 y_1^2 = (dy_1) \bullet y_1$,其中$dy_1, y_1$都是一次齐次多项式

若

f的秩$=2$.符号差$=0$.则$X = C_2 y$,(C_2可逆)使

$f = d_1 y_1^2 - d_2 y_2^2, (d_1, d_2 > 0) = (\sqrt{d}_1 y_1 + \sqrt{d}_2 y_2)$其中$\sqrt{d}_1 y_1 + \sqrt{d}_2 y_2, \sqrt{d}_2 y_1 - \sqrt{d}_2 y_2$

都是$x_1, x_2, ...x_n$的齐次一次式.

"$\Rightarrow$"设$f(x_1, x_2, ...x_n) = (a_1x_1 + a_2x_2 + ...a_nx_n)(b_1x_1 + b_2x_2 + ... + b_nx_n)$

若$\alpha = (a_1, a_2, ...a_n), \beta = (b_1, b_2, ...b_n)$线性无关, 不妨设$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \ne 0$

令$\begin{cases} y_1 = a_1x_1 + a_2x_2 + ... + a_nx_n \\ y_2 = b_1x_1 + b_2x_2 + ... + b_nx_n \\ .... \\ y_i = x_i \end{cases}$则$f = y_1 y_2$

令$\begin{cases} y_1 = z_1 + z_2 \\ y_2 = z_1 - z_2 \\ ...... \\ y_i = z_i \end{cases}$则$f = z_1^2 - z_2^2$,秩为2 符号差$=0$

若α, β线性相关, 不妨设$\beta = k\alpha$及$a_1 \ne 0$,令$\begin{cases} y_1 = a_1x_1 + a_2x_2 + ...a_nx_n \\ y_2 = x_2 \\ ...... \\ y_n = x_n \end{cases}$

则$f = ky_1^2$,秩为1

$$P_{233}7(1)A=\begin{pmatrix}99 & -6 & 24\\ -6 & 10 & -30\\ 24 & -30 & 71\end{pmatrix}$$

$p_1=99>0, p_2=12834>0, p_3=20-672-672-288-16-1960=-3588<0$

$\therefore A$正定, 二次型也正定.

$$(2)A=\begin{pmatrix}10 & 4 & 12\\ 4 & 2 & -14\\ 12 & -14 & 1\end{pmatrix}$$

$p_1=10>0. p_2=20-16=4>0, p_3=20-672-288-16-1960=-3588<0$

$\therefore A$非正定, 二次型$X'AX$非正定

(3) 判定 $f(x_1,x_2,...x_n)=\sum_{i=1}^{n}x_i^{\,2}+\sum_{1\le i<j\le n}x_ix_j$的正定性

解 $$A=\frac{1}{2}\begin{pmatrix}2 & 1 & 1 & ... & 1\\ 1 & 2 & 1 & ... & 1\\ 1 & 1 & 2 & ... & 1\\ ... & ... & ... & ... & ...\\ 1 & 1 & 1 & ... & 2\end{pmatrix}$$

由公式 $$P_k=\frac{1}{2^k}\begin{vmatrix}2 & 1 & ... & 1\\ 1 & 2 & ... & 1\\ ... & ... & ... & ...\\ 1 & ... & .. & 2\end{vmatrix}_{k阶}=\frac{1}{2K}(2-1)(2+(K-1))=\frac{K+1}{2K}>0$$

故A正定，二次型 $f(x_1,x_2,...x_n)$正定

这里顺便发现一个等式

$$\begin{vmatrix}2 & 1 & 0 & 0 & ... & 0\\ 1 & 2 & 1 & 0 & ... & 0\\ 0 & 1 & 2 & 1 & ... & 0\\ 0 & 0 & 1 & 2 & ... & 0\\ ... & ... & ... & ... & ... & ..\\ 0 & 0 & ... & ... & 1 & 2\end{vmatrix}=\begin{vmatrix}2 & 1 & 1 & ... & 1 & 1\\ 1 & 2 & 1 & ... & 1 & 1\\ 1 & 1 & 2 & ... & 1 & 1\\ ... & ... & ... & ... & ... & ...\\ 1 & 1 & 1 & ... & 2 & 1\\ 1 & 1 & 1 & ... & 1 & 2\end{vmatrix}$$

P_{233}.7④

.判别 $f(x_1,x_2,...x_n)=\sum_{i=1}^{n}x_i^{\,2}+\sum_{i=1}^{n-1}x_ix_{i+1}$,是否正定。

证:
$$A=\begin{pmatrix}1 & \frac{1}{2} & \dots & \dots & 0\\ \frac{1}{2} & 1 & \frac{1}{2} & \dots & \dots\\ \dots & \frac{1}{2} & 1 & \dots & \frac{1}{2}\\ 0 & \dots & \dots & \frac{1}{2} & 1\end{pmatrix}$$

由斜行列式, $P_k = p_k - 1 - \frac{1}{4}p_k - 2, \therefore P_k - \frac{1}{2}P_{k-1} = \frac{1}{2}(p_{k-1})\frac{1}{2_{n-1}}(p_1 - \frac{1}{2}p_0) = \frac{1}{2_k}$

$$\therefore \frac{1}{2}(P_{k-1} - \frac{1}{2}P_{k-1}) = \frac{1}{2}(\frac{1}{2_{n-1})}) = \frac{1}{2_k}$$

$$\therefore \frac{1}{2_{k-2}}(P_2 - \frac{1}{2}P_1) = \frac{1}{2_k}$$

$\therefore P_k = \frac{1}{2_{k-1}}P_1 = \frac{k-1}{2_k}$ $\quad \therefore P_k = \frac{k+1}{2^k} > 0. k = 1, 2, \dots n$

$\therefore A$正定 $\quad Cf(x_1, x_2, \dots x_n)$正定

$$P_{233}8(1)A=\begin{pmatrix}1 & t & 12\\ t & 1 & 2\\ -1 & 3 & 5\end{pmatrix}$$

$$p_1 = 1 > 0, p_2 = 1 - t^2 > 0, p_3 = 5 - 4t - 1 - 5t^2 = 4t - 5t^2 > 0$$

$$\therefore -1 < t < 1\text{且} -\frac{4}{5} < t < 0, \text{即} -\frac{4}{5} < t < 0$$

$$\therefore \text{当} -\frac{4}{5} < t < 0\text{时}, A\text{为正定, 相应二次型也正定.}$$

$P_{233}.8_{②}\ x_1{}^2 + 4x_2{}^2 + x_3{}^2 + 2tx_1x_2 + 10x_1x_3 + 6x_2x_3$

$$A=\begin{pmatrix}1 & t & 5\\ t & 4 & 3\\ 5 & 3 & 1\end{pmatrix}$$

解:

$$P_1 = 1 > 0, P_2 = 4 - t^2 > 0, \therefore t^2 < 4, \therefore -2 < t < 2$$

$$P_3 = 4 + 30t - 100 - 9 - t^2 > 0, \text{即} t^2 - 30t + 105 < 0\text{又因为}15 - 2\sqrt{30} < 15 - 2\times 6 = 3 > 2$$

$\therefore$无公共解

0

即对任何 t 都有主子式大于 0

$P_{233}.9.A$正定$\Leftrightarrow A$的主子式全大于0.

证明:$\Leftarrow$此时A的顺序主子式也大于0 。所以A正定（定理）

$\Rightarrow$任取的$i_1,i_2,\cdots i_k$行，$i_1,i_2,\cdots i_k$列作成一个k阶主子式

$$B=\begin{pmatrix} a_{i_1i_1} & a_{i_1i_2} & \dots & a_{i_1i_k} \\ a_{i_2i_1} & a_{i_2i_2} & \dots & a_{i_2i_k} \\ \dots & \dots & \dots & \dots \\ a_{i_ki_1} & a_{i_ki_2} & \dots & a_{i_ki_k} \end{pmatrix}, P=|B|$$

设 $f=X'AX$,作一个关于,$x_{i_1},x_{i_2},\dots x_{i_k}$的二次型

$g(x_{i_1},x_{i_2},\dots x_{i_k})=f(0,\dots0.x_{i_1},0,x_{i_2},0,x_{i_k},0\dots)$

$$(x_{i_1},x_{i_2},\dots x_{i_k})B\begin{pmatrix} x_{i_1} \\ x_{i_2} \\ \dots \\ \dots \\ x_{i_k} \end{pmatrix}$$

B是g的矩阵，因为任给$(\mathrm{x}_{i_1},x_{i_2},\dots x_{i_k})\neq 0$

$\therefore g(c_{i_1},c_{i_2},\dots c_{i_k})=f(0,\dots0,c_{i_1},0\dots0,c_{i_k},0,\dots)>0$

$\therefore B$为K的正定矩阵,$*$有$|B|>0$

P233,10, 证：设$A=(a_{ij})_{n\times n}$, 那么 $tE+A$的第L个顺序主子式

$$\widetilde{p}_R(t)=\begin{vmatrix} t+a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & t+a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & t+a_{kk} \end{vmatrix},=t_k+b_{k1}t_k-1+\dots b_{kk}$$

是一个t的多项式(函数),且 $\lim\limits_{t\to\infty}\widetilde{p_k}(t)=+\infty,\forall M>0$

$\therefore \exists N_k$,当$t>N_k$后，恒有$\tilde{P}_k(t)>M>0$

取$N_0=\min\{N_1,N_2,\dots,N_n\}$则当$t>N_0$,恒有

$\tilde{P}_1(t)>0$，$\tilde{P}_2(t)>0,\dots\tilde{P}_n(t)\succ 0$

$tE+A$正定

$P_{233}.11.$A正定,证明A^{-1}正定

证: $\therefore A$可逆 CA正定 C存在C C可逆使

$C'AC=E$

$\therefore (C'AC)^{-1}=E^{-1}=E$

$C^{-1}A^{-1}(C')^{-1}=E$,取$G=(C')^{-1}$,那么$G'=((C')^{-1})'=(C'')^{-1}=C^{-1}$

即$\therefore G'A^{-1}G=E$,则$A^{-1}\simeq E,\therefore A^{-1}$正定.

$P_{234}.12$考虑$(tE+A)$,因为t充分大后(10题$P_{5.77.7.2}$).$tE+A>0$

故可设$t_0>0$,且$|t_0E+A|>0$.又因为当t=0时,$|A|<0$所以

$\varepsilon\in(0,t_0),|\varepsilon E+A|=0$,所以有$X\neq 0$,使$(\varepsilon E+A)X=0$

即$X'(\varepsilon E+A)X=0.(\because x\neq 0,\therefore x'x>0.\varepsilon x'x>0)$.得到$X'AX=-\varepsilon X'X<0$

P234.13 证,设 有$f_1=X'AX,f_2=X'BX$

则由A.B正定,有 f_1,f_2 正定,即要 $x\neq 0,X'AX>0,X'BX>0$

作 $f=f_1+f_2=X'(A+B)X$

也有任 $X\neq 0,f=X'AX+X'BX>0$

所以f正定,即(A+B)正定

P$_{234}$.14 $f=X'AX\geq 0\Leftrightarrow$秩$r=$惯性指数$P$

证:设X=CY,使 $f=X'AX=y_1{}^2+y_2{}^2+...+y_p{}^2+y_{p+1}{}^2-...-y_r{}^2$

“充分性 $\Rightarrow$" 若P=r,则负系数平方项不出现

$$X_0=\begin{pmatrix}c_1\\ ...\\ c_n\end{pmatrix}\neq 0,\text{必有}y_0=C^{-1}X_0,f\text{在}X_0\text{的值为}$$

$$\therefore f_1x=x_0=X'_0AX_0=y_1{}^2+...+y_r{}^2\geq 0$$

任取$\therefore f$半正定

$$p<r,\text{取}y_0=\begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}=\varepsilon_r,x_0=cy_0=\begin{pmatrix}c_{1r}\\c_{2r}\\ \vdots\\c_{nr}\end{pmatrix}\neq 0$$

“必要性$\Rightarrow$”,反设

一方面,$f=X'_0AX_0\geq 0$

$$f=X'_0AX_0=y_0'\begin{pmatrix}1&&&&&&&&\\&\cdots&&&&&&&\\&&1&&&&&&\\&&&-1&&&&&\\&&&&\cdots&&&&\\&&&&&-1&&&\\&&&&&&0&&\\&&&&&&&\cdots&\\&&&&&&&&0\end{pmatrix}y_0=-1<0$$

另一方面,

矛盾! $\therefore p=r$

P$_{234}$.15. 证明 $$f=n\sum_{i=1}^n x_i{}^2-\left(\sum_{j=1}^n x_j\right)^2\geq 0$$

证法一:因为 $f=\sum_{j=1}^{n} {x_j}^2-2\sum_{1\le i<j\le n} x_{ij}$

恰好有 $f=\sum_{1\le i<j\le n}(x_i-x_j)^2$

故任取($(c_1,c_2,...c_n)\ne 0$,必有

$$f(c_1,c_2,...c_n)=\sum(c_i-c_j)^2\ge 0$$

$$\therefore f\text{半正定}\quad B$$

证法二:设 $f=X'AX$,则$A=\begin{pmatrix} n-1 & -1 & -1 & -1 \\ -1 & n-1 & ... & -1 \\ ... & ... & ... & ... \\ -1 & -1 & ... & n-1 \end{pmatrix}$

因此 A 的任意 k 阶主子式为

$$Q_k=\begin{vmatrix} n-1 & -1 & ... & -1 \\ 1 & n-1 & ... & -1 \\ ... & ... & ... & ... \\ -1 & -1 & ... & n-1 \end{vmatrix}_{k\text{阶}} \Rightarrow (n-1-1)^{k-1}(n-1+(k-1)(-1)$$

恒有 $Q_1,Q_2,\cdots Q_{n-1,}>0,Q_n=0$

$\therefore A$的所有主子式大于或等于0 C即$f=X'AX$半正定

P234.1 证:首先 x_1,x_2,线性无关,

(反设 x_1,x_2 线性相关,不妨设有 $X_2=kX_{1,}k\in R$

$\therefore X_2'AX_2=(kx')A(kx_1)=k_2(x_1'Ax_1)>0$

与 $x_2'Ax_2<0$矛盾。

$x(t)=x_1+t(x_2-x_1)=tx_2+(1-t)x_1\ne 0$,(对任何$t$)

$\therefore$二次型$f=x'Ax$在$x(t)$的值为

$q(t)=x'(t)Ax(t)=(tx'_2+(1-tx_i)A(tx_2+(1-t)x_1)$

$=(t(x'_2-x_1')A(t(x_2-x_1)+x_1)$

$=t_2(x_2-x_1)'A(x_2-x_1)-2tx'_1A(x_2-x_1)+x_1'Ax_1$

是t的初等连续函数, $\sigma(0)=x_1'Ax_1>0,\sigma(1)=x_2'Ax_2<0$

所有t_0,使$\phi(t_0)=0$.令$\mathrm{x}_0=x(t_0)=x_1+t_0(x_2-x_1)\ne 0$.

则有$x_0'Ax_0=\phi(t_0)=0$,证毕。

P234 补 1①化标准形. $f=x_1x_{2n}+x_2x_{2n-1}+....+x_nx_{n+1}$

解 $\begin{cases} x_1 = y_1 + y_{2n} \\ x_2 = y_2 + y_{2n-1} \\ \cdots\cdots \\ x_n = y_n + y_{n+1} \\ x_{n+1} = y_n - y_{n-1} \\ \cdots\cdots \\ x_{2n-1} = y_2 - y_{2n-1} \\ x_{2n} = y_1 - y_{2n} \end{cases}$ 即 $X = \begin{pmatrix} 1 & & & & & & & 1 \\ & 1 & & & & & 1 & \\ & & \cdots & & & \cdots & & \\ & & & 1 & 1 & & & \\ & & & 1 & -1 & & & \\ & & \cdots & & & \cdots & & \\ & 1 & & & & & -1 & \\ 1 & & & & & & & -1 \end{pmatrix}$

$Y = cy$ 则有

$f = y_1{}^2 + y_2{}^2 + \ldots + y_n{}^2 - y_{n+1}{}^2 - \ldots - y_{2n}{}^2$

验算 $C'(\frac{1}{2}\begin{pmatrix} & & & 1 \\ & & \cdots & \\ & 1 & & \\ 1 & & & \end{pmatrix})C = \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix}$

令 $H = \begin{pmatrix} & & & 1 \\ & & \cdots & \\ & 1 & & \\ 1 & & & \end{pmatrix}$, 则 $H^2 = En, A = \begin{pmatrix} 0 & \frac{1}{2}H \\ \frac{1}{2}H & 0 \end{pmatrix}, C = \begin{pmatrix} E & H \\ H & -E \end{pmatrix}$

$$\therefore C'AC = \begin{pmatrix} E' & H' \\ H' & -E' \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2}H \\ \frac{1}{2}H & 0 \end{pmatrix}\begin{pmatrix} E & H \\ H & -E \end{pmatrix} = \begin{pmatrix} E & H \\ H & -E \end{pmatrix}\begin{pmatrix} \frac{1}{2}E & -\frac{1}{2}H \\ \frac{1}{2}H & \frac{1}{2}E \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$$

P234 补 1②化标准形 $f = x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n$

解:若设 $y_1 = \frac{1}{2}(x_1 + x_2 + x_3), y_2 = \frac{1}{2}(x_1 - x_2 + x_3)$, 则

$y_1{}^2 - y_2{}^2 = x_1x_2 + x_2x_3$

(1) (1) 若n是偶数,则
$$\begin{cases} y_1 = \frac{1}{2}(x_1 + x_2 + x_3) \\ y_2 = \frac{1}{2}(x_1 - x_2 + x_3) \\ y_3 = \frac{1}{2}(x_3 + x_4 + x_5) \\ y_4 = \frac{1}{2}(x_3 - x_4 + x_5) \\ \ldots\ldots \\ y_{n-3} = \frac{1}{2}(x_{n-3} + x_{n-2} + x_{n-1}) \\ y_{n-2} = \frac{1}{2}(x_{n-3} - x_{n-2} + x_{n-1}) \\ y_n = \frac{1}{2}(x_{n-1} - x_n) \end{cases}$$

$$c_1 = \frac{1}{2}\left(\begin{array}{cc|c|c|cc|cc} 1 & -1 & 1 & & & & & \\ 1 & -1 & 1 & & & & & \\ \hline & & 1 & 1 & 1 & & & \\ & & 1 & -1 & 1 & & & \\ \hline & & & & \cdots & & & \\ \hline & & & & 1 & 1 & & \\ & & & & 1 & -1 & & \\ \hline & & & & & & 1 & 1 \\ & & & & & & 1 & -1 \end{array}\right)$$

即, $Y=C_1X$

显然 $\left|C_1\right| = \frac{1}{2^n}(-2)^{\frac{1}{2}} \neq 0, 令C = C_1^{-1}$

则 X=CY 使

$$f = y_1^2 - y_2^2 + y_3^2 - y_4^2 + \ldots. + y_{n-3}^2 - y_{n-2}^2 + y_{n-1}^2 - y_n^2$$

$\Delta(ii)$ 若n为奇数,同理

补P_{234}.1③)(也可直接证明,或归纳证明)

P234 补 1.④ $f = \sum_{i=1}^{n}(x_i - \overline{X})^2, \overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{n-1} \\ y_n \end{pmatrix} = y = \frac{1}{n}\begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & n-1 & -1 \\ 0 & 0 & \dots & \dots & 1 \end{pmatrix} X = \frac{1}{n}C_3 X \text{或} X = (\frac{1}{n}c_3)^{-1} y$$

$$= \begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 2 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} y$$

令 .

$\because y_i = x_i - \frac{1}{n}\sum_{i=1}^{n} x_i = x_i - \overline{x}, y_n = x_n,$ $\sum y_i = \sum x_i - (n-1)\overline{x} = \overline{x}$

$\therefore f = \sum_{i=1}^{n-1} y_i^{\ 2} + (x_n - \overline{x})^2 = \sum_{i=1}^{n-1} y_i^{\ 2} + (y_n - \sum_{i=1}^{n} y_i)^2 = 2(\sum_{i=1}^{n-1} y_i^{\ 2} + \sum_{1\le i<j\le n-1} y_i y_j)$

参照P_{234}.1③(5.75.5.3)令

$$Z = C_4 Y = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} & \dots & \frac{1}{3} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \frac{1}{n-1} & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix} Y$$

则 $f = 2z_1^{\ 2} + \frac{3}{2} z_2^{\ 2} + \frac{4}{3} z_3^{\ 2} + \dots + \frac{n}{n+1} z_{n-1}^{\ 2}$

其中 $x = (\frac{1}{n}c_3)^{-1} y = (\frac{1}{n}c_3)^{-1} c_4^{\ -1} y = n(c_4 c_3)^{-1} y$

矩阵验算略.

P_{234}补 2,不妨设 秩(A)=r,且 $\begin{vmatrix} a_{11} & \dots & a_{1r} \\ \dots & \dots & \dots \\ a_{r1} & \dots & a_{rr} \end{vmatrix} \neq 0(\text{左上角}r\text{阶子式})$

$$\therefore \text{作} X = C_1 Y, \text{其中} C_1 = \begin{pmatrix} a_{11} & \dots & a_{1r} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{r1} & \dots & a_{rr} & \dots & a_{rn} \\ 0 & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

那么, $f = y_1^{\ 2} + y_2^{\ 2} + \dots y_r^{\ 2} + \delta_{r+1}^{\ \ 2} + \dots + \delta_s^{\ 2} (\delta_i \text{为} y_1, \dots y_r \text{的一次式})$

作一个$f_1(y_1,y_2,\ldots y_r)=f$被为$r$元二次型�那么�任取�$c_1,c_2,\ldots c_r)\neq 0$

那么,必有$f_1(c_1,c_2,\ldots c_r)=f(c_1,\cdots c_{r,}\cdots 0)>0,\therefore f_1$是一个$r$元正定二次型

$$\begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_r \end{pmatrix}=G\begin{pmatrix} z_1 \\ z_2 \\ \ldots \\ z_r \end{pmatrix}\text{使}f_1(y_1,y_2,\ldots y_r)=c_1{}^2+c_2{}^2+\ldots+z_r{}^2$$

$$\text{令}\begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{pmatrix}=\begin{pmatrix} G & o \\ o & E_{n-r} \end{pmatrix}\begin{pmatrix} z_1 \\ \ldots \\ z_r \\ \ldots \\ z_n \end{pmatrix},C_2=\begin{pmatrix} G & o \\ o & E_{n-r} \end{pmatrix}\text{可逆}$$

则$f=y_1{}^2+\ldots+y_r{}^2+\delta_{r+1}{}^2+\ldots+\delta_s{}^2$

且$X=C_1Y=C_1C_2Z=CZ(C=C_1C_2$可逆�

必有$\therefore f$的正惯性指数$=r=$秩(A)

P234补 3(先讲补 2), $f=l_1{}^2+\ldots+l_p{}^2-l_{p+1}{}^2-\ldots-l_{p+q}{}^2$

证:设$l_i=a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n$

并设X=CY,(C可逆)使 $f=y_1{}^2+y_2{}^2+\ldots+y_{p'}{}^2-y_{p'+1}{}^2-\ldots-y_{p'+q'}{}^2$

$$p^{'}>p\text{作线性方程组}\left\{\begin{array}{l} l_1=0 \\ \ldots \\ l_p=0 \\ \ldots \\ y_{p'+1}=0 \\ \ldots \\ y_n=0 \end{array}\right\}$$

那么　反设

$y_i=b_{i1}x_i+\ldots+b_{in}x_n$

$Y=C^{-1}X$

共有 $p+(n-p')=n-(p'-p)<n$的方程

存在非零时,

$$X_0=\begin{pmatrix} c_1 \\ \ldots \\ c_n \end{pmatrix}\neq 0,y_0=c^{-1}X_0\neq 0,\because y_{p'+1}=\ldots=y_n=0\therefore y_0\text{的前}P'\text{个分量不全为}0$$

$\therefore$一方面$f=-l_{p+1}{}^2-\ldots,-l_{p+q}{}^2\leq 0$

另一方面 $f=y_1{}^2+\ldots+y_{p'}{}^2>0(i$不全为$0)$

矛盾,所有 $p'\leq p$

同理,负惯性指数$q' \leq q$

另推论:如本例形式二次型,例$p+q \geq r$(秩)

p_{235}补4证明： $\because T'AT = A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, T = \begin{pmatrix} E & X \\ o & E \end{pmatrix}$

$$= \begin{pmatrix} E & X \\ o & E \end{pmatrix} \begin{pmatrix} A_{11} & A_{11}X + A_{12} \\ A_{21} & A_{21} + A_{22} \end{pmatrix} + \begin{pmatrix} E & o \\ X' & E \end{pmatrix} \begin{pmatrix} A_{11} & A_{11}x + A_{12} \\ A_{21} & A_{21}x + A_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{11}x + A_{12} \\ x'A_{11} + A_{21} & * \end{pmatrix}$$

$A_{11}x + A_{12} = 0$则$x = -A_{11-1}A_{12}, (\because A_{12} = A_{21})$

$\therefore x'A_{11} + A_{21} = (-A_{12}', A_{11}^{-1})A_{11} + A_{21} = -A_{21}A_{11}^{-1}A_{11} + A_{21} = 0$

设

取 $T = \begin{pmatrix} E & -A_{11}^{-1}A_{12} \\ o & E \end{pmatrix}$即合要求

P_{235},补 5,若n=1,显然A=0

若 n=2,A=0,显然

$$A \neq o,则 \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}', \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}' \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}成立$$

【】

由于A_2仍为反对称

故归纳假设A_2:

$$T'_2 A_2 T_2 = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & \cdots & & & & & \\ & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \cdots & \\ & & & & & & & 0 \end{pmatrix}$$

$\therefore$ 取 $C = P_{(s,t)}P_{(t,2)}T\begin{pmatrix} \frac{1}{a} & & & \\ & 1 & & \\ & & \cdots & \\ & & & 1 \end{pmatrix}\begin{pmatrix} E_2 & 0 \\ 0 & T_2 \end{pmatrix}$ 则

$$C^{'}AC = \begin{pmatrix} 0 & 1 & & & & & & & \\ -1 & 0 & & & & & & & \\ & & \cdots & & & & & & \\ & & & 0 & 1 & & & & \\ & & & -1 & 0 & & & & \\ & & & & & 0 & & & \\ & & & & & & \cdots & & \\ & & & & & & & 0 \end{pmatrix}$$

证毕.

P_{235} 补,6 由习题第 10 题(5.77.7.2),一定存在 $C_1 > 0$, 当 $t > c_2$ 时

C_1E-A 永为正定

$C > \max\{C_1, C_2\}$ 那么同时 $CE + A, CE - A$ 正定,

即要 $X_0 \neq 0, x_0{}'(CE + A)X_0 > 0 \Rightarrow 2CX_0{}'X_0 < X_0{}'AX_0$

$X_0{}'(CE - A)X_0 > 0 \Rightarrow CX{}'_0 X_0 < X_0{}'AX_0$

即 $-CXo'Xo < X'_0 AXo < CXo'Xo$

取 $\therefore$ 对每个 X, 有 $\left|X'AX\right| < CX'X$

P_{235} 补 7, 1) $T = \begin{pmatrix} 1 & & & * \\ & 1 & & \cdots \\ & & 1 & \\ & \cdots & & \cdots \\ 0 & & & 1 \end{pmatrix}$, $B = T'AT$

将,T分块 $T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}, A = \begin{pmatrix} A_1 & A_2 \\ A_{2'} & A_3 \end{pmatrix}$ 其中, T_1, A_1 为 K 阶方阵.

则 B 的第 K 个顺序主子式的矩阵为

$$T'AT = \begin{pmatrix} T_1{}' & 0 \\ T_3{}' & T_2{}' \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_2{}' & A_3 \end{pmatrix}\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} T_1{}' & 0 \\ T_3{}' & T_2 \end{pmatrix}\begin{pmatrix} A_1T_1 & * \\ A_2{}'T_1 & * \end{pmatrix} = \begin{pmatrix} T_1{}'A_1T_1 & * \\ * & * \end{pmatrix}$$

的左上角k阶方阵,即为 $T_1{}'A_1T_1$

$\therefore B$ 的第k个顺序主子式 $= \left|T_1{}'A_1T_1\right| = \left|T_1\right|^2\left|A_1\right| = \left|A_1\right|$, 为A的第k个顺序主子式.证毕

2)归纳证明,*n=1* 显然,设 *n-1* 成立,考虑 *n* 情形

设 $A = \begin{pmatrix} A_1 & \alpha \\ \alpha' & a_{\cos} \end{pmatrix}$ 由 A_1 满足条件, 存在 T_1 特殊上三角, 使 $D_1 = T_1{}'A_1T_1$ 为对角

设 $G=\begin{pmatrix} T_1 & 0 \\ 0 & 1 \end{pmatrix}$ 仍为特殊上三角�使 $G'A'G=\begin{pmatrix} D_1 & T_1'\alpha \\ \alpha' T_1 & a_{nn} \end{pmatrix}=B$

$\because |D_1|=|T_1|^2|A_1|\neq 0 \therefore D_1$ 可逆,

又 $H=\begin{pmatrix} E_{n-1} & -D^{-1} & T_1'\alpha \\ 0 & 1 & \end{pmatrix}$ 仍为特殊上三角, 且

$H'BH=\begin{pmatrix} E_{n-1} & 0 \\ -\alpha' T_1 D_1^{-1} & 1 \end{pmatrix}\begin{pmatrix} D_1 & 0 \\ \alpha'\beta T_1 & a_{nn-x} \end{pmatrix}=\begin{pmatrix} D_1 & 0 \\ 0 & 6 \end{pmatrix}=D$,为对角矩阵

故取,C=GH 仍为特殊上三角,且 C’AC=D 为对角,证毕.

3)

$\because A$ 的顺序主子式 $P_1 \quad CP_2 ..CP\ n$ 全大于0, 故存在特殊上三角 T 使, $D=T'AT$ 为对角, 于是

D 与 A 的顺序主子式值相等, 设 $D=\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ldots & \\ & & & d_n \end{pmatrix}$

则 $d_1,d_2,\ldots d_k=p_k>0, k=1,2,\ldots n,$ 推出 $d_1,d_2,\ldots d_n>0$

所以 D 正定,即 A 正定,证毕.

$$f=\begin{vmatrix} A & y \\ y' & 0 \end{vmatrix}=\left|\begin{pmatrix} A & y \\ y' & 0 \end{pmatrix}\begin{pmatrix} E & -A^{-1}y \\ 0 & 1 \end{pmatrix}\right|=\begin{vmatrix} A & 0 \\ y' & -y'A^{-1}y \end{vmatrix}=(-|A|)\cdot y'A^{-1}y$$

P_{236}补 8, 1),

$\because A$ 正定 $\quad i$ 已知 $\quad j\ .CA^{-1}$ 正定 $\quad i$ 见习题第11题 $P5.76.6.4$), 故对任一组 $y\neq 0$ 值.

$y'A^{-1}>0, \therefore f(y_1,y_2,\ldots y_n)=(-|A|)y'A^{-1}y<0, (\because |A|>0)$

$\therefore f$ 是负定二次型

2)设 $A=\begin{pmatrix} A_1 & \alpha \\ \alpha' & a_{nn} \end{pmatrix}, B_1=\begin{pmatrix} A_1 & \partial \\ \partial' & 0 \end{pmatrix}, B_2=\begin{pmatrix} A_1 & 0 \\ \alpha' & a_{nn} \end{pmatrix}$

$\therefore |A|=|B_1|+|B_2|=(-|A_1|)\partial' A_1^{-1}\partial+a_{nn}|A_1|\leq a_{nn}|A_1|=q_{nn}p_{n-1}$

$\because A1$ 仍然证定 $\ .C|A_1|\leq a_{n-1n-1}p_{n-1}$

如此下去 $\quad C$ 则 $|A|\leq a_{nn}p_{n-1}\leq a_{nn}q_{n-1n-2}\leq a_{nn}\ldots a_{33}a_{22}p_1=a_{nn}\ldots a_{22}a_{11}$

3) 即 $|A|\leq a_{11}a_{22},\ldots a_{nn}$

作 $A=T'T=T'ET$, 则 A 正定且 $a_{ii}=\sum_{k=1}^{n}t_{ki}^2$

4) $\therefore |A|=|T|^2\leq\prod_{i=1}^{n}a_{ii}=\prod_{i=1}^{n}\sum_{k=1}^{n}t_{ki}^2=\prod_{i=1}^{n}(t_{1i}^2+t_{2i}^2+\ldots+t_{ni}^2).$

P_{236}补9（必要性）

$$A \geq 0 \Rightarrow C\text{可逆}, C'AC = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \geq 0. \therefore di \geq 0$$

$\therefore |C|^2 |A| = d_1 d_2 \cdots d_n \geq 0 \therefore |A| \geq 0.$

作$g(x_{i_1}, \ldots x_{i_2}, \ldots x_{i_k}) = f(0 \ldots x_{i_1}, \ldots 0 x_{i_2}, \ldots x_{i_k} 0)$

$$\text{则}g(x_{i_1}, \ldots x_{i_k})\text{半正定}\ , g\text{的矩阵为}\begin{pmatrix} a_{i_1 i_1} & \ldots & a_{i_1 i_k} \\ \ldots & \ldots & \ldots \\ a_{i_k i_1} & \ldots & a_{i_k i_k} \end{pmatrix} = A_1$$

$\therefore A_1 \geq 0 \therefore |A_1| \geq 0$

$$D = \begin{vmatrix} a_{11+\lambda} & a_{12} & \ldots & a_{1n} \\ a_2 1 & a_{22+\lambda} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn+\lambda} \end{vmatrix} = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_k \lambda^{n-k} + \ldots + a_n \text{中}\lambda^{n-k}\text{的系数}a_k$$

这样取到在 0 中主对角线上任取n-k项中的 λ^{n-k} 的系数 a_k 项所在的行和列,得一个K级子式(含入)DK,由于是 λ^{n-k}的系数 C故K的子式D中只能取所有的常数项 C即令 DK中的$\lambda = 0$,这正是D的一个K级主子式, 要是λ^{n-k}的系数中的一元, 故a_k为D的所有k阶主子式之和, 如

$a_1 = a_{11} + a_{22} + \ldots + a_{nn}$等.

现在考虑任意$\varepsilon > 0, A + \varepsilon E$,它的$m$阶顺序主子式, 为$A$的右上角的m阶方阵, A_r作:

$|A_k + \varepsilon Ek| = \varepsilon^k + b_1 \varepsilon^{k-1} + \ldots + b_k$

由于ε^{k-i}的系数b_i是A_k的一切i阶主子式之和, 而A_k的主子式仍为A的主子式,

由充分条件, $b_i \geq 0, \therefore |A_k + \varepsilon E_k| \geq \varepsilon_k > 0$

因此$A + \varepsilon E$正定, 故对任何$X \neq 0, X'(A + \varepsilon E) X > 0$

$\therefore \lim\limits_{\varepsilon \to \infty} X'(A + \varepsilon E) Z = X'AX \geq 0$(边续性)

所以A半正定.

第六章　线性空间习题解答

P267.1 设$M \subseteq N$,证明:$M \cap N = M, M \cup N = N$

证：$\forall x \in M \Rightarrow x \in N = M \Rightarrow x \in M \cap N \Rightarrow M \subseteq M \cap N$

$\forall x \in M \cap N \Rightarrow x \in M \Rightarrow M \cap N \subseteq M, \therefore M \cap N \subseteq M$

$\therefore M \cap N = M$

又$\forall x \in N \Rightarrow x \in M \bigcup N \Rightarrow N \subset M \bigcup N$

$\forall x \in M \bigcup N \Rightarrow x \in N$或$x \in M \subseteq N \Rightarrow x \in N$

$\therefore M \bigcup N = N$

P267.2 证：① $M \bigcap (N \bigcup L) = (M \bigcap N) \bigcup (M \bigcap L)$

② $M \bigcup (N \bigcap L) = (M \bigcup N) \bigcap (M \bigcup L)$

①：

证(1)：$x \in$左$\Leftrightarrow x \in M$且$x \in N \bigcup L \Leftrightarrow x \in M$且$(x \in N$或$x \in L) \Leftrightarrow x \in M \bigcap N$或$x \in M \bigcap L$

$\Leftrightarrow x \in$右$\Rightarrow$反过来。证毕

证(2)：$x \in$左$\Leftrightarrow \in M$或$x \in N \bigcap L \Leftrightarrow x \in M$或$(x \in N$且$x \in L)$

$\Leftrightarrow x \in M \bigcup N$且$x \in M \bigcup L \Leftrightarrow x \in$右。证毕

P_{267}.3①不做成,因为 2 个n次多项式相加不一定是n次多项式,如

$(x^u + x + 1) + (-x^u + x - 2) = 2x - 1$

$f(A) + g(A) = h_1(A), (h_1(x) = f(x) + g(x)$为多项式)

②做成,因为$kf(A) = h_2(A), (h_2(x) = kf(x)$为多项式)

③做成. 因为实对称(反对称, 上三角,下三角)之和之倍数仍为实对称(反对称, 上三角,下三角)故做成线性空间

④不做成,设$V = \{\alpha \mid \alpha$为平面上不平行β的向量$\}$

⑤不做成,违反定义 3.(5)$\because 1\alpha = \alpha$，但这里$1\alpha = 0$。取$\alpha \neq 0$即得矛盾。

P267.3⑤

$$(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + a_1 a_2)$$

$$k \circ (a_1, b_1) = (ka_1, kb_1 + \frac{1}{2}k(k-1)a_1{}^2$$

解: 显然V非空 1^0

以及 2 个封闭的代数运算 2^0

验证 3^0 先设 $\alpha = (a_1, b_2), \beta = (a_2, b_2), r = (a_3, b_3)$, 及$k, t \in R$

$(1)\alpha \oplus \beta = \beta \oplus \alpha = (a_2 + a_1, b_2 + b_1 + a_2 a_1)$

$(2)(\alpha \oplus \beta) + r = ((a_1 + a_2) + a_3, (b_1 + b_2 + a_1 a_2) + b_3 + (a_1 + a_2)a_3$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots = (a_1 + a_2 + a_3, b_1 + (b_2 + b_3 + a_2 a_3)$

$\ldots.\alpha \oplus (\beta \oplus r) = (a_1 + (a_2 + a_3), b_1 = (b_2 + (b_2 + b_3 + a_2 a_3) + a_1(a_2 + a_3)$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots = (a_1 + a_2 + a_3, b_1 + b_2 + b_3 + a_2 a_3 + a_1 a_2 + a_1 a_3) = (\alpha + \beta) + r$

$(3)0 = (0,0), \alpha + 0 = (a_1 + 0, b_1 + 0 + a_1 0) = (a_1, b_1) = \alpha$

$(4)\alpha$的负为$-\alpha = (-a_1, a_1{}^2 - b_1)$

$\ldots\ldots\ldots.\alpha \oplus (-\alpha) =)a_1 + (-a_1), b_1 + (a_1{}^2 - b_1) + a_1(-a_1)) = (0,0) = 0$

$(5)1 \circ \alpha = (1 \circ a_1, 1 \circ b_1 + \frac{1}{2} 1 \circ (1-1)a_1{}^2) = (a_1, b_1) = \alpha$

$(6)k \circ (l \circ \alpha) = k \circ (la_1, lb_1 + \frac{1}{2}l(l-1)a_1{}^2$

$\ldots\ldots\ldots\ldots\ldots = (kla_1, k(lb_1 + \frac{1}{2}k(k-1)a_1{}^2) + \frac{1}{2}k(k-1)(la_1)^2)$

$= (kla_1 + klb + \frac{1}{2}kla_1^2(l-1+(k-1))$

= （kla_1, klb_1+ $\frac{1}{2}kl((k-1)a_1^2)$

=kl∘ α

(7)(k+l)∘ α **= （（k+1）a_1,(k+l)b_1+** $\frac{1}{2}(k+l)(k+l-1)a_1^2)$

=((k+1)a_1,(k+l)b_1+ $\frac{1}{2}(k^2+l^2+2kl-k-l)a_1^2)$

$= (ka_1 + la_1, kb_1 + \frac{1}{2}k(k-1)a_1^2 + (b_1 + \frac{1}{2})l(l-1)a_1^2 + ka_1 \cdot la_1)$

$= k \circ \alpha \oplus l \circ \alpha$

(8)

$k \circ (\alpha \oplus \beta) = k \circ (a_1 + a_2, b_1 + b_2 + a_1 a_2) = (k(a_1 + a_2), k(b_1 + b_2 + a_1 a_2 + \frac{1}{2}k(k-1)(a_1 + a_2)^2)$

$= (ka_1 + ka_2, kb_1 + \frac{1}{2}k(k-1)a_1^2 + kb_2 + \frac{1}{2}k(k-1)a_2^2 + ka_1 a_2 + k(k-1)a_1 a_2)$

$= (ka_1 + ka_2, (kb_1 + \frac{1}{2}k(k-1)a_1^2) + (kb_2 + \frac{1}{2}k(k-1)a_2^2 + (k^2 a_1 a_2))$

$= (ka_1, kb_2 + \frac{1}{2}k(k-1)a_1^2) \oplus (ka_2 kb_2 + \frac{1}{2}k(k-1)a_2^2) = \alpha \oplus \beta$

满足 **3,** 故 **V** 是一个线性空间

不做成。违反分配律，$\forall \alpha \neq 0$, 则会有 $\alpha = 2.\alpha = (1+1).\alpha = 1.\alpha + 1.\alpha = \alpha + \alpha$

⑥ $\Rightarrow \alpha = 0$, 矛盾

$P_{267,3}$⑧V=R^+ P=R ***a*** $\oplus$ *b=ab* $k \circ a = a^k$

解:V非是①关于 $\oplus$ ∘封闭②

任取**a.b.c** $\in R^+, k, l \in R$

(1)*a* $\oplus$ ***b=b*** $\oplus$ ***a=ba***

(2)*(a* $\oplus$ ***b)*** $\oplus$ ***c=(ab)c=a(bc)=a*** $\oplus$ ***(b*** $\oplus$ ***c)***

(3)零元 ***0=1, a*** $\oplus$ ***0=a•1=a***

(4)负元-a=$\frac{1}{a}$,a $\oplus$ (-a)=a•$\frac{1}{a}$=1=0

(5)*1*∘*a=a*^{1}*=a*

(6)k∘(l∘a)=k∘(a^1)=(a^1)^k=a^{lk}=(lk)∘a

(7)*(k+l)*∘*a=a*$^{(k+l)}$*=a*^{k}•*a*^{l}*=a*^{k} $\oplus$ ***a^l=k*∘ *a*** $\oplus$ ***1*∘ *a***

(8)k∘ *(a* $\oplus$ ***b)=k*∘*(ab)=(ab)*^{k}*=a*^{k}*b*^{k}**

= *a*^{k} $\oplus$ ***b*^{k}= *k*∘ *a*** $\oplus$ ***k*∘ *b***

都成立,故**R$^+$**关于 $\oplus$ ∘做成**R**上的向量空间

P_{268}.4①k0=0

证:设 $k0=\alpha,则\alpha=k0=k(0+0)=k0+k0=\alpha+\alpha$

$\therefore \alpha=\alpha+(-\alpha)=0$

即 k0=0

4② $k(\alpha-\beta)=k\alpha-k\beta$

$$\because 0=\alpha+(-\alpha)=\alpha+(-1)\alpha=[1+(-1)].\alpha=0.\alpha=0$$

$$\therefore (-1)\alpha=-\alpha$$

$$故k(\alpha-\beta)=k(\alpha+(-1)\beta)=k\alpha+k(-1)\beta=k\alpha+(-1)(k\beta)$$

$$=k\alpha+(-(k\beta))=k\alpha-k\beta$$

P_{268},5 实函数空间F中,0 是 0 函数 0(x),$\forall$ x$\in$定义域O(x)=0,

于是**k•1+**$l\cdot\cos^2 t+m\cdot\cos 2t$

$=k\cdot 1+l\cos^2 t+m(2\cos^2 t-1)$

$=(k-m)\cdot 1+(l+2m)\cos^2 t$

可取,m=1,k=1,l=-2,则

$1\cdot 1+(-2)\cos^2 t+1\cdot\cos 2t=0\cdot(x)$

∴ $1.\cos^2 t,\cos 2t$ 线性相关

$P_{268.6}$在P[x]中,0 元是 0 多项式(即系数全为 0 的多项式)

证:∵ $(f_1,f_2,f_3)=1,(f_1,f_2)\neq 1,(f_2,f_3)\neq 1,(f_2,f_1)\neq 1,$

设 $a_1 f_1(x)+a_2 f_2(x)+a_3 f_3(x)=0,不妨$ 设 $a_1\neq 0$

$$\therefore\ f_1(x)-(-\frac{a_2}{a_1})f_2(x)+(-\frac{a_3}{a_1})f_3(x)$$

$\because (f_2,f_3)\neq 1,被$(f$_2$(x), f$_3$(x))=d(x),

那么*d(x)*整除 f_2,f_3 **的组合,故** $d(x)\mid f_1(x),$ **于是有**

$$d(x)\mid(f_1(x),f_2(x),f_3(x))$$

与 $(f_1,f_2,f_3)=1$ **矛盾!**

P268,7① $\varepsilon_1=(1,1,1,1),\varepsilon_2=(1,1,-1,-1),\varepsilon_3=(1,-1,1,-1),\varepsilon_4=(1,-1,-1,1),\xi=(1,2,1,1)$

设 $\xi=x_1\varepsilon_1+x_2\varepsilon_2+x_3\varepsilon_3+x_4\varepsilon_4$ 得方程解

$$\begin{pmatrix}1&1&1&1\\1&1&-1&2\\1&-1&1&1\\1&-1&-1&1\end{pmatrix}\Rightarrow\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&1&1&-1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}^{-1}\begin{pmatrix}1\\2\\1\\1\end{pmatrix}=\begin{pmatrix}\because A^2=4E\\\therefore A^{-1}=\frac{1}{4}A\end{pmatrix}$$

$$=\frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}1\\2\\1\\1\end{pmatrix}=\frac{1}{4}\begin{pmatrix}5\\1\\-1\\-1\end{pmatrix}=\begin{pmatrix}5/4\\1/4\\-1/4\\-1/4\end{pmatrix}$$

坐标为$\left(\frac{5}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\right)$

$P_{268}.7.(2)\varepsilon_1=(1,1,0,1),\varepsilon_2=(2,1,3,1),\varepsilon_3=(1,1,0,0),\varepsilon_4=(0,1,-1,-1)$，$\xi=(0,0,0,1)$

设$\xi=x_1\varepsilon_1+x_2\varepsilon_2+x_3\varepsilon_3+x_4\varepsilon_4$，得

$$\left\{\begin{aligned}&x_1+x_2+x_3=0\\&x_1+x_2+x_3+x_4=0\\&x_2+x_4=0\\&x_1+x_2=x_4=0\end{aligned}\right\}\quad\begin{pmatrix}1&2&1&0&0\\1&1&1&1&0\\0&3&0&-1&0\\1&1&0&-1&1\end{pmatrix}\to\begin{pmatrix}1&2&1&0&0\\0&-1&0&1&0\\0&-3&0&-1&0\\0&-1&-1&-1&1\end{pmatrix}$$

$$\to\begin{pmatrix}1&0&1&2&0\\0&1&0&-1&0\\0&0&0&2&0\\0&0&-1&-2&1\end{pmatrix}\to\begin{pmatrix}1&0&0&0&1\\0&1&0&-1&0\\0&0&1&2&-1\\0&0&0&1&0\end{pmatrix}\to\begin{pmatrix}1&0&0&0&1\\0&1&0&0&0\\0&0&1&0&-1\\0&0&0&1&0\end{pmatrix}$$

唯一解得$x_1=1,x_2=0,x_3=-1,x_4=0$

$\therefore\xi=\varepsilon_1-\varepsilon_2$

在此基下的 坐标为$(1,0,-1,0)$

P_{268}. 8① $P^{n\times n}$的一组是$E_{ij},i.j=1,2,\ldots,n,$共有n^2个(矩阵)元素

它们线性无关 $\because\sum_{i,j=1}^{n}a_{ij}E_{ij}=0\Rightarrow A=(a_{ij})=0\Rightarrow\forall i,j,a_{ij}=0$

且任何 $B=(b_{ij})\in P^{n\times n}$，则$B=\sum_{i,j=1}^{n}b_{ij}E_{ij}$

$\dim P^{n\times n}=n^2$，它的一个基是$E_{ij},i,j=1,2,\ldots,n$

8② $P^{n\times n}$中全体对称矩阵集合S(P)，它的一个基是$E_{ij}+E_{ji},i\le j$

$$\dim S(P)=\frac{1}{2}n(n+1)$$

$P^{n\times n}$中全体对称矩阵集合K(P)，它的一个基是$E_{ij}-E_{ji},i<j$

$$\dim K(P)=\frac{1}{2}n(n-1)$$

$P^{n\times n}$中全体上三角矩阵集合$U(T)$，它的一个基是$E_{ij},i\le j$

$$\dim U(T)=\frac{1}{2}n(n+1)$$

$P^{n\times n}$ 中全体真下 $三角矩阵集合 D^{+}(T)$,它的一个基是$E_{ij}, i>j$

$$\dim D(T)=\frac{1}{2}n(n-1)$$

8②中，θ,零元是1，取一个$a>0, a\neq 1$,则$a\in IR^{+}$

那么$\forall b\in R^{+}$,取$k=\log_a b(\because b=k\circ a=a^k \Rightarrow \lg b=k.\lg a$

$\therefore b=(\log_a b)\circ a=a^{\log_a b}$

所以a是R^{+}的一个基$\dim_R R^{+}=1$

$$P_{268}.8(4), V=\left\{ f(A) \ \middle|\ f(x)\in R[x], A=\begin{pmatrix}1 & & \\ & \omega & \\ & & \omega^2\end{pmatrix}\right., \omega=\frac{-1+\sqrt{3}i}{2}$$

解：因为$\omega^3=1$

$$A^2=\begin{pmatrix}1 & & \\ & \omega^2 & \\ & & \omega^4\end{pmatrix}=\begin{pmatrix}1 & & \\ & \omega^2 & \\ & & \omega\end{pmatrix}.A^3=\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}=E$$

所以

故任设 $f(x)\in R[x], f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

则 $f(A)=(a_0+a_3+a_6+\cdots)E+(a_1+a_4+a_7+\cdots)A+(a_2+a_5+a_8+\cdots)A^2$

故 $f(A)=b_0E+b_1A+b_2A^2$

∴E, A, A^2可表示V中所有元素。

$$xE+yA+zA^2=0\Rightarrow\begin{cases}x+y+z=0\\ x+\omega^1 y+\omega^2 z=0\\ x+\omega^2 y+\omega E=0\end{cases}$$

如果

$$\begin{vmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{vmatrix}=3(\omega^2-\omega)\neq 0,所以x=y=z=0只有零解$$

∵系数行列式

即，**E**、**A**、A^2 线性无关，由定理 **1**

dimV=3, 它的一个基是**E**、**A**、A^2

P269.9① $\varepsilon_1=(1,0,0,0),\varepsilon_2=(0,1,0,0),\varepsilon_3=(0,0,1,0),\varepsilon_4=(0,0,0,1),\xi=(x_1,x_2,x_3,x_4)$

$\eta_1=(2,1,-1,1),\eta_2=(0.3.1.0)\eta_3=(5.3.2.1),\eta_4=(6.6.1.3),$

解 $\xi=x_1\varepsilon_1+x_2\varepsilon_2+x_3\varepsilon_3+x_4\varepsilon_4$

$\therefore \varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\to\eta_1,\eta_2,\eta_3,\eta_4,\qquad y=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)Ay$

$$y=A^{-1}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}4/9 & 1/3 & -1 & -1/9\\ 1/27 & 4/9 & 1/3 & -23/27\\ 1/3 & 0 & 0 & -2/3\\ -7/27 & -1/9 & 1/3 & -6/27\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

∴坐标y为

（A^{-1}计算附下页）

$$(A,E)\to\begin{pmatrix}2&0&5&6\\1&3&3&6\\-1&1&2&1\\1&0&1&3\end{pmatrix}\to\begin{pmatrix}1&0&1&3\\0&1&3&4\\0&3&2&3\\0&0&3&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&1&0&-1\\1&0&0&-2\end{pmatrix}\to\begin{pmatrix}1&0&1&3\\0&1&3&4\\0&0&-7&-9\\0&0&3&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0&-6\\0&1&0&4\\0&0&-1&-9\\0&0&3&0\end{pmatrix}\begin{pmatrix}2&1&-3&-7\\-1&0&1&3\\2&1&-3&-8\\1&0&0&-2\end{pmatrix}\to\begin{pmatrix}1&0&0&-6\\0&1&0&4\\0&0&1&9\\0&0&0&-27\end{pmatrix}$$

$$\begin{pmatrix}2&1&-3&-7\\-1&0&1&3\\-2&-1&3&8\\7&3&-9&-26\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\to\begin{pmatrix}4/9&1/3&-1&-11/9\\1/27&4/7&-23/27&-23/27\\1/3&0&-2/3&-2/3\\-7/29&-1/9&\frac{1}{3}&26/27\end{pmatrix}$$

P269. 9. (2) 求由 $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\to\eta_1,\eta_2,\eta_3,\eta_4$ 的过渡矩形, 并求多在 $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ 下的坐标.

$\varepsilon_1=(1,2,-1,0)$　　　$\eta_1=(2,1,0,1)$

$\varepsilon_2=(1,-1,1,1)$　　　$\eta_2=(0.1,2,2)$

$\varepsilon_3=(-1,2,2,1,1)$　　　$\eta_3=(-2,1,1,2)$

$\varepsilon_4=(-1,-1,0,1)$　　　$\eta_4=(1,3,1,2)$　　　$\xi=(1,0,0,0)$

解 $\because(\eta_1,\eta_2,\eta_3,\eta_4)=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)T$ 　因此.

$(\eta_1,\eta_2,\eta_3,\eta_4,\xi)=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)(T,X)$

$\therefore(T,X)=A^{-1}B$

$$\therefore B=\begin{pmatrix}2&0&-2&1&1\\1&1&1&3&0\\0&2&1&1&0\\1&2&2&2&0\end{pmatrix}=A\begin{pmatrix}1&1&-1&-1\\2&-1&2&-1\\-1&1&1&0\\0&1&1&1\end{pmatrix}(T,X)$$

$$\therefore\left(\begin{array}{cccc|ccccc}1&1&-1&-1&2&0&-2&1&1\\2&-1&-2&-1&1&1&1&3&0\\-1&1&1&0&0&2&1&1&0\\0&1&1&1&1&2&2&2&0\end{array}\right)\to\left(\begin{array}{cccc|ccccc}1&1&-1&-1&2&0&-2&1&1\\0&-3&4&1&-3&1&5&1&-2\\0&2&0&-1&2&2&-1&2&1\\0&1&1&1&1&2&2&2&0\end{array}\right)?$$

$$\to\left(\begin{array}{cccc|ccccc}1&0&-2&-2&1&-2&-4&-1&1\\0&1&1&1&1&2&2&2&0\\0&0&-2&-3&1&-2&-5&-2&1\\0&0&7&4&0&7&11&7&-2\\0&0&1&-5&0&1&-4&1&1\end{array}\right)\to\left(\begin{array}{cccc|ccccc}1&0&0&-12&1&0&-12&1&3\\0&1&0&6&1&1&6&1&-1\\0&0&1&-5&0&1&-4&1&1\\0&0&0&-13&0&0&-13&0&3\end{array}\right)$$

$$\rightarrow\left(\begin{array}{cccc:ccccc}1&0&0&0&1&0&0&1&3/13\\0&1&0&0&1&1&0&1&5/13\\0&0&1&0&0&1&1&1&-2/13\\0&0&0&0&0&0&1&0&-3/13\end{array}\right)$$

$$T=\begin{pmatrix}1&0&0&1\\1&1&0&1\\0&1&1&1\\0&0&1&0\end{pmatrix}$$

因此: 过渡矩阵

令 在$\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$下的坐标为$\left(\frac{5}{13},\frac{5}{13},-\frac{2}{13},-\frac{3}{13}\right)$

P269.9③ $\varepsilon_1=(1,1,1,1)\varepsilon_2=(1,1,-1,-1),\varepsilon_3=(1,-1,1,-1)\varepsilon_4=(1,-1,-1,1)$

$\eta_1=(1,1,0,1),\eta_2=(2,1,3,1),\eta_3=(1,1,0,0),\eta_4=(1,-1,-1,1)$

求$\xi=(1,0,0,-1)$在$\eta_1,\eta_2,\eta_3,\eta_4$下的坐标

解　若　$(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)T=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$那么

$$T=\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}+\begin{pmatrix}1&2&1&0\\1&1&1&1\\0&3&0&-1\\1&1&0&-1\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}1&2&1&0\\1&1&1&1\\0&3&0&-1\\1&1&0&-1\end{pmatrix}$$

$$=\frac{1}{4}\begin{pmatrix}3&-1&2&-1\\1&-1&2&3\\-1&3&0&-1\\1&-1&0&-1\end{pmatrix}$$

而$\xi=(\eta_1,\eta_2,\eta_3,\eta_4)y=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)x$

$$=(\eta_1,\eta_2,\eta_3,\eta_4)TY$$

$$\therefore Y=T^{-1}X\qquad X=A^{-1}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

不如直接解出

$$\xi=(\eta_1,\eta_2,\eta_3,\eta_4)y\quad\therefore\begin{pmatrix}1&2&0&1\\1&1&1&0\\0&3&-1&0\\1&1&-1&-1\end{pmatrix}\rightarrow\begin{pmatrix}1&2&1&0\\0&-1&0&1\\0&3&0&-1\\0&0&-1&-2\end{pmatrix}\begin{pmatrix}1\\-1\\0\\-1\end{pmatrix}$$

y

$$\begin{pmatrix}1&0&1&2\\0&1&0&-1\\0&0&0&2\\0&0&1&2\end{pmatrix}\begin{pmatrix}-1\\1\\-3\\1\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&2\\0&0&0&2\end{pmatrix}\to\begin{pmatrix}-2\\1\\1\\-3\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}-2\\-1/2\\4\\-3/2\end{pmatrix}$$

故 $\xi=-2\eta_1-\dfrac{1}{2}\eta_2+4\eta_3-\dfrac{3}{2}\eta_4$在该基下坐标为$\left(-2,-\dfrac{1}{2},4,-\dfrac{3}{2}\right)$

P269.10.设 $\xi=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)x=(\eta_1,\eta_2,\eta_3,\eta_4)X$ 作$A=\begin{pmatrix}2&0&5&6\\1&3&3&6\\-1&1&2&1\\1&0&1&3\end{pmatrix}$

$$\begin{pmatrix}1&0&5&6\\1&2&3&6\\-1&1&1&1\\1&0&1&2\end{pmatrix}\to\begin{pmatrix}1&0&1&2\\0&2&2&4\\0&1&2&3\\0&0&4&4\end{pmatrix}\to\begin{pmatrix}1&0&1&2\\0&1&1&2\\0&0&1&1\\0&0&4&4\end{pmatrix}\to\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\\0&0&0&0\end{pmatrix}\quad\therefore$$

解 $x=k\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}$

只要k≠0 即可，取k=1 即有$\xi=\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4=(1,1,1,-1)$

P269.11， $V_1=R_R,V_2={R^+}_R$ 规定$B:R\to R^+,a\to e^a$（自然对数）

则B是$1-1$的和映上以$\left(a\neq b\Rightarrow e^a\neq e^b,\text{正定}b\text{原为}\ln b\right)$

又$\because\delta(a+b)=e^{a+b}=e^a\cdot e^b=e^a\oplus e^b=\delta(a)\oplus\delta(b)$

$\delta(ka)=e^{ka}=\left(e^a\right)^k=k\bullet e^a=k\bullet\delta(a)$

故δ就是同构，$R\cong R^+$其实任取一个数d(d>0,d≠)代替e均可：

$(p_{269})12$设$V_1\subseteq V_2,V_1\leq V,V_2\leq V$，且$\dim V_1=\dim V_2$

证明 $V_1=V_2$

只须证$V_1\supseteq V_2$

证：设$\dim V_1=\dim V_2=r$，且$\alpha_1,\alpha_2,\cdots,\alpha_r$为$V_1$的个基任取$\beta\in V_2$

$\alpha_1,\alpha_2,\cdots,\alpha_r\in V_2$且在$V_2$中线性无关.

因为$\dim V_2=r$，故V_2中这$r+1$个向量$\beta,\alpha_1,\alpha_2,\cdots,\alpha_r$，线性相关，由临界定理.

$$\beta\leftarrow\alpha_1,\alpha_2,\cdots,\alpha_r\Rightarrow\beta\in V_1$$

即 $V_2 \subseteq V_1$中（得证）

$(P_{269,}13), C(A)=\left\{x\in p^{n\times n} \middle| AX=XA\right\}\subseteq P^{n\times n}$

1）$\because 0\in C(A)\neq\phi$

$\forall X,Y\in C(A)\Rightarrow A(X+Y)=AX+AY=XA+YA=(X+Y)A$

$\Rightarrow A(kX)=k(AX)=k(XA)=X(kA)$

$\therefore x+y,kA\in C(A)$即C（A）$\leqslant P^{n\times n}$

2）$\because \forall x\in p^{n\times n}$,有$XE=EX$,故$X\in C(E)$

$\therefore P^{n\times n}\subseteq C(E)$但$C(E)\subseteq P^{n\times n}$

$\therefore$当$A=E$时,$C(A)=C(E)=P^{n\times n}$

3）$\forall x=(x_{ij})\in P^{n\times n}$ 由于 $A=\begin{pmatrix}1&&&\\&2&&\\&&\cdots&\\&&&n\end{pmatrix}$

有X$\in C(A)\Leftrightarrow XA=AX$

$\Leftrightarrow XA$第i行j列元素，$jx_{ij}=AX$第i行j列元素ix_{ij}，

$(\forall,ij)$

$\Leftrightarrow(\forall i,\ j)$，$xi_j(i-j)=0$

$\Leftrightarrow i\neq j$时$xi_j=0$,若$i=j$,则$i=j$,则x_{ii}任意

$$\Leftrightarrow X=\begin{pmatrix}x_{11}&&&\\&x_{22}&&\\&&\cdots&\\&&&x_{nn}\end{pmatrix}=\sum_{i=1}^{n}x_{ii}E_{ii}$$

$E_{11,}E_{22},\cdots,E_{nn}$线性无关

此时C（A）是全体对角矩阵，E_{11}，E_{22}，…，E_{nn}是它的一个基，故 dimC(A)=n

P270.14 设 $x=\begin{pmatrix}a&b&c\\d&e&f\\q&h&i\end{pmatrix}$ $Ax=\begin{pmatrix}a&b&c\\b&e&f\\3a+d+2g&3b+e+2h&3c+f+2i\end{pmatrix}$

$XA=\begin{pmatrix}a+3c&b+c&2c\\d+3f&e+f&2f\\q+3i&h+i&2i\end{pmatrix}$ $\therefore AX=XA\Rightarrow a=a+3c\Rightarrow c=0,d=d+3f\Rightarrow f=0$

且 $\begin{cases}3a+d+2g=g+3i\\3b+e+2h=h+i\\2i=2i\end{cases}$ $\therefore\begin{cases}3a+d+q=3i\\3b-e+h=i\\i,a,d,b,e\text{任意}\end{cases}$

$\therefore$依次取$(a,b,d,e,i)=\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5$得基元素

$$\begin{pmatrix}1&0&0\\0&0&0\\-3&0&0\end{pmatrix}\ \begin{pmatrix}0&1&0\\0&0&0\\0&-3&0\end{pmatrix}\ \begin{pmatrix}0&0&0\\1&0&0\\-1&0&0\end{pmatrix}\ \begin{pmatrix}0&0&0\\0&1&0\\0&-1&0\end{pmatrix}\ \begin{pmatrix}0&0&0\\0&0&0\\3&1&1\end{pmatrix}$$

dimC(A)=5

p270.15 $c_1c_3\neq 0,\ c_1d+c_2\beta+c_3r=0\Rightarrow\ \alpha=-\dfrac{c_2}{c_1}\beta-\dfrac{c_3}{c_1}r,\ r=\dfrac{c_1}{c_3}\alpha-\dfrac{c_2}{c_3}\beta$

$$\therefore \alpha.\beta \overleftrightarrow{\ } \beta.r\Rightarrow L(\alpha.\beta)=L(\beta.r)$$

P270.16① $\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{pmatrix}=\begin{pmatrix}2&1&3&1\\1&2&0&1\\-1&1&-3&0\\1&1&1&1\end{pmatrix}\Rightarrow\begin{pmatrix}1&1&1&-1\\0&1&-1&0\\0&2&-2&1\\0&-1&1&-1\end{pmatrix}\to\begin{pmatrix}1&1&1&1\\0&1&-1&0\\0&0&0&1\\0&0&0&-1\end{pmatrix}$

$\to\begin{pmatrix}1&1&1&1\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\therefore\alpha_4,\alpha_2,\alpha_3$线性无关

$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=3$故基为$\alpha_2,\alpha_3,\alpha_4$,

P270.16② $\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{pmatrix}=\begin{pmatrix}2&1&3&-1\\-1&1&-3&1\\4&5&3&-1\\1&5&-3&1\end{pmatrix}\to\begin{pmatrix}2&1&3&-1\\1&2&0&0\\2&4&0&0\\3&5&0&0\end{pmatrix}\to\begin{pmatrix}2&1&3&-1\\1&2&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

$\therefore$秩$(\alpha_1,\alpha_2,\alpha_3\alpha_4)=2,\alpha_1,\alpha_2$是一个极大无关组

$\therefore\dim L(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=2,\alpha_1,\alpha_2$是它的一个基

P270.17 $\begin{pmatrix}3&2&-5&4\\3&-1&3&-3\\3&5&-13&11\end{pmatrix}\to\begin{pmatrix}3&2&-5&4\\0&-3&8&-7\\0&3&-8&7\end{pmatrix}\to\begin{pmatrix}1&2/3&-5/3&4/3\\0&1&-8/3&7/3\\0&0&0&0\end{pmatrix}$

$\to\begin{pmatrix}1&0&1/9&-2/9\\0&1&-8/3&7/3\\0&0&0&0\end{pmatrix}$

系数矩阵为A，秩（A）=2 基础解系含 4-2=2 个向量，可为$\begin{pmatrix}-1\\24\\9\\0\end{pmatrix}\begin{pmatrix}2\\-21\\0\\9\end{pmatrix}$

∴解空间的维数为 2，基底一个是

$$(-1,24,9,0),(2,21,0,9)$$

（P270，18，①）解设$V_1=L(\alpha_1,\alpha_2)V_2=L(\beta_1,\beta_2)$

若设 $r=x_1\alpha_1+x_2\alpha_2=x_3\beta_1+x_4\beta_2$即$r\in V_1\bigcap V_2$

则 $\begin{cases} x_1-x_2-2x_3-x_4=0 \\ 2x_1+x_2+x_3+x_4=0 \\ x_1+x_2-3x_4=0 \\ x_2-x_3-7x_4=0 \end{cases}$ $\begin{pmatrix} 1 & -1 & -2 & -1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & -3 \\ 0 & 1 & -1 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 & -1 \\ 0 & 3 & 5 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & -1 & -7 \end{pmatrix}$

$$\to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -2 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

有非零解如$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} -1 \\ 4 \\ -3 \\ 1 \end{pmatrix}$ 即

$\because$秩$(\alpha_1,\alpha_2,\beta_1,\beta_2)=3$所以$\dim(V_1\bigcap V_2)=2+2-3=1$维

它的一个基是 $r=(-5,2,3,4)$

（P270，18，②）解：设V_1=L（α_1,α_2），$V_2=L(\beta_1,\beta_2)$则由

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

由此秩$(\alpha_1,\alpha_2,\beta_1,\beta_2)=4\therefore\dim(V_1+V_2)=\dim L(\alpha_1,\alpha_2,\beta_1,\beta_2,)=4$而

$\dim V_1=\dim V_2=2$，所以$\dim(V_1\bigcap V_2)=2+2-4=0$

此时$V_1\bigcap V_2$没有基.

（P270，18，③）$解:设V_1=L(\alpha_1,\alpha_2,\alpha_3),V_2=L(\beta_1,\beta_2)$

由$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}=\begin{pmatrix} 1 & +2 & -1 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ & & & \end{pmatrix}\Rightarrow\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 4 & -2 \\ 0 & 2 & 0 & -3 \\ & & & \end{pmatrix}$秩为3 $\therefore\dim V_1=3$ 显有$\dim V_2=2$

设$r=x_1\alpha_1+x_2\alpha_2+x_3\alpha_3=x_4\beta_2\in V_1\bigcap V_2$则得

$$\begin{cases} x_1+3x_2-x_3-2x_4+x_5=0 \\ 2x_1+x_2-5x_4-2x_5=0 \\ -x_1+x_2+x_3+6x_4+7x_5=0 \\ -2x_1+x_2-x_3+5x_4-3x_5=0 \end{cases}$$

$$\begin{pmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ -1 & 1 & 1 & 7 \\ -2 & 1 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -5 & 2 & -4 \\ 0 & 4 & 0 & 8 \\ 0 & 7 & -3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 5 & 13 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 11 & 27 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 00 \\ 0 & 1 & 0 & 00 \\ 0 & 0 & 1 & 00 \\ 0 & 0 & 0 & 11 \end{pmatrix}$$ 秩为3：即秩$\left(\alpha'_1,\alpha'_2,\alpha'_3,\beta'_1,\beta'_2,\right)=4$

$\therefore \dim\left(V_1+V_2\right)=4$故$\dim\left(V_1\cap V_2\right)=5-4=1$

取方程组一个非零解$\left(x_1,x_2,x_3,x_4,x_5\right)=(3,-1,-2,1,0)$

即$\beta_1=r=3\alpha_1-\alpha_2-2\alpha_3,\in V_1\cap V_2$是一个所求的基

P270.19 $x_1,+x_2+\cdots+x_n=0$的空间V_1 $\dim V_1=n$一秩$(1,1,1,\cdots,1)=n-1$

$x_1=x_2=\cdots=x_n=0$的解空间V_2 ， $A=\begin{pmatrix} 1-1 & & \\ & 1-1 & \\ & & 1-1 \end{pmatrix}$

$x_1=x_2=\cdots=x_n$的解空间V_2

$\dim V_2=n$一秩$(A)=n-(n-1)=1$

若 $\xi=\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)\in V_1\cap V_2$, 则$\left.\begin{array}{l}\Rightarrow \alpha_1=\alpha_2=\cdot=\alpha_n\overset{\Delta}{=}\alpha \\ \Rightarrow \alpha_1,\alpha_2+\cdot+\alpha_n\left(=n\alpha\right)=0\end{array}\right\}\Rightarrow \alpha=0$

$\therefore \xi=(\alpha,\alpha,\cdots,\alpha)=0$即由定理8推论$V_1+V_2$是直和

$\because \dim\left(V_1\oplus V_2\right)=(n-1)+1=n$ 是$V_1+V_2\subseteq P^n$,

$\dim\left(V_1\oplus V_2\right)=\dim p^n$(由12题6.88.8.3)$P^n=V_1\oplus V_2$

P271，20，设$V=V_1\oplus V_2,V_1=V_{11}\oplus V_{12}$那么$V=V_{11}\oplus V_{12}\oplus V_2$

证一：设$0=\alpha_{11}+\alpha_{12}+\alpha_2\because V=V_1\oplus V_2$

$\therefore \alpha_2=0,\alpha_{11}+\alpha_{12}\in V_1$也为0 即

$O=\alpha_{11}+\alpha_{12}$为$V_{11}\oplus V_{12}$的直和分解或故

$\alpha_{11}=0,\alpha_{12}=0,$所以0有唯一分解式$0=0_{11}+0_{12}+0_2$

$\therefore V=V_{11}\oplus V_{12}\oplus V_2$

证二：$\dim V=\dim V_1+\dim V_2=\left(\dim V_{11}+\dim V_{12}\right)+\dim V_2$

证毕

P271.21 $\because V=L\left(\alpha_1,\alpha_2,\cdots,\alpha_m\right)$ 作$W_i=L\left(\alpha_i\right)$

若$\forall\beta\in V$ $\beta=\sum_{i=1}^{n}\beta_i=\sum_{i=1}^{n}r_i$ $\beta_i V_i\in W_i$ 可设$\beta=b_i\alpha_i,y_i=-\frac{c_2}{c_3}\beta$

$\Rightarrow 0=\Sigma\left(\beta_i-r_i\right)=\Sigma\left(b_i-c_i\right)d_i,$无关性$\Rightarrow b_i-c_i=0\therefore b_i=c_i$

因此 $\beta_i=r_i$ $\therefore\beta$的分解式唯一，$\therefore V=W_1\oplus W_2\oplus\cdots\oplus W_n$

P271.22$\because$ $W_i=\sum_{j=1}^{i-1}V_j\subseteq\sum_{j\neq i}^{s}V_j=V_i$ 故若$\sum_{i=1}^{s}V_i$为直和则$r_i\bigcap V_i=\{0\}$

$\therefore V_i\bigcap W_i=\{0\}$从而必要性显然.

反过来证充分性

若$\sum_{i=1}^{s}V_i$不是直和，有$\alpha_1,\alpha_2,\cdots,\alpha_s,\alpha_i\in V_i$不全为0，且$0=\alpha_1+\alpha_2+\ldots+\alpha_s$

α_k为$\alpha_s,\alpha_{s-1},\cdots,\alpha_2,\alpha_1$中第一个不为 0 的向量故

$$0\neq\alpha_k=\left(-\alpha_1\right)+\left(-\alpha_2\right)+\ldots+\left(-\alpha_{k-1}\right)\subseteq\sum_{j=1}^{k-1}V_j=W_K$$

显然若k=1$\Rightarrow\alpha_1\neq 0,\alpha_2=\cdots=\alpha_S=0,$而$0-\alpha_1+0+0+\cdot+0$矛盾$\therefore k\geq 2$

又$\alpha_k\in V_k$从而$V_k\bigcap W_k\geq\neq\{0\}$与已知矛盾，故

$$\sum_{i=1}^{s}V_i=V_1\oplus V_2\oplus\cdots\oplus V_S$$

P271.23②当平面经过原点是线性子空间，不经过原点则不是

$\because$若$0\in$平面 $\alpha\in$平面

则$k\alpha\left(k\neq 1\right)\notin$平面

23②L_1+L_2生成直线$\left(当L_1=L_2\right)$

生成直线$\left(当L_1\neq L_2\right)$

$L_1+L_2+L_3$生成直线$\left(当L_1=L_2+L_3\right)$

生成直线$\left(当L_1,L_2,L_3,共面\right)$

生成空间$R^3$$\left(当L_1,L_2,L_3不共面\right)$

23②当然不一定有　如右图

$V+V$ 不x平面　y为平面中线

$y\subseteq x$

但 $y\bigcap V=0,Y\bigcap V=0,\therefore y\neq\left(y\bigcap V\right)+\left(y\bigcap V\right)=0$

P271 补 1①因为 $f_i\left(x\right)=\dfrac{f\left(x\right)}{x-a_i}$　　$f\left(x\right)=\left(x-a_1\right)(x-a_2).......\left(x-a_n\right)$

$f_i\left(a_k\right)=0\left(k\neq i\right)$ 或 $f_i\left(a_i\right)=\prod\limits_{(j\neq i)}\left(a_i-a_j\right)$

如果n=1,则 $f_1(x)=1$ 显然$(\neq 0)$线性无关

如果n≥2，而 $f_1(x),f_2(x),\cdots,f_n(x)$ 线性相关,则不妨设 $f_1(x)=\sum\limits_{i=2}k_if_i(x)$

但是在 $x=a_1$ 处值,右边恒为 0,左边为 $f_1(a_1)=\prod\limits_{j=2}^{n}(a_1-a_j)\neq 0$

矛盾$\therefore f_1(x),f_2(x),...,f_n(x)$线性无关,而dimP[X]=n

以及单个 $f_i(x),次数\leq n-1,\therefore f_i(x)\in p[x]_n.故诸f_i(x)形成基$

P271 补 1② $x^n=1的单数根为\omega,\omega^2,\omega^3,...,\omega^n=1,(\omega本原的)$ $a_i=\omega^i$

$$\therefore f_i(x)=\frac{f(x)}{x-a_i}=\frac{x^{n-1}}{x-w^i}=\frac{\omega^u-(\omega^i)^u}{x-\omega^i}=x^{n-1}+\omega^ix^{u-2}+\omega^ix^{u-3}+...+\omega^{(n-1)i}$$

$$(f_1(x),f_2(x),...,f_n(x))=(1,x,x^2,,x^{n-1})\begin{pmatrix}1&1&...&1&1\\\omega&\omega^2&...&\omega^{n-1}&1\\\omega^2&\omega^4&...&\omega^{2(n-1)}&1\\...&...&...&...&...\\\omega^{n-1}&\omega^{2(n-1)}&...&\omega^{(n-1)^2}&1\end{pmatrix}$$

$$\therefore T=\begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ \omega & \omega^2 & \ldots & \omega^{n-1} & 1 \\ \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)^2} & 1 \end{pmatrix}$$

（P271 补 2）因 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 线性无关，设秩 $(A)=r,\exists,p,Q$ 可逆使

$$A=P\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}_{n\times s}Q$$

所以 $(\beta_1,\beta_2,\cdots,\beta_s)=(\alpha_1,\alpha_2,\cdots,\alpha_n)A$

$$=(\alpha_1,\alpha_2,\cdots,\alpha_n)P\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}Q=(r_1,r_2,\cdots,r_n)\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}Q=(r_1,r_2,\cdots,r_n)\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}Q$$

$=(r_1,r_2,...,r_r,0,...,0)$

$\therefore \beta_1,\beta_2,\cdots,\beta_S\leftarrow r_1,r_2,\cdots,r_r$ 但Q可逆 $(r_1,\cdots,r_r,0,\cdots,0)=(\beta_1,\cdots,\beta_s)Q^{-1}$

$\therefore r_1,r_2,\cdots,r_r\leftarrow \beta_1,\beta_2,\cdots,\beta_s$ 由定理了

$\dim(L(\beta_1,\beta_2,\cdots,\beta_s))=dim(L(r_1,r_2,\cdots,r_r))=$ 秩 $(r_1,r_2,\cdots,r_r)=r=$ 秩 (A)

P271 补 3，设 $f(x_1,x_2,\cdots,x_m)=x'AX$

由秩 $(f)=n,f$ 的符号差为S，那么f的攒性指数 $p=\frac{n+s}{2}$

f的负惯性指数 $q=\frac{1}{2}(n-s)$

非退化线性替换，使

$$f(x_1\cdots x_n)=g(y_1\cdots y_n)=y^2{}_1+\cdots+y_p{}^2-y_{m+1}{}^2-\cdots-y^2{}_n$$

作 n 维向量

$$y_1=\begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix}=\varepsilon_1+\varepsilon_{p+1},y_2=\begin{pmatrix}0\\1\\0\\0\\1\\0\end{pmatrix}=\varepsilon_2+\varepsilon_{p+2},\cdots,y_q=\varepsilon_q+\varepsilon_{p+q}=\varepsilon_q+\varepsilon_n$$

那么$g\left(y_1,\cdots, y_n\right)$在$y_i$处的值为 0 且，若

$$y_o=b_1y_1+b_2y_2+\cdots+b_qy_q$$

则$g\left(y_1,\cdots,y_n\right)$在$y_o$的值为$b_1{}^2+b_2{}^2+\cdots+b_q{}^2-b_1{}^2-b_q{}^2=0$

于是$V_2=L\left(y_1,y_2,\cdots y_q\right)$及$V_1=L\left(cy_1,cy_2,\cdots,cy_q\right)$

$\therefore f$在V_1中任意处$\sum_{i=1}^{a}q_i\left(cy_i\right)$的值，等于q在$\sum_{i=1}^{q}a_2y_i$的值为 0

$\because y_1,y_2,\cdots,y_q$线性无关，$\therefore x_1=cy_1,x_2=cy_2,\cdots x_t=cy_q$线性无关

$\therefore \dim V_1=q=\frac{1}{2}\left(n-s\right)=\frac{1}{2}\left(n-/s/\right)$

b^o如果$s<0$则$p<q$作$x=cy, f=g\left(y_1,\cdots,y_n\right)$为规范形

这对可取$y_1,y_2,\cdots,y_p$生成V_2且V_2要化g

同时作$V_1=g\left(x_1,x_2,\cdots,x_p\right)=L\left(y_1,y_2cy_p\right)$

$\left(同a^o\right)V_1即使得f\ /V_1=0且$

$$\dim V_1=P=\frac{1}{2}\left(n+s\right)=\frac{1}{2}\left(n-/s/\right)$$

$\left(P271补4\right)$证法一：　　$\because V_1\neq V,V_2\neq V$

故若$V_1\subseteq V_2则取一\alpha\notin V_2\left(\alpha必存在\right)即可$

若$V_1\supseteq V_2则任取一\alpha\notin V_1\left(\alpha也存在\right)即可$

若$V_1\notin V_2,V_2\notin V_1则可取\alpha\in V_1,\alpha\notin V_2和\beta\in V_2,\beta\notin V_1$

$\because \alpha+\beta\in V_2\Rightarrow\beta\in V_1矛盾,\alpha+\beta\in V_2也矛盾\therefore\alpha+\beta\notin V_1,\notin V_2即为所求.$

证法 2：若$V_1\subseteq V_2或V_2\subseteq V_1,同上显然$

当$V_1\subseteq V_2,V_2\not\subseteq V_1时,取\alpha\notin V_1,\beta\in V_1,\beta\in V_2,考虑$一切的$\alpha+k\beta$如右图

$\left\{\alpha+k\beta\right\}$理解为$V_1$的平行体.

断言$\left(a\right)若有\alpha\in P,\alpha+k\beta\in V_1\left(b\right)$至多存在一个k，使$\alpha+k\beta\in V_2$

证（a）有$k\in p,\alpha+k\beta\in V_1\because\beta\in V_1\Rightarrow\alpha=\left(\alpha+k\beta\right)-k.\beta\in V_1,$矛盾！

（b）若有$k_1\neq k_2\in p使\alpha+k_1\beta\in V_2,\alpha+K_2\beta\in V_2则,k_1\beta-k_2\beta\in V_2$

$\Rightarrow (k_1 - k_2)\beta \in V_2 \Rightarrow \beta \in V_2$ 矛盾！

故若 $k_o \in p\alpha + k_o\beta \notin V_2$，则 $\alpha + k_o\beta$ 即为所求，$\notin V_1, \notin V_2$，

若 $\alpha + k_o\beta \in V_2$ 则任取 $k_1 \neq k_o$ 有 $\alpha + k_1\beta \notin V_1, \notin V_2$ 即为所求

证法二虽然思想复杂，却可以把问题做大

（P272）补 5

$s = 1$ 虽然 $(\because V_1$ 非平凡，$\therefore V_1 \neq V)$

$s = 2$ 命题已证，即第 4 题设S=k时命题成交，考虑s=k+1 时，$V_1, V_2, \cdots V_k, V_{k+1}$ 皆非平凡了空间对于 $V_1, V_2, \cdots, V_k$ 任取 $\alpha \notin V_{k+1}$ $(\because V_{K+1} \neq V)$ 考虑一切 $\alpha + k\beta$

同样（类似 4 题证法 $(P_6, 92, 12, 3)$），$\forall k \in p, \alpha + k\beta \in V_{k+1}$ 对每个 V_i $(i = 1.2 \cdots k)$ 至多只须一个使，$\alpha + k_i\beta \in V_i$

取 r_o 为不同于 $r_1, r_2, \cdots, r_k, 1$ 的任一数即

$$t_o \in p - \{t_1, t_2, \cdots, V_K, V_{k+1}\}$$

那么 $l_o \notin V_1, V_2, \cdots, V_K, V_{k+1}$ $\therefore \alpha + t_o\beta$ 即为所求

第九章 第九章 欧几里得空间习题解答

P394.1.1

A 正定 $\therefore (\alpha, \alpha) = \alpha A \alpha' \geq 0 ("=" \Leftrightarrow \alpha = 0)$ 非负性证得

由矩阵失去，线性性成立，再由 $(\beta, \alpha) = \beta A \alpha' = (\beta A \alpha')' = \alpha' A' \beta = (\alpha, \beta)$ 对称性成立，是一个内积

P394.1.2 $\left(\varepsilon_i,\varepsilon_j\right)=(0\cdots\overset{i}{1}0\cdots 6)A\begin{pmatrix}6\\1\\1\\9\end{pmatrix};=\alpha_{ij}$

$\therefore\varepsilon_i,\varepsilon_j,\cdots\varepsilon_n$的度量矩阵即为A

P394.1.2 $|(\alpha,\beta)|\le|\alpha||\beta|$

$$\because(\alpha,\beta)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_iy_j$$

$$\therefore c-s-B\text{不等式为}|(\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_iy_j)|\le\sqrt{(\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j)(\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}y_iy_j)}$$

$P393.2$ ①, α =(2,1,3,2), β =(1,2,-2,1)

$\therefore|\alpha|=\sqrt{18}=\sqrt[3]{2},|\beta|=\sqrt{10},(\alpha,\beta)=0,\therefore\alpha\perp\beta$

$\therefore\langle\alpha,\beta\rangle=\dfrac{\pi}{2}$

$P393.2$ ②, α =(1,2,2,3), β =(3,1,5,1)

$|\alpha|=\sqrt{18}=\sqrt[3]{2},|\beta|=\sqrt{36}=6,(\alpha,\beta)=18$

$\therefore(\alpha,\beta)=\text{arc cos}\dfrac{(\alpha,\beta)}{|\alpha||\beta|}=arc\ \cos\dfrac{18}{\sqrt[3]{2},6}=arc\ \cos\dfrac{\sqrt{2}}{2}=\dfrac{\pi}{4}$

$P393.2$ ③, α =(1,1,1,2), β =(3,1,-1,0)

$|\alpha|=\sqrt{7}\quad |\beta|=\sqrt{11}\quad (\alpha,\beta)=3$

$\therefore\langle\alpha,\beta\rangle=\text{arc cos}\dfrac{3}{\sqrt{77}}=70°0'30"38$

P393. 3 $\because|\alpha+\beta|\le|\alpha|+|\beta|$

$\therefore d(\alpha,\gamma)=|\alpha-\gamma|=|(\alpha-\beta)+(\beta-\gamma)|\le|\alpha-\beta|+|\beta-\gamma|$

$=d(\alpha,\beta)+d(\beta,\gamma)$

P393.4 在R^4中求一单位向量与(1, 1, −1), (1, −1, 1−, 1), (2, 1, 1, 3)正交

解设所求

$\alpha=(x_1,x_2,x_3,x_4)$则$\sum x_i^2=1$,且

x与各向量的内积为0得

$$\begin{cases} x_1+x_2-x_3+x_4=0 \\ x_1-x_2-x_3+x_4=0 \\ 2x_1+x_2+x_3+3x_4=0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 2 & 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & +1 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \text{令} x_4=3, \text{得}$$

$x_1=4, x_2=0, x_3=-1$

$\alpha=\dfrac{1}{\pm\sqrt{26}}(-4,0,-1,3)$,　　(单位化)

$P393.5$①证:因为$(\gamma,\alpha_i)=0, i=1.2\cdots n$,而$\alpha_1,\alpha_2\cdots\alpha_n$是一个基

$\therefore(\gamma,\gamma)=(\gamma,\sum_{i=1}^{n}k_i\alpha_i)=\sum_{i=1}^{n}k_i(\gamma,\alpha_i)=0.$

因此,必有$\gamma=0.$

$P393.5$②证，$\because(\gamma_1,\alpha_i)=(\gamma_2,\alpha_i),\quad i=1.2\cdots n,$

$\therefore(\gamma_1-\gamma_2,\alpha_i)=0, i=1.2\cdots n$

由第①小题：$\gamma_1-\gamma_2=0$,故$\gamma_1=\gamma_2$

$$\because(\alpha_1,\alpha_2,\alpha_3)=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{pmatrix}$$

P393.6

而$\dfrac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{pmatrix}$是正交矩阵,所以$\alpha_1,\alpha_2,\alpha_3$是标准正交基

$P393.7$

$\alpha_1=\varepsilon_1\varepsilon_s,\alpha_2=\varepsilon_1-\varepsilon_2+\varepsilon_4/\varepsilon_3=2\varepsilon_1+\varepsilon_2+\varepsilon_3$

解$:\beta_1=\alpha_1$

$$\beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=\alpha_2-\frac{1}{2}\beta_1=\frac{1}{2}\varepsilon_1-\varepsilon_2+\varepsilon_4-\frac{1}{2}\varepsilon_5=\frac{1}{2}(\varepsilon_1-2\varepsilon_2+2\varepsilon_4-\varepsilon_5)$$

$$\beta_3=\alpha_3-\frac{2}{2}\beta_1-\frac{1}{10}\beta_2=\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_5$$

再正交化称:

$$\eta_1=\frac{1}{\sqrt{2}}(\varepsilon_1+\varepsilon_5)$$

$$\eta_2=\frac{1}{\sqrt{10}}(\varepsilon_1-2\varepsilon_2+2\varepsilon_4-\varepsilon_5)$$

$$\eta_3=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_5)$$

P394.8，解：

$$\begin{pmatrix}2&1&-1&1&-3\\1&1&-1&0&1\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\\X_4\\X_5\end{pmatrix}=0\rightarrow\begin{pmatrix}1&0&0&1&-4\\0&1&-1&-1&5\end{pmatrix}X=0$$

解出：

$$\eta_1=\begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}\qquad\eta_2=\begin{pmatrix}-1\\1\\0\\1\\0\end{pmatrix}\qquad\eta_3=\begin{pmatrix}4\\-5\\0\\0\\1\end{pmatrix}$$

Schmidt:

$$\beta_1=\begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}\qquad\beta_2=\eta_2-\frac{1}{2}\beta_1=\frac{1}{2}\begin{pmatrix}-2\\1\\-1\\2\\0\end{pmatrix}\qquad\beta_3=\eta_3+\frac{5}{2}\beta_1+\frac{13}{10}\begin{pmatrix}-2\\1\\-1\\2\\0\end{pmatrix}=\frac{1}{5}\begin{pmatrix}7\\-6\\6\\13\\5\end{pmatrix}$$

单位化便得到解空间的标准正交基：

$$\varepsilon_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \qquad \varepsilon_2 = \begin{pmatrix} -2/\sqrt{10} \\ 1/\sqrt{10} \\ -/\sqrt{10} \\ 2/\sqrt{10} \\ 0 \end{pmatrix} \qquad \varepsilon_3 = \frac{1}{\sqrt{315}} = \begin{pmatrix} 7 \\ -6 \\ 6 \\ 13 \\ 5 \end{pmatrix}$$

P394.9 $(f,g) = \int_{-1}^{1} f(x)g(x)dx$

已知 $\alpha_1 = 1, \quad \alpha_2 = x, \quad \alpha_3 = x^2, \quad \alpha_4 = x^3$

解：$\beta_1 = \alpha_1 = 1$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1 = x - \frac{\int_{-1}^{-} xdx}{*} x$$

$$\beta_3 = \alpha_2 - \frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1 - \frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2 = x^2 - \frac{\frac{2}{3}}{2}1 - 0 - = x^2 - \frac{1}{3}$$

$$\beta_4 = \alpha_4 - \frac{(\alpha_4,\beta_1)}{(\beta_1,\beta_1)}\beta_1 - \frac{(\alpha_4,\beta_2)}{(\beta_2,\beta_2)}\beta_2 - \frac{(\alpha_4,\beta_3)}{(\beta_3,\beta_3)}\beta_3 = \alpha_4 - 0 - \frac{\frac{2}{5}}{\frac{2}{3}}x = x^3 - \frac{3}{5}x$$

$$又 \because (\beta_1,\beta_1) = 2 \quad |\beta_1| = \sqrt{2}, \quad (\beta_2,\beta_2) = \frac{2}{3} \quad |\beta_2| = \frac{2}{\sqrt{6}}$$

$$(\beta_3,\beta_3) = \int_{-1}^{+1}(x^4 - \frac{2}{3}x^2 + \frac{1}{9})dx = \frac{8}{45} \quad |\beta_3| = \frac{4}{\sqrt[3]{10}}$$

$$(\beta_4,\beta_4) = \int_{-1}^{1}(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2)dx = \frac{8}{175} \quad |\beta_4| = \frac{4}{\sqrt[5]{14}}$$

单位化标准正交基

$$\gamma_1 = \frac{1}{\sqrt{2}}, \quad \gamma_2 = \frac{\sqrt{6}}{2}x, \quad \gamma_3 = \frac{\sqrt{10}}{4}(3x_2 - 1), \quad \gamma_4 = \frac{\sqrt{14}}{4}(5x^3 - 3x)$$

$P396.17.4$

$$A = \begin{pmatrix} -1 & -3 & 3 & -3 \\ -3 & -1 & -3 & 3 \\ 3 & -3 & -1 & -3 \\ -3 & 3 & -3 & -1 \end{pmatrix} \qquad A4E = \begin{pmatrix} -3 & -3 & 3 & -3 \\ -3 & 3 & -3 & 3 \\ 3 & -3 & 3 & -3 \\ -3 & 3 & -3 & 3 \end{pmatrix}$$

$\because$ 秩(A+4E)=1 $\quad \therefore \lambda_1 = 4$至少为$Acy3$重根，而

$-(4+4+4)+\lambda_2 = Tr(A) = -4. \Rightarrow \lambda_2 = 8$

$\lambda_1 = -4$解(A+4E)x=o，即$\chi_{1-}\chi_2 + x_3 - x_4 = 0$

得正交基础体系

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\quad\begin{pmatrix}1\\-1\\-2\\0\end{pmatrix}\quad\begin{pmatrix}1\\-1\\1\\3\end{pmatrix}$$

单位化为

$$\frac{1}{12}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\quad\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\-2\\0\end{pmatrix}\quad\frac{1}{\sqrt{12}}\begin{pmatrix}1\\-1\\-\\3\end{pmatrix}$$

$\lambda_2$8解(A-8E)x=0.得解取自A+4E的一列

$$\begin{pmatrix}3\\-3\\3\\-3\end{pmatrix}$$

单位化为

$$\frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

$$令T=\begin{matrix}1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/2\\ 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{12} & -1/2\\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/2\\ 0 & 0 & 3/\sqrt{12} & -1/2\end{matrix}\qquad 则T'AT=T^{-1}AT=\begin{pmatrix}-4 & & & \\ & -4 & & \\ & & -4 & \\ & & & 8\end{pmatrix}$$

$P395.10.1\qquad 0\in V_1\neq\varnothing$

$$\forall\beta_1,\beta_1\in V_1\qquad\left.\begin{matrix}(\beta_1,\beta_2,\alpha)=(\beta_1,\alpha)+(\beta_2,\alpha)=0\Rightarrow\beta_1+\beta_2\in V_1\\(k\beta_1,\alpha)=k(\beta_1,\alpha)=0\Rightarrow k\beta_1\in V\end{matrix}\right\}\therefore V_1\leq V.$$

P395.10.2 $\quad\because\alpha\neq0\qquad\therefore\alpha\notin V_1\qquad$ 故$\dim V_1\leq n-1.$

将α扩充为V的一个正交基$\alpha_1=\alpha,\alpha_2,\alpha_3\cdots\alpha_n$,那么.

$\alpha i\in V_1(i\geq2)\qquad\therefore L(\alpha_2,\alpha_3\cdots\alpha_n)\leq V_1\Rightarrow\dim V_1\geq n-1.$

$\therefore\dim V_1\geq n-1.$

P394，11①设两个基：$\varepsilon_1,\varepsilon_{2,}\cdots\varepsilon_n$及$\eta_1,\eta_2,\cdots\eta_n$,它们的度量矩阵分别为A和B,并设

$$(\eta_1,\eta_2,\cdots,\eta_n)=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)C$$

任设$\alpha,\beta\in V,\alpha=(\varepsilon_1,\cdots,\varepsilon_n)X_1=(\eta_1,\eta_2,\cdots,\eta_n)X_2$

$$\beta=(\varepsilon_1,\cdots,\varepsilon_n)Y_1=(\eta_1,\eta_2,\cdots,\eta_n)Y_2$$

所以$X_1=CX_2,Y_1=CY_2$

$$(\alpha,\beta)=X_2^{'}BY_2=X_1^{'}AY_1=X_2^{'}(C'AC)Y_2$$

$\therefore C'AC=B$(合同)

P394.11②，

取V的一个基$\alpha_1,\alpha_2,\cdots\alpha_n$,其度量矩阵为$A$,因为A正交,故存在矩阵C,使

C'AC=E

做基$(\eta_1,\eta_2,\cdots,\eta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)$C,那么$\eta_1,\eta_2,\cdots\eta_n$的度量矩阵为$C'AC=E$,
因此$\eta_1,\eta_2,\cdots,\eta_n$为标准正交基.

P394.12,$\alpha_1,\alpha_2,\cdots,\alpha_m\in V$　　$\alpha_{ij}=(\alpha_i,\alpha_j)$记:

$$G(\alpha_1,\alpha_2,\cdots,\ \alpha_m)=(\alpha_{ij})_{m\times m}$$

称$G(\alpha_1,\alpha_2,\cdots,\ \alpha_m)$为$\alpha_1,\alpha_2,\cdots,\ \alpha_m$的Gram矩阵

称$|G(\alpha_1,\alpha_2,\cdots,\ \alpha_m)|$为$\alpha_1,\alpha_2,\cdots,\ \alpha_m$的Gram行列式

证明$\alpha_1,\alpha_2,\cdots,\ \alpha_m$线性无关$\Leftrightarrow\left|G(\alpha_1,\alpha_2,\cdots,\ \alpha_m)\right|\neq 0$

证:若m=1, α_1线性无关$\Leftrightarrow(\alpha_1,\ \alpha_1)>0\Leftrightarrow|G(\alpha_1)|=|\alpha_1|^2\neq 0$,成立

若$m>1$,而$|G(\alpha_1,\alpha_2,\cdots,\ \alpha_n)|=0$

不妨设$A=(\beta_1,\beta_2,\cdots,\ \beta_m)$

$$\Leftrightarrow\beta_j=\sum_{\substack{k=1\\k\neq j}}^{m}c_k\beta_k\Leftrightarrow\alpha_{ij}=\sum_{k\neq j}c_k a_{ik}=\sum_{k\neq j}c_k(\alpha_i,\alpha_k)$$

$$\Leftrightarrow(\alpha_i,\alpha_j-\sum_{k\neq j}c_k,\alpha_k)=0,\therefore\Leftrightarrow\gamma=0,i=1,2,\cdots m.$$

$$\because\gamma=\alpha_j-\sum_{k\neq j}c_k\alpha_k\in L(\alpha_1,\alpha_2,\cdots,\alpha_m),$$

$$\Leftrightarrow\alpha_j=\sum_{k\neq j}ck\alpha_k\Leftrightarrow\alpha_1,\alpha_2,\cdots,\alpha_m\text{线性相关}$$

$$|G(\alpha_1)|=|\alpha_1|^2$$

$$|G(\alpha_1,\alpha_2)|=\begin{vmatrix}(\alpha_1,\alpha_2),(\alpha_1,\alpha_2)\\(\alpha_2,\alpha_1),(\alpha_2,\alpha_2)\end{vmatrix}=\begin{vmatrix}|\alpha_1|^2 & |\alpha_2||\alpha_1|\cos\theta\\|\alpha_1||\alpha_2|\cos\theta & |\alpha_2|^2\end{vmatrix}$$

$$=|\alpha_1|^2|\alpha_2|^2\,(1-\cos^2\theta)=(|\alpha_1||\alpha_2|\cos\theta)^2$$

类似地：

$$|G(\alpha_1,\alpha_2,\alpha_3)|=(\text{平行六面体积})^2$$

P394，13，设：
$$A=\begin{pmatrix}\alpha_n & \alpha_n & \cdots & \alpha_{1n}\\ 0 & \alpha_{22} & \cdots & \alpha_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha_{nn}\end{pmatrix}$$

因为A正交，故A'A=E，令A=$(\beta_1,\beta_2,\cdots\beta_n)$

由第 1 行列，$\alpha_{11}^2=1,\alpha_{11}=\pm1$

由β_1与其余各列正交，$\beta_1\perp\beta_j$（j>1），（β_1，β_j）$=a_{11}\alpha_{1j}=0\Rightarrow a_{1j}=0(j>1)$

$$\therefore A=\begin{pmatrix}\pm1 & 0\\ 0 & A_1\end{pmatrix}$$

其中A_1仍为上三角正交矩阵，但阶数少 1，故可用归纳法给出证明，且n=1 时显然为真，由归纳法原理，证毕。

P394，14①，设$A=(\alpha_1,\alpha_2,\cdots,\alpha_n)$，则$\alpha_1,\alpha_2,\cdots,\alpha_n$做成$\mathbb{R}^n$的一个基，用Schmidt方法把它们正交化$\varepsilon_1,\varepsilon_2,\cdots\varepsilon_n$，由定理2($P9,130,4.1$)

$$L(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_i)=L(\alpha_1,\alpha_2,\cdots,\alpha_i),\forall_i$$

$$\therefore(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}t_{11} & t_{12} & \cdots & t_{1n}\\ & t_{22} & \cdots & t_{in}\\ & & \ddots & t_{n-1n}\\ & & & t_{nn}\end{pmatrix},t_{ii}>0$$

$$\text{令}Q=(\varepsilon_1,\varepsilon_2,\cdots\varepsilon_n)\text{正交},T_1=\begin{pmatrix}t_{11} & \cdots & t_{1n}\\ & \ddots & \\ 0 & & t_{nn}\end{pmatrix},T=T_1^{-1}$$

$$\therefore Q=AT_1$$

$$A=QT_1^{-1}=QT$$

(唯一性)若$A=QT=Q_2T_2$

$\therefore Q_2^{-1}Q = T_2T^{-1}$

$\therefore$ 上三角矩阵T_2T为正交矩阵$Q_2^{-1}Q$

$\because T_2, T$的对角线皆大于0, $\therefore T_2T^{-1}$的对角线皆大于0, 由13题（见P19, 138, 10.1）

T_2T^{-1}=E，$\therefore$ T_2=T，满秩

$\therefore$ Q_2=Q

P394，14②，$\because$ A正交，则存在C可逆使

A=C'C

而 C 可逆，由①，有 C=QT，Q 正交 T 上三角。

$\therefore$ A=C'C=T'Q'QT=T'ET=T'T

P395，15①，A$\alpha = \alpha - 2(\eta,\alpha)\eta$　　　$\eta \in V$为一单位向量

$\because (A\alpha, A\beta) = (\alpha - 2(\eta,\alpha)\eta, \beta - 2(\eta,\beta)\eta)$

$= (\alpha,\beta) - 2(\eta,\alpha)(\eta,\beta) - 2(\eta,\beta)(\alpha,\eta) + 4(\eta,\alpha)(\eta,\beta)(\eta,\eta)$

$= (\alpha,\beta)$　　　$\therefore A$保持内积

又$A(k\alpha + l\beta) = k\alpha + l\beta - (2\eta, k\alpha + l\beta)\eta$

$= k(\alpha - 2(\eta,\alpha))\eta + l(\beta - 2(\eta,\beta))\eta = kA\alpha + lA\beta$

$\therefore A$为线性的, 故A是正交变换

P395，15②，以η为起点,扩充一个标准正交基, $\varepsilon_1 = \eta, \varepsilon_2, \varepsilon_3, \cdots \varepsilon_n$

则$A\varepsilon_1 = \varepsilon_1 - 2\varepsilon_1 = -\varepsilon_1$　　　$A\varepsilon_i = \varepsilon_i - 0 = \varepsilon_i (i \ge 2)$

$$\therefore A(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$\therefore |A| = -1$, 为第二类的

P395，15③，$\because 1$至少为A的$n-1$重特征值, 特征子空间V, $\dim V_1 = n-1$,

取V_1的标准正交基$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1}$扩充为$V$的标准正交基$\varepsilon_1, \cdots, \varepsilon_{n-1}, \varepsilon_n$

$\because A\varepsilon_i = \varepsilon_i (i = 1, 2, \cdots n-1)$, 而$A\varepsilon_n$与$A\varepsilon_i$正交$(i = 12\cdots n-1)$

$\therefore A\varepsilon_n \in (L(A\varepsilon_1, A\varepsilon_2, \cdots, A\varepsilon_{n-1}))^{\perp} = (L(\varepsilon_1, \varepsilon_2, \cdots \varepsilon_n))^{\perp} = L(\varepsilon_n)$

$A\varepsilon_n = \pm\varepsilon_n (\because A$正交), 若$A\varepsilon_n = \varepsilon_n \Rightarrow \dim V_1 = n$矛盾, $\therefore A\varepsilon_n = -\varepsilon_n$

$$\forall \alpha \in V,\quad \alpha = \sum_{i-1}^{n} x_i\varepsilon_i$$

$$A\alpha = \sum_{i-1}^{n-1} x_i\varepsilon_i - x_n\varepsilon_n = \alpha - 2x_n\varepsilon_n = \alpha - 2(\varepsilon_n, \alpha)\varepsilon_n$$

是一个镜面反射

$P395,16,$若$A'=-1,\quad \lambda_0$为A的特征值$,X_0$为其特征向量

$$X_0 \neq 0, \qquad \therefore \overline{X_0}'X_0 \neq 0$$

$$AX_0=\lambda_0 X_0 \qquad A\overline{X_0}=\overline{AX_0}=\overline{\lambda_0 X_0}=\overline{\lambda_0}\,\overline{X_0}$$

$$\therefore \lambda_0\overline{X_0}'X_0=\overline{X_0}'(\lambda_0 X_0)=\overline{X_0}'(AX_0)=-(\overline{X_0}'\overline{A}')X_0=-\left((\overline{AX_0})'\right)X_0$$

$$=-\left(\overline{AX_0}\right)'X_0=-\left(\overline{\lambda_0 X_0}'\right)X_0=-\overline{\lambda_0}\quad(\overline{X_0}'X_0)$$

$\therefore$(由$\overline{X_0}'X_0\neq 0$) $\quad\lambda_0=-\overline{\lambda_0} \qquad \lambda_0=0$或纯虚数

$$|\lambda E-A|=\begin{vmatrix}\lambda-2 & 2 & 0\\ 2 & \lambda-1 & 2\\ 0 & 2 & \lambda\end{vmatrix}=\lambda^3-3\lambda^2-6\lambda+8=(\lambda-1)(\lambda-4)(\lambda+2)$$

P395，17①：

$$\lambda_1=1,\lambda_2=4,\lambda_3=-2$$

解：(A−λ_1E)X=0

(A−E)X=0, X=(2, 1, −2)', $\varepsilon_1=\frac{1}{3}(2,1,-2)'$

解$:(A-4E)X=0,X=(2,-2,1)',\quad \varepsilon_2=\frac{1}{3}(2,-2,1)'$

解$:(A+2E)X=0,X=(1,2,2)',\quad \varepsilon_2=\frac{1}{3}(1,2,2)'$

令$T=\frac{1}{3}\begin{pmatrix}2 & 2 & 1\\ 1 & -2 & 2\\ -2 & 1 & 2\end{pmatrix}$

则$:T'AT=T^{-1}AT=\begin{pmatrix}1 & & \\ & 4 & \\ & & -2\end{pmatrix}$

P395，17②，A=$\begin{pmatrix}2 & 2 & -2\\ 2 & 5 & -4\\ -2 & -4 & 5\end{pmatrix}$

解：①：$\because A-\lambda E,(\lambda$用数1代$)$, A−$E$的秩为1

$$\therefore|\lambda E-A|=(\lambda-1)^2(\lambda-\lambda_2),\therefore \lambda_2=Tr(A)-\lambda_1-\lambda_1=10$$

$2°,(A-E)(A-10E)=0$

$$\lambda_1=1,\text{解}:x_1+2x_2-2x_3=0,\text{得}:\eta_1=\begin{pmatrix}1\\ -1/2\\ 0\end{pmatrix},\eta_2=\begin{pmatrix}1\\ 2\\ \frac{5}{2}\end{pmatrix}$$

$$\varepsilon_1=\begin{pmatrix}2/\sqrt{5}\\ -1/\sqrt{5}\\ 0\end{pmatrix},\qquad \varepsilon_2=\begin{pmatrix}2/\sqrt{45}\\ 4/\sqrt{45}\\ 5/\sqrt{45}\end{pmatrix}$$

$$\lambda_2=10,\text{取}A-E\text{的一列}:\eta_3=\begin{pmatrix}1\\2\\-2\end{pmatrix}\qquad \varepsilon_2=\begin{pmatrix}1/3\\ 2/3\\ -2/3\end{pmatrix}$$

$$3°,\text{取}T=\begin{pmatrix}2/\sqrt{5} & 2/\sqrt{45} & 1/3\\ -1/\sqrt{5} & 4/\sqrt{45} & 2/3\\ 0 & 5/\sqrt{45} & -2/3\end{pmatrix}$$

$$\text{则}:T'AT=T^{-1}AT=\begin{pmatrix}1 & & \\ & 1 & \\ & & 10\end{pmatrix}$$

$$A=\begin{pmatrix}0&0&4&1\\0&0&1&4\\4&1&0&0\\1&4&0&0\end{pmatrix}$$

P395,17③,

$$\text{解}:|A-\lambda E|=\begin{vmatrix}-\lambda&0&4&1\\0&-\lambda&1&4\\4&1&-\lambda&0\\1&4&0&-\lambda\end{vmatrix}=(5-\lambda)\begin{vmatrix}1&0&4&1\\1&-\lambda&1&4\\1&1&-\lambda&0\\1&4&0&-\lambda\end{vmatrix}=(5-\lambda)(3-\lambda)\begin{vmatrix}1&0&4&1\\1&-\lambda&1&4\\1&1&-\lambda&0\\0&1&-1&1\end{vmatrix}-$$

$$=(\lambda-3)(\lambda-5)\begin{vmatrix}1&0&4&1\\0&-\lambda&1&4\\4&1&-\lambda&0\\1&4&0&-\lambda\end{vmatrix}=(\lambda-3)(\lambda-5)\begin{vmatrix}-\lambda&-3&3\\1&\lambda-4&-1\\1&-1&1\end{vmatrix}$$

$$=(\lambda-3)(\lambda-5)\begin{vmatrix}-\lambda&-3-\lambda&3+\lambda\\1&-\lambda-3&-1-1\\1&0&0\end{vmatrix}=(\lambda-3)(\lambda-5)(\lambda+3)\begin{vmatrix}-1&1\\-\lambda-3&-2\end{vmatrix}$$

$$=(\lambda-3)(\lambda+3)(\lambda-5)(\lambda+5)$$

$$\lambda_1=3,\lambda_2=-3,\lambda_3=5,\lambda_4=-5$$

解:(A-$\lambda_1 E$)X=0

(A-3E)X=0,得:X=$\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$,$\varepsilon_1=(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})'$

解:(A$+$3E)X=0,得:X=(1,-1,-1,1)'$,\therefore\varepsilon_2=(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})'$

解:(A-5E)X=0,得:X=(1,1,1,1)'$,\therefore\varepsilon_3=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})'$

解:(A$+S\in$)X=0,得:X=(1,1,-1,-1)'$,\therefore\varepsilon_4=(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})'$

令:T=$\frac{1}{2}\begin{vmatrix}-1&1&1&1\\1&-1&1&1\\-1&-1&1&-1\\1&1&1&-1\end{vmatrix}$

则:T'AT=T^{-1}AT=$\begin{pmatrix}3&&&\\&-3&&\\&&5&\\&&&-5\end{pmatrix}$

$$A=\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$$

P395,17⑤:

解:秩(A)=1, $\therefore\lambda_1=0$为A的3重根(特征根)

$\lambda_1+\lambda_1+\lambda_1+\lambda_2=Tr(A),\therefore\lambda_2=4$

$\therefore A(A-4E)=0$

对于$\lambda_1=0$,解:$x_1+x_2+x_3+x_4=0$

得:

$$\alpha_1=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\1\\-2\\0\end{pmatrix},\alpha_3=\begin{pmatrix}1\\1\\1\\-3\end{pmatrix}\text{单位化}:\varepsilon_1=\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\\0\end{pmatrix},\varepsilon_2=\begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\\0\end{pmatrix},\varepsilon_3=\begin{pmatrix}1/\sqrt{12}\\1/\sqrt{12}\\1/\sqrt{12}\\-3/\sqrt{12}\end{pmatrix}$$

$$\text{对于}\lambda_2=4,\text{其解为}A\text{的一列}:\alpha_4=\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\varepsilon_4=\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$$

令T= $\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/2\\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/2\\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/2\\ 0 & 0 & -3/\sqrt{12} & 1/2\end{pmatrix}$

$$\text{则}T^{-1}AT=T'AT=\begin{pmatrix}0&&&\\&0&&\\&&0&\\&&&4\end{pmatrix}$$

P395，18①f=X' $f=X'\begin{pmatrix}1&-2&0\\-2&2&-2\\0&-2&3\end{pmatrix}X,B\begin{pmatrix}1&-2&0\\-2&2&-2\\0&-2&3\end{pmatrix}$

另解法，作正交矩阵，H= $\begin{pmatrix}&&1\\&1&\\1&&\end{pmatrix},(H^{-1}=H'=H)$

$$\therefore A=H'BH-E=\begin{Bmatrix}3&-2&0\\-2&2&-2\\0&-2&1\end{Bmatrix}E=\begin{Bmatrix}2&-2&0\\-2&1&-2\\0&-2&0\end{Bmatrix}\text{即为170中的}A(\text{见}P9,139,11,4)$$

$$\text{取}T=\frac{1}{9}\begin{pmatrix}2&2&1\\1&-2&2\\-2&1&2\end{pmatrix}\text{正交, 则T'AT=}\begin{pmatrix}1&&\\&4&\\&&-2\end{pmatrix}=T'(H'BH-E)T$$

$$\therefore(HT)'B(HT)=\begin{pmatrix}1&&\\&4&\\&&-2\end{pmatrix}+T'ET=\begin{pmatrix}2&0&0\\0&5&0\\0&0&-1\end{pmatrix},HT=\frac{1}{3}\begin{pmatrix}-2&1&2\\1&-2&2\\2&2&1\end{pmatrix}$$

$$\begin{cases} X_1 = -\frac{2}{3}y_1 + \frac{1}{3}y_2 + \frac{2}{3}y_3 \\ X_2 = \frac{1}{3}y_1 - \frac{2}{3}y_2 + \frac{2}{3}y_3 \\ X_3 = \frac{2}{3}y_1 + \frac{2}{3}y_2 + \frac{1}{3}y_3 \end{cases}$$

即令 则$f = 2y_1^2 + 5y_2^2 - y_3^2$

$$f = X'\begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix}X, B = \begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix}$$

P395,18②

又∵ A=-B+3E即为 17②中的A（见P9,138,10,3）

$$取T = \begin{pmatrix} 2/\sqrt{5} & 2/\sqrt{45} & 1/3 \\ -1/\sqrt{5} & 4/\sqrt{45} & 2/3 \\ 0 & 5/\sqrt{45} & -2/3 \end{pmatrix}, 则T'AT = T^{-1}(3E-B)T = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$

$$\therefore T^{-1}BT = T^{-1}(3E)T - \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix} = \begin{pmatrix} 2 & & \\ & 2 & \\ & & -7 \end{pmatrix}$$

令 $\begin{cases} x_1 = \frac{2}{\sqrt{5}}y_1 + \frac{2}{\sqrt{45}}y_2 + \frac{1}{3}y_3 \\ x_2 = -\frac{1}{\sqrt{5}}y_1 + \frac{4}{\sqrt{45}}y_2 + \frac{2}{3}y_3 \\ x_3 = \frac{5}{\sqrt{45}}y_2 - \frac{2}{3}y_3 \end{cases}$ ，则 $f = 2y_1^2 + 2y_2^2 - 7y_3^2$

$$f = X'\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}X, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

P395，18③,

$$解|\lambda E - B| = \lambda^2 - 1 = (\lambda-1),(\lambda+1), \therefore \varepsilon_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$令T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, 则T_1正交, 且$$

$$T_1'BT_1 = T_1^{-1}BT_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$令T=\begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix}也正交,\therefore T'AT=T^{-1}AT=\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$令\begin{cases} x_1=\dfrac{1}{\sqrt{2}}y_1+\dfrac{1}{\sqrt{2}}y_2 \\ x_2=\dfrac{1}{\sqrt{2}}y_1-\dfrac{1}{\sqrt{2}}y_2 \\ x_3=\dfrac{1}{\sqrt{2}}y_3+\dfrac{1}{\sqrt{2}}y_4 \\ x_4=\dfrac{1}{\sqrt{2}}y_3-\dfrac{1}{\sqrt{2}}y_4 \end{cases}$$

$则f=y_1^2-y_2^2+y_3^2-y_4^2$

P395，18④，

$$f=X'\begin{pmatrix} 1 & -1 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 3 & -2 & 1 & -1 \\ -2 & 3 & -1 & 1 \end{pmatrix}X=X'AX$$

$$解:|\lambda E-A|=\begin{vmatrix} \lambda-1 & 1 & -3 & 2 \\ 1 & \lambda-1 & 2 & -3 \\ -3 & 2 & \lambda-1 & 1 \\ -2 & -3 & 1 & \lambda-1 \end{vmatrix}=(\lambda-1)\begin{vmatrix} 1 & 1 & -3 & 2 \\ 1 & \lambda-1 & 2 & -3 \\ 1 & 2 & \lambda-1 & 1 \\ 1 & -3 & 1 & \lambda-1 \end{vmatrix}$$

$$=(\lambda-1)(\lambda-7)\begin{vmatrix} 1 & 1 & -3 & -2 \\ 1 & \lambda-1 & 2 & -3 \\ 1 & 2 & \lambda-1 & 1 \\ 0 & 1 & -1 & 1 \end{vmatrix}=(\lambda-1)(\lambda-7)\begin{vmatrix} 1 & -2 & -3 & -1 \\ 1 & \lambda+1 & 2 & -1 \\ 1 & \lambda+1 & \lambda-1 & \lambda \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$=(\lambda-1)(\lambda-7)(-1)(-1)(\lambda+3)\begin{vmatrix} 1 & -2 & -1 \\ 1 & \lambda+1 & -1 \\ 1 & \lambda+1 & \lambda \end{vmatrix}$$

$$=(\lambda-1)(\lambda-7)\begin{vmatrix} 1 & -2 & -1 \\ 0 & \lambda+3 & 0 \\ 0 & \lambda+3 & \lambda+1 \end{vmatrix}=(\lambda-1)(\lambda-7)(\lambda+1)(\lambda+3)$$

$\therefore \lambda_1=1,\lambda_2=7,\lambda_3=-1,\lambda_4=-3$

$(A-\lambda_1E)X=(A-E)X=0$, 得$X=(1, 1, 1, 1)'$, $\varepsilon_1=\dfrac{1}{2}(1,1,1,1)'$

$(A-\lambda_2E)X=(A-7E)X=0$, 得$X=(1, -1, 1, -1)'$, $\varepsilon_2=\dfrac{1}{2}(1,-1,1,-1)'$

$(A-\lambda_3E)X=(A+E)X=0$, 得$X=(1, 1, -1, -1)'$, $\varepsilon_3=\dfrac{1}{2}(1,1,-1,-1)'$

$(A-\lambda_4E)X=(A+3E)X=0$, 得$X=(1, -1, -1, 1)'$, $\varepsilon_4=\dfrac{1}{2}(1,-1,-1,1)'$

$$\therefore 令: T=\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}, 则 T'AT=T^{-1}AT=\begin{pmatrix}1&&&\\&7&&\\&&-1&\\&&&-3\end{pmatrix}$$

$$\left\{\begin{pmatrix}X_1\\X_2\\X_3\\X_4\end{pmatrix}=T\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}\right., 则$$

$$f=y_1^2+7y_2^2-y_3^2-3y_4^2$$

P395,19,$\because$ A实对称，存在正交矩阵T，使

$$T'AT=T^{-1}AT=\begin{pmatrix}\lambda_1&&&\\&\lambda_2&&\\&&\ddots&\\&&&\lambda_n\end{pmatrix}=D$$

其中λ_1，$\lambda_2,\cdots\lambda_n$为$A$的所有特征根

A正交$\Leftrightarrow D$正交$\Leftrightarrow$正惯性指数=n$\Leftrightarrow \lambda_i>0(\forall_i=1,2\cdots n)$

P396.20 “充分性”，设λ_1为A的实特征根，取λ_1的单位特征向量ε_1，扩充为$\mathbb{R}^n$的标准正交基$\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$, 取$T_1=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)$为正交矩阵

$$T_1^{-1}AT_1=T_1^1AT_1=\begin{pmatrix}\lambda_1&\alpha\\&A_1\end{pmatrix}$$

$\because|\lambda E-A|=(\lambda-\lambda_1)|\lambda E-A_1|$, 故$A_1$的特征根全为实根, 且阶数少, 故由归纳假设（n=1，显然成立），存在T_2正交。

$B_2=T_2A_1T_2$为上三角矩阵

作正交矩阵T,T_3

$$T_3=\begin{pmatrix}1&0\\0&T_2\end{pmatrix}, T=T_1T_3$$

那么, $T'AT=T^{-1}AT=T_3^{-1}(T_1^{-1}AT_1)T_3=T_3^{-1}\begin{pmatrix}\lambda_1 & \alpha \\ 0 & A_1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & T_2\end{pmatrix}$

$$\begin{pmatrix}\lambda_1 & \alpha T_2 \\ 0 & T_2^1 A_1 T_2\end{pmatrix}=\begin{pmatrix}\lambda_1 & \beta \\ 0 & B_1\end{pmatrix}$$

为上三角矩阵

"必要性",若$T'AT=T^{-1}AT=B$为上三角

$\because A,T\in\mathbb{R}^{n\times n},\therefore B\in\mathbb{R}^{n\times n}$

$\because|\lambda E-A|=|\lambda E-B|=\prod_{i=1}^{n}(\lambda-b_{ii})$全为实特征根$b_{11},b_{22},\cdots b_{nn}$,

P396，21“必要性”$T^{-1}AT=B$，则A，B相似，故特征值全部相同，

“充分性”，若A，B的特征值都由λ_1，λ_2，$\cdots\lambda_n$，

则存在，T_1，T_2正交，使

$$T_1'AT_1=T_1^{-1}AT_1=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix},\quad T_2'BT_2=T_2'BT_2\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}=D$$

$$\therefore T_1^{-1}AT_1=T_2^{-1}BT_2$$

令$T=T_1T_2^{-1}=T_1T_2'$也是正交的,且

$$T^{-1}AT=T_2(T_1^{-1}AT_1)T_2^{-1}=T_2DT_2^{-1}=B$$

P396,22,$A'=A,A^2=A$,证存在T正交, $T'AT=T^{-1}AT=\begin{pmatrix}1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}$

证:$\because$ A实对称, 故必有正交矩阵T使

$$T^{-1}AT=T'AT=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}$$

(其中特征值$\lambda_1\geq\lambda_2\geq\lambda_3\geq\cdots\geq\lambda_n$)

$\because A(A-E)=0$,最小多项式为$x(x-1)$的因式,$\therefore$特征多项式为$(x-1)^r x^{n-r}$

而λ_i为特征值,$X-\lambda_i\mid(x-1)^r X^{n-r}\Rightarrow\lambda_i=1$或0

$$\therefore\lambda_1=\lambda_2=\cdots=\lambda_r=1,\lambda_{r+1}=\cdots=\lambda_n=0$$

证毕.

$P396,23\quad A\in L(V)$为正交变换,子空间$W\le V$为A的不变子空间

$\forall\alpha\in W^{\perp}$,由于对$\forall\beta\in W$有$A\beta\in W$(必须假设dimW有限)

$\because$ dimW有限, $\therefore$ AW=W, $\forall\gamma\in$ W, 必有$\beta\in$ W, 使Aβ=γ

$\therefore (A\alpha,\gamma)=(A\alpha,A\beta)=(\alpha,\beta)=0$

因此,$A\alpha\in W^{\perp}$

故$W^{\perp}$也是A的不变子空间

(注:若dimW=∞, V=$\left\{\alpha=(x_1,x_2,\cdots,x_n\cdots)\mid x_i\text{中有限个非}0\right\}$,$W=\left\{\alpha\in V\mid x_1=\cdots=x_r=0\right\}$

$A\alpha=(0,x_1,x_2,x_3,\cdots x_n,\cdots)$.内积为对应分量之积之和,则$A$为正交变换.

W为$A-$子空间,$W^{\perp}=\left\{\alpha\in V\mid x_{r+1}=x_{r+2}=\cdots=x_n=\cdots=0\right\}$,不是$/A-$子空间

$\because \gamma=(0,0,\overset{(r个)}{\cdots},1,0,\cdots)\in W^{\perp}$,但$A\gamma=(0,\cdots,\overset{(r+1个)}{0},1,0,\cdots,0)\notin W^{\perp}$)

P396、24① “必要性”，若A反对称，在标准正交基$\varepsilon_1,\varepsilon_2,\cdots\varepsilon_n$下

$A(\varepsilon_1,\cdots\varepsilon_n)=(\varepsilon_1,\cdots\varepsilon_n)A\qquad A=(a_{ij})_{n\times n}$

则：$a_{ij}=(A\varepsilon_j,\varepsilon_i)=-(\varepsilon_j,A\varepsilon_i)=-(A\varepsilon_i,\varepsilon_j)=-a_{ij}$.

$\therefore A'=-A$

“充分性”，若$A(\varepsilon_1,\cdots\varepsilon_n)=(\varepsilon_1,\cdots\varepsilon_n)A\qquad\therefore A'=-A$

$\forall\alpha=(\varepsilon_1,\cdots\varepsilon_n)X\qquad\beta=(\varepsilon_1,\cdots\varepsilon_n)Y.\qquad A\alpha,A\beta$以坐标为AX、AY。

$\therefore (A\alpha,\beta)=(AX)'Y=X'A'Y=-X'AY=-X'(AY)=-(\alpha,A\beta)$

即，A 为反对称的

P396、242 设V_1为/A-子空间，A反对称。

W=$V^{\perp}\quad\forall\alpha\in W\quad\forall\beta\in V_1\quad\because$ A$\beta\in V_1$

$\therefore$（Aα，β）=（α，Aβ）=0

$\therefore A\alpha\in W$ 故W为A-子空间。

P397.25.设V=$V_1\oplus V_1^2$

必要性, 若α=β+γ,($\beta\in V_1,\gamma\in V_1^2$),则$\forall\xi V_1$. $\alpha-\beta\perp\beta-\xi$

$\therefore|\alpha-\xi|^2=|(\alpha-\beta)+(\beta-\xi)|^2=|\alpha-\beta|^2\ge|\alpha-\beta|^2$

即$|\alpha-\beta|\le|\alpha-\xi|$.

α在V_1的分解式, $\alpha=\alpha_1+\alpha_2\quad\alpha_1\in V_1\quad\alpha_2\in V_1^2$

于是$|\alpha-\alpha_1|\le|\alpha-\beta|\le|\alpha-\alpha_1|\qquad$又$\because\beta-\alpha_1\in V_1$

充分性，取$\therefore|\alpha-\alpha_1|^2=|\alpha-\beta|^2+|\beta-\alpha_1|^2\Rightarrow|\beta-\alpha_1|=0\Rightarrow\beta=\alpha_1$必为内射影.

$P396,26,$证 1）$(V_1+V_2)^{\perp}=V_1^{\perp}\cap V_2^{\perp}$ 和 $2)(V_1\cap V_2)^{\perp}=V_1^{\perp}+V_2^{\perp}$

证 1) $\because (V_1+V_2)^{\perp}\subseteq V_1^{\perp},V_2^{\perp}$ $\quad\therefore (V_1+V_2)^{\perp}\subseteq V_1^{\perp}\cap V_2^{\perp}$

反过来, $\forall\alpha\in V_1^{\perp}\cap V_2^{\perp}$, 则$\forall\beta\in V_1+V_2\quad \beta=\beta_1+\beta_2(\beta_i\in V_i)$

$\because \alpha\perp\beta_1,\alpha\perp\beta_2,\therefore\alpha\perp\beta_1+\beta_2=\beta\Rightarrow\alpha\in(V_1+V_2)^{\perp}$

$\therefore V_1^{\perp}\cap V_2^{\perp}\subseteq(V_1+V_2)^{\perp}$ $\quad$ 故$(V_1+V_2)^{\perp}=V_1^{\perp}\cap V_2^{\perp}$

2),由于正交补是唯一的

$\because (V_1\cap V_2)^{\perp}=\left((V_1^{\perp})^{\perp}\cap(V_2^{\perp})^{\perp}\right)^{\perp}=\left((V_1^{\perp}+V_2^{\perp})^{\perp}\right)^{\perp}=V_1^{\perp}+V_2^{\perp}$

$$P396,27\qquad A=\begin{pmatrix}0.39 & -1.89\\ 0.61 & -1.80\\ 0.93 & -1.68\\ 1.35 & -1.50\end{pmatrix}\qquad B=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$A'A=\begin{pmatrix}3.2116 & -5.4225\\ -5.4225 & 11.8845\end{pmatrix}\qquad A'B=\begin{pmatrix}3.28\\ -6.87\end{pmatrix}$$

$\therefore A'AX=A'B,\qquad |A'A|=8.76475395=d$

得$:d_x=1.728585,d_y=-4.277892$

$$\therefore X=\frac{d_x}{d}=0.197220025\approx 0.197\qquad M=\frac{d_y}{d}=-0.48867896\approx-0.488$$

设$A=(\alpha_1,\alpha_2)$,故B到子空间$W=L(\alpha_1,\alpha_2)$的垂足为$0.197\alpha_1-0.488\alpha_2$

B到W的距离为 $|B-0.197\alpha_1+0.488\alpha_2|$

$$=\left|\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-\begin{pmatrix}0.07683\\0.12017\\0.18321\\0.26595\end{pmatrix}-\begin{pmatrix}0.92232\\0.8784\\0.81984\\0.732\end{pmatrix}\right|-\left|\begin{pmatrix}0.00085\\0.00143\\-0.00305\\0.00205\end{pmatrix}\right|=\sqrt{0.000016272}=0.004033906\approx 0.00403$$

P397 补 1，设λ为A（正交）的特征值，定义AX=AX，则A为正交变换

$\because AX_0=\lambda X_0\Rightarrow|/AX_0|=|X_0|\Rightarrow|\lambda|=1\Rightarrow\lambda=\pm1,(\because\lambda\in\mathbb{R})$

P397，补 **2**， A 为 V 中正交变换，|A|=1

设A的特征值为$\lambda_1,\lambda_2,\cdots\lambda_n,\because|\lambda_i|=1$,及$f_A(x)$是一个实系数多项式,

$\therefore\lambda_1,\lambda_2,\cdots\lambda_n$中有$k$对共轭之积为1,剩下的n-2k个实根$\lambda_{i_1},\lambda_{i_2},\cdots\lambda_{i_{n-2k}}$之积为1。

$\because\lambda_{i_s}=\pm1,\therefore-1$的个数必为偶数个,而dimV为奇数,因此至少有一个特征值为实数1

P397 补 3（仿上题），∵|A|=-1，剩下的n-2k个实根之积为-1，其中必有特征值= -1。

P397,补 4：令$r=A(k\alpha+l\beta-kA\alpha-lA\beta,\quad\because A$得内积

$$\therefore(\gamma,\gamma)=A(k\alpha+l\beta),A(k\alpha+l\beta))-(A(k\alpha+l\beta),kA\alpha)-(A(k\alpha+l\beta),lA\beta)$$
$$-(kA\alpha,A(k\alpha+l\beta))+(kA\alpha,kA\alpha)+(k(A\alpha,lA\beta)-(lA\beta,A(k\alpha+l\beta))$$
$$+(lA\beta,kA\alpha)+(lA\beta,lA\beta)$$
$$=(k\alpha+l\beta,k\alpha+l\beta)-k(k\alpha+l\beta,\alpha)-l(k\alpha+l\beta,\beta)-k(\alpha,k\alpha+l\beta)+k^2(\alpha,\alpha)+kl(\alpha,\beta)$$
$$-l(\beta,k\alpha+l\beta)+kl(\beta,\alpha)+l^2(\beta,\beta)$$
$$=(k\alpha+l\beta-k\alpha-l\beta,k\alpha+l\beta-k\alpha-l\beta)=0,\therefore r=0$$
而$\forall k,l,\alpha,\beta,\ A(k\alpha+l\beta)=kA\alpha+lA\beta,\therefore A\in L(V)$,故$A$是正交变换

$P397$补5,"必要性":$(\beta_i,\beta_j)=(A\alpha_i,A\alpha_j)=(\alpha_i,\ \alpha_j)$

"充分性"(归纳法)$m=1$时,$|\alpha_1|=|\beta_1|$

作标准正交基$\varepsilon_1=\dfrac{1}{|\alpha_1|}\alpha_1,\varepsilon_2,\cdots,\ \varepsilon_n$及$\eta_1=\dfrac{1}{|\beta_1|}\beta_1,\eta_2,\cdots,\ \eta_n$,则线性变换$A:\varepsilon_i\to\eta_i$

是正交变换，则A $\alpha_1=A|\alpha_1|,\varepsilon_1=|\beta_1|$，$A\varepsilon_1=\beta_1$,而为所求设$m-1$成立,考虑m情形

由假设有正交变换A_1，$\alpha_i\to\beta_i,\alpha_m\to\tilde{\beta}_m,i=1,2,\cdots,m-1$，由于$A_1$保持内积及Gram矩阵，行列式的线性相关系。

$\beta_1,\cdots\beta_{m-1},\tilde{\beta}_m$与$\beta_1,\beta_2,\cdots\beta_{m-1},\beta_n$

任何一个局部的线性关系相同，设W=L$(\beta_1,\cdots\beta_{m-1})$

$V_1=L(\beta_1,\cdots,\ \beta_{m-1},\tilde{\beta})=W+L(\tilde{\beta}_m),V_2=L(\beta_1,\cdots,\ \beta_m)=W+L(\beta_m)$

(1)若$\tilde{\beta}_m\in W$,则$\beta_m\in W$,设W的标准正交基$\varepsilon_1,\varepsilon_2,\cdots,\ \varepsilon_r$

$\because(\tilde{\beta}_m,\beta_i)=(\beta_m,\beta_i)\Rightarrow\forall i(\tilde{\beta}_m,\varepsilon_i)=(\beta_m,\varepsilon_i)\Rightarrow\tilde{\beta}_m=\beta_m$,则$A$,即已为所求

(2)若$\tilde{\beta}_m\notin W$,则$\beta_m\notin W$,设V_1的标准正交基$\varepsilon_1,\cdots,\varepsilon_r,\tilde{\varepsilon}_{r+1},V_2$的标准正交基$\varepsilon_1,\cdots,\varepsilon_r$,
$\varepsilon_{\gamma+1}$,分别扩充为V的标准正交基,$\cdots\tilde{\varepsilon}_{\gamma+1}\cdots\tilde{\varepsilon}_n$,及$\cdots\varepsilon_{\gamma+1},\cdots\varepsilon_n$,作线性变换/A,

$\varepsilon_i\to\varepsilon_i,\tilde{\varepsilon}_j\to\begin{pmatrix}i\le r\\ j>v\end{pmatrix}$,是一个正交变换,而$\beta_m(\widetilde{\beta_m})$用$\varepsilon_1,\cdots,\varepsilon_r,\varepsilon_{r+1}$(或$\widetilde{\varepsilon_{r+1}}$)表示时的系数完全由$\beta_i,\beta_j$之间的内积确定,由充分已知条件,这些系数对应相等.

$\therefore A_2\tilde{\beta}_m=A_2(\alpha_1\varepsilon_1+\cdots+\alpha_r\varepsilon_r+\alpha_{r+1}\tilde{\varepsilon}_{r+1})=\alpha_1\varepsilon_1+\alpha_2\varepsilon_2+\cdots+\alpha_r\varepsilon_r+\alpha_{r+1}\varepsilon_{r+1}=\beta_m$

且A_2在W上不动,即,$A_2\beta_i=\beta_i(1\le i\le r)$

取$A=A_2A_1$的合成,则:$A:\alpha_i\to\beta_i\quad(i=1,2,\cdots m)$

且A为正交变换,即为所求.

$P397$补6,$\because$ A实对称,$\therefore$存在T正交,使

$$T'AT=T^{-1}AT=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix},\text{其中}\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n,\text{为}A\text{的特征值}$$

$\because A^2=E,\therefore A$的最小多项式为$x^2-1$的因式,故

A的特征多项式为$(x-1)^r(x+1)^{n-r}$,即A的特征值为1或-1

$\therefore$当$\lambda_i=\pm1$,因此$\lambda_1=\cdots=\lambda_r=1,\lambda_{r+1}=\cdots=\lambda_n=-1$

即:$T^{-1}AT=\begin{pmatrix}E_r & 0\\ 0 & -E_{n-r}\end{pmatrix}$

P397 补 7,作正交替换,X=TY,$\therefore X'X'=(Y'T')(TY)=Y'Y$

使$f=X'AX=\lambda_1y_1^2+\lambda_2y_2^2+\cdots+\lambda_ny_n^2,\qquad \lambda_1\le\lambda_2\le\cdots\le\lambda_n$

$\therefore f=X'AX\le\lambda_n(y_1^2+y_2^2+\cdots+y_n^2)=\lambda_nY'Y=\lambda_nX'X$

$f=X'AX\ge\lambda_1(y_1^2+y_2^2+\cdots+y_n^2)=\lambda_1Y'Y=\lambda_nX'X$,即得证

P397.补8设$f=X'AX$,且正交替换$X=TY$,使(λ_1=λ)

$$f=\lambda y_1^2+\lambda_2y_2^2+\cdots+\lambda_ny_n^2\qquad \text{令}Y_0=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\qquad \overline{X}=TY_0=\begin{pmatrix}\overline{x_1}\\\overline{x_2}\\\vdots\\\overline{x_n}\end{pmatrix}\in R^n$$

$$\therefore f(\overline{X})=f(\overline{x_1},\overline{x_2},\cdots,\overline{x_n})=\lambda=\lambda Y_0'Y_0=\lambda\left((T'\overline{X})'(T'\overline{X})\right)$$

$$=\lambda\left(\overline{X}'(TT')\overline{X}\right)=\lambda\left(\overline{X}'\overline{X}\right)=\lambda(\overline{x_1}^2+\overline{x_2}^2+\cdots+\overline{x_n}^2)$$

P397,补 **9**①,取 $\eta=\alpha-\beta\ne0,\eta_0=\dfrac{1}{|\eta|}\eta$

作镜面反射,$A:\xi\to\xi-2(\eta_0\xi)\eta_0,\forall\xi$

则$A\alpha=\alpha-2\left(\dfrac{1}{|\eta|}(\alpha-\beta),\alpha\right)\eta_0=\alpha-2\left(\dfrac{1}{|\eta|}(\alpha-\beta),\alpha\right)\dfrac{1}{|\eta|}\eta$

$$\because 2\left(\frac{1}{|\eta|}(\alpha-\beta),\alpha\right)\frac{1}{|\eta|}=2\frac{(\alpha-\beta,\alpha)}{(\alpha-\beta,\alpha-\beta)}=2\frac{|\alpha|^2-(\alpha,\beta)}{|\alpha|^2-2(\alpha,\beta)+|\beta|^2}=2\frac{1-(\alpha,\beta)}{2-2(\alpha,\beta)}=1$$

$\therefore A\alpha=\alpha-\eta=\beta,\qquad$ 即为所求.

P397,补 **9**②,设正交变换A,标准正交基:$\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n\to\eta_1,\eta_2,\cdots,\eta_n$

作镜面反射$B_1:\varepsilon_1\to\eta_1,\varepsilon_i\to B_1\varepsilon_i\quad$ (i>1)

$\therefore L(B\varepsilon_2,B\varepsilon_3,\cdots,B\varepsilon_n)=L(\varepsilon_1)^\perp\to L(\eta_1)^\perp=L(\eta_2,\eta_3,\cdots,\eta_n)$

不妨设B_K是一系列镜面反射使:

$\varepsilon_1\to\eta_1,\cdots,\varepsilon_k\to\eta_k,\quad \varepsilon_{k+1}\to B_k\varepsilon_{k+1},\cdots,\varepsilon_n\to B_k\varepsilon_n$

作一镜面反射 $\mathbb{C}_k:\xi_1=B_k\varepsilon_{k+1}-\eta_{k+1}\quad \xi_0=\dfrac{1}{|\xi_1|}\xi$

$\mathbb{C}_k : \alpha \to \alpha - 2(\xi_0, \alpha)\xi_0$ 使$B\varepsilon_{k+1} \to \eta_{k+1}$

$\because (\varepsilon_1, \cdots \varepsilon_k), \eta_1, \cdots \eta_k$与$\xi_1$正交, $\therefore l c_{kj}\eta_i \to \eta_i (1 \le i \le k)$

令$B_{k+1} = C_k B_k$是一系列镜面反射之积, 且

$\varepsilon_1 \to \eta_1, \cdots, \varepsilon_k \to \eta_k, \varepsilon_{k+1} \to \eta_{k+1}$

继续下去，n 步后必存一系列反射之积|B 使

$\varepsilon_1 \to \eta_1$, $\varepsilon_2 \to \eta_2$, $\cdots, \varepsilon_n \to \eta_n$

由线性变换的唯一存在性， $A = B_n$ 是一系列镜面反射之积

P397，补 10，设C可逆，使C'BC=E （$\because B$>0）

令A_1=C'AC，实对称，存在正交Q，使Q'AQ对角形

令 T=CQ，可逆，则

T'AT=Q'(C'AC)Q=Q'A·Q，对角形

T'BT=Q'(C'BC)Q=Q'EQ=E，对角形，（证毕）

$P398$,补11,设$:\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$及$\eta_1, \eta_2, \cdots, \eta_n$都为标准正交基,解

$$(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)A = (\eta_1, \eta_2, \cdots, \eta_n)$$

$\because \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$及$\eta_1, \eta_2, \cdots, \eta_n$的渡量矩阵都是单位矩阵$E$

任取$\alpha, \beta \in \mathbb{C}^n, \alpha = (\varepsilon_1, \varepsilon_2 \cdots \varepsilon_n) X_1 = (\eta_1, \eta_2 \cdots \eta_n) X_2$

$$\beta = (\varepsilon_1, \varepsilon_2 \cdots \varepsilon_n) Y_1 = (\eta_1, \eta_2 \cdots \eta_n) Y_2$$

其中$X_1 = AX_2, Y_1 = AY_2$

$$(\alpha, \beta) = (\sum x_i \varepsilon_i, \sum y_j \varepsilon_j) = \sum x_i \overline{y_j} (\varepsilon_i, \varepsilon_j) = X_1' \overline{Y_1} = X_2' A' \overline{AY_2}$$

$$= (\sum x_i' \eta_i, \sum y_j' \eta_j) = \sum x_i' \overline{y_j'} (\eta_i, \eta_j) = X_2' E \overline{Y_2}$$

由X_2, Y_2的任意性, $A' \overline{A} = E$ 故$\overline{A}' A = E$,即A为酉矩阵

$P398$补12,设A为酉矩阵,λ为其特征值,$X_0 \ne 0, AX_0 = \lambda X_0$

$$\therefore \overline{AX_0} = \overline{A}\overline{X_0} = \overline{\lambda}\overline{X_0} \qquad (\overline{AX_0})' = (\overline{\lambda X_0})' = \overline{\lambda}' \overline{X}'_0$$

$$\therefore |\lambda|^2 \overline{X_0}' X_0 = \overline{\lambda X_0}' \quad (\lambda X_0) = (\overline{AX_0})'(AX_0) = \overline{X_0}'(\overline{A}' A) X_0 = \overline{X_0}' X_0$$

$$\because \overline{X_0}' X_0 = |X_0|^2 \ne 0 \qquad \therefore |\lambda|^2 = 1 \qquad 即 |\lambda| = 1$$

$P398$,补13,设A复可逆， $A = (\alpha_1, \alpha_2, \cdots, \alpha_n)$,基$\alpha_1, \alpha_2, \cdots, \alpha_n$

用Schmidt方法，将基正交化,为$\beta_1, \beta_2, \cdots, \beta_n$,可知

$$(\beta_1, \beta_2 \cdots \beta_n) \begin{pmatrix} 1 & t_{12} & \cdots & t_{1n} \\ 0 & 1 & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = (\alpha_1, \alpha_2, \cdots, \alpha_n) = (\beta_1, \beta_2, \cdots, \beta_n) T_1$$

用将β_i单位化，$\gamma_i=\frac{1}{|\beta_i|}\beta_i$　　D=$\begin{pmatrix} |\beta_1|^{-1} & & & \\ & |\beta_2|^{-1} & & \\ & & \ddots & \\ & & & |\beta_n|^{-1} \end{pmatrix}$

可知标准正交基$(\gamma_1,\gamma_2,\cdots\gamma_n)=(\beta_1,\beta_2,\cdots\beta_n)D$

令$T_2=T_1^{-1}=\begin{pmatrix} 1 & t_{12}'' & \cdots & t_{1n}'' \\ 0 & 1 & \cdots & t_{2n}'' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$　　$D^{-1}=\begin{pmatrix} |\beta_1| & & & \\ & |\beta_2| & & \\ & & \ddots & \\ & & & |\beta_1| \end{pmatrix}$

$$\therefore A=(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\beta_1,\beta_2,\cdots,\beta_n)T_1=(\gamma_1,\cdots,\gamma_n)D^{-1}T_1$$

即为上三角且对角线上全大于 0,$(t_{ii}=|\beta_i|>0)$

其次，另设A=U_3T_3，一个分解，则，UT=U_3T_3

$$\tilde{U}=U_3^{-1}U=\overline{U}_3{}'U=T_3T^{-1}$$

为上三角的酉矩阵（类假正 13 题，P9，138，10.1 练习 13）

$$\therefore \tilde{U}\text{为对角矩阵, 对角线大于}0\Rightarrow \tilde{U}=E$$

$$\therefore T_3=T, U_3=U, (\text{唯一性证毕})$$

P398，补 14，$\overline{A}'=A$，若λ_0为特征值，则有$X_0\neq 0, AX_0=\lambda_0X_0$

$$\therefore \lambda_0\overline{X_0}'X_0=\overline{X_0}'(\lambda X_0)=\overline{X_0}'(AX_0)=(\overline{X_0}'A)X_0=(\overline{X_0}'\overline{A}')X_0$$

$$=\overline{(AX_0)}'X_0=\overline{(\lambda_0X_0)}'X_0=\overline{\lambda_0X_0}'X_0=\overline{\lambda_0}(\overline{X_0'}X_0)$$

$\because \overline{X_0}'X_0\neq 0$　　$\therefore \lambda_0=\overline{\lambda_0}$　　故$\lambda_0\in\mathbb{R}$

若A有两个特征值$\lambda\neq\mu$，且$X_1\neq 0, X_2\neq 0$，使$AX_1=\lambda X_1, AX_2=\mu X_2$

$$\therefore \lambda\overline{X}_2{}'X_1=\overline{X}_2{}'(\lambda X_1)=\overline{X}_2{}'(AX_1)=(\overline{X}_2{}'A)X_1=(\overline{X}_2'\overline{A}')X_1$$

$$=\overline{(AX_2)}'X_1=\overline{(\mu X_2)}'X_1=\overline{\mu}\overline{X_2}'X_1=\mu(\overline{X_2}'X_1)$$

$\because \lambda\neq\mu$　　$\therefore \overline{X}_2'X_1=0$

即$(X_1,X_2)=X_1'\overline{X_2}=(X_i\overline{X_2}')=\overline{X}_2'X_1=0$，　　$\therefore X_1\perp X_2$