SKETCH OF THE "ALTERNATING SQP" METHOD FOR FITTING POISSON TOPIC MODELS

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1. Some derivations. Given an $n \times p$ matrix of counts X with individual entries $x_{ij} \geq 0$, our aim is to fit a Poisson model of the counts,

$$\begin{aligned} p(x) &= \prod_{i=1}^{n} \prod_{j=1}^{p} p(x_{ij}) \\ &= \prod_{i=1}^{n} \prod_{j=1}^{p} \text{Poisson}(x_{ij}; \lambda_{ij}), \end{aligned} \tag{1.1}$$

in which the Poisson rates are given by the mixture $\lambda_{ij} = \sum_{k=1}^{K} l_{ik} f_{jk}$. Therefore, the Poisson model is specified by a $p \times K$ matrix F with entries $f_{ik} \geq 0$ (the "factors"), and an $n \times K$ matrix L with entries $l_{ik} \geq 0$ (the "loadings"). Fitting F and L is equivalent to non-negative matrix factorization with the "beta-divergence" cost function [2]. It can also be used to recover a maximum-likelihood estimate for the latent Dirichlet allocation (LDA) model [1]. So fitting this model is useful for a wide range of applications.

The log-likelihood for the Poisson model is

$$\log p(x \mid F, L) \propto \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ij} \log(\textstyle\sum_{k=1}^{K} l_{ik} f_{jk}) - \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{K} l_{ik} f_{jk}, \tag{1.2}$$

where the constant of proportionality is obtained from factorial terms in the Poisson densities. Our specific aim is to find a F and L that maximizes the log-likelihood (1.2); that is, we would like to solve

$$\begin{array}{ll} \text{minimize} & -\log p(x \mid F, L) \\ \text{subject to} & F \geq 0, L \geq 0. \end{array} \tag{1.3}$$

In the remainder, we derive an efficient approach to doing this.

Our strategy for solving (1.3) is to alternate between solving for F with L fixed, and solving for L with F fixed. When solving for F with L fixed, and vice versa, the problem naturally decomposes into a collection of much smaller subproblems that are more tractable to solve. All the subproblems are of the following form:

$$\begin{array}{ll} \text{minimize} & \phi(y; B, w) \\ \text{subject to} & y_k \geq 0 \text{ for all } k = 1, \ldots, K, \end{array} \tag{1.4}$$

in which the objective function is defined as

$$\phi(y; B, w) = -\sum_{i=1}^{n} w_i \log\left(\textstyle\sum_{k=1}^{K} b_{ik} y_k\right) + \displaystyle\sum_{i=1}^{n} \sum_{k=1}^{K} b_{ik} y_k. \tag{1.5}$$

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To see the connection between subproblem (1.4) and the original optimization problem, observe that the negative log-likelihood can be written as

$$-\log p(x \mid F, L) = \sum_{i=1}^{n} \phi(l_i; F, x_i), \tag{1.6}$$

where x_i is the ith row of X and l_i is the ith row of L, and it can additionally be written as

$$-\log p(x \mid F, L) = \sum_{j=1}^{p} \phi(f_j; L, x_i), \tag{1.7}$$

in which x_j is the jth column of X, and f_j is the jth row of F. Therefore, when F is unchanging, each row of L can be optimized separately by solving a problem of the form (1.4), and when L is fixed, each row of F can be optimized separately by solving a problem of the form (1.4).

Instead, we will solve the following optimization problem,

$$\begin{array}{ll} \text{minimize} & f(t; P, u) \\ \text{subject to} & t_k \geq 0 \text{ for all } k = 1, \ldots, K, \end{array} \tag{1.8}$$

in which the objective function is defined as

$$f(t; P, u) = -\sum_{i=1}^{n} u_i \log \left(\textstyle\sum_{k=1}^{K} p_{ik} t_k \right) + \sum_{k=1}^{K} t_k. \tag{1.9}$$

This is equivalent to the above when

$$u_k = w_k / (\sum_{k'} w_{k'})$$
$$p_{ik} = b_{ik} \times \sum_i w_i / \sum_i b_{ik}$$

REFERENCES

[1] D. M. Blei, A. Y. Ng, and M. I. Jordan, *Latent Dirichlet allocation*, Journal of Machine Learning Research, 3 (2003), pp. 993–1022.
[2] D. D. Lee and H. S. Seung, *Algorithms for non-negative matrix factorization*, in Advances in Neural Information Processing Systems 13, 2001, pp. 556–562.