Lanchester's Linear Law

Lanchester's Laws are formulae which use differential equations to describe and predict the outcome of engagements between military forces. They were first introduced by Frederick Lanchester during World War I. Depending on the circumstances of the battle, one may have to use either Lanchester's Linear Law or Lanchester's Square Law in order to get realistic results. In the following, we will see how well Lanchester's Linear Law can be applied to engagements in StarCraft II.

Lanchester's Linear Law describes a fight in a narrow pass where only a certain amount of units may fight in both armies at a time. It is built upon the following assumptions:

- The rate of attrition of both armies is equal to the number of units fighting in the other army times the relative strength of the enemy units.
- Any fallen unit is immediately replaced by units waiting to enter the battle, thus the number of fighting units remains the same.
- Since the theory uses continuous functions, it gives best results if the units are weak (die fast) and come in great numbers.

Let's say, we have two armies fighting, X and Y. They have the army sizes x_0 and y_0 at the beginning of the battle. They also have the relative strength regarding the enemy army, α and β, respectively. The relative unit strength is defined so that it gives the number of enemy units a unit can kill in a given time. At any time, n and m units, respectively, can participate in the battle.
Our goal is to find a function

$$f\colon\ \mathbb{R}^6 \to \mathbb{R};\ \ [x_0, y_0, \alpha, \beta, n, m] \to f(x_0, y_0, \alpha, \beta, n, m)$$

which gives the number of survivors of the X (or the Y) army at the end of the battle, as a function of the army sizes, relative unit strengths, and the number of fighting units.
If we introduce the functions of the army sizes with respect to time,

$$x\colon\ \mathbb{R} \to \mathbb{R};\ \ t \to x(t)$$

and

$$y\colon\ \mathbb{R} \to \mathbb{R};\ \ t \to y(t),$$

respectively, we can form the following differential equations based on the initial assumptions:

$$\frac{dx(t)}{dt} = -m\beta \tag{1}$$

and

$$\frac{dy(t)}{dt} = -n\alpha. \tag{2}$$

Dividing equation (1) by (2), we have

$$\frac{dx}{dy} = \frac{m\beta}{n\alpha}, \tag{3}$$

so

$$n\alpha dx = m\beta dy. \tag{4}$$

After integration, we get

$$nax - m\beta y = c, \tag{5}$$

where $c \in \mathbb{R}$ is the integration constant. This means that the value $n\alpha x - m\beta y$ is constant during the fight. Since the initial army sizes are x_0 and y_0, it is obvious that

$$c = n\alpha x_0 - m\beta y_0, \tag{6}$$

so

$$n\alpha x - m\beta y = n\alpha x_0 - m\beta y_0. \tag{7}$$

Also, we can see that if $n\alpha x_0 > m\beta y_0$, then the X army wins, and vice versa. If the two are identical, the armies, in theory, tie, and both armies are killed. For this reason, this value—the number of units multiplied by the relative strength—is called the *strength of the army*.
This army strength is linearly proportional to the number of units in an army, hence the name Linear Law.
If the X army wins, then at the end of the fight $y = 0$, so the number of survivors in the X army is

$$x_\mathrm{s} = x_0 - \frac{m\beta}{n\alpha} y_0.$$

If the Y army wins, then $x = 0$ at the end, so the Y survivors are

$$y_\mathrm{s} = y_0 - \frac{n\alpha}{m\beta} x_0.$$

These functions are exactly the what we were looking for (as function f).

In StarCraft II, the relative unit strength may be defined as

$$\alpha = \frac{1}{hT},$$

where h is the number of attacks the unit needs to kill an enemy unit and T is the unit's attack cooldown. The Linear Law may be applied in fights in choke points, where the number of units attacking on both sides can be considered constant. It best describes melee combat, or engagements where the attack range of both sides is identical. If it is not the case, then it should be taken into account that for a short time one army is already attacking while the other is closing the gap.

Let's apply this to the fight between Zealots and Zerglings. In a very small choke,[1] only two Zealots and three Zerglings can fight at a time, so $n = 2$ and $m = 3$ in this case. Let's take $x_0 = 20$ Zealots and $y_0 = 30$ Zerglings in said choke. Now it's clear that the Zealots should win the fight, but how many will survive?
The Zealot's relative strength against Zerglings is $\alpha = 0.27778 \frac{1}{\mathrm{gs}}$; that of the Zergling against Zealots is $\beta = 0.04105 \frac{1}{\mathrm{gs}}$. Then the army strength of the Zealots (X army) is $20 \cdot 0.27778 = 5.5556$; that of the Zerglings is $30 \cdot 0.04105 = 1.2315$, so the Zealot army is indeed much stronger. The number of survivors is

$$x_\mathrm{s} = x_0 - \frac{m\beta}{n\alpha} y_0 = 13.35,$$

so there should be about 14 surviving Zealots. However, testing with the unit tester gives an easily reproducible result of 15 survivors. It is only one more than it should, but it is more all the same. Is the formula broken?
Let us make another experiment, this time with 40 Zealots and 70 Zerglings. Now the result should be 24.48, so 25, but the result is 27. Again, more than expected.
The following table summarizes a few more experiments.

[1] See "S Choke" in *HotS Unit Tester Online* by Brandon, XGDragon and hunter, in the Arcade menu in StarCraft II

Army X	Army Y	x_0	y_0	α	β	n	m	Win	Calculated	Tested	Diff.
Zealot	Zergling	20	30	0.27778	0.04105	2	3	X	13.35	15	1
Zealot	Zergling	40	70	0.27778	0.04105	2	3	X	24.48	27	2
Zealot	DT	30	15	0.10417	0.14758	2	3	Y	0.88	1	0
Zealot	DT	20	10	0.10417	0.14758	2	3	Y	0.59	1	0
Zealot	DT 3/3/3	20	10	0.0641	0.19677	2	3	Y	5.66	7	1
DT	Zergling	10	30	0.59032	0.05526	3	3	X	7.19	10	2
DT 3/3/3	Zergling	10	60	0.59032	0.01796	3	3	X	8.17	10	1
Zealot	Zergling 3/3	20	30	0.20833	0.06842	2	3	X	5.22	7	1
Stalker	Hydralisk	20	15	0.07716	0.09524	2	2	X	1.49	3	1
Stalker	Hydralisk	30	25	0.07716	0.09524	2	2	Y	0.70	1	0
Zealot	Crackling	25	40	0.27778	0.04867	4	6	X	14.49	19	4
Zealot	DT	25	20	0.10417	0.14758	4	6	Y	8.24	12	3
Zealot	DT	25	20	0.10417	0.14758	6	8	Y	6.77	11	4
Zealot	DT	40	20	0.10417	0.14758	6	8	X	2.22	7	4
Zergling	Crackling	30	30	0.1796	0.21295	3	3	Y	4.70	1	-4

Table 1: Calculated and tested results of engagements

It seems that there are almost always more survivors than there should be. The reason why the formula might not work as expected is the following. Our goal was to determine how many *units* survive, but the formula, in fact, gives us how much *damage* is done. The two would only be identical if the focus fire were perfect (i. e. 10 units worth of damage would then kill 10 units for sure). Even then, the number of survivors should always be rounded up (10.5 units worth of damage will only kill 10 units however good the focus fire).
Because of the lack of perfectly focused fire, the last units of the losing army are more likely to distribute their damage between the winning army's fighting units. Additionally, when there are fewer units in the losing army than could fight, they are actually dealing less damage than they should according to the derivation. I believe this latter factor counts less, but all in all, there is about a 1 to 4 difference which seems to be greater the greater the fighting numbers (n and m) are.

In general, it can be said that the number of survivors in a real scenario is almost always higher than it would be calculated, but the difference is not greater than the fighting number (n) of the winning side, so the formula may be modified as:

$$x_s' \approx x_0 - \frac{m\beta}{n\alpha} y_0 + n,$$

and

$$y_s' \approx y_0 - \frac{n\alpha}{m\beta} x_0 + m.$$