Automatica 46 (2010) 211–215

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

A state-feedback approach to event-based control[☆]

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ARTICLE INFO

Article history:
Received 6 February 2009
Received in revised form
21 July 2009
Accepted 19 October 2009
Available online 14 November 2009

Keywords:
Event-based control
State-feedback control
Performance
Networked control system

ABSTRACT

This paper proposes a new method for event-based state-feedback control in which a control input generator mimics a continuous feedback between two consecutive event times. The performance of the event-based control system is evaluated by comparing this loop with the continuous state-feedback loop. An upper bound of the difference between both loops is derived, which shows that the approximation of the continuous state-feedback loop by the event-based control loop can be made arbitrarily tight by appropriately choosing the threshold parameter of the event generator.

1. Introduction to event-based control

Event-based control is a means to reduce the communication between the sensors, the controller and the actuators in a control loop by invoking a communication among these components only after an event has indicated that the control error exceeds a tolerable bound. This working principle differs fundamentally from that of the usual feedback loop, in which data are communicated continuously or at every sampling instance given by a clock. Hence, in the control schemes currently used a communication takes place also in the case of small control errors when no information feedback is necessary to satisfy the performance requirements.

This paper considers the event-based control loop shown in Fig. 1. The activities of the controller are incorporated in the control input generator and the event generator. The event generator determines the time instants t_k $(k = 0, 1, \ldots)$ at which the next communication from the event generator towards the control input generator is invoked. The control input generator determines the function $\boldsymbol{u}(t)$ for the time interval $t \in [t_k, t_{k+1})$ in dependence upon the information $\boldsymbol{x}(t_k)$ obtained at time t_k. The dashed arrow in the figure indicates that this information link is only used after an event has been generated, whereas the arrows shown as solid lines are used continuously.

The aim of this paper is to propose a new scheme of event-based control, which adapts the communication frequency to the current system performance. The communication link should only be used if the disturbance $\boldsymbol{d}(t)$ has caused an intolerable effect on the loop performance. As the main result, algorithms for the event generation and the control input generation are described for which the event-based control loop can be proved to mimic the continuous state-feedback loop with arbitrary accuracy.

In the literature, there is no uniform terminology of event-based control. Although the triggering mechanisms are called event-based sampling (Aström, 2008) or event-driven sampling (Heemels, Sandee, & van den Bosch, 2007), Lebesgue sampling (Aström & Bernhardsson, 2002), deadband sampling (Otanez, Moyne, & Tilbury, 2002), asynchronous control (Heemels et al., 1999), or level-crossing sampling (Kofman & Braslavsky, 2006), with slightly different meaning, all refer to the situation where the control action is not invoked by some clock but "on demand" by the control error or some other signal exceeding a certain threshold. After in the early papers (Arzen, 1999; Heemels et al., 1999) the event-based nature of controllers has been investigated by simulation and experiments, references Aström (2008) and Aström and Bernhardsson (2002) derived for a single-order system analytical results comparing the performance of event-based and sampled-data control loops. They showed that under certain conditions the event-based control loop can have even a better performance than the sampled-data loop, because the communication may not only be reduced in time intervals with small disturbances but also increased in time intervals with large disturbances. In Henningsson, Johannesson, and Cervin (2008), this stochastic setting of the analysis of event-based control has been extended. Deadband control proposed in Otanez et al. (2002) concerns a similar scheme. In self-triggered sampling (Anta & Tabuada, 2008), the event generator determines at event time t_k the next event time t_{k+1} in advance.

[☆] The material in this paper was partially presented at European Control Conference, Budapest 2009. This paper was recommended for publication in revised form by Associate Editor Tongwen Chen under the direction of Editor Ian R. Petersen.

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doi:10.1016/j.automatica.2009.10.035

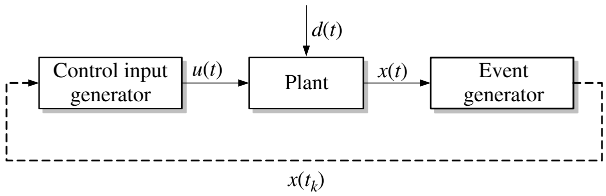

Fig. 1. Event-based control loop.

Reference Heemels et al. (2007) investigated event-based control in a more general setting where the plant may be a linear system of arbitrary dynamical order. The idea is to stick to a given sampling period only as long as the plant state is outside a set $\mathcal{B}$ around the origin of the state space and to ignore the sampling whenever the state is inside of $\mathcal{B}$. The control loop has been proved to be ultimately bounded in the sense that its state remains in a set Ω for all bounded disturbances.

A different approach is taken in Brockett and Liberzon (2000), De Persis (2008), De Persis and Isidori (2004) and Wong and Brockett (1999), where the limitation in the information exchange among the sensor, the controller and the actuator is represented by a quantisation of the signals, which likewise leads to a reduction of communication. However, in this approach the reduction lies in the information contents (measured in bits of information to be transferred) rather than in the frequency of communication.

Recent publications like Mazo and Tabuada (2008) and Wang and Lemmon (2008, 2009) investigated event-based control in the broader setting of multi-loop systems closed over the same communication network.

To concentrate the investigations of this paper on a single item, it is assumed that the plant is stable and no model uncertainties occur. Hence, the only reason to communicate information via the dashed arrow in Fig. 1 is given by the situation that the disturbance $\boldsymbol{d}$ has an intolerable effect on the plant state $\boldsymbol{x}$. The communication link is assumed to react instantaneously (at least in comparison to the time constants of the plant), so that there is no time delay between the event generation and the receipt of the data $\boldsymbol{x}(t_k)$ by the control input generator and both components work synchronously at the event times. As the prize for this low utilisation of the communication link is a higher computational effort, the paper does not impose any restriction on the computational complexity of the event generator and the control input generator.

Consequently, the disturbance behaviour of a feedback loop is the main concern of this paper. It is investigated how event-based control should be structured so that its performance matches that of a continuous state-feedback loop introduced in Section 2 in the sense that the state $\boldsymbol{x}(t)$ of the event-based control loop remains, for all time t, in a bounded surrounding $\Omega_e(\boldsymbol{x}_{SF}(t))$ of the state $\boldsymbol{x}_{SF}(t)$ of the continuous state-feedback loop. Section 3 proposes appropriate control input and event generators. It will be shown in Section 4 that the size of the set Ω_e (approximation precision) can be made arbitrarily small by appropriately choosing the threshold of the event generator (Theorem 1). Furthermore, it is investigated how the communication frequency is influenced by the disturbance $\boldsymbol{d}(t)$ (Theorem 2).

2. State-feedback control

This section describes a state-feedback control loop the behaviour of which should be approximated by the event-based control loop. The plant

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{d}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{1}$$

has the state $\boldsymbol{x} \in \mathbb{R}^n$, input $\boldsymbol{u} \in \mathbb{R}^m$ and disturbance $\boldsymbol{d} \in \mathbb{R}^p$. The disturbance is assumed to be bounded:

$$\|\boldsymbol{d}(t)\| \le d_{\max}.$$

Together with the state feedback

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t) \tag{2}$$

the closed-loop system

$$\dot{\boldsymbol{x}}_{SF}(t) = \underbrace{(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})}_{\bar{\boldsymbol{A}}}\boldsymbol{x}_{SF}(t) + \boldsymbol{E}\boldsymbol{d}(t), \qquad \boldsymbol{x}_{SF}(0) = \boldsymbol{x}_0 \tag{3}$$

results. The state-feedback matrix $\boldsymbol{K}$ has been chosen so that the closed-loop system has a satisfactory behaviour, particularly with respect to its disturbance attenuation properties (Lunze, 2008).

For the event-based control scheme proposed below, Eq. (3) shows that the control input generated by the state-feedback controller (2) at some time $t \ge t_k$ is given by

$$\boldsymbol{u}(t) = -\boldsymbol{K}\mathrm{e}^{\bar{\boldsymbol{A}}(t-t_k)}\boldsymbol{x}_{SF}(t_k) - \int_{t_k}^{t} \boldsymbol{K}\mathrm{e}^{\bar{\boldsymbol{A}}(t-\tau)}\boldsymbol{E}\boldsymbol{d}(\tau)\mathrm{d}\tau, \quad t \ge t_k. \tag{4}$$

It is important to see that the input $\boldsymbol{u}(t)$ does not only depend on the measured state $\boldsymbol{x}_{SF}(t_k)$ but also on the disturbance input $\boldsymbol{d}(t)$.

3. Event-based control

3.1. Control input generator

This section describes the main parts of the event-based control loop shown in Fig. 1. The main idea is to use a copy of the model (3) of the continuous state-feedback loop for both the event detection and the control input generation.

The structure of the control input generator is a direct consequence of the analysis described in the preceding section. For the time $t \ge t_k$ the plant under the input (4) behaves exactly like the state-feedback loop (3). However, this control law necessitates the knowledge of the disturbance $\boldsymbol{d}(t)$ for the time interval $t \in [t_k, t_{k+1})$. As the disturbance is generally unknown and immeasurable, a disturbance estimate

$$\boldsymbol{d}(t) = \hat{\boldsymbol{d}}_k \quad \text{for } t \ge t_k \tag{5}$$

is used. The control input generator applies the input

$$\boldsymbol{u}(t) = -\boldsymbol{K}\mathrm{e}^{\bar{\boldsymbol{A}}(t-t_k)}\boldsymbol{x}(t_k) - \boldsymbol{K}\bar{\boldsymbol{A}}^{-1}\left(\mathrm{e}^{\bar{\boldsymbol{A}}(t-t_k)} - \boldsymbol{I}\right)\boldsymbol{E}\hat{\boldsymbol{d}}_k, \quad t \ge t_k \tag{6}$$

to the plant, where $\boldsymbol{x}(t_k)$ is the measured state communicated from the event generator to the control input generator. Eq. (6) follows from Eq. (4) for constant disturbances (5). Note that the control input (6) is generated by the continuous state-feedback system subject to the constant disturbance (5), which is represented by the state-space model

$$\dot{\boldsymbol{x}}_s(t) = \bar{\boldsymbol{A}}\boldsymbol{x}_s(t) + \boldsymbol{E}\hat{\boldsymbol{d}}_k, \qquad \boldsymbol{x}_s(t_k^+) = \boldsymbol{x}(t_k), \quad t \ge t_k \tag{7}$$

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}_s(t), \tag{8}$$

where $\boldsymbol{x}_s(t_k^+)$ denotes the state of the control input generator after the state has been updated at the event time t_k.

The control input generator estimates the disturbance according to the following recursion:

$$\hat{\boldsymbol{d}}_0 = \boldsymbol{0}$$

$$\hat{\boldsymbol{d}}_k = \hat{\boldsymbol{d}}_{k-1} + \left(\boldsymbol{A}^{-1}\left(\mathrm{e}^{\boldsymbol{A}(t_k - t_{k-1})} - \boldsymbol{I}\right)\boldsymbol{E}\right)^{+}\left(\boldsymbol{x}(t_k) - \boldsymbol{x}_s(t_k^-)\right), \tag{9}$$

where $\boldsymbol{x}_s(t_k^-)$ is the state of the control input generator (7) before the update. $(.)^+$ denotes the pseudoinverse $\boldsymbol{H}^+ = (\boldsymbol{H}'\boldsymbol{H})^{-1}\boldsymbol{H}'$. The inverse matrix $(\boldsymbol{H}'\boldsymbol{H})^{-1}$ exists if, as usual, the number of disturbances is lower than the number of state variables ($\dim \boldsymbol{x} \ge \dim \boldsymbol{d}$) and the matrices occurring in $\boldsymbol{H}$ have full rank. Eq. (9) determines the magnitude $\hat{\boldsymbol{d}}_k$ of a constant disturbance, which has the same effect on the plant state as the disturbance $\boldsymbol{d}(t)$ in the time interval $t \in [t_{k-1}, t_k)$.

3.2. Event generator

This section proposes to generate events by comparing the measured state trajectory $\boldsymbol{x}(t)$ with the state trajectory $\boldsymbol{x}_s(t)$ of the continuous state-feedback loop (7), (8). The first event is generated at time $t_0 = 0$ and any further event whenever the measured state $\boldsymbol{x}(t)$ leaves the surrounding

$$\Omega(\boldsymbol{x}_s) = \{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_s\| \le \bar{e}\} \tag{10}$$

of the state $\boldsymbol{x}_s(t)$ with $\|.\|$ symbolising a vector norm. Hence, the $(k+1)$th event is generated if the condition

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_s(t)\| = \bar{e} \tag{11}$$

is satisfied, where $\boldsymbol{x}$ is the measured state and $\boldsymbol{x}_s$ is determined by Eq. (7). The time t at which this happens is denoted by t_{k+1} and the state of the event generator by $\boldsymbol{x}_s(t_{k+1}^-)$.

3.3. Implementation issues

In summary, the event-based control system shown in Fig. 1 consists of the following components:

- the plant (1),
- the event generator (11), which uses Eqs. (7), (9) to determine the state $\boldsymbol{x}_s(t)$ and
- the control input generator (7), (8), which also estimates the disturbance according to Eq. (9).

If the $(k+1)$th event is generated according to Eq. (11), the information $\boldsymbol{x}(t_{k+1})$ is communicated from the event generator to the control input generator (and used there with index $k \leftarrow k+1$). In this scheme, both the event generator and the control input generator need the model (7) and the disturbance estimator (9). The duplication of the estimator (9) can be avoided if the disturbance estimate $\hat{\boldsymbol{d}}_k$ is also communicated from the event generator to the control input generator.

4. Analysis of the event-based control loop

4.1. Model of the closed-loop system

This subsection investigates the behaviour of the event-based control loop in the time interval between two consecutive event times t_k and t_{k+1}. For the time period $t \in [t_k, t_{k+1})$ the plant (1) together with the control input generator (7), (8) is described by

$$\begin{pmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}_s(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{BK} \\ \boldsymbol{O} & \bar{\boldsymbol{A}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_s(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{d}(t) + \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{E} \end{pmatrix} \hat{\boldsymbol{d}}_k,$$
$$\begin{pmatrix} \boldsymbol{x}(t_k) \\ \boldsymbol{x}_s(t_k^+) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}(t_k) \\ \boldsymbol{x}(t_k) \end{pmatrix}.$$

For the analysis, the state transformation

$$\begin{pmatrix} \boldsymbol{x}_\Delta(t) \\ \boldsymbol{x}_s(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_s(t) \end{pmatrix} \tag{12}$$

is used to get the equivalent model

$$\begin{aligned} \begin{pmatrix} \dot{\boldsymbol{x}}_\Delta(t) \\ \dot{\boldsymbol{x}}_s(t) \end{pmatrix} &= \begin{pmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{O} & \bar{\boldsymbol{A}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_\Delta(t) \\ \boldsymbol{x}_s(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{d}(t) + \begin{pmatrix} -\boldsymbol{E} \\ \boldsymbol{E} \end{pmatrix} \hat{\boldsymbol{d}}_k, \\ \begin{pmatrix} \boldsymbol{x}_\Delta(t_k^+) \\ \boldsymbol{x}_s(t_k^+) \end{pmatrix} &= \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{x}(t_k) \end{pmatrix} \end{aligned} \tag{13}$$

which yields the state

$$\boldsymbol{x}(t) = \boldsymbol{x}_s(t) + \boldsymbol{x}_\Delta(t)$$

with

$$\boldsymbol{x}_s(t) = e^{\bar{\boldsymbol{A}}(t-t_k)}\boldsymbol{x}(t_k) + \bar{\boldsymbol{A}}^{-1}\left(e^{\bar{\boldsymbol{A}}(t-t_k)} - \boldsymbol{I}\right)\boldsymbol{E}\hat{\boldsymbol{d}}_k \tag{14}$$

$$\boldsymbol{x}_\Delta(t) = \int_{t_k}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{E}\,\boldsymbol{d}_\Delta(\tau)\mathrm{d}\tau \tag{15}$$

and the disturbance estimation error $\boldsymbol{d}_\Delta(t) = \boldsymbol{d}(t) - \hat{\boldsymbol{d}}_k$. Note that $\boldsymbol{x}_s(t)$ is the trajectory of the continuous state-feedback system for time $t \in [t_k, t_{k+1})$ with initial state $\boldsymbol{x}_s(t_k^+) = \boldsymbol{x}(t_k)$ and constant disturbance $\boldsymbol{d}(t) = \hat{\boldsymbol{d}}_k$.

4.2. Behaviour of the control loop

The following lemma describes an important property of the event-based closed-loop system:

Lemma 1. *The event-based control ensures that the state $\boldsymbol{x}(t)$ of the event-based control loop remains for all $t \ge 0$ in the surrounding $\Omega(\boldsymbol{x}_s(t))$ of the state $\boldsymbol{x}_s(t)$ of the system (7):*

$$\boldsymbol{x}(t) \in \Omega(\boldsymbol{x}_s(t)). \tag{16}$$

In the literature, such a set Ω is also said to be positively invariant (Heemels et al., 2007).

Proof. The state $\boldsymbol{x}_\Delta(t)$ is represented by Eq. (13)

$$\dot{\boldsymbol{x}}_\Delta(t) = \boldsymbol{A}\boldsymbol{x}_\Delta(t) + \boldsymbol{E}\boldsymbol{d}_\Delta(t), \quad \boldsymbol{x}_\Delta(t_k^+) = \boldsymbol{0}.$$

The event-based communication mechanism ensures that at the event times t_k $(k = 0, 1, \ldots)$ the state $\boldsymbol{x}_\Delta$ is reset to zero: $\boldsymbol{x}_\Delta(t_k^+) = \boldsymbol{0}$. This happens after $\boldsymbol{x}_\Delta(t)$ has satisfied the condition $\|\boldsymbol{x}_\Delta(t_k^-)\| = \bar{e}$. Hence, the state $\boldsymbol{x}_\Delta(t) = \boldsymbol{x}(t) - \boldsymbol{x}_s(t)$ is bounded

$$\|\boldsymbol{x}_\Delta(t)\| \le \bar{e}, \tag{17}$$

which proves the lemma. □

The lemma shows that the event generating mechanism works well and no additional sampled-data control scheme as proposed in Heemels et al. (2007) has to be introduced. The state $\boldsymbol{x}_\Delta(t)$ remains in the set $\Omega(\boldsymbol{x}_s(t))$ for all $t > 0$. Whenever it touches the boundary of this set, the event generation mechanism resets $\boldsymbol{x}_\Delta(t)$ such that $\boldsymbol{x}(t_k)$ becomes the center point of the set $\Omega(\boldsymbol{x}_s(t_k^+))$ at the event times t_k.

4.3. Comparison of the event-based control loop and the continuous state-feedback loop

The quality of the event-based control is evaluated now by showing that the deviation of the event-based control loop from the behaviour of the continuous state-feedback loop is bounded and can be made arbitrarily small by appropriately choosing the event generation threshold $\bar{e}$. The event-based control loop is described by Eqs. (1), (8):

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{BK}\boldsymbol{x}_s(t) + \boldsymbol{E}\boldsymbol{d}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{18}$$

The evaluation is made by first showing that the difference $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{SF}(t)$ between the states of the event-based loop and the continuous control loop is bounded and second by deriving an upper bound of $\|\boldsymbol{e}(t)\|$.

Lemma 2. *The difference $\boldsymbol{e}(t)$ between the states of the event-based control loop and the continuous state-feedback loop is bounded.*

Proof. For the state difference $\boldsymbol{e}(t)$, Eqs. (3), (18) yield

$$\dot{\boldsymbol{e}}(t) = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{e}(t) + \boldsymbol{BK}(\boldsymbol{x}(t) - \boldsymbol{x}_s(t))$$

and with $\boldsymbol{x}_\Delta(t) = \boldsymbol{x}(t) - \boldsymbol{x}_s(t)$

$$\dot{\boldsymbol{e}}(t) = \bar{\boldsymbol{A}}\boldsymbol{e}(t) + \boldsymbol{BK}\boldsymbol{x}_\Delta(t), \qquad \boldsymbol{e}(0) = \boldsymbol{0}. \tag{19}$$

As the state $\boldsymbol{x}_\Delta(t)$ is bounded according to Eq. (17) and the state-feedback loop is stable, the difference $\boldsymbol{e}(t)$ is bounded as well. □

The analysis can be made more precise by showing that the bound on $\boldsymbol{e}(t)$ depends monotonically on $\bar{e}$. To see this, the difference $\boldsymbol{e}(t)$ is represented by the convolution of $\boldsymbol{x}_\Delta(t)$ with the impulse response matrix $\boldsymbol{G}(t) = \mathrm{e}^{\bar{\boldsymbol{A}}t}\boldsymbol{BK}$ of the model (19). Hence, the inequalities

$$\|\boldsymbol{e}(t)\| \le \int_0^t \|\boldsymbol{G}(t-\tau)\| \cdot \|\boldsymbol{x}_\Delta(\tau)\| \mathrm{d}\tau$$

and

$$\|\boldsymbol{e}(t)\| \le e_{\max} \tag{20}$$

hold with

$$e_{\max} = \bar{e} \cdot \int_0^\infty \|\boldsymbol{G}(\tau)\| \, \mathrm{d}\tau. \tag{21}$$

Theorem 1. *The difference $\boldsymbol{e}(t)$ between the states of the event-based control loop and the continuous state-feedback loop is bounded from above by Eqs. (20), (21).*

As a consequence, the event-based controller proposed in this paper can be made to mimic a continuous state feedback with an arbitrary precision by accordingly choosing the event threshold $\bar{e}$. The state $\boldsymbol{x}(t)$ remains in the set

$$\boldsymbol{x}(t) \in \Omega_{\mathrm{e}}(\boldsymbol{x}_{\mathrm{SF}}(t)) = \{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_{\mathrm{SF}}(t)\| \le e_{\max}\}. \tag{22}$$

4.4. Communication frequency

The frequency of the data communication within the event-based control scheme depends upon the disturbance $\boldsymbol{d}(t)$. For a bounded disturbance the maximum communication frequency can be evaluated as follows. Assume that the transformed disturbance $\boldsymbol{d}_\Delta(t)$ is bounded

$$\|\boldsymbol{d}_\Delta(t)\| \le d_{\Delta\max} \quad \text{for } t \ge 0.$$

The difference state $\boldsymbol{x}_\Delta(t)$ is given by

$$\dot{\boldsymbol{x}}_\Delta(t) = \boldsymbol{A}\boldsymbol{x}_\Delta(t) + \boldsymbol{E}\boldsymbol{d}_\Delta(t), \quad \boldsymbol{x}_\Delta(0) = \boldsymbol{0}.$$

An event is generated whenever the equation

$$\|\boldsymbol{x}_\Delta(t)\| = \left\| \int_{t_k}^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{E}\boldsymbol{d}_\Delta(\tau)\,\mathrm{d}\tau \right\| = \bar{e}$$

holds. The minimum time $T_{\min}$ for which this condition is satisfied at time $t = t_k + T_{\min}$ is given by

$$T_{\min} = \arg\min_t \tag{23}$$

$$\text{s.t.} \left\| \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{E}\boldsymbol{d}_\Delta(\tau)\mathrm{d}\tau \right\| = \bar{e}, \quad \|\boldsymbol{d}_\Delta(t)\| \le d_{\Delta\max}.$$

The following estimation yields a bound on this time. If the upper bound

$$\int_0^t \left\|\mathrm{e}^{\boldsymbol{A}\tau}\boldsymbol{E}\right\| \mathrm{d}\tau \cdot d_{\Delta\max} \ge \left\| \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{E}\boldsymbol{d}_\Delta(\tau)\mathrm{d}\tau \right\|$$

is set to $\bar{e}$, the time $\bar{T}$ can be obtained as the upper integral bound for which the relation

$$\int_0^{\bar{T}} \left\|\mathrm{e}^{\boldsymbol{A}\tau}\boldsymbol{E}\right\| \mathrm{d}\tau = \frac{\bar{e}}{d_{\Delta\max}} \tag{24}$$

holds. This time is a lower bound of $T_{\min}$: $T_{\min} \ge \bar{T}$.

Theorem 2. *For any bounded disturbance, the minimum time $T_{\min}$ between two communication events is bounded from below by $\bar{T}$ satisfying Eq. (24).*

This theorem shows how the communication frequency depends on the disturbance. This property contrasts with sampled-data control, where the sampling frequency is chosen with respect to the plant properties (time constants) rather than the disturbance magnitude.

4.5. Closed-loop system behaviour for small disturbances

For sufficiently small disturbance $\boldsymbol{d}(t)$ the event generator (11) does not generate any event besides the initial event at $t_0 = 0$. To show this, the disturbance is written as $\bar{d}\tilde{\boldsymbol{d}}(t)$ where

$$\|\tilde{\boldsymbol{d}}(t)\| \le 1 \quad \text{for } t \ge 0 \tag{25}$$

and $\bar{d}$ is a parameter determining the disturbance magnitude.

Lemma 3. *For every bounded signal $\tilde{\boldsymbol{d}}(t)$ there exists a magnitude $|\bar{d}|$ such that for the disturbance $\boldsymbol{d}(t) = \bar{d}\tilde{\boldsymbol{d}}(t)$ the event generator does not generate any event for $t > 0$.*

Proof. According to Eq. (11) there will be no event generated as long as the inequality $\|\boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{s}}(t)\| < \bar{e}$ holds for all $t > 0$. According to Eq. (15), which is applied for $k = 0$ and $\hat{\boldsymbol{d}}_0 = \boldsymbol{0}$, this inequality is true if

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{s}}(t)\| = \left\| \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{E}\ (\bar{d}\tilde{\boldsymbol{d}}(\tau) - \hat{\boldsymbol{d}}_0)\ \mathrm{d}\tau \right\| < \bar{e}$$

and

$$\max_{t\ge0} \left\| \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{E}\ \bar{d}\tilde{\boldsymbol{d}}(\tau)\mathrm{d}\tau \right\| < \bar{e}$$

hold. A bound of the left-hand side of the last equation exists because the plant is stable and

$$\max_{t\ge0} \left\| \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{E}\ \bar{d}\tilde{\boldsymbol{d}}(\tau)\mathrm{d}\tau \right\| \le |\bar{d}| \int_0^\infty \left\|\mathrm{e}^{\boldsymbol{A}\tau}\boldsymbol{E}\right\| \mathrm{d}\tau$$

holds. Hence, for the disturbance parameter $\bar{d}$ with

$$|\bar{d}| < \frac{\bar{e}}{\int_0^\infty \|\mathrm{e}^{\boldsymbol{A}\tau}\boldsymbol{E}\|\mathrm{d}\tau} \tag{26}$$

no event is generated, which proves the lemma. □

4.6. Example

The application of this event-based control scheme to a thermofluid process reported in Lehmann and Lunze (2009) illustrates the behaviour of the event-based control loop proposed here (Fig. 2).

On the left-hand side the plant is subject to a constant disturbance $\boldsymbol{d}(t) = \bar{\boldsymbol{d}}$. An event takes place at time t_1. As at this event time the disturbance magnitude $\bar{d}$ is correctly estimated by the disturbance estimator (9) ($\hat{\boldsymbol{d}}_1 = \bar{\boldsymbol{d}}$), from this time on the event-based control loop behaves exactly like the continuous loop and, hence, $\boldsymbol{x}(t)$ (solid) and $\boldsymbol{x}_{\mathrm{s}}(t)$ (dashed) coincide. No further event occurs. The third line depicts the state $\boldsymbol{x}_{\mathrm{SF}}(t)$ of the continuous control loop, which shortly after the event at time t_1 is identical to the state $\boldsymbol{x}(t)$ of the event-based loop.

On the right-hand side of Fig. 2 the disturbance changes after the first events as shown by the dotted line in the top subplot. After the event at time t_4 the disturbance remains constant and its magnitude is estimated with sufficient accuracy. The effect of the difference $|\boldsymbol{d}_\Delta(t)| = |\hat{\boldsymbol{d}}_4 - \boldsymbol{d}(t)|$ for $t \ge t_4$ is small enough to not invoke any further event. Both components of $\boldsymbol{x}(t)$ and $\boldsymbol{x}_{\mathrm{s}}(t)$ differ but $\|\boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{s}}(t)\|$ remains smaller than the event bound $\bar{e}$. The third line showing $\boldsymbol{x}_{\mathrm{SF}}(t)$ is close to $\boldsymbol{x}_{\mathrm{s}}(t)$.

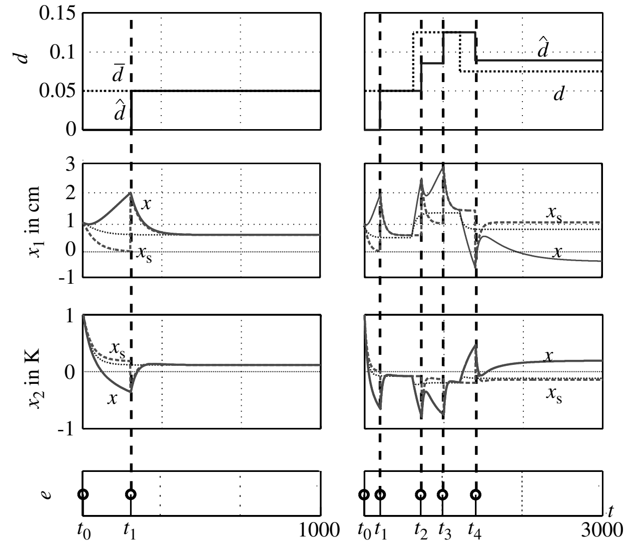

Fig. 2. Behaviour of the process model subject to two different exogenous disturbances.

5. Conclusions

The paper proposes an event-based control scheme for which the performance of the closed-loop system approximates the behaviour of a continuous state-feedback system. A communication is invoked only if the effect of the disturbance reaches a given upper bound. To evaluate the current performance and to determine the control input, both the event generator and the control input generator include a copy of the continuous state-feedback system. Between two consecutive event times the unknown disturbance is approximated by a disturbance estimate and a new event is only generated if the effect of the approximation error exceeds a given sensitivity bound.

It has been shown that the sensitivity bound $\bar{e}$ of the event generator can be chosen such that the event-based control loop approximates the continuous state-feedback loop with arbitrary precision. The communication frequency is adapted to the effect of the disturbance. If the disturbance is small enough, no further event is generated.

Acknowledgement

This work is supported by the Deutsche Forschungsgemeinschaft (DFG), grant number LU 462/26-1.

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