Event Based Control

Karl J. Åström

Department of Automatic Control LTH, Lund University, Box 118, SE-221 00 Lund, Sweden

Summary. In spite of the success of traditional sampled-data theory in computer control it has some disadvantages particularly for distributed, asynchronous, and multi-rate system. Event based sampling is an alternative which is explored in this paper. A simple example illustrates the differences between periodic and event based sampling. The architecture of a general structure for event based control is presented. The key elements are an event detector, an observer, and a control signal generator, which can be regarded as a generalized data-hold. Relations to nonlinear systems are discussed. Design of an event based controller is illustrated for a simple model of a micro-mechanical accelerometer.

1 Introduction

Computer controlled systems are traditionally based on periodic sampling of the sensors and a zero order hold of the actuators, see [31], [48], and [52]. When controlling linear time invariant systems the approach leads to closed loop systems that are linear but periodic. The periodic nature can be avoided by considering only the behavior at times that are synchronized with the sampling resulting in the *stroboscopic model.* The system can then be described by difference equations with constant coefficients. Traditional sampled data theory based on the stroboscopic model is simple, and has been used extensive in practical applications of computer controlled systems. Periodic sampling also matches the time-triggered model of real time software which is simple and easy to implement safely, see [10], [37].

Since standard sampled-data theory only considers the behavior of the system at the sampling instants, it is necessary to ensure that the inter-sample behavior is satisfactory. A simple way to do this is to also sample not only the system but also the continuous time loss function, see [1], [2] and [6]. This approach, which is called *loss function sampling*, is equivalent to minimising the continuous time behavior subject to the constraint that the control signal is piece-wise constant, see [1]. Analysis and design reduces to calculations for a time-invariant discrete system. Another approach, called *lifting*, is to describe the behavior of the the state over a whole sampling interval, see [12], [56]. Lifting also gives a description of the system which is time-invariant but there are technical difficulties because the state space is infinite dimensional.

Even if traditional sampled data control is the standard tool for implementing computer control, there are severe difficulties when dealing with systems having multiple sampling rates or systems with distributed computing. With multi-rate sampling the complexity of the system depends critically on the ratio of the sampling rates. For distributed systems the theory requires that the clocks are synchronized. In networked distributed systems it has recently been a significant interest in the effects of sampling jitter, lost samples and delays on computer controlled systems, see [11].

Event based sampling is an alternative to periodic sampling. Signals are then sampled only when significant events occurs, for example, when a measured signal exceeds a limit or when an encoder signal changes. Event based control has many conceptual advantages. Control is not executed unless it is required, control by exception, see [36]. For a camera based sensor it could be natural to read off the signal when sufficient exposure is obtained. Event based control is also useful in situations when control actions are expensive, for example when controlling manufacturing complexes, and when when it is expensive to acquire information like in computer networks. Event based control is also the standard form of control in biological systems, see [55]. A severe drawback with event based systems is that very little theory is available.

All sampled systems, periodic as well as event based share a common property that the feedback is intermittent and that control is open loop between the samples. After an event the control signal is generated in an open loop manner and applied to the process. In traditional sampled-data theory the control signal is simply kept constant between the sampling instants, a scheme that is commonly called a zero order hold (ZOH). In event based systems the generation of the open loop signal is an important issue and the properties of the closed loop system depends critically on how the signal is generated.

There has not been much development of theory for systems with event based control. There are early results on discontinuous systems, [20], [53], [54], and impulse control, see [8] and [7]. Event based systems can also be regarded as special cases of hybrid control, where the system runs open loop between the regions, [15].

This paper gives an overview of systems with event based control. Section 2 presents a number of examples of were event based control is beneficial. Section 3 which is based on [4] and [5] gives a detailed discussion of a simple example that illustrates several issues about event based control. The example shows the benefits of event based control compared with conventional periodic sampling with impulse holds and zero order holds. Section 4 presents new ideas on analysis of design of event based control. A general system structure is given, and the different subsystems are discussed. Particular emphasis is given to the design of the control signal generator which can be viewed as a generalized hold circuit. It is shown that control signal generator can be chosen so that the event based system is equivalent to a nonlinear control system. This implies that techniques for nonlinear control can be applied, [28]. The design procedure is illustrated by a simple version of a MEMS accelerometer in Section 5.

2 Examples

Event based control occurs naturally in many situations from simple servo system to large factory complexes and computer networks. It is also the dominating control principle in biological systems. Encoders are primarily event based sensors. Relay systems with on-off control and satellite thrusters are event based [20], [16]. Systems with pulse-width of pulse-frequency modulation are other examples [50], [43], [22], [21], [49]. In this case the control signal is restricted to be a positive or negative pulse of given size. The controller decides when the pulses should be applied, what sign they should have, and the pulse lengths.

Event based control is easy to implement and was therefore used in many early feedback systems. Accelerometer and gyros with pulse feedback were used in systems for inertial navigation, see [41], [17]. A typical accelerometer consists of a pendulum which is displaced by acceleration. The displacement is compensated by force sensors and a feedback which keeps the pendulum in its neutral position. The restoring force was generated by pulses that moved the pendulum towards the center position as soon as a deviation was detected. Since all correcting impulses have the same form, the velocity can be obtained simply by adding pulses. Much work on systems of this type was done in the period 1960-1990.

Systems with relay feedback [20], [53] are other examples of event based control. Here feedback occurs when certain signals exceed given limits. The sigma-delta modulator or the one-bit AD converter, [42] [9], which is commonly used in audio and mobile telephone system, is another example. It is interesting to note that in spite of their wide spread use there does not exist a comprehensive theory for design of sigma-delta modulators. Design is currently done based on extensive simulations and experiments.

When controlling automotive engines it is natural to base sampling on crank angle position rather than on time, see [24], [23] and [13]. In ship control it is natural to base control on distance travelled rather than time. It is also much more natural for the ships captain to deal with quantities like turning radius instead of turning rate. Similarly in control of rolling mills it is natural to sample based on length rather than time. Other examples of event based process control are given in [38].

A typical plant in the process industry has many production units separated by buffer tanks for smoothing production variations. It is highly desirable not too change production rates too frequently because each change will cause upsets. There are always disturbances in the production. It may also be necessary to change production when the levels in the storage tanks are close to upper and lower limits. Controlling a large production facility can be approached via event based control. Nothing is done unless there are severe upsets or when storage tanks are approaching the limits. An early attempt with event based control of a paper mill is given in [46], a more recent project is described in [47].

Control of networks are other examples of event based control. In the Internet there are a large number of routers that just forward messages. The end-to-end transmission is controlled by the transmission control protocol (TCP) that ensures that the full message is received. To achieve this some information must be added to the message as well as mechanism for resending the message and for controlling the transmission rate. The TCP protocol is thus an example of event based control, see [29] and [34].

In biological systems the neurons interact by sending pulses. Electrical stimuli changes ion concentrations in the neuron and a pulse is emitted when the potential reaches a certain level, see [33], [27]. Several efforts have been made to mimic these systems. Implementations of silicon neurons are found in [40] and the paper [14] shows how they can be used to implement simple control systems. An interesting feature is the ease of interfacing and the possibility of constructing very reliable systems by duplicating the neurons and simply adding pulses.

3 Comparison of Periodic and Event Based Control

To provide insight into the differences between periodic and event based control we will first consider a simple regulation problem where all calculations can be performed analytically. The results are based on [4] and [5], where many additional details are given. Consider a system to be controlled described by the equation

$$dx = udt + dv, \tag{1}$$

where the disturbance $v(t)$ is a Wiener process with unit incremental variance and u the control signal. It is assumed that the state x is measured without error. The object is to control the system so that the state is as close to the origin as possible. To be specific we will minimize the mean square variations

$$V = \frac{1}{T} E \int_0^T x^2(t)dt. \tag{2}$$

Conventional periodic sampling with different holds will be compared with event based control where control actions are taken only when the output is outside the interval $-a < x < a$. The effects of different data holds will also be explored. We will compare the distribution of $x(t)$ and the variances of the outputs for both control schemes.

Periodic Sampling with Impulse Hold

We will first consider the case of impulse hold. The control signal is then an impulse applied when an event occurs. Since the process dynamics is an integrator it is possible to reset the state to zero instantaneously at each event. Let t_k be the time when the event occurs, the control law then becomes

$$u(t) = -x(t_k)\delta(t - t_k), \tag{3}$$

where δ is the delta function. The control law (3) implies that an impulse which resets the control to zero is applied at each sampling interval. After the impulse the closed loop system is governed by $dx = dv$ and the variance then grows linearly until the next sampling interval occurs. Since the incremental variance is 1 the average variance over a sampling interval is $h/2$ and the minimal loss function (2) for periodic sampling and impulse hold becomes

$$V_{PIH} = \frac{1}{2}h. \tag{4}$$

Periodic Sampling with Zero Order Hold

To illustrate that the data hold influences the results we will consider the standard situation with periodic sampling and a first order hold. Let h the sampling period, the sampled system is then

$$x(t+h) = x(t) + hu(t) + e(t), \tag{5}$$

which a special case of a standard discrete system with $\Phi = 1$ and $\Gamma = h$ [6]. The mean square variance over one sampling period is

$$\begin{aligned} V &= \frac{1}{h}\int_0^h Ex^2(t)\,dt = \frac{1}{h}J_e(h) \\ &\quad + \frac{1}{h}\big(Ex^T Q_1(h)x + 2x^T Q_{12}(h)u + u^T Q_2(h)u\big) \\ &= \frac{1}{h}(R_1(h)S(h) + J_e(h)), \end{aligned} \tag{6}$$

where $Q_1(h) = h,\ \ Q_{12}(h) = h^2/2,\ \ Q_2(h) = h^3/3$, $R_1(h) = h$ and

$$J_e(h) = \int_0^h Q_{1c}\int_0^t R_{1c}\,d\tau\,dt = h^2/2 \tag{7}$$

see [2]. The function $S(h)$ is the positive solution of the Riccati equation

$$S = \Phi^T S\Phi + Q_1 - L^T RL, \quad L = R^{-1}(\Gamma^T S\Phi + Q_{12}^T), \quad R = Q_2 + \Gamma^T S\Gamma,$$

where the argument h has been dropped to get cleaner equations. The Riccati equation has the solution $S(h) = h\sqrt{3}/6$, and the controller which minimizes the loss function (2) is

$$u = -Lx = \frac{1}{h}\frac{3+\sqrt{3}}{2+\sqrt{3}}x.$$

The minimum variance with periodic sampling and a zero order hold is thus

$$V_{PZOH} = \frac{3+\sqrt{3}}{6}h. \tag{8}$$

Notice that the impulse hold gives the variance $V_{PIH} = h/2$, while a zero order hold gives the variance $V_{PZOH} = h(3+\sqrt{3})/6$. The impulse hold is thus more effective than the zero order hold.

Event Based Control

For event based control we specify a region $-a < x < a$. No control action will be taken if the state is inside this region. Control actions are only taken at events t_k when $|x(t_k)| = a$. A simple strategy is to apply an impulse that drives the state to the origin, i.e. $x(t_k+0) = 0$. With this control law the closed loop system becomes a Markovian diffusion process of the type investigated in [18]. Let $T_{\pm d}$ denote the exit time i.e. the first time the process reaches the boundary $|x(t_k)| = a$ when it starts from the origin. The mean exit time can be computed from the fact that $t - x_t^2$ is a martingale between two impulses and thus

$$h_E := E(T_{\pm d}) = E(x^2_{T_{\pm d}}) = a^2.$$

The average sampling period thus equals $h_E = a^2$.

The probability distribution of x is given by the Kolmogorov forward equation for the Markov process

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) - \frac{1}{2}\frac{\partial f}{\partial x}(d)\delta_x + \frac{1}{2}\frac{\partial f}{\partial x}(-d)\delta_x,$$

with $f(-a) = f(a) = 0$, see [18]. This partial differential equation has the stationary solution

$$f(x) = (a - |x|)/a^2, \tag{9}$$

which can be verified by direct substitution. Notice that the distribution is not Gaussian but symmetric and triangular with the support $-a \le x \le a$. The steady state variance is

$$V_{EIH} = \frac{a^2}{6} = \frac{h_E}{6}. \tag{10}$$

Comparison

Summarizing the results we find that the loss functions are given by the Equations (4), (8) and (10) that

$$V_{PZOH} = \frac{3+\sqrt{3}}{6}h, \quad V_{PIH} = \frac{h}{2}, \quad V_{EIH} = \frac{a^2}{6} = \frac{h_E}{6}. \tag{11}$$

The variances are thus related as $4.7h{:}3h{:}a^2$. It follows that for periodic sampling a zero order hold increases the variance by 50% compared with impulse hold. To compare periodic sampling with event based sampling we will choose the parameter a so that the average sampling rates are the same. For event based sampling the average sampling period was $h_E = a^2$. Equating this with h gives $a^2 = h$. An event based controller thus gives a variance that is 3 times smaller than a controller with periodic sampling.

The reason for the differences is that the event based controller acts as soon as an error is detected. The reason why impulse holds give smaller variances than a zero order hold is because it is advantageous to act decisively as soon as a deviation is detected. In the particular case two thirds of the improvement is thus due to sampling and one third due to the impulse hold.

The behavior of the closed loop systems obtained with the different control strategies are illustrated by the simulation results shown in Figure 1. Simulation is performed by approximating continuous behavior by fast sampling. The same noise sequence is used in all simulations. The states are shown in

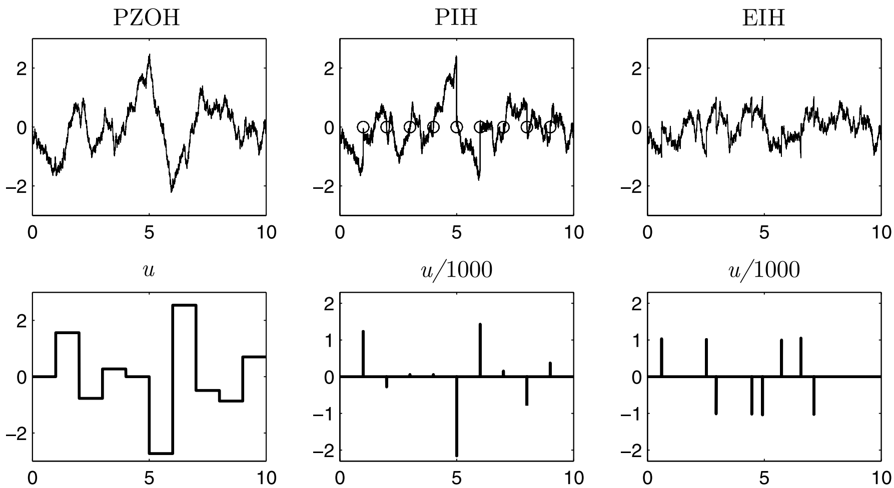

Fig. 1. Simulation of an integrator with periodic sampling and first order hold (*left*) periodic sampling and impulse hold (*center*) and event based control with impulse hold (*right*). The *top plots* shows the state and the control signals are shown in the *lower plots*. The number of samples are the same in all cases

the upper plots and the corresponding control signals in the lower plots. The left plots show results for periodic sampling with a zero order hold (ZOH), the center plots correspond to periodic sampling with impulse hold (IH) and the plots to the right show event based control with impulse hold EIH. The improved performance with event based control is clearly visible in the figure. Notice that the process state stays within the bounds all the time. In the plot of the process state for periodic sampling with impulse hold we have also marked the state at the sampling times with circles. With impulse sampling the state is reset instantaneously as is clearly seen in the figure. The advantage of the impulse hold is apparent by comparing the behavior of zero order hold and impulse sampling at time $t = 5$ where the state has a large positive value.

4 A General Structure

A block diagram of a system with event based control is shown in Figure 2. The system consists of the process, an event detector, an observer, and a control signal generator. The event detector generates a signal when an event occurs, typically when a signal passes a level, different events may be generated for up- and down-crossings. The observer updates the estimates when an event occurs and passes information to the control signal generator which generates the input signal to the process. The observer and the control signal generator run open loop between the events, the absence of an event is however information that can be used by the observer [26]. A simple special case is when the full state of the process is transmitted when an event occurs.

The control strategy is a combination of feedback and feedforward. Feedback actions occur only at the events. The actuator is driven by the control signal generator in open loop between the events. A consequence is that the behavior of the system is governed by the control signal generator. Design of the control signal generator is therefore a central issue.

It is interesting to compare with a conventional sampled data system where the events are generated by a clock and the behavior of the system is primarily determined by the control law. Such a system can also be represented by

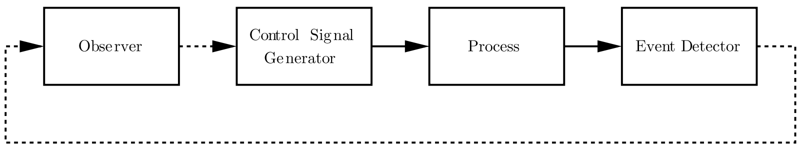

Fig. 2. Block diagram of a system with event based control. *Solid lines with filled arrows* denotes continuous signal transmission and the *dashed lines with open arrows* denotes event based signal transmission

Figure 2 but there is a block representing the control law inserted between the sampler and the control signal generator. For a conventional sampled system the behavior of the closed loop system is essentially determined by the control algorithm, but in an event based controller the behavior is instead determined by the control signal generator. Therefore it makes sense to use another name, even if the control signal generator can be regarded as a generalized hold. The different elements of the system in Figure 2 will now be discussed.

The Control Signal Generator

The design of hold circuits was discussed in the early literature on sampled-data systems, see [48] and [31]. When computer control became widely used the zero order hold became the standard solution. Linear holds were sometimes used when a smooth control signal was desired, [6]. There was a renewed interest of generalised data hold circuits around 1990 when it was discovered that the properties of a system can be changed significantly, [32]. The solutions proposed often led to irregular control signals which excited high frequency modes and gave poor robustness [19]. A properly designed control signal generator can however give improved performance as is shown in [45].

There has recently been a renewed interest in generalized hold in order to obtain systems that are insensitive to sampling jitter [44]. This is a central issue for event based systems where sampling can be very irregular. The difference between the different holds can be illustrated by a simple example. Consider the following first order system

$$\frac{dx}{dt} = x - u.$$

An unstable system is chosen because the differences will be more pronounced. Assume that an event based controller is designed based on an average sampling time t_0. We will compare an ordinary zero order hold and a hold with exponential decay. Consider the situation when $x(0) = 1$. A straightforward calculation shows that the control signals and the state behavior are given by

$$\begin{aligned} u_{PZOH} &= \frac{1}{1-e^{-t_0}}x(0) & \qquad x_{PZOH} &= \frac{1-e^{t-t_0}}{1-e^{-t_0}}x(0) \\ u_{EH} &= \frac{(a+1)e^{-t}}{1-e^{-(a+1)t_0}}x(0) & \qquad x_{EH} &= \frac{e^{-at}-e^{t-(a+1)t_0}}{1-e^{-(a+1)t_0}}x(0)\,. \end{aligned}$$

Figure 3 shows the state and the control signal for the different holds. The zero order hold gives a constant control signal but the exponential hold generates a control signal that is large initially and then decays. The behavior of the state is also of interest, all holds give the desired state $x = 0$ at the nominal time $t = t_0 = 2$ but the rate of change of the state at t_0 is quite different. The zero order hold gives the largest rate and the exponential hold with the fastest decay gives the smallest rate.

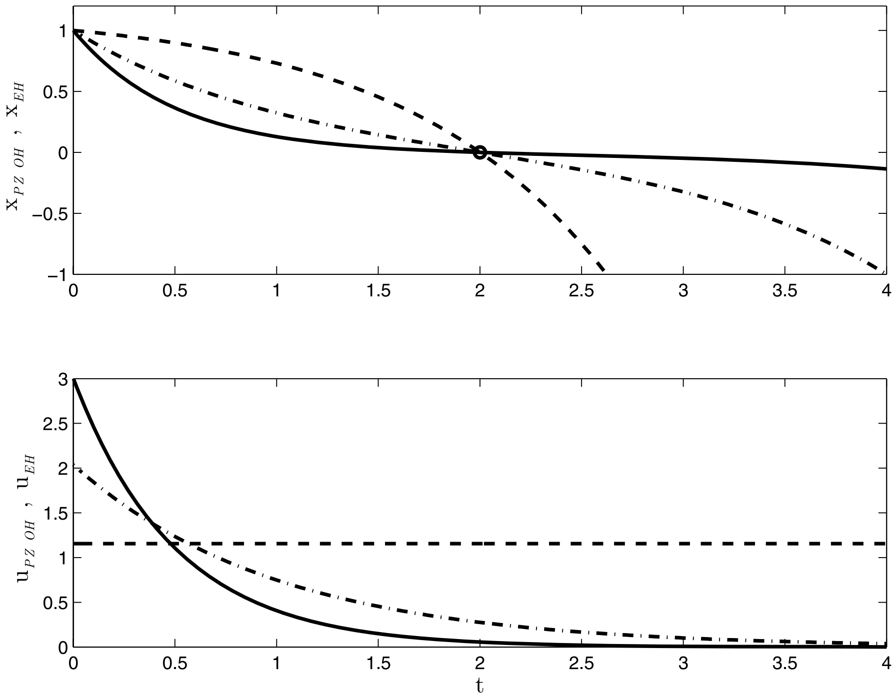

Fig. 3. Behavior of state and control variables after an event for systems with zero order hold (*dashed*) and exponential holds with $a = 1$ (*dash-dotted*) and $a = 2$ (*solid*). The signal is sampled at time $t = 0$ and the control signal is then applied in an open-loop manner. Notice that the states for both systems is the same at the nominal sampling time $t_0 = 2$, but that the rate of change of the state is much smaller with exponential hold.

The simple example shows that the hold circuit has important consequences. Data holds where the control signal decays after the event have the advantage that the behavior of the system is robust to changes in the times events occur. We can thus conclude that for event based control it is desirable to have holds that give control signals which are large initially and decay fast. The impulse hold where the control signal is an impulse is an extreme case. Holds with large control signals may however be undesirable for systems with poorly damped resonant modes. In the system discussed in Section 3 impulse sampling gave better performance than a zero order hold.

Since the hold circuit is important it is natural that it should be matched to the process and we will therefore briefly discuss methods for designing control signal generators circuits. There are many ways to do this, in fact many methods for control system design can be used. Consider for example the system described by

$$\frac{dx}{dt} = Ax + Bu, \tag{12}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$. Design of the control signal generator is essentially the problem of finding an open loop control signal that drives the system from its state at the time of the event to a desired state. In regulation problems the desired state is typically a given constant for example $x = 0$. This can always be done in finite time if the system is reachable. There are many ways to generate the signal, we can for example use a dead-beat controller which drives the state to zero in finite time. Optimal control and model predictive control are other alternatives that are particularly useful when there are restriction on the control signal.

In this paper we will determine a state feedback $u = -Lx_c$ for the system (12) which drives the state to zero. Such a control signal can be generated from the dynamical system

$$\frac{dx_c}{dt} = (A - BL)x_c, \quad u = -Lx_c, \tag{13}$$

which is initialized with the process state at the event or the estimated process state. Notice that x_c is the controller state and that the control is applied in an open-loop manner like a feedforward signal.

Relations to Nonlinear Control

There is an advantage to generating the control signal from Equation 13. Assuming that there are no model errors and no disturbances, the controller state x_c is then equal to the process state x and the open loop control is identical to a closed loop control of the process with the control law $u = -Lx$. Since the control signal generator is a dynamical system the system can be analysed using nonlinear control theory which is highly attractive, see [28], [30]. An example where this idea is elaborated will be given in Section 5.

The Observer

When the state of the process is not measured directly it is suitable to reconstruct it using an observer. The observer problem for event based control is not a standard problem. Consider for example a system described by

$$dx = Axdt + Budt + dv, \quad dy = Cx\,dt + de, \tag{14}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}$ and v and e are Wiener processes of appropriate dimensions with incremental covariances R_1dt and R_2dt. Assume that an event is generated when the magnitude of the output y exceeds a. If no event is obtained we clearly have information that the measured signal is in the interval $-a < y < a$, when an event occurs we obtain a precise measurement of y. This is clearly a non-standard information pattern.

An ad hoc solution is to use the following approximate Kalman filter

$$\begin{aligned}\frac{d\hat{x}}{dt} &= A\hat{x} + Bu + K(t)(0 - C\hat{x})\\ K(t) &= P(t)CR_2^{-1}\\ \frac{dP}{dt} &= AP + PA^T - PC^T R_2^{-1} CP + R_1,\end{aligned}$$

when no event occurs. A reasonable assumption for the measurement noise is $R_2 = a^2/(12t_s)$, where t_s is the average sampling rate, or the time elapsed since the last event. Recall that $a^2/12$ is the variance of a uniform distribution over the interval $(-a,\ a)$. When an event occurs the state is estimated by

$$\begin{aligned}\hat{x}^+ &= \hat{x} + PC(\bar{R}_2 + CPC^T)^{-1}(y_e - C\hat{x})\\ P^+ &= P - PC(\bar{R}_2 + CPC^T)^{-1}C^T P,\end{aligned}$$

where $y_e = a$ or $-a$ depending on which boundary is crossed, and $\bar{R}_2$ is the variance of the detection error. The superscript + denotes the values immediately after the detection. A simple assumption is $\bar{R}_2 = 0$.

The filtering problem can be solved exactly by computing the conditional distribution of the state x of (14) given that $-a < y < a$. This problem is discussed in [26], where it is shown that at least for the accelerometer example in Section 5 the approximate Kalman filter discussed in Section 4 gives similar results. In [25] and [26] it is also shown that the exact solution to the filtering problem has interesting properties. For example it is shown that the conditional distribution is log-concave under quite general conditions.

5 An Example

As an illustration we will consider a simple model for a MEMS accelerometer, which consists of a mass supported a weak spring with a detector which detects small deviations of the mass from the reference position and a capacitive force actuator. In extreme cases the sensing is done by measuring tunnelling current via a very narrow tip, see [39], [35].

Neglecting the spring coefficient the system becomes a double integrator. Using normalized parameters the system can be described by the model

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{15}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x. \tag{16}$$

It is assumed that the largest control signal is 1 and that an event is detected when $|y(t)| = a$. We also assume that the direction of the crossing is detected. In reality only position information is available, for simplicity we will here assume that the full state is available when an event occurs.

The state feedback approach is used to generate the control signal. Assuming a linear feedback $u = -Lx$ the closed loop system becomes

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = \begin{pmatrix} 0 & 1 \\ -l_1 & -l_2 \end{pmatrix} x \\ u &= -\begin{pmatrix} l_1 & l_2 \end{pmatrix} x. \end{aligned} \tag{17}$$

The characteristic polynomial for the closed loop system is

$$s^2 + l_2 s + l_1.$$

Since the largest control signal is one it is natural to choose $l_1 = a^{-1}$. This means that full control action is used if the mass has zero velocity, and the output is at the boundary $|x_1| = a$ of the detection zone. Choosing $l_2 = 2/\sqrt{a}$ gives critical damping. The settling time after a disturbance is then of the order of $5\sqrt{a}$. With $a = 1$ we then get $l_1 = 1$ and $l_2 = 2$ and a settling time of 5.

When an event occurs the system (17) is initialized with the state equal to the state of the process and the control signal is then generated by running (17) in open loop. In this case the control signal generator is thus matched to the dynamics of the system. Since the actuator has limitations it is useful to limit the control signal, see [51]. The control signal then becomes

$$u = -\mathrm{sat}(l_1 \hat{x}_1 + l_2 \hat{x}_2).$$

If the directions of the crossings are known it is possible to add a refinement by applying the full control power when the output leaves the band $-a < y < a$ and to apply the signal from the hold circuit when the signal enters the band again. This gives a relay action which ensures that the system quickly enters the detection zone. The complete nonlinear control law then becomes

$$u = \begin{cases} 1 & \textit{if} \quad x_1 \le -1 \\ -\mathrm{sat}(l_1 \hat{x}_1 + l_2 \hat{x}_2) & \textit{if} \quad -a < x_1 < -a \\ -1 & \textit{if} \quad x_1 \ge a\,. \end{cases} \tag{18}$$

The control signal generator is thus implemented by applying this nonlinear feedback law to the system (15) and running it in open loop. Another way to generate the hold signal is to observe that the pulse shape is given by

$$u = \begin{cases} 1 & \textit{if} \quad x_1 \le -1 \\ -\mathrm{sat}(\pm p_1(t) + p_2(t) v^*) & \textit{if} \quad -a < x_1 < -a \\ -1 & \textit{if} \quad x_1 \ge a\,, \end{cases}$$

where $p_1(t)$ and $p_2(t)$ are given functions of time which can be computed off line, v^* the velocity when x_1 enters the region $-a < x_1 < 1$, and the sign of the p_1-term depends on which side it enters from. A table look-up procedure can thus also be used to generate the control signal.

Equivalent Nonlinear System

If there are no disturbances and no modeling errors and if the control signal is generated by (13), the system with event based control behaves like the system (15) with the nonlinear feedback (18). Since this system is of second order it is straightforward to analyze its behavior. A phase plane of the system is shown in Figure 4 for $a = 1$. The phase plane is symmetric so it suffices to discuss half the region. It follows from (18) that $u = -1$ in the shaded region. The equations can then be integrated analytically in that region, and we find that the trajectories are given by the parabolic curves

$$x_1 = x_1(0) + \frac{1}{2}x_2^2(0) - \frac{1}{2}x_2.$$

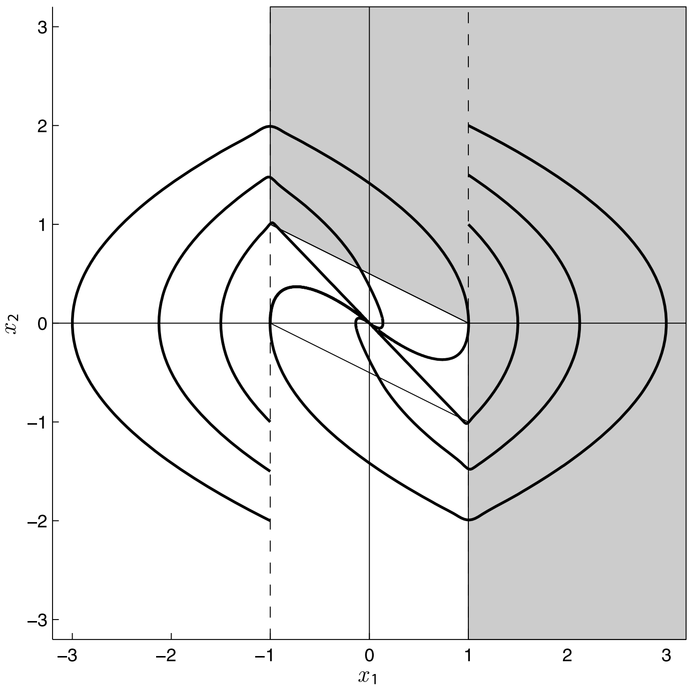

Fig. 4. Phase plane for the system when there are no modeling errors and no disturbances. The control signal is $u = -1$ in the *shaded zone*

A solution starting at $x_1(0) = -1$ with $x_2(0) \geq \sqrt(2)$ intersects the line $x_1 = 1$ at $x_2 = \pm\sqrt{x_2^2(0) - 2}$. The velocity x_2 decreases for each passage of the region $-1 < x_1 < 1$. Trajectories starting at $x_1(0) = -1$ with $1 \leq x_2(0) \leq \sqrt{2}$ enter the linear region, and trajectories starting at $x_1(0) = -1$ with $x_2(0) < 1$ remain in the rhomboid region. The origin is globally asymptotically stable and all trajectories entering the rhomboid region around the origin will remain in that region.

Response to Random Accelerations

The control law (18) implies that the system works like a system with relay feedback when the output is outside the detection region. This feedback attempts to force the output into the detection region. The action of the signal generator is to generate a signal that drives the state to the nominal zero position. This means that the signal generator provides a damping action.

The consequences of event based control is clearly seen in the response to a random acceleration. A simulation of such a case is shown in Figure 5. Notice

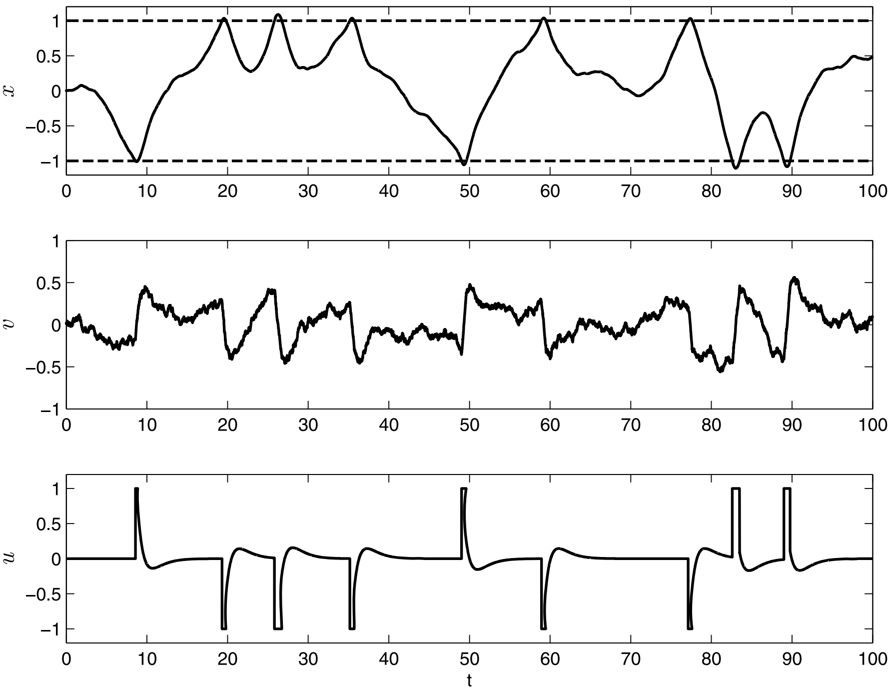

Fig. 5. Simulation of a MEMS accelerometer with event based feedback and random acceleration

that the deviation is kept between the limits even if the events are quite sparse. Also notice the shape of the pulses. They maintain the maximum values as long as the deviation is outside the detector limit, they jump at the events when the output enters the detection region and then they decay gracefully to zero. In this case it is essential to apply full restoring force when the output is outside the detection band. If this is not done the system may diverge after a large disturbance.

Response to a Constant Acceleration

The previous analysis is useful in order to understand the basic behavior of the system. It is, however, more relevant to explore the real purpose of the system by investigating how it responds to a constant acceleration Figure 6 shows another simulation when there is a constant acceleration. The top curve in the figure shows the position of the mass $y = x_1$, the center curve is the control signal u and the bottom curve is the integral of the acceleration error

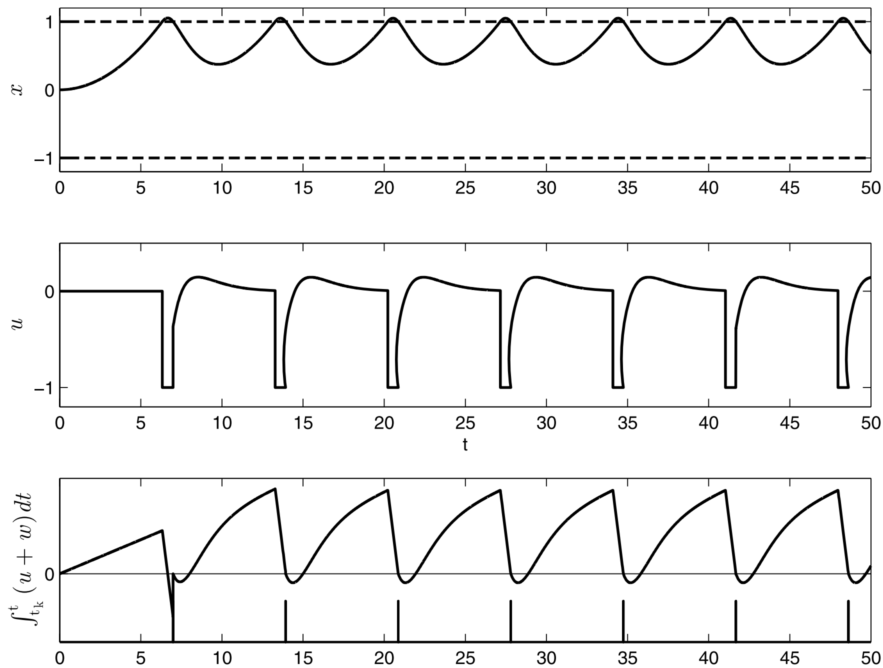

Fig. 6. Response to a constant acceleration $w = 0.05$. The *upper plot* shows position of the mass, the *middle plot* shows the control signal u and the *lower plot* shows the integral of the control error. The events are also indicated in the lower plot

e_a. The integral is reset to zero at the events formed by the up-crossings. The events are also marked in the figure.

Let the acceleration be w, and let the control signal be u. Assume that t_k and t_{k-1} are consecutive events where the same boundary is crossed. It then follows that

$$x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} (w(\tau) + u(\tau)) d\tau = x(t_k) + \int_{t_k}^{t_{k+1}} (t_{k+1} - t)(w(t) + u(t)) dt.$$

Since $x(t_k) = x(t_{k-1})$ it follows that

$$\int_{t_k}^{t_{k+1}} \int_{t_k}^{t} u(t) dt = \int_{t_k}^{t_{k+1}} (t_{k+1} - t) u(t) dt = - \int_{t_k}^{t_{k+1}} (t_{k+1} - t) w(t) dt.$$

The double integral of the control signal between two consecutive events is thus a weighted average of the acceleration in the interval between the events.

A similar analysis shows that if the events represents crossings of different boundaries a correction term $4a/(t_{k+1} - t_k)^2$ should be added. The time resolution is given by the width of the detection interval. If there is a periodic solution we have $v(t_{k+1}) = v(t_k)$ and in this case we find that the integral of the control signal over an interval between two events is the average of the acceleration during that interval. This is illustrated in the lower plot in Figure 6 which shows the integral of the acceleration error. The integral is reset to zero at the events formed by the up-crossings. The events are also marked in the figure.

A phase plane is shown in Figure 7. The figure shows that the limit cycle is established quickly. Also notice the parabolic shape of the trajectories for

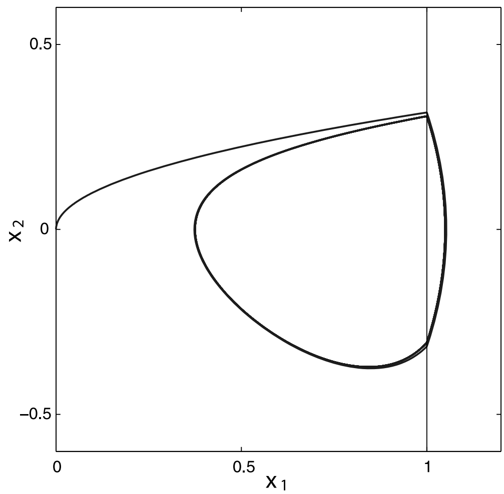

Fig. 7. Phase plane for the simulation in Figure 6

$x_1 > 1$. Exact conditions for limit cycles and their local stability can be computed using the method in [3].

6 Conclusions

Even if periodic sampling will remain the standard method to design computer controlled systems there are many problems where event based control is natural. Several examples of event based control were given in Section 2. In Section 3 we investigated a simple example where the event based and periodic control could be compared analytically. The example showed that the performance with event based control was superior to periodic sampling, and it also shown that that the control signal generation is important. A general architecture of an event based controller was discussed in Section 4. Key elements of the system are the event detector, the observer, and the control signal generator. Several ways to design the control signal generator was discussed. One method has the advantage that the event based system is equivalent to a nonlinear system. The results were illustrated with design of a controller for a simplified version of a MEMS accelerator. Even if event based systems have been used for a long time, the field is still at its infancy and there are many challenging theoretical problems that are not solved.

References

1. K.J. Åström. *On the choice of sampling rates in optimal linear systems.* Technical report rj-243, IBM Research, San José CA, 1963.
2. K.J. Åström. *Introduction to Stochastic Control Theory.* Academic Press, New York, 1970. Dover reprint published in 2006.
3. K.J. Åström. Oscillations in systems with relay feedback. In W. Miller, editor, *IMA Workshop on Adaptive Control.* Springer-Verlag, New York, 1993.
4. K.J. Åström and B. Bernhardsson. Comparison of periodic and event based sampling for first-order stochastic systems. *Proc. of the 14th IFAC World Congress*, pages 301–306, 1999.
5. K.J. Åström and B. Bernhardsson. Comparison of Riemann and Lebesque sampling for first order stochastic systems. *Proc. of the 41st IEEE Conf. on Decision and Contr.*, 9:2011–2016, 2002.
6. K.J. Åström and B. Wittenmark. *Computer-Controlled Systems.* Prentice Hall, 3rd edition, 1997. First published in 1984 edition.
7. J.P. Aubin. *Impulse differential inclusions and hybrid system: A viability approach.* Lecture notes, University of California, Berkeley, 1999.
8. A. Bensoussan and J.-L. Lions. *Impulse control and quasi-variational inequalities.* PhD thesis, Gauthier-Villars, Paris, 1984.
9. G. Bourdopoulos, A. Pnevmatikakis, V. Anastassopoulos, and T.L. Deliyannis. *Delta-Sigma Modulators: Modeling, Design and Applications.* Imperial College Press, London, 2003.

10. G.C. Buttazzo. *Hard Real-time Computing Systems: Predictable Scheduling Algorithms and Applications*. Kluwer, Amsterdam, 1997.
11. A. Cervin, D. Henriksson, B. Lincoln, J. Eker, and K.-E. Årzén. How does control timing affect performance? *IEEE Control Systems Magazine*, 23(3):16–30, 2003.
12. T. Chen and B.A. Francis. *Optimal Sampled-Data Control Systems*. Springer Verlag, New York, 1995.
13. C. Dase, J. Sullivan, and B. MacCleery. Motorcycle control prototyping using an FPGA-based embedded control system. *IEEE Control Systems Magazine*, 26(5):17–21, 2003.
14. S.S.P. DeWeerth, L. Nielsen, C.A. Mead, and K.J. Åström. A simple neuron servo. *IEEE Trans. Neural Networks*, 2(2):248–251, 1991.
15. M.D. DiBenedetto and A. Sangiovanni-Vincentelli. *Hybrid systems in computation and control*. Springer Verlag, New York, 2001.
16. S.J. Dodds. Adaptive, high precision, satellite attitude control for microprocessor implementation. *Automatica*, 17(4):563–573, 1981.
17. S.S. Draper, W. Wrigley, and J. Hovorka. *Inertial Guidance*. Pergamon Press, Oxford, 1960.
18. W. Feller. Diffusion processes in one dimension. *Trans. Am. Math. Soc.*, 55:1–31, 1954.
19. A. Feuer and G.C. Goodwin. Generalized sample hold functions – frequency domain analysis of robustness, sensitivity, and intersample behavior. *IEEE Trans. on Automat. Contr.*, 39(5):1042–1047, 1994.
20. I. Flügge-Lotz. *Discontinuous and Optimal Control*. McGraw-Hill, New York, 1968.
21. P.M. Frank. A continuous-time model for a PFM-controller. *IEEE Trans. on Automat. Contr.*, 25(5):782–784, 1979.
22. B. Friedland. Modeling systems for pulse-width modulated control. *IEEE Trans. on Automat. Contr.*, 21:739–746, 1976.
23. L. Guzzella and C. Onder. Past, present and future of automotive control. In M. C. Smith B. A. Francis and J. C. Willems, editors, *Control of Uncertain Systems: Modeling, Approximation and Design*. Springer-Verlag, Heidelberg, 2006.
24. E. Hendricks, M. Jensen, A. Chevalier, and T. Vesterholm. Problems in event based engine control. *Proc. of the 1994 Amer. Contr. Conf.*, 2:1585–1587, 1994.
25. T. Henningsson. *Logarithmic concave observers*. Master's thesis isrn lutfd2/tfrt--5747--se, Department of Automatic Control, Lund University, Sweden, 2005.
26. T. Henningsson and K.J. Åström. Log-concave observers. *Proc. of Mathematical Theory of Networks and Systems*, 2006.
27. R.K. Hobbie. *Intermediate Physics for Medicine and Biology*. Springer Verlag, 1997.
28. A. Isidori. *Nonlinear Control Systems*. Springer Verlag, New York, 3rd edition, 1995. First published 1985.
29. V Jacobson. Congestion avoidance and control. *Proc. of SIGCOMM '88*, pages 314–329, 1988.
30. M. Johansson. *Piecewise Linear Control Systems*. Springer Verlag, Heidelberg, 1999.
31. E.I. Jury. *Sampled-Data Control Systems*. John Wiley & Sons, New York, 1958.
32. P.T. Kabamba. Control of linear systems using generalized sampled-data hold functions. *IEEE Trans. on Automat. Contr.*, 32(9):772–783, 1987.

33. J. Keener and J. Sneyd. *Mathematical Physiology.* Springer Verlag, New York, 2001.
34. F.P. Kelley. Stochastic models of computer communication. *J. Royal Statistical Society*, B47(3):379–395, 1985.
35. M. Khammash, L. Oropeza-Ramos, and K.L. Turner. Robust feedback control design of an ultra-sensitive, high bandwidth tunneling accelerometer. *Proc. of the 2005 Amer. Contr. Conf.*, 6:4176–4180, 2005.
36. H. Kopetz. Should responsive systems be event triggered or time triggered? *IEICE Trans. on Information and Systems*, E76-D(10):1525–1532, 1993.
37. H. Kopetz. Time-triggered real-time systems. *Proc. of the 15th IFAC World Congress*, 2002.
38. W.H. Kwon, Y.H. Kim, S.J. Lee, and K.N. Paek. Event-based modeling and control for the burnthrough point in sintering control. *IEEE Trans. on Contr. Systems Tech.*, 7(1):31–41, 1999.
39. C.H. Liu and T.W. Kenny. A high-precision, wide-bandwidth micromachined tunneling accelerometer. *IEE/IEEE Journal of Microelectromechanical Systems*, 10(3):425–433, 2001.
40. C.A. Mead. *Analog VLSI and Neural Systems.* Addison-Wesley, Reading, Massachusetts, 1989.
41. J.E. Miller. The pulsed integrated pendulous accelerometer (pipa). In W. Miller, editor, *Space Navigation Guidance and Control.* Springer-Verlag, Technivision, Maidenhead, England, 1996. Agardograph 105.
42. S.R. Norsworthy, R. Schreier, and G. Temes. *Delta-Sigma Data Converters.* IEEE Press, New York, 1996.
43. M. Nougaret. A design method for first-order pulse-width-modulated systems. *Int. J. of Control*, 15(3):541–549, 1972.
44. K. Ohno, M. Hirata, and R. Horowitz. A comparative study of the use of the generalized hold function for HDDS. *Proc. of the 42nd IEEE Conf. on Decision and Contr.*, pages 5967–5972, 2003.
45. K. Ohno, M. Hirata, and R. Horowitz. A comparative study of the use of the generalized hold function for HDDS. *IEEE/ASME Transactions on Mechatronics*, 10(1):26–33, 2005.
46. B Pettersson. Production control of a complex integrated pulp and paper mill. *Tappi*, 52(11):2155–2159, 1969.
47. J. Pettersson, L. Ledung, and X. Zhang. Decision support for pulp mill operations based on large-scale optimization. *Control Systems*, 2006.
48. J.R. Ragazzini and G.F. Franklin. *Sampled-data Control Systems.* McGraw-Hill, New York, 1958.
49. H. Sira-Ramirez. A geometric approach to pulse-width modulated control in nonlinear dynamical systems. *IEEE Trans. on Automat. Contr.*, 34(2):184–187, 1989.
50. R.A. Skoog. On the stability of pulse-width-modulated feedback systems. *IEEE Trans. on Automat. Contr.*, 13(5):532–538, 1968.
51. A.R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Trans. on Automat. Contr.*, 41(9):1256–1270, 1996.
52. Y.Z. Tsypkin. *Theory of Impulse Systems.* State Publisher for Physical and Mathematical Literature, Moscow, 1958.
53. Y.Z. Tsypkin. *Relay Control Systems.* Cambridge University Press, Cambridge, UK, 1984.

54. V.I. Utkin. Discontinuous control systems: State of the art in theory and applications. *Proc. of the 10th IFAC World Congress*, 1987.
55. H.R. Wilson. *Spikes, Decisions, and Actions: The Dynamical Foundations of Neuroscience*. Oxford University Press, 1999.
56. Y. Yamamoto. A retrospective view on sampled-data control systems. *CWI Quarterly*, 9(3):261–276, 1996.