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Self-triggered state feedback control of linear plants under bounded disturbances

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Abstract— **This paper addresses the problem of self-triggered state feedback control for linear plants under bounded disturbances. In a self-triggered scenario, the controller is allowed to choose when the next sampling time should occur and does so based on the current sampled state and on a priori knowledge about the plant. The main contribution of this paper is the definition of a triggering strategy that allows for the consideration of a class of controllers that is much larger than that of static controllers with a zero-order hold of the last state measurement. This is done by resorting to a model-based control architecture whereby a model of the plant is used to simulate the time evolution of the plant's state in between sampling times when no information about the plant's actual state is available. An illustrative example with simulation results shows how the new triggering strategy along with a model-based controller design yield a clear gain in performance when compared to other strategies proposed so far.**

I. INTRODUCTION

In the design and analysis of controllers for continuous time systems, it is usually assumed that the state of the system (or its outputs) is available continuously. In practice, however, controllers must be implemented in digital devices and, for this reason, there is a need to develop control techniques for systems whose state measurements are not available continuously. The special case where measurements are available periodically (periodic sampling) has been studied extensively in the literature. The relaxation of the periodic sampling restriction to accommodate sampled-data systems with non-uniform sampling has also been considered. Such systems arise for example in the study of networked control systems, where the sampling intervals are viewed as an exogenous signal that can be deterministic but bounded, or stochastic with a known distribution.

It is important to remark that while in some cases of practical interest the sampling times are not known in advance, in an event-based scenario the controller is often allowed to choose the next sampling time (also, know as update or release time, e.g., in the area of networked control systems), which effectively works as an extra degree of freedom in the design process. Controller design for such systems must therefore produce some kind of triggering or scheduling law.

In event-triggered control (see Fig. 1), an event detector is responsible for testing that a triggering condition (basically, a function of the plant's state) is observed, in which case a sampling event is triggered. The advantage of this approach versus a periodic sampling strategy is that in practice only a small loss of performance is observed, while the number of samples required may be drastically reduced. For a discussion on the advantages of event-based control, the reader is referred to [1]–[4] and the references therein.

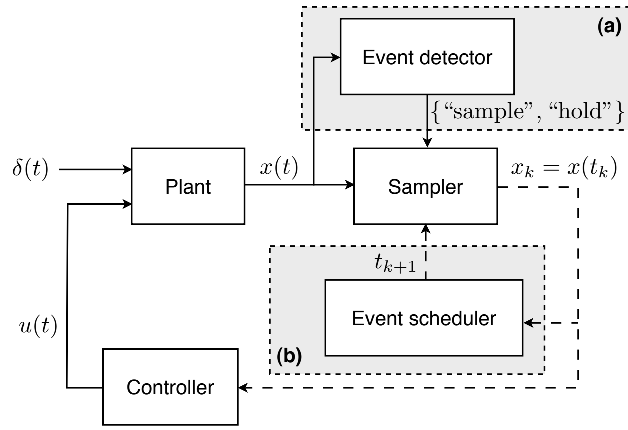

Fig. 1. Triggered control systems: event-triggered (block **(a)** active); self-triggered (block **(b)** active). The state and control input of the plant are represented by $x(t)$ and $u(t)$, respectively, and $\delta(t)$ denotes a bounded disturbance. Solid lines denote continuous-time signals while dashed lines denote discrete-time signals.

In order to avoid monitoring the state constantly, as required in event-triggered control, self-triggered strategies have been proposed [5]–[9]. Instead of continuously testing a triggering condition, an event scheduler (see Fig. 1) computes when the next sampling event should occur, based on the current sampled state and on knowledge about the plant dynamics.

This paper addresses the problem of self-triggered control for linear plants under bounded disturbances. Its main contribution is the definition of a triggering method that departs considerably from that given in [7], [8]. The new triggering condition allows for the consideration of a class of controllers that is much larger than that of static controllers with a zero-order hold of the last state measurement considered in [7], [8]. This is done by resorting to a model-based control architecture akin to that proposed in [10], whereby a model of the plant is used to simulate the evolution of the plant's state in between sampling times when no information about the plant's actual state is available. A clear improvement in performance is observed in simulation. This strategy has been used in [11] in the context of event-based controllers.

The paper is organized as follows. Section II introduces the class of plants under consideration and reviews the stability properties that can be achieved with continuous control. The

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self-triggered approach is described in Section III, along with a brief review of the triggering mechanism proposed in [7], [8]. The model-based control architecture is introduced in Section IV. The main contribution of this paper is the new triggering method proposed in Section V. We also addressed its stability analysis and practical implementation. In Section VI, through an illustrative example, the performance of both periodic and self-triggered controllers is compared and discussed. Finally, concluding remarks are given in Section VII.

II. LINEAR PLANT WITH DISTURBANCES

Consider a linear time-invariant plant with state $x \in \mathbb{R}^n$ such that $x(t_0) = x_0 \in \mathbb{R}^n$ and, for all $t \geq t_0$,

$$\dot{x}(t) = Ax(t) + Bu(t) + \delta(t) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are plant matrices, $u(t) \in \mathbb{R}^m$ is the control input and $\delta(t) \in \mathbb{R}^n$ is a bounded exogenous disturbance. By bounded disturbance we mean that the $\mathcal{L}_\infty$ norm of $\delta(t)$ is bounded, that is, there exists $\delta_b \geq 0$ such that $\|\delta\|_{\mathcal{L}_\infty} = \sup_{t\geq t_0} \|\delta(t)\| \leq \delta_b$, where $\|v\|$ denotes the Euclidean norm of $v \in \mathbb{R}^n$. The pair (A, B) is assumed to be controllable. The presence of $\delta(t)$ makes it impossible to render the origin of (1) asymptotically stable. Consider the following stability notions borrowed from [12]. The solutions of (1) are *globally uniformly ultimately bounded* (GUUB) with ultimate bound b if there exists a positive constant b, and for every $a > 0$, there is $T = T(a, b) \geq 0$, independent of t_0, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \forall t \geq t_0 + T.$$

The system (1) is said to be (exponentially) *input-to-state stable* (ISS) if there exist positive constants κ, λ, and γ such that, for any initial state $x(t_0)$, the solution $x(t)$ satisfies

$$\|x(t)\| \leq \kappa e^{-\lambda(t-t_0)}\|x(t_0)\| + \gamma \sup_{t_0\leq\tau\leq t} \|\delta(\tau)\|$$

for all $t \geq t_0$. Notice that ISS implies GUUB. If the system (1) is ISS and the disturbance $\delta(t)$ converges to zero as t goes to infinity, then so does the state $x(t)$.

A proper choice of state feedback control law will render the closed-loop system ISS. Given a positive definite matrix Q, let P and K be such that P is positive definite and the Lyapunov equation

$$A_{cl}^\top P + PA_{cl} + Q = 0, \tag{2}$$

is satisfied with $A_{cl} = A + BK$ (this is always possible because the pair (A, B) is assumed to be controllable). Under continuous feedback of the form $u(t) = Kx(t)$, it is possible to show that the closed loop system is ISS by using standard analysis tools (see, e.g., [12, Theorem 4.19]).

Here, we sketch a proof that shows that the closed-loop systems is GUUB, as it will be useful later in the text. Consider the Lyapunov function

$$V(x) = x^\top Px. \tag{3}$$

Let $a_1 = \lambda_{\min}(P)$, $a_2 = \lambda_{\max}(P)$, and $c = \lambda_{\min}(Q)$. Then,

$$\begin{aligned}\dot{V}(x(t)) &= x^\top(A_{cl}^\top P + PA_{cl})x + 2x^\top P\delta\\ &= -x^\top Qx + 2x^\top P\delta\\ &\leq -c\|x\|^2 + 2a_2\|x\|\delta_b\\ &\leq -(1-\theta)c\|x\|^2, \forall\|x\| \geq r(\theta),\end{aligned} \tag{4}$$

where $0 < \theta < 1$ and

$$r(\theta) = \frac{2a_2\delta_b}{\theta c}. \tag{5}$$

It follows from (4) that, for all $x_0 \in \mathbb{R}^n$, the function V defined in (3), along the solutions of (1) with feedback law $u(t) = Kx(t)$, satisfies

$$V(x(t)) \leq V(x_0)e^{-\gamma(\theta)(t-t_0)}, \forall t_0 \leq t < t_0 + T \tag{6}$$

and

$$V(x(t)) \leq V_b(\theta), \forall t \geq t_0 + T \tag{7}$$

for some finite $T \geq 0$, where $\lambda_0 = \frac{c}{a_2}$,

$$\gamma(\theta) = (1-\theta)\lambda_0 \tag{8}$$

$$V_b(\theta) = a_2r^2(\theta) = a_2\Big(\frac{2a_2\delta_b}{\theta c}\Big)^2. \tag{9}$$

Note that (6) and (7) are equivalent to

$$V(x(t)) \leq \max\{V(x_0)e^{-\gamma(\theta)(t-t_0)}, V_b(\theta)\}, \forall t \geq t_0. \tag{10}$$

In the absence of disturbances ($\delta_b = 0$), the closed-loop system becomes exponentially stable with decay rate λ_0 (in this case, θ can be taken equal to 0).

III. SELF-TRIGGERED CONTROL

In this section, we will restrict ourselves to measuring the plant's state only at certain time instants $\{t_k\}$ that we shall refer to as sampling times. One possibility is to have sampling times that are equally spaced by a given period, that is, $t_{k+1} = t_k + T_s$ where $T_s > 0$ is a fixed sampling period. In this paper, we follow a different strategy where the sampling intervals are not set beforehand, but are instead path dependent, that is, they depend on the plant's state evolution. Moreover, we assume that a sampling action can be executed or triggered at anytime.

Consider the linear plant (1) but with a sampled-data controller

$$u(t) = f(t, x_k) \text{ for } t \in [t_k, t_{k+1}) \tag{11}$$

where x_k denotes $x(t_k)$. We do not impose that u should be kept constant between sampling times, hence its dependence on time. In event-triggered sampling, the next sampling time is defined implicitly as a condition on the plant's state:

$$t_{k+1} = \min\{t > t_k : h(t, x(t); t_k, x_k) = 0\}. \tag{12}$$

The function h is chosen in order to meet desired stability and performance requirements. If the solution $\phi(t; t_k, x_k)$ of (1) with control input (11) can be computed for all $t \geq t_k$ only from knowledge about the plant and x_k, one can further write

$$t_{k+1} = \min\{t > t_k : \tilde{h}(t; t_k, x_k) = 0\},$$

with $\tilde{h}(t;t_k,x_k) = h(t,\phi(t;t_k,x_k);t_k,x_k)$. In an ideal situation, we would arrive at an explicit formula for the next sampling time t_{k+1} as a function of the current measured state, that is, $t_{k+1} = g(t_k,x_k)$. This would yield a self-triggered sampling strategy where, at time $t = t_k$, the controller determines when the next sampling time t_{k+1} should occur based on the current (measured) state and on knowledge about the plant dynamics. However, in general, such g cannot be obtained in closed-form. Nonetheless, it is possible to solve (12) approximately while still guaranteeing desired stability and performance requirements. In what follows, we review the control law and triggering method proposed in [7], [8].

A. Review of previous triggering

Since system (1) is ISS with continuous feedback, the question addressed by the authors in [8] is whether it is possible to design a feedback law for $u(t)$ and a triggering procedure that generates $\{t_k\}$ such that the closed-loop system is ISS. The proposed solution keeps the control input constant and equal to Kx_k between sampling times, that is, $u(t) = Kx_k$ for all $t \in [t_k, t_{k+1})$. Thus, the closed-loop system dynamics become

$$\dot{x}(t) = Ax(t) + BKx_k + \delta(t).$$

Assume for the time being that $\delta(t) \equiv 0$. Consider the function V, defined in (3), whose rate of decay is $\lambda_0 > 0$ under continuous-time control. Let $\lambda \in \mathbb{R}$ be such that $0 < \lambda < \lambda_0$, and let

$$S_0(t;t_k,x_k) = V(x_k)e^{-\lambda(t-t_k)}. \tag{13}$$

The triggering condition is defined as in (12) with

$$h(t,x(t);t_k,x_k) = V(x(t)) - S_0(t;t_k,x_k). \tag{14}$$

If this condition is enforced for all $t \geq t_0$, then

$$V(x(t)) \leq V(x_0)e^{-\lambda(t-t_0)}, \forall t \geq t_0.$$

In order to conclude that V tends to zero with rate of decay λ, it is still necessary to prove that the sequence of sampling times tends to infinity. To simplify the notation we will omit the last two arguments of h and write $h(t,x(t))$. The function h is differentiable and satisfies $h(t_k,x_k) = 0$ and $\dot{h}(t_k,x_k) < 0$. These properties guarantee the existence of a minimum interval of time between sampling times, that is, for all $x_k \in \mathbb{R}^n$ there exists a positive constant $\tau_{\min}$ such that $\tau_k = t_{k+1} - t_k \geq \tau_{\min}$ where τ_k denotes the sampling interval. This ensures that the sequence of sampling times tends to infinity as time goes to infinity. In [7] it is shown that

$$\tau_{\min} = \min\{\tau > 0 : \det M_\lambda(\tau) = 0\}, \tag{15}$$

where $M_\lambda(\tau) = Ce^{F^\top \tau}C^\top PCe^{F\tau}C^\top - e^{-\lambda\tau}P$,

$$F = \begin{bmatrix} A+BK & -BK \\ -A-BK & -BK \end{bmatrix} \text{ and } C = [I_n \quad 0_n].$$

In [8], the authors extend the previous results to the case where bounded disturbances are present using the same triggering mechanism. The closed-system is shown to be ISS.

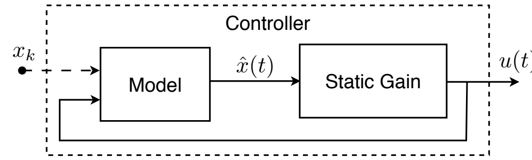


Fig. 2. Model-based controller for self-triggered architecture.

IV. MODEL-BASED CONTROL

Regarding the previous choice of feedback law, we point out that holding the value of u constant between sampling times is just one possibility. If we allow the input to be time-varying, and not just piecewise constant, improvement in the control of the plant should to be expected. We therefore consider the inclusion of a plant model in the controller as in model-based control [10]. The model simulates the plant's state evolution in between sampling times when no information about the plant's actual state is available. The controller takes the form (see Fig. 2)

$$\begin{aligned} &\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \ \hat{x}(t_k) = x(t_k) \\ &u(t) = K\hat{x}(t). \end{aligned}$$

The input u produced by this kind of controller is no longer piecewise constant, in general. The controller that keeps the control input constant and equal to Kx_k, is obtained by making $\hat{A}$ and $\hat{B}$ equal to zero. We will refer to this controller as the *zero-order hold* (ZOH) controller. Another possibility is to take $\hat{A} = A$ and $\hat{B} = B$, obtaining what we shall refer to as the *exact model-based* (EMB) controller. Assuming perfect knowledge of A and B does not impose any added restriction since, to compute the next sampling time, we need to know exactly A and B. Nonetheless, in general, we will not have exact knowledge of A and B, which raises questions about robustness to parameter uncertainty. Such issues are not addressed in this paper but provide fertile ground for future research.

The plant's time response for $t \geq t_k$ can be written as

$$x(t) = x_{\mathrm{n}}(t) + v(t)$$

where $x_{\mathrm{n}}(t)$ represents the nominal closed-loop time response in the absence of disturbances and $v(t)$ accounts for the disturbance's impact on the state evolution. A few computations yield

$$x_{\mathrm{n}}(t) = \left[e^{A(t-t_k)} + \int_{t_k}^{t} e^{A(t-s)}BKe^{(\hat{A}+\hat{B}K)(s-t_k)}\mathrm{d}s\right]x(t_k) \tag{16}$$

$$v(t) = \int_{t_k}^{t} e^{A(t-s)}\delta(s)\mathrm{d}s. \tag{17}$$

Notice that $x_{\mathrm{n}}(t)$ can be computed for all $t \geq t_k$ only from the knowledge of x_k, the system dynamics, and the controller.

It is important to note that in the absence of disturbances, after a single sampling instant occurs, the state $\hat{x}$ of the EMB controller will be equal to the actual state x of the plant for all time. Hence, applying the previous triggering mechanism would yield a closed-loop system with no triggers, or better yet, a periodic EMB controller of period $\tau_{\max}$. One advantage of the new triggering is that both ZOH and EMB controllers yield true self-triggered controllers.

V. NEW TRIGGERING

In [7], the performance specification is based on the plant without disturbances, that is, the function $S_0(t)$ defined in (13) is a decaying exponential since the closed-loop system is expected to be exponentially stable. This is no longer the case when disturbances are present and therefore this performance specification used in [8] is not matched to the plant. The new triggering accounts for the expected performance and thus depends on the disturbance bound δ_b. The basic idea unfolds in three steps.

First, we modify the performance function $S_0(t)$ to resemble the expected continuous-time performance. Taking into account (10), for $t \geq t_k$ our performance specification is

$$S(t) = \max\{V(x_k)e^{-\gamma(\vartheta)(t-t_k)}, V_b(\theta)\} \tag{18}$$

where $\gamma(\cdot)$ and $V_b(\cdot)$ are defined in (8) and (9), respectively, and $0 < \theta < \vartheta < 1$. The condition $\theta < \vartheta$ implies that $\gamma(\theta) > \gamma(\vartheta)$, that is, S has a smaller rate of decay than $V(x(t))$ with continuous-time control. Stated intuitively, S decreases exponentially when x_k is far from the origin, and it becomes constant when x_k is close to the origin.

Next, we look for a function $U(t, x)$ such that

$$V(x(t)) \leq U(t, x_{\rm n}(t)),$$

for all $t \geq t_k$ and $k \geq 0$. Notice that U does not depend on the disturbance $\delta(t)$ but will depend on its norm bound δ_b.

Finally, the triggering function is modified to yield

$$h(t, x_{\rm n}(t)) = U(t, x_{\rm n}(t)) - S(t).$$

We will show that this strategy renders the closed-loop system GUUB, which is a weaker property then ISS. Nevertheless, simulations results given in Section VI that a gain in performance is achieved.

A. Bounding function $U(t, x)$

We begin by deriving a bound for the norm of $v(t)$ defined in (17), that results in

$$\begin{aligned}\|v(t)\| &= \left\| \int_{t_k}^{t} e^{A(t-s)}\delta(s)\mathrm{d}s \right\| \leq \int_{t_k}^{t} \|e^{A(t-s)}\delta(s)\|\mathrm{d}s \\ &\leq \int_{t_k}^{t} \|e^{A(t-s)}\|\|\delta(s)\|\mathrm{d}s \leq \delta_b \int_{t_k}^{t} \|e^{A(t-s)}\|\mathrm{d}s \\ &\leq \delta_b \int_{t_k}^{t} e^{\sigma(t-s)}\mathrm{d}s = \frac{\delta_b}{\sigma}(e^{\sigma(t-t_k)} - 1) =: \beta(t)\end{aligned} \tag{19}$$

where σ is such that $\|e^{At}\| \leq e^{\sigma t}$ for $t \geq 0$. Possible choices are $\|A\|$, $\frac{1}{2}\lambda_{\max}(A + A^\top)$, or any value in between [13].

Since $x(t) = x_{\rm n}(t) + v(t)$, for all $t \geq t_k$, we have that

$$\begin{aligned}V(x(t)) &= x^\top(t)Px(t) \\ &= (x_{\rm n}^\top(t) + v^\top(t))P(x_{\rm n}(t) + v(t)) \\ &= V(x_{\rm n}(t)) + v^\top(t)P\,(2x_{\rm n}(t) + v(t)) \\ &\leq V(x_{\rm n}(t)) + \beta(t)\,(2\|Px_{\rm n}(t)\| + \lambda_{\max}(P)\beta(t)) \\ &= U(t, x_{\rm n}(t))\end{aligned}$$

where the bounding function is thus defined as

$$U(t, x) := V(x) + \mu(t, x)$$

with $\mu(t, x) = \beta(t)\,(2\|Px\| + \lambda_{\max}(P)\beta(t))$. Note that, if $\delta_b = 0$, then $v(t) \equiv 0$, $V_b = 0$, and $\mu(t) \equiv 0$, and therefore $x(t) = x_{\rm n}(t)$, $S(t) = S_0(t)$, and $U(t, x) = V(x)$. This means that, in the absence of disturbances, both triggering techniques (the one in [7], [8] and the proposed one) generate the same sequence of sampling times.

B. Stability Analysis

What keeps us from claiming GUUB right away, is the fact that we do not know at this stage if there exists $\tau_{\min} > 0$ such that $\tau_k \geq \tau_{\min}$ for all $k \geq 0$. If this were true, $S(t)$ would eventually be constant and equal to $V_b(\theta)$. Since the triggering condition ensures that $V(t, x(t)) \leq U(t, x_{\rm n}(t)) \leq S(t)$, for all $t \geq t_0$, GUUB would follow. Next, we show that such a $\tau_{\min}$ indeed exists.

We start by computing the time derivative of $U(t, x_{\rm n}(t))$ at $t = t_k$. Using the fact that $x_{\rm n}(t_k) = x_k$, $u(t_k) = Kx_k$, $\beta(t_k) = 0$, $\dot{\beta}(t_k) = \delta_b$, and (2), we conclude that

$$\dot{U}(t_k, x_k) = -x_k^\top Qx_k + 2\delta_b\|Px_k\|. \tag{20}$$

We can further bound (20) as

$$\begin{aligned}\dot{U}(t_k, x_k) &\leq -\lambda_{\min}(Q)\|x_k\|^2 + 2\lambda_{\max}(P)\delta_b\|x_k\| \\ &\leq -(1-\theta)\lambda_{\min}(Q)\|x_k\|^2, \forall \|x_k\| \geq r(\theta)\end{aligned} \tag{21}$$

where $r(\theta)$ is defined in (5). The fact that

$$\{x \in \mathbb{R}^n : \|x\| \geq r(\theta)\} \supseteq \{x \in \mathbb{R}^n : V(x) \geq V_b(\theta)\},$$

and that $\theta < \vartheta$, yields

$$\dot{U}(t_k, x_k) \leq -\gamma(\theta)V(x_k) < -\gamma(\vartheta)V(x_k) = \dot{S}(t_k) \tag{22}$$

for all x_k such that $V(x_k) \geq V_b(\theta)$.

Depending on whether $V(x_k)$ is larger, equal, or smaller than $V_b(\theta)$, three different scenarios are possible (see Fig. 3).

If x_k is such that $V(x_k) > V_b(\theta)$ (Fig. 3(a)), then

$$V(x_k) = U(t_k, x_k) = S(t_k),$$

which together with (22), implies that there exists $\tau_{\min,1} > 0$ such that

$$U(t, x_{\rm n}(t)) < S(t)$$

for all $t \in [t_k, t_k + \tau_{\min,1})$.

If x_k is such that $V(x_k) = V_b(\theta)$ (Fig. 3(b)), then

$$\begin{aligned}V(t_k) = U(t_k, x_k) = S(t_k) = V_b(\theta) \\ \dot{U}(t_k, x_k) \leq -\gamma(\theta)V(x_k) < 0 = \dot{S}(t_k).\end{aligned}$$

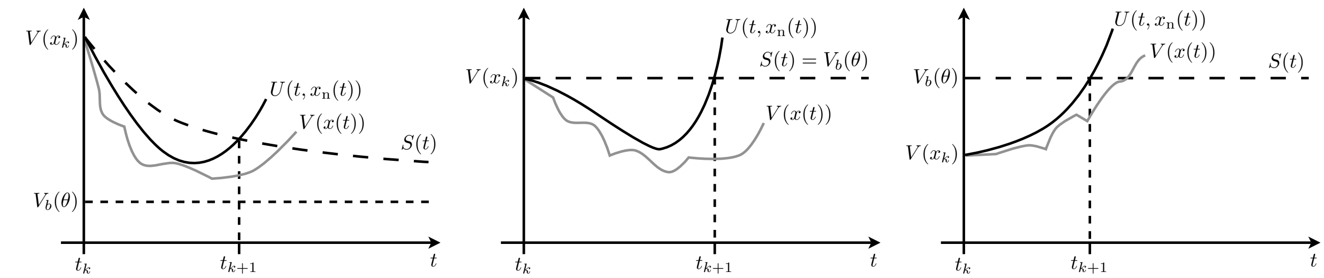
Therefore, there exists $\tau_{\min,2} > 0$ such that

$$V(x(t)) \leq U(t, x_{\rm n}(t)) < S(t) = V_b(\theta)$$

for all $t \in [t_k, t_k + \tau_{\min,2})$.

If x_k is such that $V(x_k) < V_b(\theta)$ (Fig. 3(c)), then

$$\begin{aligned}V(x_k) = U(t_k, x_k) < V_b(\theta) = S(t_k) \\ |\dot{U}(t_k, x_k)| \leq \omega, \dot{S}(t_k) = 0\end{aligned}$$



(a) $V(x_k) > V_b(\theta)$. (b) $V(x_k) = V_b(\theta)$. (c) $V(x_k) < V_b(\theta)$.

Fig. 3. Possible triggering scenarios after sampling instant t_k.

for some positive constant ω. Therefore, there exists $\tau_{\min,3} > 0$ such that

$$V(x(t)) \le U(t, x_n(t)) < S(t) = V_b(\theta)$$

for all $t \in [t_k, t_k + \tau_{\min,3})$.

In summary, the solutions of the closed-loop sampled-data system convergence in finite time to the set $\{x \in \mathbb{R}^n : V(x) \le V_b(\theta)\}$ and remain inside it. Moreover, the sequence of sampling intervals $\{\tau_k\}_{k=0}^{+\infty}$ is bounded below by $\tau_{\min} = \min\{\tau_{\min,1}, \tau_{\min,2}, \tau_{\min,3}\}$.

C. Implementation

Unlike [7], [8], we do not have an effective way of computing any of the $\tau_{\min,i}$ (because the bounding function U depends on the norm of x_k), and therefore, we shall use in the new triggering the value of $\tau_{\min}$ given by (15) but with λ replaced by $\lambda_0(1-\theta)$. We will now explain in greater detail the gridding procedure used to evaluate the triggering function $h(t, x)$, that can be either $h_{\mathrm{VS}_0}(t, x) = V(x) - S_0(t)$ for the triggering in [7], [8], or $h_{\mathrm{US}}(t, x) = U(t, x) - S(t)$ for the new triggering. Mimicking the same gridding strategy of [7], [8], the function h is evaluated on a grid of equally spaced points between $\tau_{\min}$ and $\tau_{\max}$ with spacing Δ (Δ and $\tau_{\max}$ are design parameters). Each point of the grid is indexed by l with $L_{\min} \le l \le L_{\max}$, where $L_{\min} = \lfloor \frac{\tau_{\min}}{\Delta} \rfloor$ and $L_{\max} = \lfloor \frac{\tau_{\max}}{\Delta} \rfloor$. The next sampling time is given by $t_{k+1} = t_k + \Delta l^*$ where

$$l^* = \min\{L_{\min} \le l \le L_{\max} : h(t_k + l\Delta, x_n(t_k + l\Delta)) \le 0\}.$$

VI. AN ILLUSTRATIVE EXAMPLE

In order to compare both types of triggering mechanisms discussed, we shall use a linearized model of an unstable batch reactor process used in [7], [8] to demonstrate their triggering method. The plant is modeled as a linear system where

$$A = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}.$$

A state feedback controller is designed to place the continuous time closed-loop poles at $\{-3 \pm i1.2, -3.6, -3.9\}$, yielding a rate of decay of $\lambda_0 = 0.8253$ and the gain matrix

$$K = \begin{bmatrix} 0.1006 & -0.2469 & -0.0952 & -0.2447 \\ 1.4099 & -0.1966 & 0.0139 & 0.0823 \end{bmatrix}.$$

Table I shows, for different values of the ratio λ/λ_0, the values of $\tau_{\min}$ and the parameter Δ used in the simulations.

TABLE I

VALUES OF $\tau_{\min}$ AND Δ FOR DIFFERENT VALUES OF λ/λ_0

λ/λ_0	$\tau_{\min}$ (ms)	Δ (ms)	$L_{\min}\Delta$ (ms)
0	43.6	10	40
0.1	39.8	10	30
0.2	35.9	10	30
0.3	31.9	10	30
0.4	27.8	10	20
0.5	23.5	10	20
0.6	19.2	5	15
0.7	14.7	5	10
0.8	10.0	5	5
0.9	5.1	5	5

The maximum sampling interval is $\tau_{\max} = 720\,\mathrm{ms}$. The following five methods are compared:

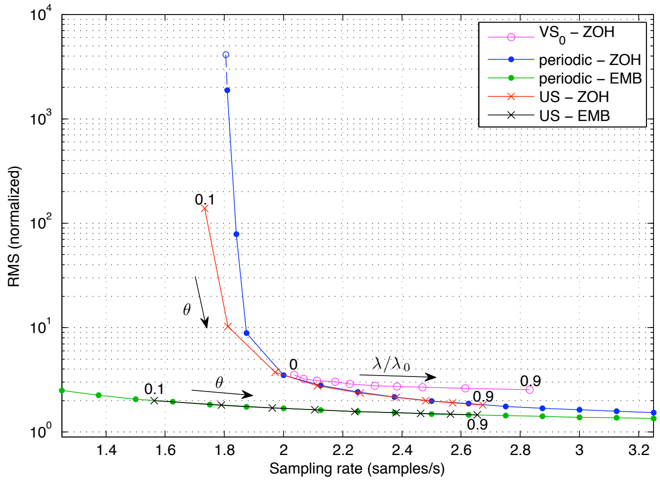
- periodic sampling with ZOH and EMB controllers (`periodic-{ZOH, EMB}`);
- self-triggered sampling of [8] with ZOH controller and adjustable parameter λ ($\mathtt{VS}_0$`-ZOH`);
- new self-triggered sampling with ZOH and EMB controllers, and adjustable parameter θ (`US-{ZOH, EMB}`).

When using methods `US-*`, we choose for $\tau_{\min}$ and Δ the values in Table I that corresponds to $\lambda/\lambda_0 = 1 - \theta$. In simulation, we have observed that the sampling intervals of the `US-*` never went below $\tau_{\min}$. We also have chosen $\sigma = 8.2202$ in (19) and $\vartheta = 1.1\theta$ in (18).

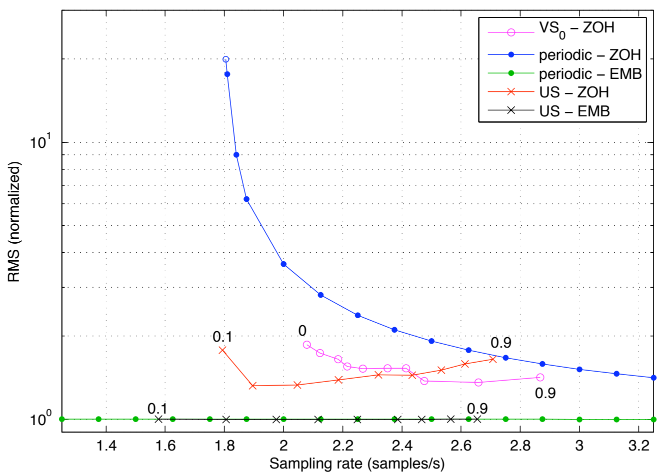
A series of Monte Carlo simulations were carried out to compare the performance of all sampling schemes. Each simulation had a total duration of $T_{\mathrm{sim}} = 500\,\mathrm{s}$ and a total of $N_{\mathrm{sims}} = 10$ simulations were carried out for each case. The disturbance $\delta(t)$ was generated by multiplying two signals: an uniformly distributed signal over the interval $[0, \delta_b]$ with $\delta_b = 10$ and uniformly distributed unit vectors in $\mathbb{R}^4$. The initial state x_0 was randomly generated with a Gaussian distribution of zero mean and covariance $\mathtt{diag}(10^3, 10^3, 10^3, 10^3)$. The criterion used to compare the performance of all sampling schemes is the root mean square (RMS) of the state response defined as

$$\mathrm{RMS}(t_1, t_2) = \sqrt{\frac{1}{N_{\mathrm{sims}}} \sum_{i=1}^{N_{\mathrm{sims}}} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x_i^\top(t) x_i(t) \mathrm{d}t}$$

where $x_i(t)$ is the time response of the closed-loop system of ith simulation. This time response has a transient phase followed by a noisy steady-state phase. Therefore, we split the time axis into two intervals, $[0,\ 100]\,\mathrm{s}$ (transient) and



(a) RMS from 100 s to 500 s (steady-state).



(b) RMS from 0 s to 100 s (transient).

Fig. 4. RMS values versus sampling rate for different sampling methods (average sampling rate for the case of self-triggered methods).

$[100,\ 500]$ s (steady-state), and computed RMS values for both intervals. The results are shown in Fig. 4 for different sampling methods and also continuous-time control. The RMS values have been normalized by the RMS value with continuous control which is 134.8492 and 1.1921 for transient and steady-state intervals, respectively. These values are plotted against the sampling rate of the different methods, which for the case of self-triggered methods correspond to the average sampling rate. The sampling rate of methods $\mathtt{VS_0\text{-}ZOH}$ and `US-`$\star$ is adjusted by changing λ and θ, respectively. The first point of method `periodic-ZOH` which has a sampling rate of 1.805 samples/s is unstable and therefore its RMS value grows to infinity when the simulation time goes to infinity.

In the steady-state interval (Fig. 4(a)), the $\mathtt{VS_0\text{-}ZOH}$ method is outperformed by all other methods with similar sampling rate. The proposed self-triggered controllers, `US-`$\star$, match the performance of their periodic equivalents, `periodic-`$\star$ (method `US-ZOH` is actually better than `periodic-ZOH` when the latter is close to instability). The performance is similar because the sampling intervals of `US-`$\star$ vary slightly around a certain value in steady-state.

In the transient interval (Fig. 4(b)), the $\mathtt{VS_0\text{-}ZOH}$ and `US-ZOH` triggering methods outperform their periodic counterparts in the range of sampling rates shown. The RMS values of `US-ZOH` increase with θ because higher values imply a smaller desired rate of decay when the state is far from the origin (check (18)). It also increases at $\theta = 0.1$ since for this value $S(t)$ becomes constant sooner ($V_b(\theta)$ is larger), that is, most of the time interval $[0,\ 100]$ s is already steady-state in which $\theta = 0.1$ has a worse performance. The RMS values of `periodic-EMB` and `US-EMB` are basically the same and almost match the one with continuous control (impossible to see in the presented scale).

In either case, transient or steady-state, and as expected, using a EMB controller instead of a ZOH controller clearly improves the closed-loop performance.

VII. Conclusions

In this paper we have developed a new triggering strategy for linear plants under bounded disturbances that represents a valuable alternative to the triggering mechanism proposed in [8]. The proposed self-triggered controller is shown to be globally uniformly ultimately bounded. An illustrative example with simulation results shows how the new triggering strategy, along with a model-based control architecture akin to that proposed in [10], yield a clear gain in performance when compared to previous strategies.

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