# Derivation of an event-triggering rule for the fast and saturating controller

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## 1 The problem

We have a continuous controller proposed in [1], where a Lyapunov function $V(\mathbf{x})$ is defined, as well as a resulting control signal, $\boldsymbol{\tau}$, defined as

$$\boldsymbol{\tau} = \mathbf{T}(\mathbf{q}) - \mathbf{D}(\mathbf{x})\boldsymbol{\omega}. \tag{1}$$

Where the state $\mathbf{x}$ is the system attitude, $\mathbf{x} = [\mathbf{q}\ \boldsymbol{\omega}]$, and $V(\mathbf{x})$ is given by the sum of the kinetic energy of the system, $E_{rot}(\boldsymbol{\omega})$, with an artificial potential energy $E_{pot}(\mathbf{q})$. Its time derivative is

$$\dot{V}(\mathbf{x}) = \boldsymbol{\omega}^\top \boldsymbol{\tau} - \mathbf{T}(\mathbf{q})^\top \boldsymbol{\omega}$$

and, by applying (1), we get

$$\dot{V}(\mathbf{x}) = -\boldsymbol{\omega}^\top \mathbf{D}(\mathbf{x})\boldsymbol{\omega}, \tag{2}$$

where $\mathbf{D}(\mathbf{x})$ is a positive semi-definite matrix, ensuring $\dot{V}(\mathbf{x}) \leq 0$.

The challenge now is to find an event-triggering rule that allows for a non-periodic update of the control signal and at the same time does not permit the derivative of the Lyapunov function to grow above zero. I will use the subscript '$k$' when refering to a variable at time $t = t_k$ and no index when refering to a variable at the present time, so simplify the notation.

## 2 Definitions

### 2.1 State error

Assuming that the last sampling instant is given by $t_k$, the state evolution can be given by $\mathbf{x} = \mathbf{x}_k + \mathbf{e}$, where the error $\mathbf{e}$ is defined as

$$\mathbf{e} = \mathbf{x} - \mathbf{x}_k = \begin{bmatrix} \mathbf{q} - \mathbf{q}_k \\ \boldsymbol{\omega} - \boldsymbol{\omega}_k \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{q}} \\ \hat{\boldsymbol{\omega}} \end{bmatrix} \tag{3}$$

## 2.2 Attitude quaternion

The proposed controler parameterizes the attitude using a quaternion, $\mathbf{q}$, that simbolizes the attitude error with respect to a given reference. Since it prioritizes the thrust direction alignment over the yaw, it further decomposes the quaternion into the product of two other quaternions, $\mathbf{q_{xy}} = [q_x \; q_y \; 0 \; q_p]^\top$ and $\mathbf{q_z} = [0 \; 0 \; q_z \; q_w]^\top$. From these two quaternions we can extract the displacement angle of the thrust axis, $\varphi = 2\arccos(q_p)$ and the yaw error angle, $\vartheta = 2\arccos(q_w)$.

## 2.3 Auxiliary function

Several auxiliary functions are proposed in [1]. In particular, $\Lambda_{\epsilon_l}^{\epsilon_u}(\epsilon)$ is of interest. It is defined as

$$\Lambda_{\epsilon_l}^{\epsilon_u}(\epsilon) = \begin{cases} \epsilon \;\text{ if } 0 \leq \epsilon \leq \epsilon_l \\ \epsilon_l \;\text{ if } \epsilon_l < \epsilon \leq \epsilon_u \\ \epsilon_l \frac{\epsilon-\pi}{\epsilon_u-\pi} \;\text{ if } \epsilon_u < \epsilon \leq \pi \end{cases} \tag{4}$$

## 2.4 Artificial torque field

The torque field generated by the artificial potencial energy, $\mathbf{T}(\mathbf{q})$, is given by the sum of four fields,

$$\mathbf{T}(\mathbf{q}) = \mathbf{T}_\varphi^\varphi(\mathbf{q}) + \mathbf{T}_\varphi^\vartheta(\mathbf{q}) + \mathbf{T}_\perp^\vartheta(\mathbf{q}) + \mathbf{T}_\mathbf{z}^\vartheta(\mathbf{q}) \tag{5}$$

where

$$\begin{aligned}
\mathbf{T}_\varphi^\varphi(\mathbf{q}) &= \frac{c_\varphi \Lambda_{\varphi_l}^{\varphi_u}(\varphi)}{\sqrt{1-q_p^2}} \begin{bmatrix} q_x \\ q_y \\ 0 \end{bmatrix} \\
\mathbf{T}_\varphi^\vartheta(\mathbf{q}) &= \frac{c_\vartheta \cos^3\left(\frac{\varphi}{2}\right) \sin\left(\frac{\varphi}{2}\right) \int_0^\vartheta \Lambda_{\vartheta_l}^{\vartheta_u}(\epsilon) d\epsilon}{\sqrt{1-q_p^2}} \begin{bmatrix} q_x \\ q_y \\ 0 \end{bmatrix} \\
\mathbf{T}_\perp^\vartheta(\mathbf{q}) &= \frac{q_z c_\vartheta \cos^3\left(\frac{\varphi}{2}\right) \sin\left(\frac{\varphi}{2}\right) \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta)}{\sqrt{1-q_w^2}\sqrt{1-q_p^2}} \begin{bmatrix} q_y \\ -q_x \\ 0 \end{bmatrix} \\
\mathbf{T}_\mathbf{z}^\vartheta(\mathbf{q}) &= -\frac{q_z c_\vartheta \cos^4\left(\frac{\varphi}{2}\right) \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta)}{\sqrt{1-q_w^2}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

and, since

$$\begin{aligned}
\sin\left(\frac{\varphi}{2}\right) &= \sqrt{1-q_p^2} \\
\cos^3\left(\frac{\varphi}{2}\right) &= q_p^3 \\
\cos^4\left(\frac{\varphi}{2}\right) &= q_p^4
\end{aligned}$$

the torque field (5) becomes

$$\mathbf{T}(\mathbf{q}) = \begin{bmatrix} \left( \dfrac{c_\varphi \Lambda_{\varphi_l}^{\varphi_u}(\varphi)}{\sqrt{1-q_p^2}} - q_p^3 c_\vartheta \displaystyle\int_0^\vartheta \Lambda_{\vartheta_l}^{\vartheta_u}(\epsilon) d\epsilon \right) q_x + \dfrac{q_z q_p^3 c_\vartheta \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta) q_y}{\sqrt{1-q_w^2}} \\ \left( \dfrac{c_\varphi \Lambda_{\varphi_l}^{\varphi_u}(\varphi)}{\sqrt{1-q_p^2}} - q_p^3 c_\vartheta \displaystyle\int_0^\vartheta \Lambda_{\vartheta_l}^{\vartheta_u}(\epsilon) d\epsilon \right) q_y - \dfrac{q_z q_p^3 c_\vartheta \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta) q_x}{\sqrt{1-q_w^2}} \\ \dfrac{q_z q_p^4 c_\vartheta \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta)}{\sqrt{1-q_w^2}} \end{bmatrix} \tag{6}$$

# 3 Rule derivation

Using an event-triggering rule, $\boldsymbol{\tau} = \boldsymbol{\tau}_k$, equation (2) does not hold anymore. Instead, we have

$$\dot{V}(\mathbf{x}) = \boldsymbol{\omega}^\top \boldsymbol{\tau}_k - \mathbf{T}(\mathbf{q})^\top \boldsymbol{\omega}$$

and, by recalling $\mathbf{x} = \mathbf{x}_k + \mathbf{e}$, together with (3) and (1)

$$\dot{V}(\mathbf{x}_k + \mathbf{e}) = \boldsymbol{\omega}^\top \left[ \mathbf{T}(\mathbf{q}_k) - \mathbf{T}(\mathbf{q}_k + \hat{\mathbf{q}}) \right] - \boldsymbol{\omega}_k^\top \mathbf{D}(\mathbf{x}_k) \boldsymbol{\omega}_k - \hat{\boldsymbol{\omega}}^\top \mathbf{D}(\mathbf{x}_k) \boldsymbol{\omega}_k.$$

The term $-\boldsymbol{\omega}_k^\top \mathbf{D}(\mathbf{x}_k)\boldsymbol{\omega}_k = \dot{V}(\mathbf{x}_k)$ is always $\leq 0$, as it is constant and obtained as in [1]. $\hat{\boldsymbol{\omega}}^\top \mathbf{D}(\mathbf{x}_k)\boldsymbol{\omega}_k$ has $\hat{\boldsymbol{\omega}}$ as the only variable, making it easy to monitor. The subtraction $\mathbf{T}(\mathbf{q}_k) - \mathbf{T}(\mathbf{q}_k + \hat{\mathbf{q}})$ is the operation that is harder to compute, since it involves constantly doing the computations (6). It is possible to linearize $\mathbf{T}(\mathbf{q})$ around $\mathbf{q}_k$ to obtain an expression that is linear in an error term:

$$\mathbf{T}(\mathbf{q}_k + \hat{\mathbf{q}}) \simeq \mathbf{T}(\mathbf{q}_k) + \nabla \mathbf{T}(\hat{\mathbf{q}}) \tag{7}$$

where

$$\nabla \mathbf{T}(\hat{\mathbf{q}}) = \left.\frac{\partial T}{\partial q_x}\right|_{t=t_k} \hat{q}_x + \left.\frac{\partial T}{\partial q_y}\right|_{t=t_k} \hat{q}_y + \left.\frac{\partial T}{\partial q_z}\right|_{t=t_k} \hat{q}_z + \left.\frac{\partial T}{\partial q_p}\right|_{t=t_k} \hat{q}_p + \left.\frac{\partial T}{\partial q_w}\right|_{t=t_k} \hat{q}_w$$

Using (7), $\dot{V}(\mathbf{x}_k + \mathbf{e})$ becomes

$$\dot{V}(\mathbf{x}_k + \mathbf{e}) = -\boldsymbol{\omega}^\top \nabla \mathbf{T}(\hat{\mathbf{q}}) + \dot{V}(\mathbf{x}_k) - \hat{\boldsymbol{\omega}}(t)^\top \mathbf{D}(\mathbf{x}_k) \boldsymbol{\omega}_k$$

and, by ensuring

$$\boldsymbol{\omega}^\top \nabla \mathbf{T}(\hat{\mathbf{q}}) + \hat{\boldsymbol{\omega}}(t)^\top \mathbf{D}(\mathbf{x}_k) \boldsymbol{\omega}_k \leq \dot{V}(\mathbf{x}_k) \tag{8}$$

we get $\dot{V}(\mathbf{x}(t)) \leq 0$.

# 4 Simulation

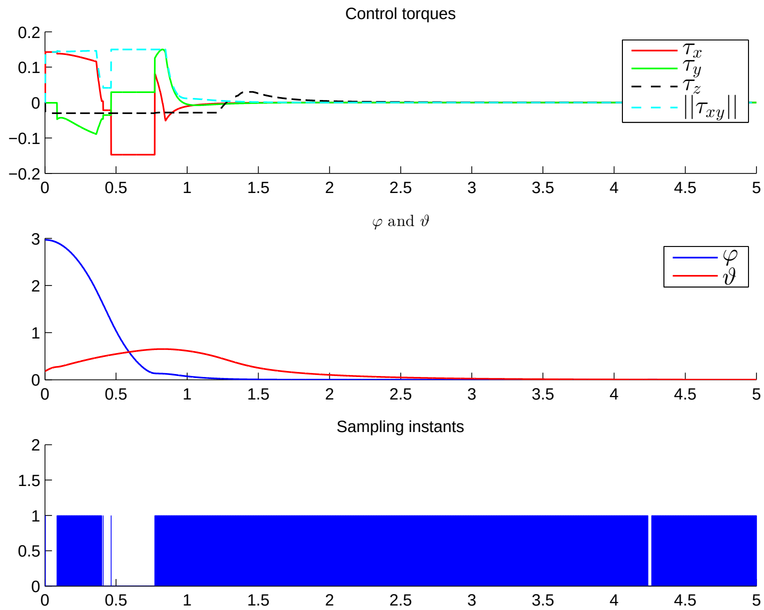


Figure 1: Resulting control torques and system behavior using rule (8)

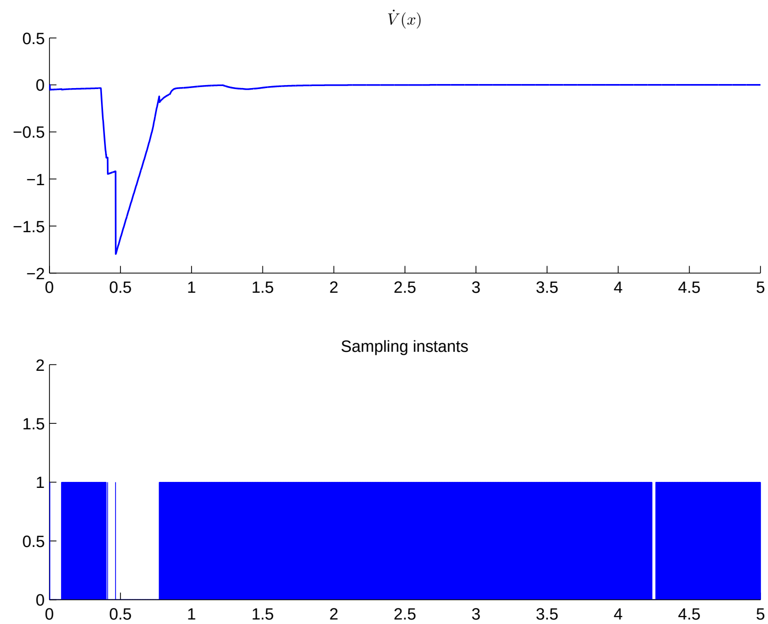


Figure 2: $\dot{V}(\mathbf{x})$ evolution

# References


[1] O. Fritsch, B. Henze, and B. Lohmann, "Fast and saturating attitude control for a quadrotor helicopter," 2013.

# A $\nabla \mathbf{T}(\hat{\mathbf{q}})$ derivation

We need to compute the partial derivatives that composes $\nabla \mathbf{T}(\hat{\mathbf{q}})$ in order to apply (8). With:

$$
\begin{aligned}
A &= \sqrt{1 - q_{w_k}^2} \\
B &= c_\vartheta \int_0^{\vartheta_k} \Lambda_{\vartheta_l}^{\vartheta_u}(\epsilon) d\epsilon \\
C &= \frac{c_\varphi}{\sqrt{1 - q_{p_k}^2}} \\
D &= \frac{c_\vartheta}{A} \\
E &= \arccos(q_{w_k}) \\
F &= \arccos(q_{p_k})
\end{aligned}
$$

we have:

$$
\begin{aligned}
\left.\frac{\partial T}{\partial q_x}\right|_{t=t_k} &= \begin{bmatrix} C\Lambda_{\varphi_l}^{\varphi_u}(\varphi_k) - q_{p_k}^3 B \\ -q_{z_k} q_{p_k}^3 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ 0 \end{bmatrix} \\
\left.\frac{\partial T}{\partial q_y}\right|_{t=t_k} &= \begin{bmatrix} q_{z_k} q_{p_k}^3 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ C\Lambda_{\varphi_l}^{\varphi_u}(\varphi_k) - q_{p_k}^3 B \\ 0 \end{bmatrix} \\
\left.\frac{\partial T}{\partial q_z}\right|_{t=t_k} &= \begin{bmatrix} q_{y_k} q_{p_k}^3 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ q_{x_k} q_{p_k}^3 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ q_{p_k}^4 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \end{bmatrix}.
\end{aligned}
$$

$\left.\frac{\partial T}{\partial q_p}\right|_{t=t_k}$ and $\left.\frac{\partial T}{\partial q_w}\right|_{t=t_k}$ will depend on the value of $\varphi$ and $\vartheta$, respectively, due to (4). Since $\frac{\partial \varphi}{\partial q_p} = -\frac{2}{\sqrt{1-q_p^2}}$, we have, for $0 \le \varphi < \varphi_l$:

$$\left.\frac{\partial T}{\partial q_p}\right|_{t=t_k} = \begin{bmatrix} \left(\frac{-4CF}{A} + \frac{q_{p_k}C}{A^2} - 3q_{p_k}^2 B\right) q_{x_k} + 3q_{z_k} q_{y_k} q_{p_k}^2 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ \left(\frac{-4CF}{A} + \frac{q_{p_k}C}{A^2} - 3q_{p_k}^2 B\right) q_{y_k} - 3q_{z_k} q_{x_k} q_{p_k}^2 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ 4q_{z_k} q_{p_k}^3 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \end{bmatrix}$$

for $\varphi_l \le \varphi < \varphi_u$:

$$\left.\frac{\partial T}{\partial q_p}\right|_{t=t_k} = \begin{bmatrix} \left(\varphi_l C - 3q_{p_k}^2 B\right) q_{x_k} + 3q_{z_k} q_{y_k} q_{p_k}^2 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ \left(\varphi_l C - 3q_{p_k}^2 B\right) q_{y_k} - 3q_{z_k} q_{x_k} q_{p_k}^2 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ 4q_{z_k} q_{p_k}^3 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \end{bmatrix}$$

and, for $\varphi_u \le \varphi < \pi$:

$$\left.\frac{\partial T}{\partial q_p}\right|_{t=t_k} = \begin{bmatrix} \left(\frac{C\varphi_l q_{p_k}(2F-\pi)}{A^2(\varphi_u-\pi)} - \frac{2C\varphi_l}{(\varphi_u-\pi)A} - 3q_{p_k}^2 B\right) q_{x_k} + 3q_{z_k} q_{y_k} q_{p_k}^2 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ \left(\frac{C\varphi_l q_{p_k}(2F-\pi)}{A^2(\varphi_u-\pi)} - \frac{2C\varphi_l}{(\varphi_u-\pi)A} - 3q_{p_k}^2 B\right) q_{y_k} - 3q_{z_k} q_{x_k} q_{p_k}^2 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \\ 4q_{z_k} q_{p_k}^3 D \Lambda_{\vartheta_l}^{\vartheta_u}(\vartheta_k) \end{bmatrix}$$

For $\vartheta = 2\arccos(q_w)$ we get $\frac{\partial \vartheta}{\partial q_w} = -\frac{2}{\sqrt{1-q_w^2}}$ and $\frac{\partial (\vartheta)^2}{\partial q_w} = -\frac{8\arccos(q_w)}{\sqrt{1-q_w^2}}$.

When $0 \le \vartheta < \vartheta_l$, $\int_0^{\vartheta(t)} \Lambda_{\vartheta_l}^{\vartheta_u}(\epsilon) d\epsilon = \frac{\vartheta^2}{2}$ and, as such:

$$\left.\frac{\partial T}{\partial q_w}\right|_{t=t_k} = \begin{bmatrix} 4q_{p_k}^3 q_{x_k} D + \frac{2q_{w_k} q_{z_k} q_{p_k}^3 q_{y_k} DE}{A^2} - \frac{2q_{z_k} q_{p_k}^3 q_{y_k} C}{A} \\ 4q_{p_k}^3 q_{y_k} D - \frac{2q_{w_k} q_{z_k} q_{p_k}^3 q_{x_k} DE}{A^2} + \frac{2q_{z_k} q_{p_k}^3 q_{x_k} C}{A} \\ \frac{2q_{z_k} q_{p_k}^4 q_{w_k} DE}{A^2} - \frac{2q_{z_k} q_{p_k}^4 D}{A} \end{bmatrix}$$

When $\vartheta_l \le \vartheta < \vartheta_u$, $\int_0^{\vartheta(t)} \Lambda_{\vartheta_l}^{\vartheta_u}(\epsilon) d\epsilon = \frac{\vartheta_l^2}{2} + \vartheta_l(\vartheta - \vartheta_l)$:

$$\left.\frac{\partial T}{\partial q_w}\right|_{t=t_k} = \begin{bmatrix} 2\vartheta_l q_{p_k}^3 q_{x_k} D + \frac{\vartheta_l q_{w_k} q_{z_k} q_{p_k}^3 q_{y_k}(2E-\vartheta_l) D}{A^2} - \frac{2\vartheta_l q_{z_k} q_{p_k}^3 q_{y_k} D}{A} \\ 2\vartheta_l q_{p_k}^3 q_{y_k} D - \frac{\vartheta_l q_{w_k} q_{z_k} q_{p_k}^3 q_{x_k}(2E-\vartheta_l) D}{A^2} + \frac{2\vartheta_l q_{z_k} q_{p_k}^3 q_{x_k} D}{A} \\ \frac{\vartheta_l q_{z_k} q_{p_k}^4 q_{w_k}(2E-\vartheta_l) D}{A^2} - \frac{2\vartheta_l q_{z_k} q_{p_k}^4 D}{A} \end{bmatrix}$$

Finally, for $\vartheta_u \leq \vartheta \leq \pi$, the integral becomes $\dfrac{\vartheta_l^2}{2} + \vartheta_l\left(\vartheta_u - \vartheta_l\right) + \dfrac{\vartheta_l\left(\vartheta^2 - \vartheta_u^2\right)}{2\left(\vartheta_u - \pi\right)} + \dfrac{\vartheta_l \pi\left(\vartheta_u - \vartheta\right)}{\vartheta_u - \pi}$ and the last partial derivative is

$$\left.\frac{\partial T}{\partial q_w}\right|_{t=t_k} = \begin{bmatrix} -\dfrac{\vartheta_l q_{p_k}^3 q_{x_k}\left(2\pi - 4E\right) D}{\vartheta_u - \pi} + \dfrac{\vartheta_l q_{w_k} q_{z_k} q_{p_k}^3 q_{y_k}\left(2E - \pi\right) D}{\left(\vartheta_u - \pi\right) A^2} - \dfrac{2\vartheta_l q_{z_k} q_{p_k}^3 q_{y_k} D}{\left(\vartheta_u - \pi\right) A} \\ -\dfrac{\vartheta_l q_{p_k}^3 q_{y_k}\left(2\pi - 4E\right) D}{\vartheta_u - \pi} - \dfrac{\vartheta_l q_{w_k} q_{z_k} q_{p_k}^3 q_{x_k}\left(2E - \pi\right) D}{\left(\vartheta_u - \pi\right) A^2} + \dfrac{2\vartheta_l q_{z_k} q_{p_k}^3 q_{x_k} D}{\left(\vartheta_u - \pi\right) A} \\ \dfrac{\vartheta_l q_{z_k} q_{p_k}^4 q_{w_k}\left(2E - \pi\right) D}{\left(\vartheta_u - \pi\right) A^2} - \dfrac{2\vartheta_l q_{z_k} q_{p_k}^4 D}{\left(\vartheta_u - \pi\right) A} \end{bmatrix}$$