AVL Tree:

每个操作**O(log n)**

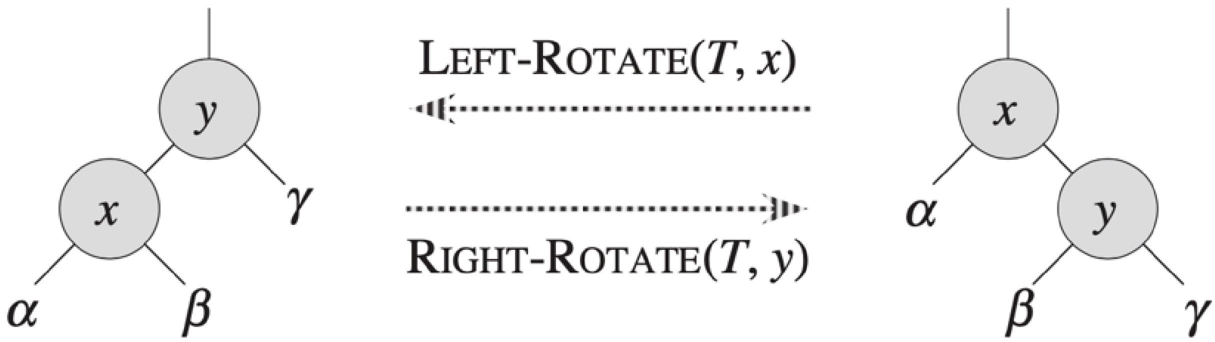
Definition

An empty binary tree is height balanced.（定义深度为-1）
If T is a nonempty binary tree with T_L and T_R as its left and right subtrees,
T is balanced iff:
(1) T_L and T_R are height balanced;
(2) $|h_L - h_R| \leq 1$

The balance factor **BF(node)** = $h_L - h_R$. In an AVL tree, BF(node) $\in -1, 0, 1$

Rotation 旋转

- Time Complexity O(1)
- 从叶子往上找到第一个bug节点
- 左高右旋右高左旋
- LL，RR，LR，RL
- Left Rotate 左旋
- Right Rotate 右旋



Insertion

Trouble Maker节点在：
左子树的**左**子树：右旋 **LL rotation**
右子树的**右**子树：左旋 **RR rotation**
左子树的**右**子树：从下往上先左旋再右旋 **LR rotation**
右子树的**左**子树：从下往上先右旋再左旋 **RL rotation**

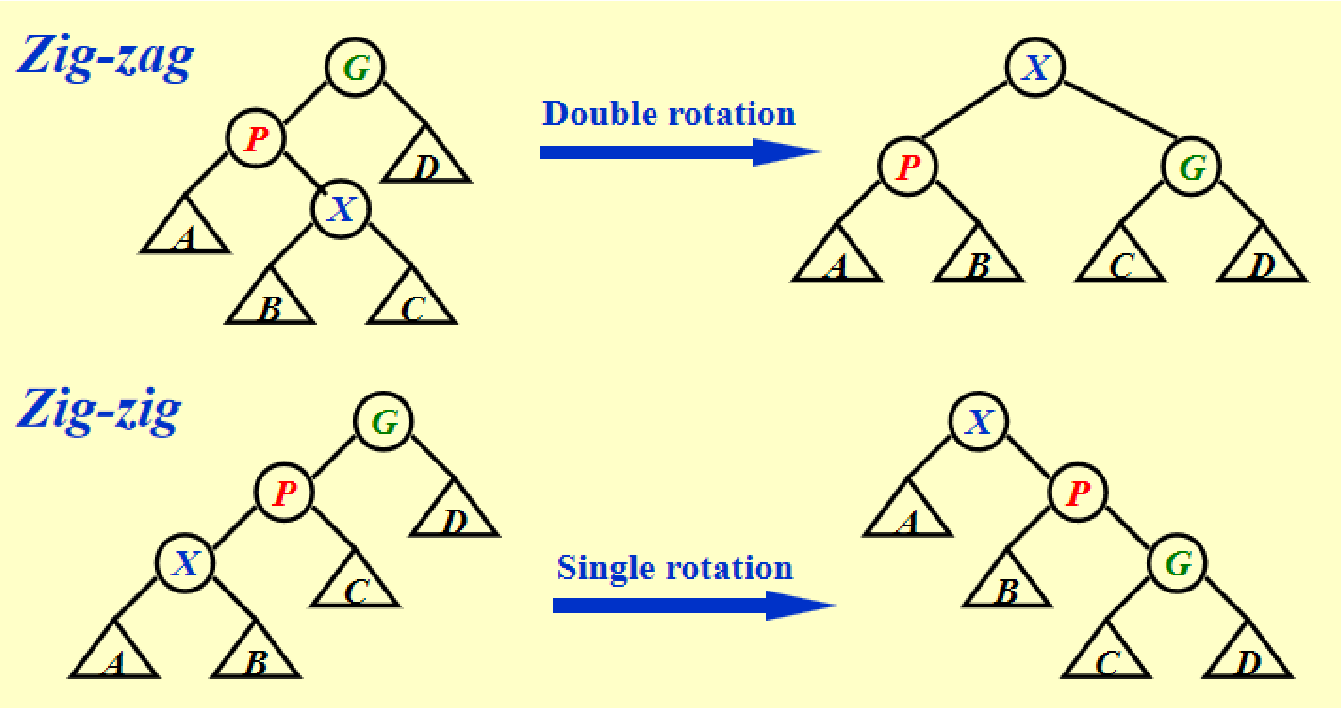
深度为h的树最少结点数$n_h = n_{h-1} + n_{h-2} + 1 = F(h+3) - 1 for h \geq -1$，F为斐波那契数。
$F_i \approx \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^i$
$n_h = O(ln n)$

Splay Tree

Definition

连续M次操作总共不超过$O(Mlogn)$
摊还成本 Amortized cost $\mathbf{O(logN)}$!!!
IDEA: After a node is accessed, it is pushed to the ROOT by a series of AVL tree rotations.
总是把最坏操作拎到根节点

- zig-zag：拎当前节点*2
- zig-zig：先拎父节点再转这个节点



Splaying not only moves the accessed node to the root, but also roughly halves the depth of most nodes on the path.

Deletion

Step 1: Find X (and then X becomes the root)
Step 2: Remove X
Step 3: FindMax(T_L) (and the largest element becomes the rootof T_L)
Step 4: Make T_R the right child of the root of T_L

Time Analysis

- worst-case bound
- amortized bound
- average-case bound

$worst-casebound \geq amortizedbound \geq average-casebound$

方法:

- Aggregate analysis
- Accounting method
- Potential method

Amortized Analysis

Target: 连续M次操作为O(Mlog N)

Aggregate analysis

Idea： Show that for all n, a sequence of n operations takes worst-case time T(n) in total. In the worst case, the average cost, or amortized cost, per operation is therefore T(n)/n.

Accounting method

amortized cost $\hat{c}_i$
actual cost c_i
$\hat{c}_i - c_i = credit(>=0)$
We MUST have:
$T_{amortized} = \frac{\sum_{i=1}^{n}\hat{c}_i}{n} \geq \frac{\sum_{i=1}^{n}c_i}{n}$

- average case bound

Potential method

(for amortized analysis)
$\hat{c}_i - c_i = Credit = \Phi(D_i) - \Phi(D_{i-1})$

$$\sum_{i=1}^{n}\hat{c}_i = \sum_{i=1}^{n}(c_i + \Phi(D_i) - \Phi(D_{i-1}))$$

$$= (\sum_{i=1}^{n}c_i) + \Phi(D_n) - \Phi(D_0)$$

势能函数初始 **全局最小**，$\Phi(D_n) - \Phi(D_0) \geq 0$。

Splay Tree

$\Phi(T) = \Sigma Rank(i)$
$Rank(i) = logS(i)$
S(i) is the number of decendants of i including itself

- **Zig**

$$\begin{aligned} &\hat{c}_i \\ &= 1 + R2(x) - R1(x) + R2(P) - R1(P) \\ &<= 1 + R2(x) - R1(x) \end{aligned}$$

- **Zig-zag**

$$
\begin{aligned}
&\hat{c}_i \\
&= 2 + R2(x) - R1(X) + R2(P) - R1(P) + R2(G) - R1(G) \\
&= 2 - R1(X) + R2(P) - R1(P) + R2(G) \\
&= (2 + R2(P) + R2(G)) - R1(X) - R1(P) \\
&<= 2(R2(x) - R1(x))
\end{aligned}
$$

- theory: If a + b <= c, then log(a) + log(b) <= 2log(c) - 2.
- **Zig-zig**

$$
\begin{aligned}
&\hat{c}_i \\
&= 2 + R2(x) - R1(x) + R2(P) - R1(P) + R2(G) - R1(G) \\
&<= 3(R2(x) - R1(x)) \\
&\mathbf{S1(x) + S2(G) = S2(x)} \\
&=> \hat{c}_i = 2 - R1(x) + R2(P) - R1(P) + R2(G) \\
&<= 2 - R1(x) + R2(P) - R1(P) + 2 * R2(x) - 2 - R1(x) \\
&= 2(R2(x) - R1(x)) + R2(P) - R1(P) \\
&<= 3(R2(X) - R1(X))
\end{aligned}
$$

Therorem

The amortized time to splay a tree with root T at node X is at most 3(R(T) - R(x)) + 1 = O(log N)

Red-Black Tree

Definition

1. 非红即黑
2. 根黑
3. 叶子节点（外部）（哨兵节点）NIL为黑（所有“叶子”的子节点都是NIL）
4. 红节点的儿子是黑色，黑节点的儿子可以是黑色
5. 对每一个节点，到所有后代的路径中包含相同数量的黑色节点

black-height (bh)

从x到叶子节点的黑节点个数（不含x），bh(Tree) = bh(root)
有N个内部节点的树最高 2ln(N+1)

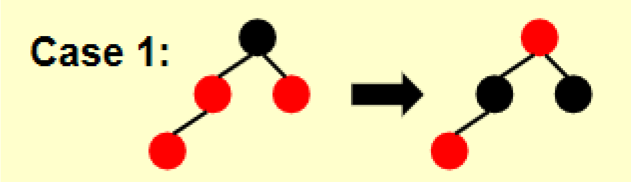
- 1. $sizeof(x) >= 2^{bh(x)} - 1$ (prove by induction)
- 2. $bh(Tree) \geq h(Tree)/2$

Insertion

插入并染红 -> 破坏2，4

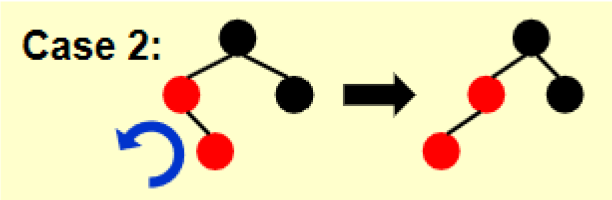
case1:
将父节点染黑 -> 父节点的兄弟染黑 -> 将爷爷染红 锅甩给爷爷（只改色不旋转）
一路甩到根 -> 根染黑
甩到根的儿子 -> 那没事了

O(log N)

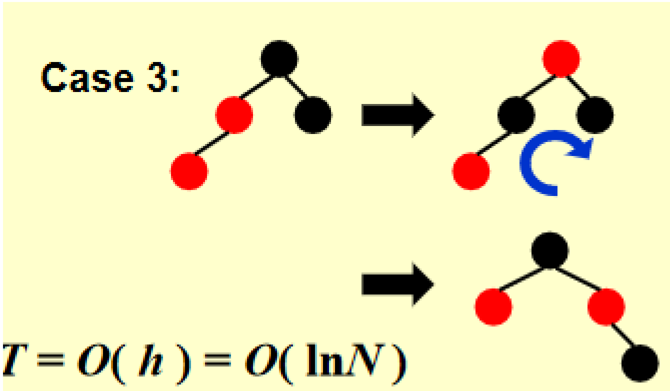


“叔叔你怎么已经黑了？”

case2:
我在右边 -> 绕父节点左旋变成case3 （加上case3一共旋转2次）
O(1)



case3:
我在左边 -> 染完了，不平衡了，向右转 （转1次）
O(1)



最多旋转2次！

Deletion

- 总之是删了个叶子
  （直接删目标叶子or用后代叶子的值替代目标节点然后删叶子）

1. 红叶子，无事发生

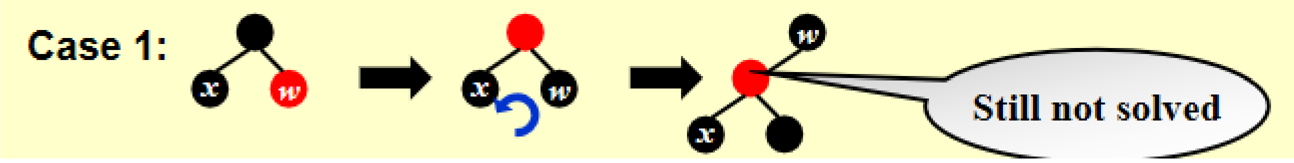
2. 黑叶子 -> 身上叠两层黑（很黑） -> 破坏1

(1) **父节点红，兄弟黑**

-> 父亲变黑兄弟变红 -> ok了，把叶子删掉

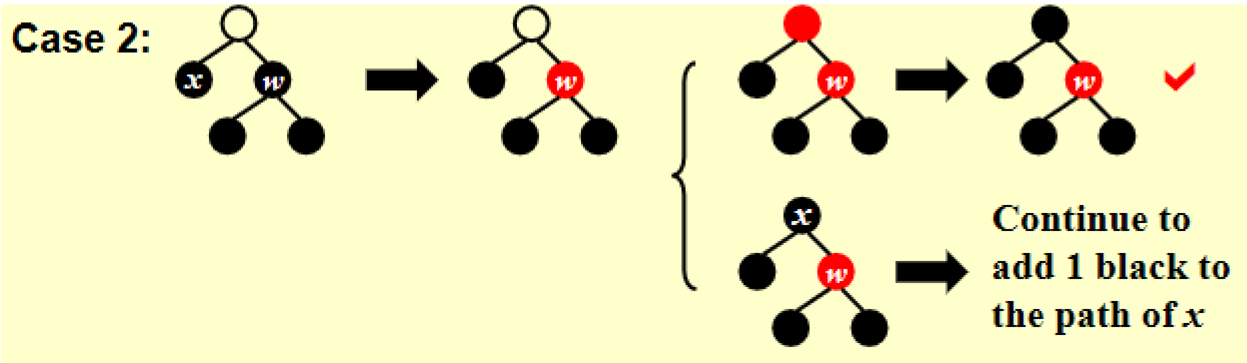
(2) **case1：兄弟红（父亲一定黑）**

-> 父亲变红，兄弟变黑，兄弟转上去，黑侄子变兄弟 -> 同上一种情况

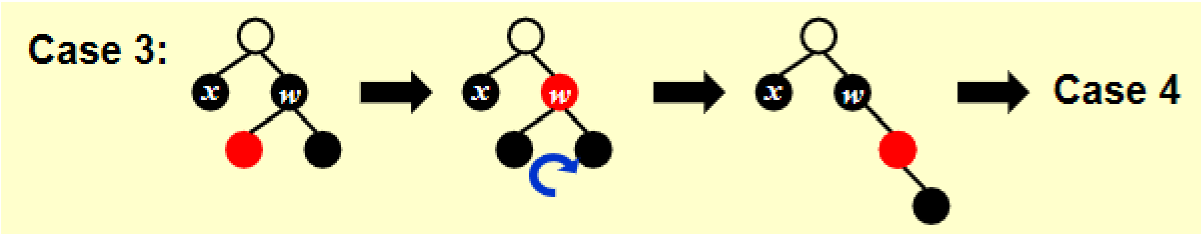


(3) **兄弟为黑：**

- **case2：侄子均黑**
  -> 兄弟变红，黑色传给父亲 -> 父亲红变黑 or 父亲已经是黑色，向上甩锅

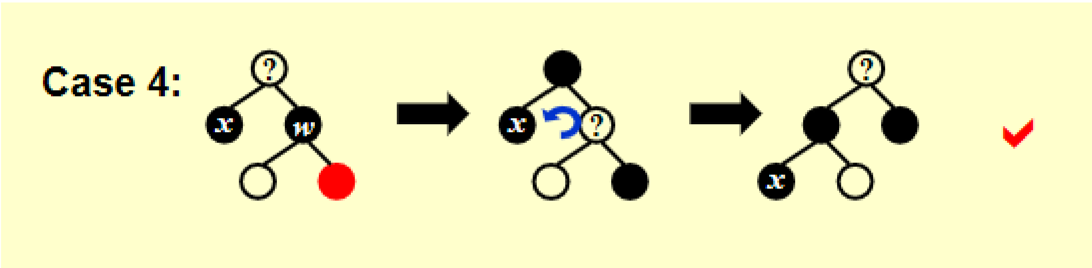


- **case3：近侄子红，远侄子黑**
  -> 近侄子染黑，转成兄弟 -> 变成case4



- **case4：近侄子随便，远侄子红**

  -> 父节点与兄弟节点交换颜色，远侄子染黑 -> 兄弟节点转上去 -> 删除



B+ Tree

Definition

A B+ Tree of order M:

- root is either a leaf or has 2~M children
- all nonleaf nodes have between $\lceil \frac{M}{2} \rceil$ and M children
- all leaves are at the same depth
  B+ Tree of order 4: "2-3-4 tree"
  真实数据在叶子里，父节点保存从第二个子树开始每个子树的最小元素
  M通常为3~4

Find

和BST差不多

Insertion

找位置 O(log N); 插入节点内 O(M); 总共O(log N)

- 节点内满了 -> 向上递归分裂

Deletion

- 能删就直接删
- 节点键不够 -> 与兄弟合并
- 合并后向上递归

Inverted File Index 倒排表

Term-Document Incidence Matrix

稀疏矩阵表示文档及其包含的词
-> 词典terms + 出现次数times + 出现文档documents
查找时按照出现次数从小到大取交集

- 词干处理 Word Stemming （时态不管）
- 停用词 Stop Words （a an the等高频词）
- 定义偏序关系（如词典序），用树处理

Distribute indexing

- Solution 1: Term-partitioned index
- Solution 2: Document-partitioned index
  第一种方案效率高，因为可以同时查询多个词
  第二种方案鲁棒性高，因为不容易完全崩掉啥也查不到（最多查不全）

Dynamic indexing

维护两套索引 main index 和 auxiliary index
向 auxiliary index 添加新内容，定期合并到 main index
删除时打上“删除了！”的标签
两套 index 都要查找

Posting list

记录单词出现的位置
-> 数字太大
-> 保存相邻两个位置的差值

Thresholding

重要性排序

Leftist Heaps 左偏树/左偏堆

Definition

Npl(NULL) = -1
Npl(X) = min{Npl(X) + 1 for C in children of X}
npl(null path legth) 左 >= 右 for all nodes
左偏性（leftist heap property）： For every node X in the heap, the npl of the left child is at least as large as that of the right child.

Theorem

A leftist tree with r nodes on the right path must have at least $2^r - 1$ nodes.

Merge

merge H1 & H2:
H1 == NULL -> H2;
H2 == NULL -> H1;
取根节点键较小的作为H1,
H1为单个节点 -> H1->left = H2
else

- H1->right = Merge(H1->right, H2)
- swap H1->left H1->right if necessary
- H1->npl = H1->right->npl + 1

$T = (logN)$

Delete

delete the root
merge subtrees
T = O(log N)

Skew Heaps 斜堆

Target

连续M次操作O(M * logN)
$T_{amortized} = O(logN)$

Merge

空 + ？ -> ?
非空 + 非空 -> 取较小的根，新根的右儿子与另一树合并
ALWAYS 交换左右儿子

Insertion

新节点与旧树merge

Deletion

删除根节点，左右子树合并

amortized analysis

- heavy：右子树节点数 > 左子树
- light：左子树节点数 > 右子树
- D_i = root of the resulting tree
- 势能函数 $\Phi(D_i)$ = num of heavy nodes
- 仅原树右支上的节点状态改变
- l: light; h: heavy
  在右支上：
  合并前 $\Phi_0 = h1 + h2 + h$
  合并后 $\Phi_N \leq l1 + l2 + h$
- $T_{worst} = l1 + h1 + l2 + h2$
  $T_{amortized} = T_{worst} + \Phi_N - \Phi_0 \leq 2(l1 + l2) = O(logN)$

Binomial Queues

Structure

- a **collection** of heap-ordered trees, known as a **forest**
- each heap-ordered tree is a binomial tree
- a binomial tree of height 0 is a one-node tree
- a binomial tree B_k of height k is formed by attaching a B_{k-1} to the root of another B_{k-1}
- B_k consists a root with k children B_1 B_k, has 2^k nodes, and has C_k^d(or $\binom{k}{d}$)) nodes at depth d

Find Min

choose from $\lceil logN \rceil$ roots, $T_p = O(logN)$

Merge

树的高度不重复 -> 保留
两森林中有相同高度（k）的二项堆 -> 合并为高度k+1的树
类似二进制加法，第i位代表高度为i的树，1 + 1 -> 进位
使用左儿子右兄弟存储每棵树。为了方便合并两棵树，兄弟应当从高到矮

Delete Min

找到最小的根节点
把它删掉，它的儿子加入队列等待合并
原队列与新队列合并

Time Analysis

Claim: A binomial queue of N elements can be built by N successive insertions in O(N) time

- 势能分析
  高开销 -> 降低树的数量；低开销 -> 树变多
  以树的数量作为势能
  C_i::= cost of $\mathbf{i}$th insertion
  Φ_i::= num of trees after $\mathbf{i}$th insertion
  $C_i + (\Phi_i - \Phi_{i-1}) = 2$
  $\sum_{i=1}^{N}(C_i + \Phi_i - \Phi_{i-1}) = 2N$
  $\sum_{i=1}^{N} C_i = 2N - \Phi_N \leq 2N = O(N)$

$T_{worst} = O(logN), T_{amortized} = 2$

Backtracking 回溯算法

Usually using depth first search.
UNDO if fail !!!

Eight queens

Turnpike Reconstruction Problem

Given N(N-1)/2 distances, reconstruct a point set of N from the distances.

Stick Problem

Cut a number of small sticks of the same length into sections at random and put back together. Find the minimum possible length of th original sticks.
Suppose i th section is put into the X[i]th stick.
The i th layer in the game tree is the place of the i th section.
Candidates are the divisors of the total length.

剪枝办法：

- 长度比最长木棍还短

三子棋

MiniMax Strategy (probably with alpha-beta pruning that linits the searching to only $O(\sqrt{N})$, N is the size of the full game tree)

Eight digits

A digit board with eight digits and an empty space, aiming to put them into the right order. Find the minimum number of moves.
剪枝办法：

- 忽略空格交换的约束条件，最少情况仍大于已有情况则剪掉

Devide and Conquer 分治

Definition

Recursively:

- Devide the problem into a number of sub-problems
- Conquer the sub-problems by solving them recursively
- Combine the solutions to the sub-problems into the solution for the original problem

Time Analysis

$T(N) = a * T(N/b) + f(N)$

- $f(N) = cN : T(N) = O(NlogN)$
- $f(N) = cN^2 : T(N) = O(N^2)$

Wrong guess:
$T(N) = 2\lfloor \frac{N}{2} \rfloor + N = O(N)$
Suppose T(m) <= cm for all m <= N/2,
then T(N) <= cN + N = O(N) **(FALSE!)**

Master Method 主方法

$$T(N) = \begin{cases} \Theta(N^{log_b a}) & f(N) = O(N^{log_b a - \xi}) \\ \Theta(N^{log_b a} logN) & f(N) = \Theta(N^{log_b a}) \\ \Theta(f(N)) & f(N) = \Omega(N^{log_b a + \xi}) \end{cases}$$

Another form:

$$T(N) = aT(N/b) + \Theta(N^k log^p N)$$
$$T(N) = \begin{cases} O(N^{log_b a}) & a > b^k \\ O(N^k log^{p+1} N) & a = b^k \\ O(N^k log^p N) & a < b^k \end{cases}$$

Dynamic Programming 动态规划

状态：**唯一**表达一个子问题的解；
状态+状态转移方程确定一个算法
要按照规模递减~（比较好写代码）

- 定义状态
- 以第一步或最后一步为例思考如何切割，写出状态转移方程
- 检查最优子结构是否成立（原问题最优解是否建立在子问题最优解之上）

斐波那契数

矩阵乘法排序

$M_{a*b} \cdot M_{b*c}$的时间复杂度为$O(abc)$
对于$M_{10*20} * M_{20*50} * M_{50*1} * M_{1*100}$，不同乘法顺序的时间复杂度度不同
计算$M_1 * M_2 * M_3 * ... * M_n$：

- 暴力枚举的方案数b_n：
  对最外面两个括号$(M_1 * ... * M_k) * (M_{k+1} * ... M_n)$，两侧括号位置已定，枚举中间")("的位置。
  然后分成两个子问题 $M_1 * ... * M_k$ 和 $M_{k+1} * ... M_n$

$$b_n = \sum_{i=1}^{n-1} b_i b_{n-i} = O(\frac{4^n}{n\sqrt{n}})$$

- 动态规划：
  $F[N] = \sum F[i] * F[N-i]$（不能唯一确定）
  => 用$F[N][i]$表示从i开始长度为N的序列的最小代价（唯一确定）
  考虑$(M_1 M_2)(M_3 M_4)$：总代价为$F[2][1] + F[2][3] + R_1 * C_2 * C_4$
  则

$$F[N][i] = min_{1 \le k < N}\{F[k][i] + F[N-k][i+k] + R_i * C_{i+k-1} * C_{i+N-1}\}$$

Optional Binary Search Tree

Given N words $w_1 < w_2 < ... < w_N$, and the probability of searching for each w_i is p_i. Arrange these words in a BST that minimize the expected total access time.
$F[N][i]$表示规模为N，从w_i开始的子问题

$$F[N][i] = min_{0 \le k \le N}\{\sum_{j=i}^{i+N-1} p_j + F[k][i] + F[N-k-1][i+k+1]\}$$

All-Pairs Shortest Path

Product Assembly

$F[N][i]$表示装前N个零部件后在第i条生产线（$i \in \{0, 1\}$）

$$F[N][i] = min\{F[N-1][0] + t_{0->i,i}, F[N-1][1] + t_{1->i,i}\}$$