# Contents



# Section 13.2: Line Integrals

## Symbol Chart

| Closed Line Integral |
| --- |
| $\oint_C$ |

**Definition**

Let $\vec{r}$: I → $\mathbb{R}^3$, $r(\vec{t})$ = $\langle x(t), y(t), z(t)\rangle$ Let *f*: $(D) \leq \mathbb{R}^3$ → $\mathbb{R}$ be a continuous function of the curve $\vec{r} \subseteq (D)$

<u>Problem:</u> Determine the total mass of a solid lying along the image of the curve $\vec{r}$, with density *f*.

Assume *f* $\geq$ 0. Let $\Delta$ be a division of the curve into *n* sub-arcs.

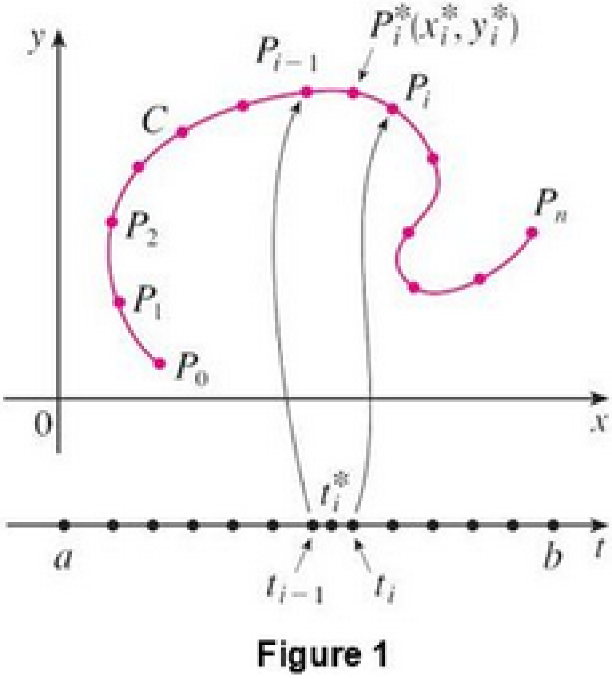


**Figure 1:** Line Integral Subarcs

<u>Define:</u> The line integral of function *f* along $(C)$ = the image of the $\vec{r}$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x, y, z)\Delta S_k$$

The above form can be rewritten in the Riemann Integral with the form of:

$$\oint_C f(x, y, z)ds$$

**Solving**

1. Find a parameterization for $(C)$:

   $r(\vec{t})$ = $\langle x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\rangle$

   a $\leq$ t $\leq$ b

2. Compute the arc length [speed]

$$ds = \sqrt{\frac{d(x)}{d(t)}^2 + \frac{d(y)}{d(t)}^2 + \frac{d(z)}{d(t)}^2}\, dt$$

3. Set up the Riemann Integral

$$\int_a^b f(x(t), y(t), z(t))ds$$

**Example 1**

$$\oint_C x^2y + 2ds = I$$

Where $(C)$ is the upper half of the circle $x^2 + y^2 = 1$

Solution:

$$\vec{r} : [0, \pi] \to \mathbb{R}^2$$

$$r(\vec{t}) = \cos(t)\vec{i} + \sin(t)\vec{j}$$

$$ds = \sqrt{(\frac{d(x)}{d(t)})^2 + (\frac{d(y)}{d(t)})^2} = \sqrt{(\sin(t))^2 + (\cos(t))^2} = 1dt$$

$$I = \int_0^\pi (2 + \cos^2(t)sin(t)) * 1dt = 2t - \left.\frac{\cos^3(t)}{3}\right|_{t=0}^{\pi}$$

$$\boxed{2\pi + \frac{2}{3}}$$

**Example 2**

Integrate

$$f(x, y, z) = x + \sqrt{y} - z^2$$

over the linear path from A(0,0,0) to B(1,1,1). Need a parameter for the line segment from A to B

Solution:

$$\vec{AB} = \vec{OB} - \vec{OA} = \langle 1, 1, 1\rangle - \langle 0, 0, 0\rangle$$

$$[AB] : x = 0 + 1(t), y = 0 + 1(t), z = 0 + 1(t) | 0 \leq t \leq 1$$

$$ds = \sqrt{1 + 1 + 1} = \sqrt{3}dt$$

. This is the magnitude of $\vec{AB}$

$$\int_0^1 (t + \sqrt{t} + -t^2)\sqrt{3}dt$$

$$[\frac{t^2}{2} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^3}{3}]|_{t=0}^1 \sqrt{3} = \boxed{\frac{5\sqrt{3}}{6}}$$

### Example 3

Find the mass of a thin wire lying along the curve

$$r(\vec{t}) = \sqrt{2}\vec{i} + \sqrt{2}\vec{j} + (4 - t^2)\vec{k}|0 \leq t \leq 1$$

<u>Solution:</u>

$$r(\vec{t})' = \langle \sqrt{2}, \sqrt{2}, -2t \rangle$$

$$|r(\vec{t})'| = \sqrt{4t^2 + 2 + 2} = 2\sqrt{t^2 + 1}$$

$$M = \int_0^1 3t * 2\sqrt{t^2 + 1}dt = 3 * \frac{(t^2 + 1)^{\frac{3}{2}}}{\frac{3}{2}}|_{t=0}^1 = \boxed{2(2\sqrt{2} - 1)}$$

## Work Type Integrals

$$\boxed{W = \int_a^b \vec{F} \ \cdot \ \vec{T} \ ds}$$

### Definition

Let $\vec{F}$ be a continuous vector field defined along a smooth curve $(C)$ given by $r(\vec{t})$ | a $\leq$ t $\leq$ b .

Then, the line integral of $\vec{F}$ along $(C)$ is:

$$\oint_C \vec{F} \ \cdot \ \vec{T}\, ds = \oint_C \vec{F} \ \cdot \ d\vec{r} = \int_a^b F\overrightarrow{r(\vec{t})} \cdot \ r(\vec{t})' dt$$

### Line Integrals of Vector Fields

- Unit tangent depends on parameterization/orientation

#### Definition

Let $\vec{F}$ be a continuous vector field along a curve $(C)$: $r(\vec{t}) = \langle x(t), y(t), z(t) \rangle$ with $x, y, z$ differentiable.

Then the work type integral is:

$$W = \oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C \vec{F}\, d\vec{r} = \boxed{\int_a^b \overrightarrow{Fr}(\vec{t}) \cdot r(\vec{t})'\, dt}$$

#### Notations

$\overrightarrow{Fr}(\vec{t}) \leftarrow$ represents the vector field F at the point $r(\vec{t})$

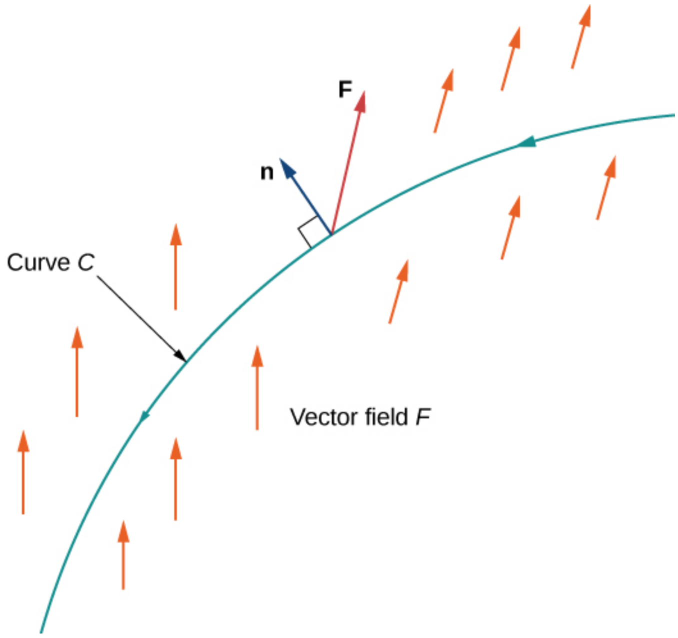


**Figure 2:** Flux of Vector Field

**Example**

Find the work done by

$$F(\vec{x,y}) = x^2\vec{i} - xy\vec{j}$$

moving along the quarter-circle

$$r(\vec{t}) = \cos(t)\vec{i} + \sin(t)\vec{j}\ |0 \leq t \leq \frac{\pi}{2}$$

<u>Solution:</u>

We know:

| x(t) | y(t) |
|---|---|
| $\cos(t)$ | $\sin(t)$ |

Replace all x and y values accordingly to achieve two vectors:

$$\vec{F}(\vec{r}(t)) = \langle \cos^2(t), -\sin(t)\cos(t) \rangle$$
$$r(\vec{t})' = \langle -\sin(t)\cos(t) \rangle$$

Set up the integral and solve:

$$W = \int_0^{\frac{\pi}{2}} \langle \cos^2(t), -\sin(t)\cos(t) \rangle \ \cdot\ \langle -\sin(t)\cos(t) \rangle$$
$$W = \int_0^{\frac{\pi}{2}} (-\sin(t)cos^2(t) - \sin(t)cos^2(t))\ dt$$
$$W = -2\int_0^{\frac{\pi}{2}} (sin(t)cos^2(t))$$
$$W = -2(\frac{-\cos^3(t)}{3}|_{t=0}^{\frac{\pi}{2}})$$
$$\boxed{-\frac{2}{3}}$$

This object is slowing down, as work can be both positive and negative.

**Example 2**

Evaluate

$$\oint_C \vec{F}\, d\vec{r}$$

where:

$$\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$$

over $(C)$ :

$$\begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$$

$$0 \leq t \leq 1$$

Solution:

$$W = \int_0^1 \langle t^3, t^5, t^4 \rangle \ \cdot \ \langle 1, 2t, 3t^2 \rangle dt = \int_0^1 (t^3 + 2t^6 + 3t^6) dt$$

$$W = \int_0^1 (5t^6 + t^3)$$

$$\frac{t^4}{4} + \frac{5t^7}{7} |_{t=0}^{1}$$

$$\boxed{\frac{27}{28}}$$