Multivariable Calculus Concepts

Semester One 2018-19

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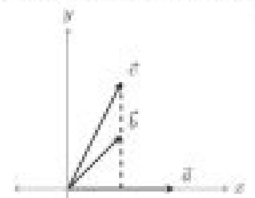
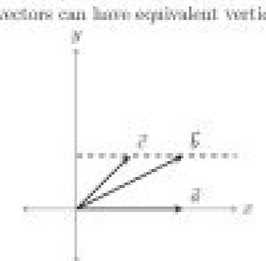
Vectors

Parallelogram Vector Addition

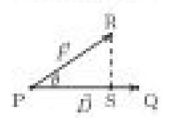
If $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$, that does not mean that $\vec{b}=\vec{c}$ because two separate vectors can have the same horizontal component



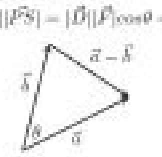
Also, if $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, $\vec{b}$ does not have to be equal to $\vec{c}$ because two separate vectors can have equivalent vertical components



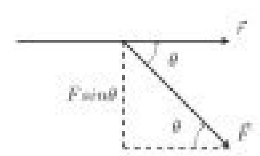
Dot Products



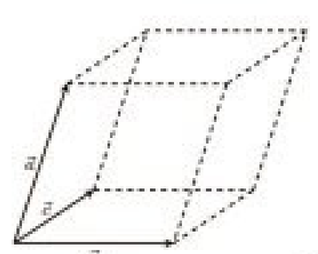
$$|\vec{PS}|=|\vec{F}|cos\theta$$
$$W=|\vec{D}||\vec{PS}|=|\vec{D}||\vec{F}|cos\theta=\vec{D}\cdot\vec{F}$$



$$|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|cos\theta$$
$$\vec{a}\cdot\vec{b}=\frac{|\vec{a}|^2+|\vec{b}|^2-|\vec{a}-\vec{b}|^2}{2}$$
$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$$



$$\vec{\tau}=(|\vec{r}||\vec{F}|sin\theta)\vec{n}$$
$$\vec{\tau}=(|\vec{r}||\vec{F}|sin\theta)\vec{n}$$
$$\vec{a}\times\vec{b}=(|\vec{a}||\vec{b}|sin\theta)\vec{n}$$
$$\vec{a}=<a_1,a_2,a_3>, \vec{b}=<b_1,b_2,b_3>$$
$$\vec{a}\times\vec{b}=\begin{bmatrix}\vec{i}&\vec{j}&\vec{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{bmatrix}=<(a_2b_3-a_3b_2),-(a_1b_3-a_3b_1),(a_1b_2-a_2b_1)>$$



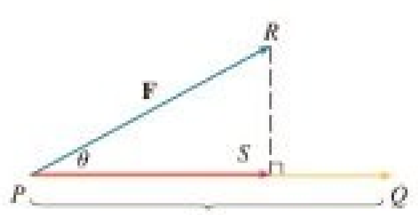
$$V=|\vec{a}cos(\theta)|\vec{b}\times\vec{c}|=\vec{a}\cdot(\vec{b}\times\vec{c})$$

Unit Vectors

A unit vector is a vector that has a length of 1.

The unit vector of $\vec{a}$:

$$\vec{u}=\frac{\vec{a}}{|\vec{a}|}$$



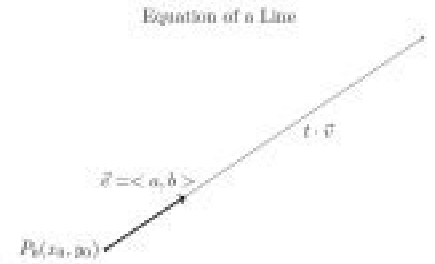
$$\boldsymbol{F}=\vec{PR}$$
$$\boldsymbol{D}=\vec{PQ}$$

The work done by $\mathbf{F}$ is defined as the magnitude of the displacement, $|\mathbf{D}|$, multiplied by the magnitude of the applied force in the direction of the motion

$$|\vec{PS}|=|\mathbf{F}|cos\theta$$

So the work is defined to be $W=|\mathbf{D}|(|\mathbf{F}|cos\theta)=|\mathbf{D}||\mathbf{F}|cos\theta$

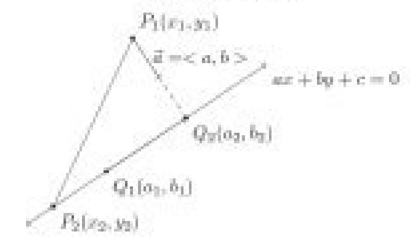
Equation of a Line



$$\vec{r}=P_0+t\vec{v}=<x_0,y_0>+t<a,b>=<x_0+ta,y_0+tb>$$
$$x=x_0+ta$$
$$y=y_0+tb$$

Space Curves, Lines, and Planes

Distance to a Line



Since the slope of $ax+by+c=0$ is $-\frac{a}{b}$, the slope of a $\perp$ line is $\frac{b}{a}$, therefore $<a,b>\perp \vec{Q_1Q_2}$.

$$\text{so } <a,b>\cdot<a_2-a_1,b_2-b_1> = 0$$

Proof:

$$aa_2-aa_1+bb_2-bb_1=0$$
$$aa_1+bb_1=-c=aa_2+bb_2$$
$$aa_2-aa_1+bb_2-bb_1=0$$

The magnitude of the scalar projection of $\vec{P_1P_2}$ onto $\vec{a}$ is the distance between P_1 and the line.

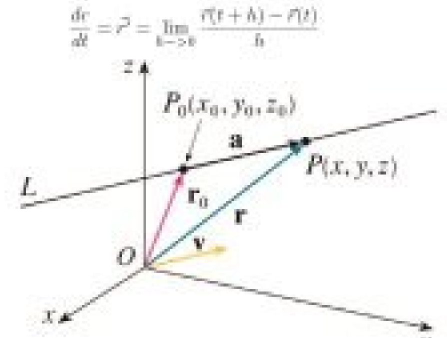
$$|comp_{<a,b>}\vec{P_1P_2}|=\frac{|<x_1-x_2,y_1-y_2>\cdot<a,b>|}{\sqrt{a^2+b^2}}=\frac{|ax_1-ax_2+by_1-by_2|}{\sqrt{a^2+b^2}}$$
$$=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$
$$\text{since } -ax_2-by_2=+c$$

Three Space Distances

$$\sqrt{x^2+y^2+z^2}=D$$

Derivative of a Vector

$$\frac{dr}{dt}=\vec{r}'=\lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$



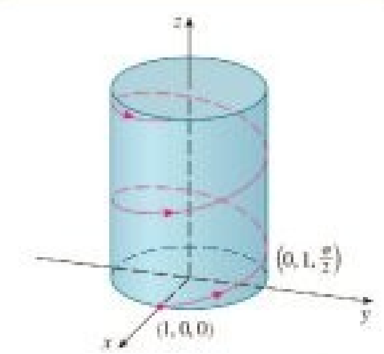
$$\vec{r}=\vec{r_o}+t\vec{v}$$

Let P be a point not on the plane that passes through the points Q, R and S. Show that the distance d from P to the plane is

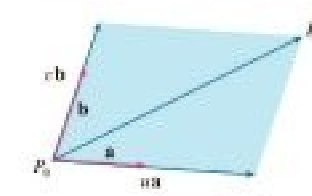
$$d=\frac{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}{|\mathbf{a}\times\mathbf{b}|}$$

where $\mathbf{a}=\vec{QR}$, $\mathbf{b}=\vec{QS}$, and $\mathbf{c}=\vec{QP}$.

The distance is defined as $|\vec{PS}|=d$. But refering to triangle PQS, $d=|\vec{PS}|=|\vec{PS}|sin\theta=|\mathbf{b}|sin\theta$. But θ is the angle between $\vec{QP}=\mathbf{b}$ and $\vec{QR}=\mathbf{a}$. Thus by definition of cross product, $sin\theta=\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$, and so $d=|\mathbf{b}|sin\theta=\frac{|\mathbf{b}||\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$



$$\vec{r(t)}=\cos t\boldsymbol{i}+\sin t\boldsymbol{j}+t\boldsymbol{k}$$



$$\boldsymbol{r}=\boldsymbol{r}_o+u\boldsymbol{a}+v\boldsymbol{b}$$

Curvature Proof

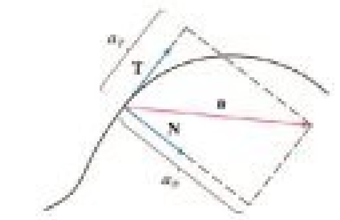
$T=\frac{r'}{\lvert r'\rvert}$	Definition	(1)
$\lvert r'\rvert=\frac{ds}{dt}$	$s(t)=\int_a^t\lvert r'(u)\rvert du$, First Theorem of Calculus	(2)
$r'=\lvert r'\rvert T$	1, Algebra	(3)
$\lvert r'\rvert T=\frac{ds}{dt}T=r'$	2, 3, Algebra	(4)
$r''=\frac{d^2s}{dt^2}T+\frac{ds}{dt}T'$	4, Product rule	(5)
$T\times T=0$	Property of cross product	(6)
$r'\times r''=(\frac{ds}{dt})^2(T\times T')$	4, 5, 6, $\vec{a}\times c\vec{a}=c(\vec{a}\times\vec{a})$	(7)
$\lvert r'\times r''\rvert=(\frac{ds}{dt})^2\lvert T\rvert\lvert T'\rvert\sin(\frac{\pi}{2})=(\frac{ds}{dt})^2\lvert T'\rvert$	7, definition of cross, $\lvert T\rvert=1$ so, T is orthogonal to T'	(8)
$\lvert T'\rvert=\frac{\lvert r'\times r''\rvert}{(\frac{ds}{dt})^2}$	8, Algebra	(9)
$\lvert T'\rvert=\frac{\lvert r'\times r''\rvert}{\lvert r'\rvert^2}$	2, 9, Algebra	(10)
$\kappa=\frac{\lvert T'\rvert}{\lvert r'\rvert}$	$\kappa=\lvert\frac{dT}{ds}\rvert$ chain rule to $\lvert\frac{T'}{\frac{ds}{dt}}\rvert=\frac{\lvert T'\rvert}{\lvert r'\rvert}$	(11)
$\kappa=\frac{\lvert r'\times r''\rvert}{\lvert r'\rvert^3}$	10, 11, Algebra	(12)

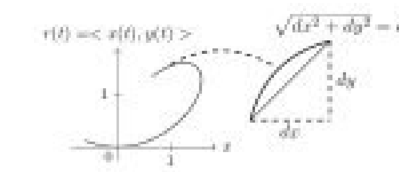
Tangential and Normal Components of Acceleration

$\vec{T}(t)=\frac{\vec{r}'(t)}{\lvert\vec{r}'(t)\rvert}=\frac{\vec{v}(t)}{\lvert\vec{v}(t)\rvert}=\frac{\vec{v}}{v}$	Definition	(13)
$\vec{v}=v\vec{T}$	Definition, $\vec{r}'(t)=v(t)$	(14)
$\vec{v}=v\vec{T}$	2, Algebra	(15)
$\vec{v}'=\vec{a}=v'\vec{T}+v\vec{T}'$	3, Product rule	(16)
$\kappa=\frac{\lvert\vec{T}'\rvert}{\lvert\vec{r}'\rvert}=\frac{\lvert\vec{T}'\rvert}{v}$	Definition, chain rule	(17)
$\lvert\vec{T}'\rvert=\kappa v$	5, Algebra	(18)
$\vec{N}=\frac{\vec{T}'}{\lvert\vec{T}'\rvert}$	Definition	(19)
$\vec{T}'=\lvert\vec{T}'\rvert\vec{N}=\kappa v\vec{N}$	6, 7, Algebra	(20)
$\vec{a}=v'\vec{T}+\kappa v^2\vec{N}$	4, 8, Algebra	(21)
$\vec{a}=a_T\vec{T}+a_N\vec{N}$	Where $a_T=v'$ and $a_N=\kappa v^2$	(22)

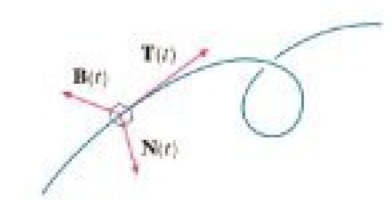
Normal Component of Acceleration

$\vec{a}=v'\vec{T}+\kappa v^2\vec{N}$	Definition	(23)
$\vec{v}=v\vec{T}$	Definition	(24)
$\vec{v}\cdot\vec{a}=v\vec{T}\cdot(v'\vec{T}+\kappa v^2\vec{N})$	1, 2, Substitution	(25)
$\vec{v}\cdot\vec{a}=vv'\vec{T}\cdot\vec{T}+\kappa v^3\vec{T}\cdot\vec{N}$		(26)
$\vec{v}\cdot\vec{a}=vv'$	4, $\vec{T}\cdot\vec{T}=1$ and $\vec{N}\cdot\vec{T}=0$	(27)
$a_T=v'=\frac{\vec{v}\cdot\vec{a}}{v}=\frac{r'(t)\cdot r''(t)}{\lvert r'(t)\rvert}$	5, Definition, Algebra	(28)
$a_N=\kappa v^2=\frac{\lvert r'(t)\times r''(t)\rvert}{\lvert r'(t)\rvert^3}\lvert r'(t)\rvert^2$	Previously proved theorem, definitions	(29)
$a_N=\frac{\lvert r'(t)\times r''(t)\rvert}{\lvert r'(t)\rvert}$	7, Algebra	(30)



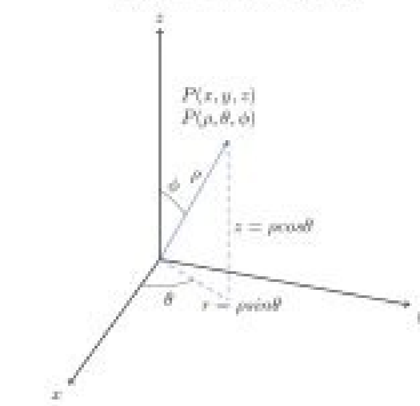


$$\frac{dx}{dt}=f'(t)$$
$$dx=f'(t)dt, dy=g'(t)dt$$
$$dL=\sqrt{dx^2+dy^2}=\sqrt{(f'(t)dt)^2+(g'(t)dt)^2}=\sqrt{(\frac{dx}{dt})^2+\frac{dy}{dt})^2}$$
$$L=\int dL=\int_a^b\sqrt{(\frac{dx}{dt})^2+\frac{dy}{dt})^2}=\int_a^b|r'(t)|dt$$
$$s(t)=\int_a^b|r'(t)|dt$$

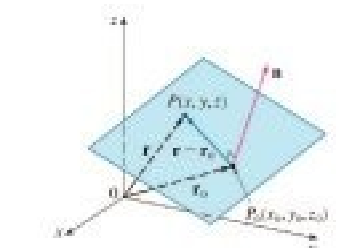


$$\vec{T(t)}=\frac{\vec{r'(t)}}{|\vec{r'(t)}|}$$
$$\vec{N(t)}=\frac{\vec{T'(t)}}{|\vec{T'(t)}|}$$
$$\vec{B(t)}=\vec{T(t)}\times\vec{N(t)}$$
$$\kappa=\frac{|d\vec{T}|}{|ds|}$$

Spherical Coordinate System



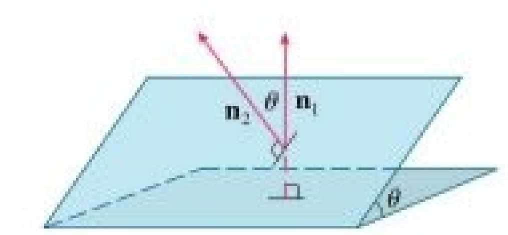
$$r=\rho sin\phi$$
$$x=rcos\theta=\rho cos\theta sin\phi$$
$$y=rsin\theta=\rho sin\theta sin\phi$$
$$z=\rho cos\phi$$
$$\rho^2=x^2+y^2+z^2$$



$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r_o})=0$$
$$\mathbf{r}=<x,y,z>$$
$$\mathbf{r_o}=<x_o,y_o,z_o>$$
$$\mathbf{n}=<a,b,c>$$

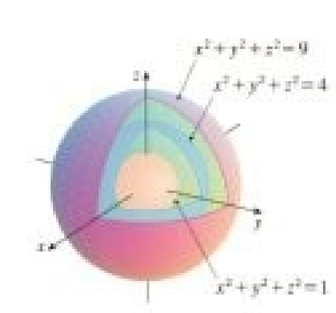
So the equation of a plane is

$$a(x-x_o)+b(y-y_o)+c(z-z_o)=0$$



$$cos\theta=\frac{\mathbf{n_1}\cdot\mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}$$

Three Dimensional Shapes

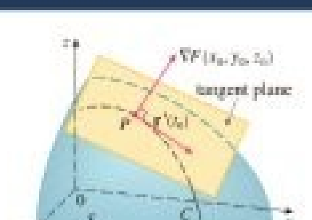


$$f(x,y,z)=x^2+y^2+z^2$$

Given s is a level surface with equation $F(x,y,z)=k$, a level surface, of the function F, and $P(x_o,y_o,z_o)$ is a point on s. C is any curve that lies on s and passes through P, and C is described by $r(t)=<x(t),y(t),z(t)>$ with $r_o=<x_o,y_o,z_o>$ corresponding to P. Any point on C must satisfy the equation of $F(x(t),y(t),z(t))=k$.

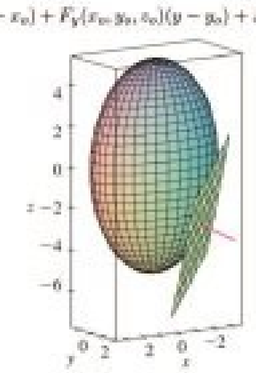
Differentiating both sides with the chain rule you get

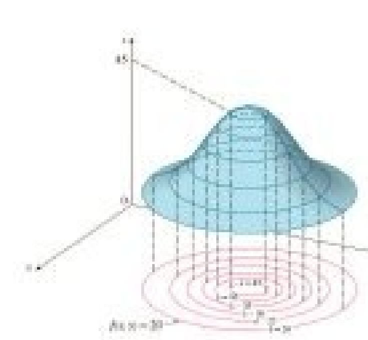
$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}=0$$
$$<\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}>\cdot<\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}>=0$$
$$\nabla F\cdot\vec{r'(t)}=0$$
$$t=t_o$$
$$\nabla F(x_o,y_o,z_o)\cdot\vec{r'(t_o)}=0$$

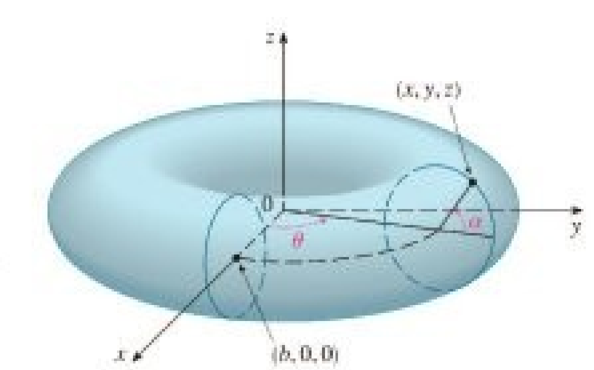


So the gradient vector at P is perpindicular to the tangent vector $\vec{r'(t)}$. Given that $F(x(t),y(t),z(t))=k$ defines a level curve of the surface S, then the gradient vector is always perpindicular to all level surfaces. So the normal vector to a level curve is $\nabla F(x_o,y_o,z_o)$. Therefore, the equation of a tangent plane at point P is given by

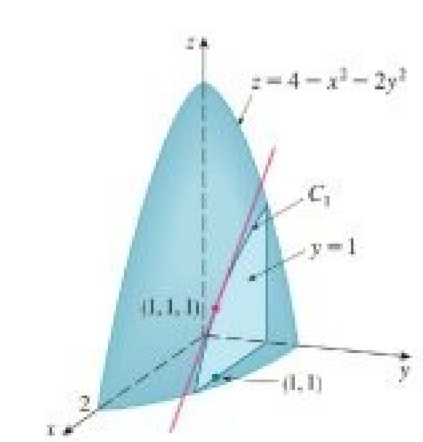
$$F_x(x_o,y_o,z_o)(x-x_o)+F_y(x_o,y_o,z_o)(y-y_o)+F_z(x_o,y_o,z_o)(z-z_o)=0$$







$$x=bcos(\theta)+acos(\alpha)cos(\theta)$$
$$y=bsin(\theta)+acos(\alpha)sin(\theta)$$
$$z=asin(\alpha)$$



The tangent line on C is $\frac{\partial F}{\partial x}$

Directional Derivatives and Gradient

Gradient Vectors and Directional Derivative Equations

$$\nabla f(x,y)=<f_x(x,y),f_y(x,y)>$$
$$D_uf(x,y)=\nabla f(x,y)\cdot u$$

Partial Derivatives

$$g(x)=f(x,b)$$
$$f_x(a,b)=g'(a)$$

Partial Derivatives using limit definition of a derivative

$$f_x(x,y)=lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$
$$f_y(x,y)=lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Tangent Plane to a Level Surface

$$\nabla f(x_0,y_0,z_0)\cdot<x-x_0,y-y_0,z-z_0>=0$$

Tangent Line to a Level Curve

$$\nabla f(x_0,y_0)\cdot<x-x_0,y-y_0>=0$$

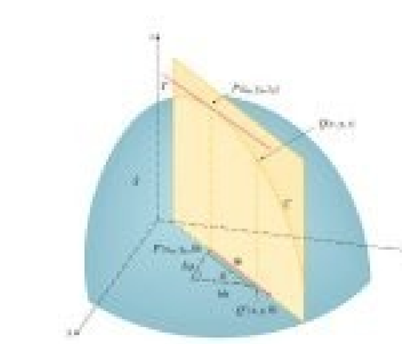
Directional Derivative

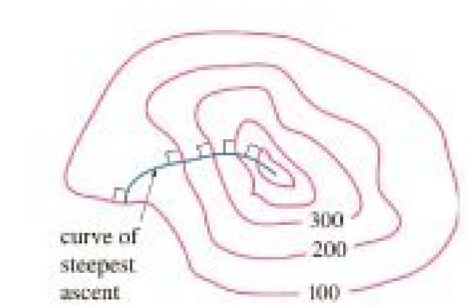
If $g(h)=f(x_o+ha,y_o+hb)$, then by the the definition of a derivative

$$g'(0)=\lim_{h\to 0}\frac{f(x_o+ha,y_o+hb)-f(x_o,y_o)}{h}=D_uf(x_o,y_o)$$

Or by the chain rule with $x=x_o+ha$ and $y=y_o+hb$

$$g'(h)=\frac{\partial f}{\partial x}\frac{dx}{dh}+\frac{\partial f}{\partial y}\frac{dy}{dh}$$
$$=f_x(x_o,y_o)a+f_y(x_o,y_o)b$$
$$Duf(x_o,y_o)=\nabla f(x_o,y_o)\cdot\vec{u}$$





The direction of the gradient vector, $\nabla f(x,y,z)$, is the direction of steepest ascent because D_u is maximized with $\nabla f\cdot\vec{u}$ when u is in the direction of ∇f