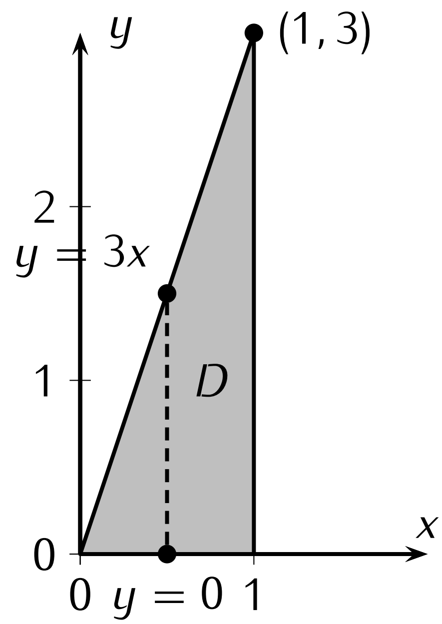
Lecture 19 - Monday, May 12

DOUBLE INTEGRALS OVER GENERAL REGIONS (§15.3)

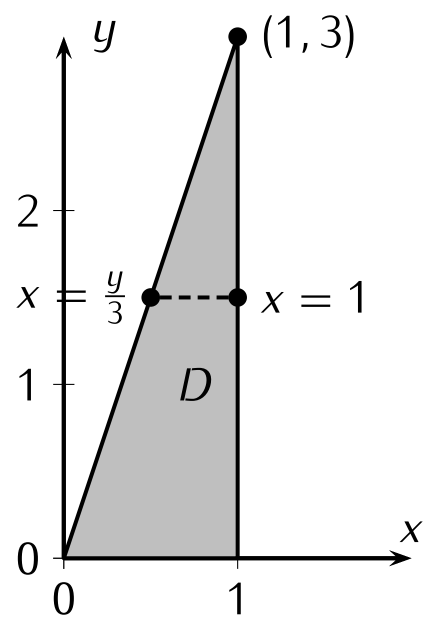
Example: What is the integral of $f(x, y) = 2 - 3x + xy$ over the triangle R that is spanned by $(0, 0), (1, 0), (1, 3)$?

Integration order 1:



$$\int_0^1 \left(\int_0^{3x} 2 - 3x + xy\,dy \right) dx$$
$$= \int_0^1 2y - 3xy + \frac{1}{2}xy^2 \Big|_{y=0}^{3x} dx$$
$$= \int_0^1 6x - 9x^2 + \frac{9}{2}x^3 dx = \frac{9}{8}$$

Integration order 2:

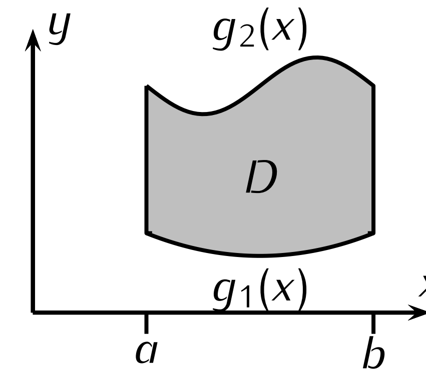


$$\int_0^3 \left(\int_{y/3}^1 2 - 3x + xy\,dx \right) dy$$
$$= \int_0^3 2x - \frac{3}{2}x^2 + \frac{1}{2}x^2 y \Big|_{x=y/3}^{1} dy$$
$$= \int_0^3 \left(-\frac{1}{18}y^3 + \frac{1}{6}y^2 - \frac{1}{6}y + \frac{1}{2} \right) dy = \frac{9}{8}$$

Idea: Choose the integration boundaries so that they represent the region.

Case I: Consider **region** of the form

$$D = \left\{ (x, y) : \begin{array}{c} a \leq x \leq b; \\ g_1(x) \leq y \leq g_2(x) \end{array} \right\}$$

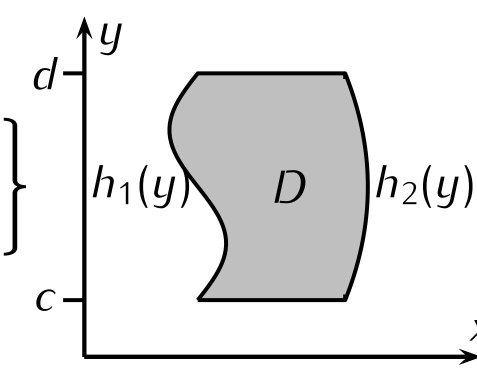


Then the signed volume under f on D is

$$\iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\; dy\; dx$$

Case II: Consider **region** of the form

$$D = \left\{ (x, y) : \begin{array}{c} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{array} \right\}$$



Then the signed volume under f on D is

$$\iint_D f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\; dx\; dy$$

Example (Final exam, Spring 2013)

Compute the double integral

$$\int_0^{\sqrt{2}} \int_{y^2}^{2} y^3 e^{x^3} dx dy$$

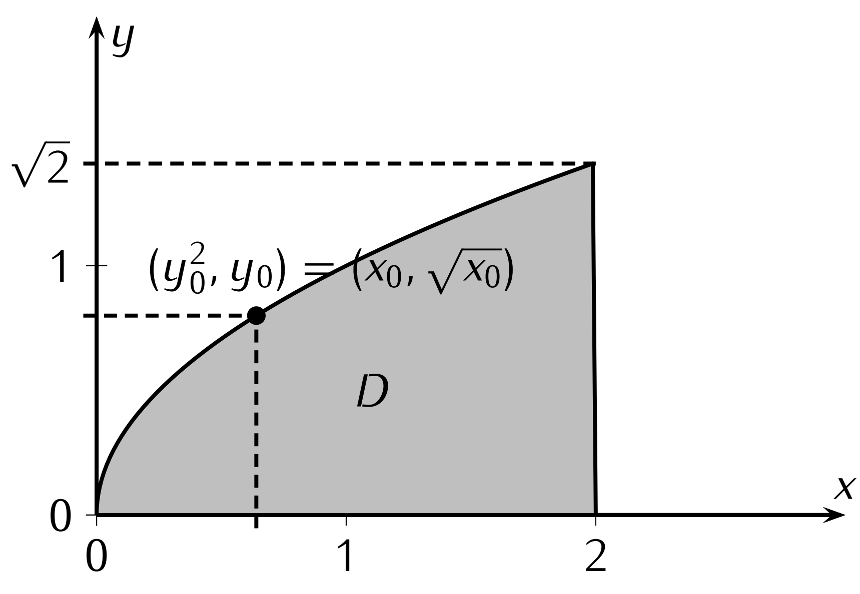
Question: What is $\int e^{x^3} dx$?

Answer: No expression with basic functions exists.

Solution: Invert the integration order!

- **Step 1:** Make a picture of the region

$$D = \{(x, y) : y^2 \le x \le 2,\ 0 \le y \le \sqrt{2}\}$$



- **Step 2:** Observe

$$D = \{(x, y) : 0 \le x \le 2, 0 \le y \le \sqrt{x}\}$$

Example (cont.)

- **Step 3:** Invert integration order and integrate

$$\begin{aligned}
\int_0^{\sqrt{2}} \int_{y^2}^{2} y^3 e^{x^3} dx dy &= \int_0^2 \Big(\int_0^{\sqrt{x}} y^3 e^{x^3} dy \Big) dx \\
&= \int_0^2 e^{x^3} (\frac{1}{4} y^4) \Big|_{y=0}^{y=\sqrt{x}} dx \\
&= \frac{1}{4} \int_0^2 e^{x^3} x^2 dx \\
&\overset{(*)}{=} \frac{1}{12} \cdot e^{x^3} \Big|_{x=0}^{2} = \frac{1}{12}(e^8 - 1)
\end{aligned}$$

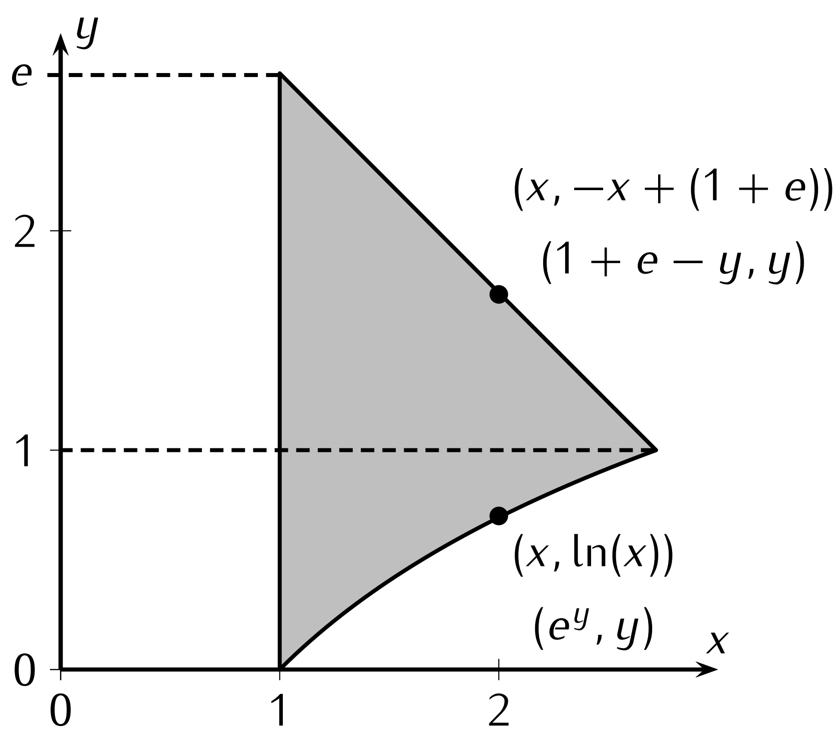
$(*)$ Integration by substition: Choose $f(u) := e^u$, $g(x) = x^3$. Then $F(u) = \int e^u du = e^u$ and $g'(x) = 3x^2$. Hence

$$\int \frac{1}{3} x^2 e^{x^3} dx = \int g'(x) \cdot f(g(x)) dx = F(g(x)) = e^{x^3}$$

Example (Final exam, Autumn 2011)

Sketch the region of integration and change the order of integration

$$\int_1^e \int_{\ln(x)}^{-x+(1+e)} dy\,dx$$



Calculation:

$$\begin{aligned} (I) \quad y &= -x + (1+e) \Leftrightarrow x = -y + 1 + e \\ (II) \quad y &= \ln(x) \Leftrightarrow x = e^y \end{aligned}$$

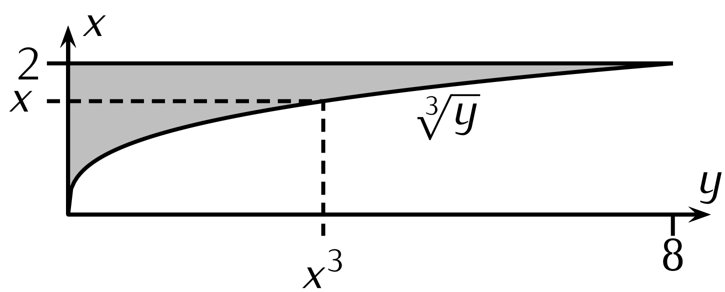
Finally the integral with reversed int. order is

$$\int_0^1 \Big(\int_1^{e^y} f(x,y)dx \Big) dy + \int_1^e \Big(\int_1^{1+e-y} f(x,y)dx \Big) dy$$

Example (Midterm II, Aut. '12, Loveless, Ex 3b)
Switching the order of integration, evaluate

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,dx\,dy$$

Remark: $\int \sqrt{x^4+1}dx$ has no closed formula with elementary functions.



$$\begin{aligned}\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,dx\,dy &= \int_0^2 \left(\int_0^{x^3} \sqrt{x^4+1}dy\right)dx \\ &= \int_0^2 \sqrt{x^4+1}x^3 dx \\ &= \frac{2}{12}(x^4+1)^{3/2}\Big|_0^2 =\end{aligned}$$

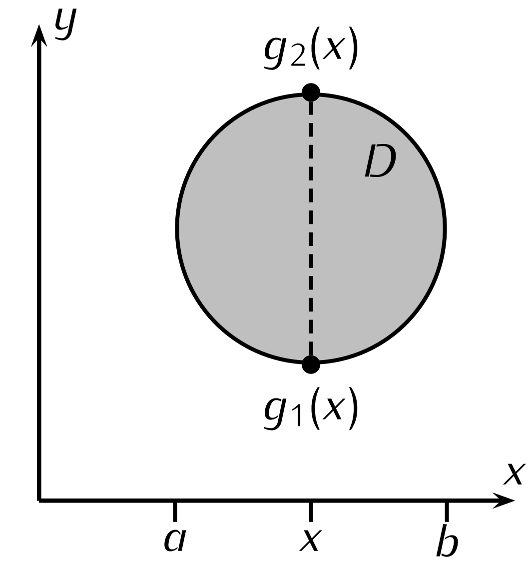
Integration by substitution: $f(u) := \sqrt{u}$, $g(x) := x^4+1$. Then $\int f(u)du = \frac{2}{3}u^{3/2}$ and $g'(x) = 4x^3$. Hence

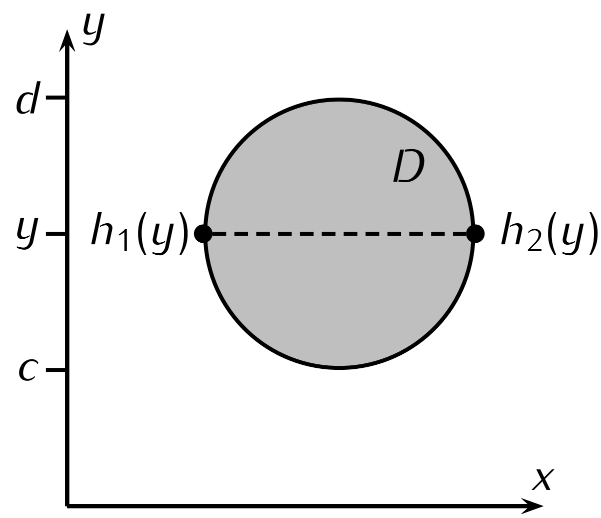
$$\int 4x^3\sqrt{x^4+1}dx = \int g'(x)\cdot f(g(x))dx = F(g(x)) = \frac{2}{3}(x^4+1)^{3/2}$$

<u>SUMMARY: INTEGRATION OVER REGIONS</u>

For a **region**

$$D = \left\{ (x,y) : \begin{array}{c} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array} \right\} = \left\{ (x,y) : \begin{array}{c} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{array} \right\}$$





One has

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Effects of changing the integration order:

- "Difficulty" of integral may change dramatically
- Might need to split D into several regions

Hints: Better make a picture of D!