Math 125 - Fall 2017 Version F

Name: SOLUTIONS Date: 11/8

Chapter 7, Sections 1-8
Chapter 8, Section 1
Chapter 9, Sections 1, 2, 3

Trigonometric Identities

- **Pythagorean Identities:**

$$\sin^2 t + \cos^2 t = 1$$

$$\tan^2 t + 1 = \sec^2 t \quad \text{and} \quad 1 + \cot^2 t = \csc^2 t$$

- **Double-Angle Formulas:**

$$\sin 2t = 2 \sin t \cos t$$

$$\cos 2t = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = 2\cos^2 t - 1$$

$$\tan 2t = \frac{2\tan t}{1 - \tan^2 t}$$

- **Negative Angle Identities:**

$$\sin(-t) = -\sin t, \qquad \cos(-t) = \cos t, \qquad \tan(-t) = -\tan t$$

- **Cofunction Identities:**

$$\sin t = \cos\left(t - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - t\right) \qquad \cos t = \sin\left(t + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} - t\right)$$

- **Sum and Difference Identities:**

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta \qquad \cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \sin\phi\cos\theta \qquad \cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

Instructions: You must solve each question completely, explaining your reasoning. Partial credit will be awarded for answers that are incorrect, but show progress towards a correct solution. You will not receive credit if you do not clearly show how you are obtaining your answers. Grading will be based on the solution and your write-up. Do all the work on the exam.

1. (10 points) Consider the function: $y = 2\sin(2t+8) - 3 \Rightarrow y = 2\sin(2(t+4)) - 3$

(a) Determine the maximum, minimum, amplitude, period, and horizontal shift.

5

Sol. Midline: $y = -3$

$\boxed{\text{Amplitude} = |A| = |2| = 2}$

Maximum $= -3 + 2 = \boxed{-1}$

Minimum $= -3 - 2 = \boxed{-5}$

Period: $P = \frac{2\pi}{B} = \frac{2\pi}{2} = \boxed{\pi}$

$\boxed{\text{H.S.} \Rightarrow 4 \text{ units to the left}}$

(b) Describe in words how to obtain the graph $y = 2\sin(2t+8) - 3$ from the graph $y = \sin(2t)$?

5

Sol. From the graph of $y = \sin(2t)$:

(a) Horizontal Shift Left by 4 units.

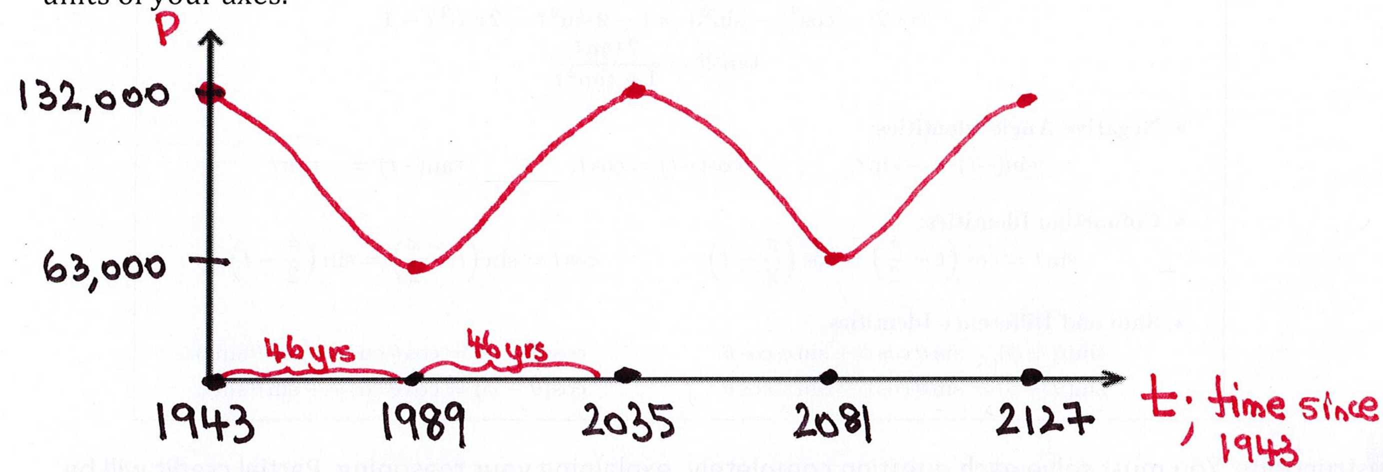
(b) Vertical Stretch by factor of 2.

(c) Vertical Shift down by 3 units.

2. (16 points) A caribou population in a national park dropped from a high of 132,000 in 1943 to a low of 63,000 in 1989, and has risen since then. Scientists hypothesize that the population follows a sinusoidal cycle affected by predation and other environmental conditions, and that the caribou will again reach their previous high.

(a) Carefully sketch a graph that represents 2 periods of the change in the caribou population. Be sure to include all necessary information to best describe the situation. Be sure to label the units of your axes.

4



(b) What are the values for the period, amplitude, midline, maximum and minimum?

5 Sol. Period $= \boxed{92 \text{ years}}$

Amplitude $= |132000 - 97500|$ $\boxed{= 34500}$

Midline: $y = \frac{132000+63000}{2} \Rightarrow \boxed{y = 97500}$

Max.: $\boxed{132,000}$

Min: $\boxed{63000}$

(c) Predict the next year when the population will again be 132,000.

2 Sol. The population will again reach 132,000 in the year of

$\boxed{2035}$

(d) Construct a formula to represent the sinusoidal function.

5 Sol $y = A\cos B(t-h) + K$

$$\boxed{\cancel{y}\ P(t) = 34500\cos \frac{\pi}{46}(t) + 97500}$$

where t is since 1943.

$P = \frac{2\pi}{B}$

$92 = \frac{2\pi}{B}$

$\frac{2\pi}{92} = \frac{92B}{92}$

$\pi/46 = B$

3. (12 points) Find the exact value of the following without a calculator. If it is undefined, enter "undefined". Make sure you clearly show how you arrived at your solution.

6

(a) $\cos 990^\circ$ $\Rightarrow 990^\circ = 720^\circ + 270^\circ = 2$ rotations $+ 0.75$ rotations.

Sol.

$\cos 990^\circ = \cos 270^\circ = \boxed{0}$.

6

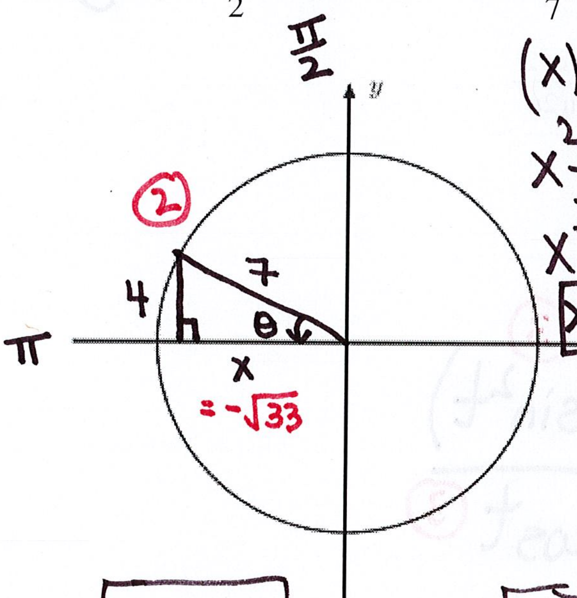
(b) $\sin 765^\circ$ $\Rightarrow 765^\circ = 720^\circ + 45^\circ = 2$ rotations $+ 45^\circ$

Sol.

$\sin 765^\circ = \sin 45^\circ = \boxed{\dfrac{\sqrt{2}}{2}}$

4. (10 points) If $\dfrac{\pi}{2} \le \theta \le \pi$ and $\sin\theta = \dfrac{4}{7}$, find exact values for the other five trigonometric functions.

10



$(x)^2 + (4)^2 = (7)^2$

$x^2 + 16 = 49$

$x^2 = 33$

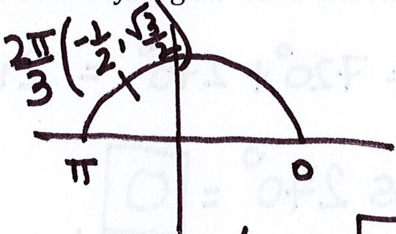
$\boxed{x = -\sqrt{33}}$ 3

$\cos\theta = \boxed{\dfrac{-\sqrt{33}}{7}}$ 1 $\sec\theta = \boxed{\dfrac{-7}{\sqrt{33}}}$ 1 $\cot\theta = \boxed{\dfrac{-\sqrt{33}}{4}}$ 1

$\tan\theta = \boxed{\dfrac{4}{-\sqrt{33}}}$ 1 $\csc\theta = \boxed{\dfrac{7}{4}}$ 1

5. (12 points) Find the exact values of the following quantities without using a calculator. Indicate your reasoning. Use radian measure for any angles.

(a) $\cos^{-1}\left(\frac{-1}{2}\right)$



3 Sol: $0 \le \cos^{-1} \le \pi$

Thus, $\cos^{-1}\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3}}$

(b) $\cos\left[\cos^{-1}\left(\frac{-1}{2}\right)\right]$

4 Sol: $\cos\left(\frac{2\pi}{3}\right) = \boxed{\frac{-1}{2}}$

(c) $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$ $\cos^{-1}\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3}}$; based upon the result in part(a) !!

5 Sol.

6. (10 points) Prove the following identity algebraically:

$$\tan t = \frac{1-\cos 2t}{\sin 2t}$$

Sol: $\tan t = \frac{1-\cos 2t}{\sin 2t}$ 3

10

$= \frac{1-(1-2\sin^2 t)}{2\sin t\cos t}$ 3

$= \frac{\cancel{1}-\cancel{1}+2\sin^2 t}{2\sin t\cos t}$ 4

$= \frac{\cancel{2}\sin^{\cancel{2}} t}{\cancel{2}\cancel{\sin t}\cos t} = \frac{\sin t}{\cos t} = \tan t$!!

10

7. (10 points) Solve for θ, an angle in a right triangle, if $5\cos(2\theta)+6=2\cos(2\theta)+7$. Find the degree to 3 decimal places.

Sol. $5\cos(2\theta)+6=2\cos(2\theta)+7$

$\quad -2\cos(2\theta)-6 \quad -2\cos(2\theta)-6$ 3

$\frac{3\cos(2\theta)}{3}=\frac{1}{3}$

$\cos(2\theta)=\frac{1}{3}$

$2\theta=\cos^{-1}\left(\frac{1}{3}\right)$ 4

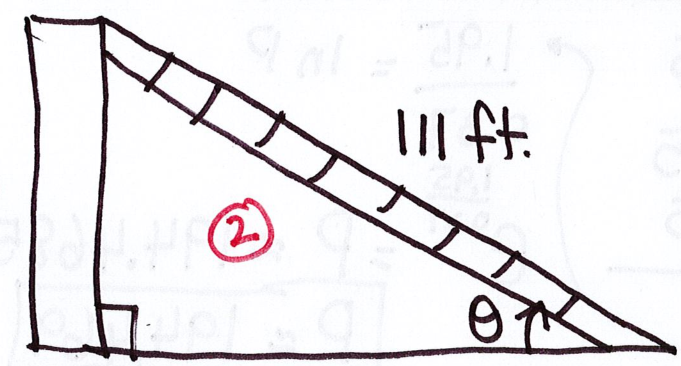
$\frac{2\theta}{2}=\frac{70.5288}{2}$

$\boxed{\theta=35.264^\circ}$ 3

10

8. (10 points) A fire department's longest ladder is 111 feet long, and the safety regulation states that they can use it for rescues up to 105 feet off the ground. What is the maximum safe angle of elevation for the rescue ladder? Round to the nearest degree.

Sol.



$\sin\theta=\frac{105}{111}$ 4

$\theta=\sin^{-1}\left(\frac{105}{111}\right)$

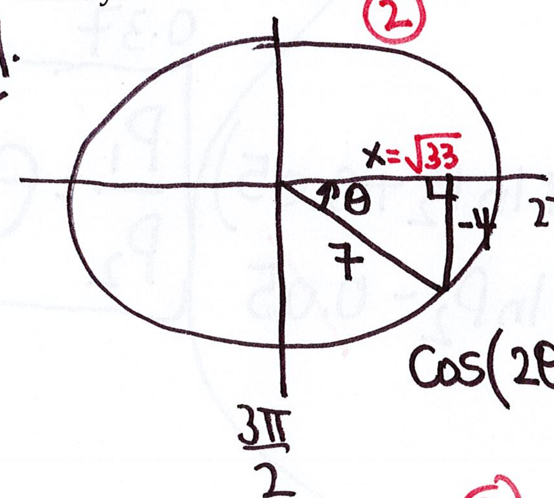
$\theta=71.0754$

$\boxed{\theta\approx 71^\circ}$ 4

10

9. (10 points) If $\frac{3\pi}{2}<\theta<2\pi$ and $\sin(\theta)=\frac{-4}{7}$, find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ exactly.

Sol.



$x^2+(-4)^2=(7)^2$

$x^2+16=49$ 2

$x^2=33$

$\boxed{x=\sqrt{33}}$

$\sin(2\theta)=2\sin\theta\cos\theta$

$=2\left(\frac{-4}{7}\right)\left(\frac{\sqrt{33}}{7}\right)$ 2

$\boxed{=\frac{-8\sqrt{33}}{49}}$

$\cos(2\theta)=1-2\sin^2\theta$

$=1-2\left(\frac{-4}{7}\right)^2$ 2

$=1-2\left(\frac{16}{49}\right)=1-\frac{32}{49}=\boxed{\frac{17}{49}}$

$\tan(2\theta)=\frac{\sin(2\theta)}{\cos(2\theta)}$ 2

$=\frac{-8\sqrt{33}}{\cancel{49}}\cdot\frac{\cancel{49}}{17}$

$\boxed{=\frac{-8\sqrt{33}}{17}}$

5

Bonus: Psychologists have found that the average walking speed, w, in feet per second, of a person living in a city of population P, in thousands of people, is given by the function

$$w = 0.37 \ln P + 0.05.$$

(a) The population of *San Antonio, Texas is about 1,236,249.* Find the average walking speed of people living in *San Antonio.*

Sol. $W = 0.37 \ln P + 0.05$

1 $W = 0.37 \ln(1236.249) + 0.05$

$\boxed{W = 2.68 \text{ ft/sec}}$

(b) A sociologist measures the average walking speed in a city to be approximately 2.0 feet/second. Use this information to estimate the population of the city.

Sol $W = 0.37 \ln P + 0.05$

2 $2.0 = 0.37 \ln P + 0.05$

$-0.05 \qquad -0.05$

$\frac{1.95}{0.37} = \frac{0.37 \ln P}{0.37}$

$\frac{1.95}{0.37} = \ln P$

$e^{\frac{1.95}{0.37}} = P \approx 194.4685144$ thous.

$\boxed{P \approx 194{,}469}$

(c) Let $w1$ and $w2$ be the average walking speeds in two different cities with populations $P1$ and $P2$, respectively. Using logarithm properties, find a simplified formula for the difference $w1 - w2$ in terms of $P1$ and $P2$.

2

Sol $W_1 = 0.37 \ln P_1 + 0.05$

$W_2 = 0.37 \ln P_2 + 0.05$

$W_1 - W_2 = 0.37 \ln P_1 + 0.05 - (0.37 \ln P_2 + 0.05)$

$W_1 - W_2 = 0.37 \ln P_1 + \cancel{0.05} - 0.37 \ln P_2 - \cancel{0.05}$

$W_1 - W_2 = 0.37(\ln P_1 - \ln P_2)$

$\boxed{W_1 - W_2 = 0.37 \ln\left(\frac{P_1}{P_2}\right)}$

$\frac{W_1 - W_2}{0.37} = \ln\left(\frac{P_1}{P_2}\right)$

$\boxed{\frac{P_1}{P_2} = e^{\frac{W_1 - W_2}{0.37}}}$