Section 5.4 - Changing the Order of Integration

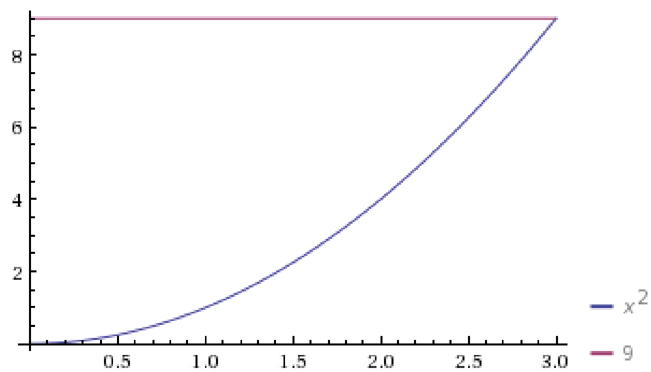
Problem 1. Evaluate the integral by first reversing the order of integration,

$$\int_{x=0}^{x=3}\int_{y=x^2}^{y=9} x^3 e^{y^3}\,dy\,dx.$$

Solution. Even if we tried to integrate with respect to y first, we cannot do it. We can't just switch either. In order to integrate with respect to x , we can't have x's in the limits. So, to reverse the order, it is best to first sketch the region. Notice first that our region has two properties:

$$0 \le x \le 3 \qquad x^2 \le y \le 9.$$

We then can draw the region:



Since we want to integrate with respect to x first, we will need limits for x as functions of y and we need constant bounds for y. Looking at the picture, we get

$$0 \le y \le 9 \qquad 0 \le x \le \sqrt{y}.$$

With this information, we can now set up our new integral and hopefully be able to solve it!

$$\begin{aligned}
\int_{x=0}^{x=3}\int_{y=x^2}^{y=9} x^3 e^{y^3}\,dy\,dx &= \int_{y=0}^{y=9}\int_{x=0}^{x=\sqrt{y}} x^3 e^{y^3}\,dx\,dy \\
&= \int_{y=0}^{y=9} \left(\frac{1}{4}x^4 e^{y^3}\right)\Bigg|_{x=0}^{x=\sqrt{y}} dy \\
&= \int_{y=0}^{y=9} \frac{1}{4}y^2 e^{y^3}\,dy \\
&= \frac{1}{12}e^{y^3}\Bigg|_{y=0}^{y=9} \\
&= \frac{1}{12}\left(e^{729}-1\right).
\end{aligned}$$

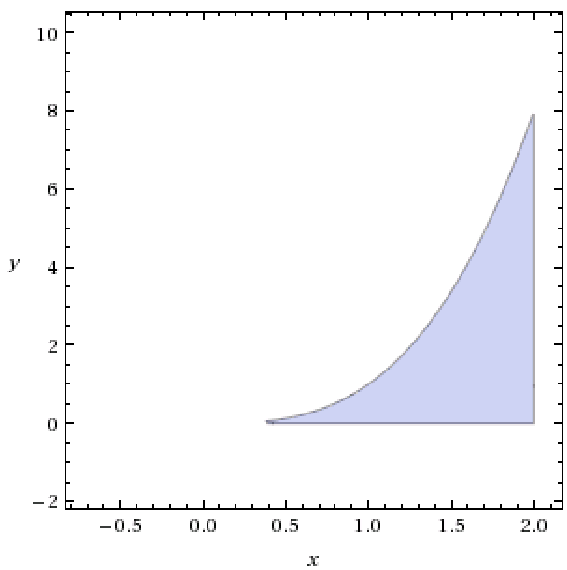
Problem 2. Evaluate the integral by first reversing the order of integration,

$$\int_{y=0}^{y=5} \int_{x=\sqrt[3]{y}}^{x=2} \sqrt{x^4+1}\, dx\, dy.$$

Solution. The region is described by the following two inequalities:

$$0 \le y \le 8 \qquad \sqrt[3]{y} \le x \le 2.$$

We sketch the region and get the picture



Now we can find our new inequalities and we get that

$$0 \le x \le 2 \qquad 0 \le y \le x^3.$$

With this information, we can now set up our new integral and evaluate:

$$\begin{aligned}
\int_{y=0}^{y=5} \int_{x=\sqrt[3]{y}}^{x=2} \sqrt{x^4+1}\, dx\, dy &= \int_{x=0}^{x=2} \int_{y=0}^{y=x^3} (x^4+1)^{\frac{1}{2}}\, dy\, dx \\
&= \int_{x=0}^{x=2} y(x^4+1)\Big|_{y=0}^{y=x^3} dx \\
&= \int_{x=0}^{x=2} x^3(x^4+1)^{\frac{1}{2}}\, dx \\
&= \frac{1}{6}\left(17^{\frac{3}{2}}-1\right).
\end{aligned}$$

Note: These notes and problems are meant to follow along with *Vector Calculus* by Jerrold Marsden and Anthony Tromba, Sixth Edition. The pictures were generated using Wolfram Alpha.