

# HW2

## 4.2

画出以下 MA(2) 模型的自相关函数图，系数设为：

(a)

$$\theta_1 = 0.5, \theta_2 = 0.4$$

(b)

$$\theta_1 = 1.2, \theta_2 = -0.7$$

(c)

$$\theta_1 = -1, \theta_2 = -0.6$$

```
In [1]: # 加载必要的库
library(stats)

# 设置随机种子以保证结果可复现
set.seed(123)

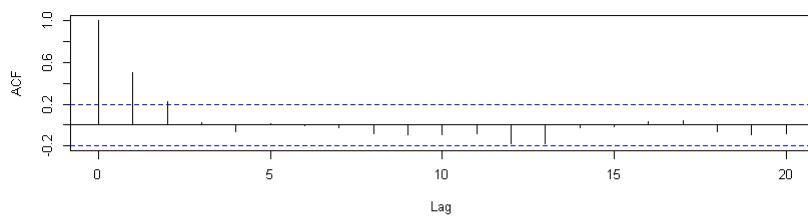
# 定义MA(2)模型的系数
theta_a <- c(0.5, 0.4)
theta_b <- c(1.2, -0.7)
theta_c <- c(-1, -0.6)

# 模拟MA(2)模型的时间序列数据
ma_a <- arima.sim(model = list(ma = theta_a), n = 100)
ma_b <- arima.sim(model = list(ma = theta_b), n = 100)
ma_c <- arima.sim(model = list(ma = theta_c), n = 100)

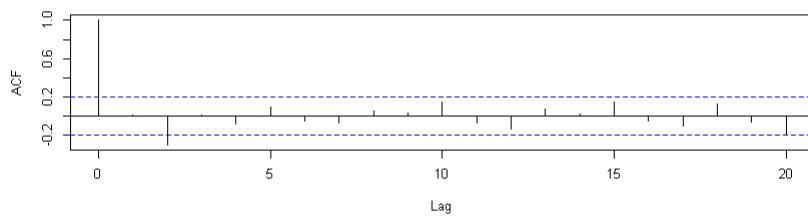
par(mfrow = c(3, 1))

acf(ma_a, main = expression((a):~MA(2)~with~theta[1] == 0.5~and~theta[2] == 0.4))
acf(ma_b, main = expression((b):~MA(2)~with~theta[1] == 1.2~and~theta[2] == -0.7))
acf(ma_c, main = expression((c):~MA(2)~with~theta[1] == -1~and~theta[2] == -0.6))
```

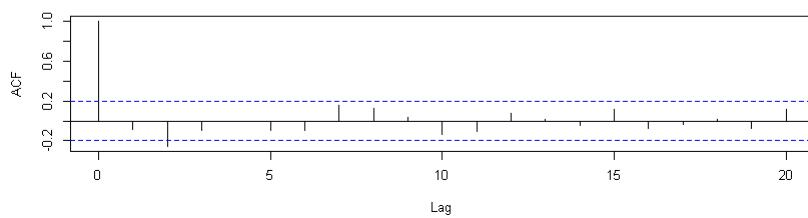
(a): MA(2) with  $\theta_1 = 0.5$  and  $\theta_2 = 0.4$



(b): MA(2) with  $\theta_1 = 1.2$  and  $\theta_2 = -0.7$



(c): MA(2) with  $\theta_1 = -1$  and  $\theta_2 = -0.6$



## 4.2 AR 代码实现

4.4 证明：当  $\theta$  被  $\frac{1}{\theta}$  替代时，MA(1) 过程的自相关函数不变

Proof.

$$\text{MA}(1): Y_t = e_t - \theta e_{t-1}, \text{ 且 } \text{Corr}(Y_t, Y_s)$$

$$\textcircled{1} |t-s| \geq 2, \text{ Cov}(Y_t, Y_s) = \text{Cov}(e_t - \theta e_{t-1}, e_s - \theta e_{s-1}) = 0$$

$$\textcircled{2} |t-s|=1, \text{ 若 } t < s, \text{ 则 } s=t+1. \text{ 则 } \text{Cov}(Y_t, Y_s) = \text{Cov}(Y_t, Y_{t+1})$$

$$\text{原式} = \text{Cov}(e_t - \theta e_{t-1}, e_{t+1} - \theta e_t) = -\theta \text{Var}(e_t)$$

$$\therefore \text{Corr}(Y_t, Y_s) = \frac{\text{Cov}(Y_t, Y_s)}{(\text{Var}(Y_t)\text{Var}(Y_s))^{1/2}} = \frac{-\theta \text{Var}(e_t)}{\text{Var}(Y_t)^{1/2} \text{Var}(Y_{t+1})^{1/2}} = \frac{-\theta \sigma^2}{\text{Var}(Y_t)}$$

$$\textcircled{3} |t-s|=0, \text{ Var}(Y_t) = \text{Cov}(e_t - \theta e_{t-1}, e_t - \theta e_{t-1}) = \text{Var}(e_t) + \theta^2 \text{Var}(e_{t-1})$$

$$\text{Var}(Y_t) = (1+\theta^2)\sigma^2$$

$$\therefore \text{综上, Corr}(Y_t, Y_s) = \begin{cases} 0, & |t-s| \geq 2 \\ -\frac{\theta}{1+\theta^2}, & |t-s|=1 \\ 1, & |t-s|=0 \end{cases}$$

则将  $\theta$  替换为  $\frac{1}{\theta}$  时,  $0=0, -\frac{\theta}{1+(1/\theta)^2} = -\frac{\theta}{1+\theta^2}, |t|=1$ , 证明自相关函数不变

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4.25 考虑 AR(1) 过程, 满足  $Y_t = \phi Y_{t-1} + e_t$ , 其中  $\phi$  为常数,  $\{e_t\}$  为一个白噪声过程

并且  $e_t$  独立于过去值  $\{Y_{t-1}, Y_{t-2}, \dots\}$ . 令  $Y_0$  为均值  $\mu_0$ 、方差  $\sigma_0^2$  的随机变量.

(a). 证明: 当  $t > 0$  时, 可写成

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots + \phi^{t-1} e_1 + \phi^t Y_0$$

(b). 证明: 当  $t > 0$  时, 有  $E(Y_t) = \phi^t \mu_0$ .

(c). 证明: 当  $t > 0$  时,

$$\text{Var}(Y_t) = \begin{cases} \frac{1-\phi^{2t}}{1-\phi^2} \sigma_e^2 + \phi^{2t} \sigma_0^2, & \phi \neq 1 \\ t \sigma_e^2 + \sigma_0^2, & \phi = 1 \end{cases}$$

(d). 假设  $\mu_0=0$ , 试证: 若  $\{Y_t\}$  平稳, 则必有  $|\phi| < 1$

(e). 仍假设  $\mu_0=0$ , 证明: 若  $\{Y_t\}$  平稳, 则  $\text{Var}(Y_t) = \sigma_e^2 / (1-\phi^2)$ , 因此必有  $|\phi| < 1$

Proof.

(a). 对  $t > 0$ , 有  $Y_t = e_t + \phi Y_{t-1} = e_t + \phi(e_{t-1} + \phi Y_{t-2}) = e_t + \phi e_{t-1} + \phi^2 Y_{t-2}$   
 $= e_t + \phi e_{t-1} + \phi^2(e_{t-2} + \phi Y_{t-3}) = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 Y_{t-3}$   
 $= \dots = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots + \phi^{t-1} e_1 + \phi^t Y_0$ . 10 证

(b). 由 (a), 有  $E(Y_t) = E(e_t) + \phi E(e_{t-1}) + \phi^2 E(e_{t-2}) + \dots + \phi^{t-1} E(e_1) + \phi^t E(Y_0)$   
 $\Rightarrow E(Y_t) = o_1 + \dots + o_t + \phi^t E(Y_0) = \phi^t \mu_0$

(c). 若  $\phi < 0$ ,  $Var(Y_t) = Var(e_t + \phi e_{t-1} + \dots + \phi^{t-1} e_1 + \phi^t Y_0) = ((1 + \phi^2 + \dots + \phi^{2t-2}) \sigma_e^2 + \phi^{2t} \sigma_0^2)$   
若  $\phi \neq 1$ ,  $Var(Y_t) = \frac{(1 - \phi^2)(1 + \phi^2 + \dots + \phi^{2t-2})}{1 - \phi^2} \cdot \sigma_e^2 + \phi^{2t} \sigma_0^2 = \frac{1 - \phi^{2t}}{1 - \phi^2} \sigma_e^2 + \phi^{2t} \sigma_0^2$

② 若  $\phi = 1$ ,  $Var(Y_t) = t \sigma_e^2 + \sigma_0^2$   
 $\Rightarrow Var(Y_t) = \begin{cases} \frac{1 - \phi^{2t}}{1 - \phi^2} \sigma_e^2 + \phi^{2t} \sigma_0^2 \\ t \sigma_e^2 + \sigma_0^2 \end{cases}$ , 得证.

(d). 假设  $\phi = 1$ , 即  $Y_t = Y_{t-1} + e_t$ , 换言之,  $Y_t = Y_{t-2} + e_{t-1} + e_t = \dots = Y_0 + \sum_{k=1}^t e_k$   
此时我们任取  $t > 0$ , (a), (b), (c) 中条件仍然满足, 则

对  $0 < s < t$ , 有  $Y_t = Y_{t-s} + e_s = Y_{t-s} + e_{t-s} + e_t = Y_s + (e_{s+1} + \dots + e_t)$

则对  $0 < s < t$ ,  $\gamma_{t,s} = Cov(Y_s, Y_t) = Cov(Y_s, Y_s + (e_{s+1} + \dots + e_t))$

则  $\gamma_{t,s} = Var(Y_s) = s \sigma_e^2 + \sigma_0^2$  与  $s$  相关, 与  $\phi$  线性相关. 故必有  $\phi \neq 1$ .

(e). 仍在  $0 < s < t$  处观察时间序列, 由 (d), 有  $\gamma_{t,s} = Var(Y_s) = \frac{1 - \phi^{2t}}{1 - \phi^2} \sigma_e^2 + \phi^{2t} \sigma_0^2$   
 $\gamma_{t,s} = \frac{\sigma_e^2}{1 - \phi^2} + \phi^{2t} (\sigma_0^2 - \frac{\sigma_e^2}{1 - \phi^2}) = \frac{\sigma_e^2}{1 - \phi^2} + \phi^{2s} \cdot \phi^{2(t-s)} (\sigma_0^2 - \frac{\sigma_e^2}{1 - \phi^2})$  与  $s$  无关

则必有  $\sigma_0^2 - \frac{\sigma_e^2}{1 - \phi^2} = 0$ . 由  $\gamma_{t,s} = \frac{\sigma_e^2}{1 - \phi^2}$  得证

此时有  $1 - \phi^2$  必为正,  $1 - \phi^2 > 0$ , 即  $|\phi| < 1$

## 4.10

画出下列每个 ARMA 模型的自相关函数：

(a)

ARMA(1, 1),  $\phi = 0.7, \theta = 0.4$

(b)

ARMA(1, 1),  $\phi = 0.7, \theta = -0.4$

```
In [3]: # 加载必要的库
library(stats)

# 定义ARMA模型的参数
phi_a <- 0.7
theta_a <- 0.4
phi_b <- 0.7
theta_b <- -0.4

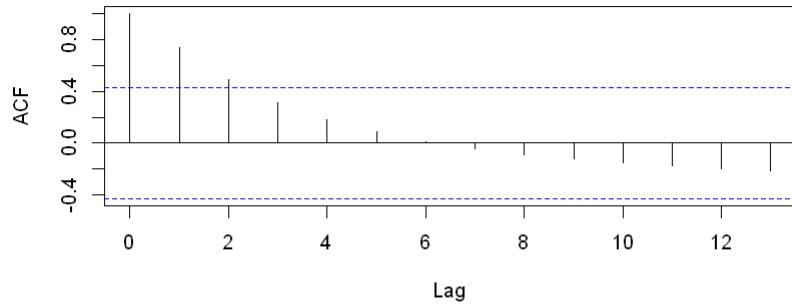
# 生成ARMA(1,1)模型的理论自相关函数
acf_a <- ARMAacf(ar = phi_a, ma = theta_a, lag.max = 20)
acf_b <- ARMAacf(ar = phi_b, ma = theta_b, lag.max = 20)

# 绘制自相关函数图
par(mfrow = c(2, 1)) # 设置绘图区域为2行1列

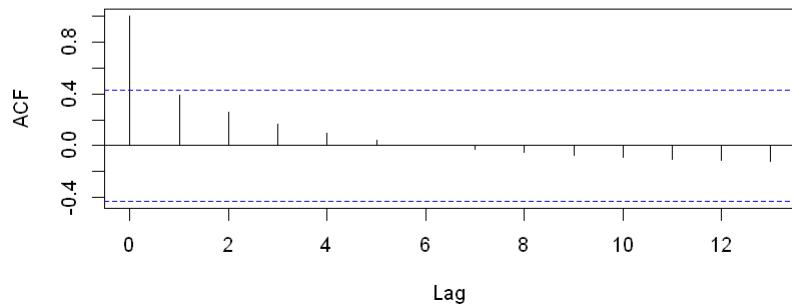
# 绘制第一个模型的自相关函数
acf(acf_a, main = expression((a):~ARMA(1, 1)~with~phi == 0.7, theta == 0.4)), xla

# 绘制第二个模型的自相关函数
acf(acf_b, main = expression((b):~ARMA(1, 1)~with~phi == 0.7, theta == -0.4)), xla
```

(a): ARMA(1, 1) with  $\phi = 0.7$



(b): ARMA(1, 1) with  $\phi = 0.7$



## 4.10 见 R 程序实现

4.11 对 ARMA(1,2) 模型  $Y_t = 0.8 Y_{t-1} + e_t + 0.7 e_{t-1} + 0.6 e_{t-2}$ , 则有:

$$(a). \text{ 当 } k > 2 \text{ 时, } p_k = 0.8 p_{k-1}$$

$$(b). p_2 = 0.8 p_1 + 0.6 \sigma_e^2 / \gamma_0$$

Proof.

(a). 该 TS 中,  $\phi_0 = 0$ ,  $\phi_1 = 0.8$ , AR 特征方程  $\phi(x) = 1 - 0.8x \Rightarrow \phi(x) = 0$  之解为  $x = \frac{1}{0.8}$

落在单位圆外, 故此该 ARMA 序列平稳. 因而由平稳性质, 可得:

$$\gamma_k = \text{Cov}(Y_t, Y_{t-k}) = \text{Cov}(0.8 Y_{t-1} + e_t + 0.7 e_{t-1} + 0.6 e_{t-2}, Y_{t-k}) \stackrel{k>2}{=} 0.8 \gamma_{t-1}$$

$$\Rightarrow p_k = 0.8 p_{k-1}, \text{ 得证}$$

$$(b). Y_{t-1} = 0.8 Y_{t-2} + e_t + 0.7 e_{t-1} + 0.6 e_{t-2}$$

$$\text{Cov}(Y_t, Y_{t-1}) = \gamma_1 = \text{Cov}(Y_t, 0.8 Y_{t-2} + e_t + 0.7 e_{t-1} + 0.6 e_{t-2}) = 0.8 \gamma_2 + \text{Cov}(Y_t, e_t)$$

$$= 0.8 \gamma_2 + \text{Cov}(0.8 Y_{t-1} + e_t + 0.7 e_{t-1} + 0.6 e_{t-2}, e_t) = 0.8 \gamma_2 + 0.6 \sigma_e^2$$

$$\Rightarrow \gamma_1 / \gamma_0 = 0.8 \gamma_2 / \gamma_0 + \sigma_e^2 / \gamma_0 \Rightarrow p_1 = 0.8 p_2 + \frac{0.6 \sigma_e^2}{\gamma_0^2}, \text{ 得证}$$

4.22 证明陈述 “ $1 - \phi_1 x - \phi_2 x^2 - \dots - \phi_p x^p = 0$  的根的绝对值大于 1” 等价于

陈述 “ $x^p - \phi_1 x^{p-1} - \dots - \phi_p = 0$  的根的绝对值小于 1”. (提示: 如果  $G$  是一个

方程的根, 则  $1/G$  是另一个方程的根)

Proof.

非零

① “ $\Rightarrow$ ”: 对于  $G$  为  $Eq_1$  的解, 则  $|G| > 1$ , 此时  $\frac{1}{G}$  必为  $Eq_2$  的根, 这一结论又有

$|\frac{1}{G}| < 1$ ; 若  $Eq_2$  存在  $x=0$  的根, 则也有其绝对值小于 1. “ $\Rightarrow$ ”成立

② “ $\Leftarrow$ ”: 若  $G$  为  $Eq_2$  的解, 若  $G \neq 0$ , 则  $\frac{1}{G}$  必为  $Eq_1$  的根. 注意到  $Eq_1$  不存在零解 ( $\neq 0$ )

$$\text{因为 } Eq_1 \Leftrightarrow x^p \left( \left(\frac{1}{x}\right)^p - \phi_1 \left(\frac{1}{x}\right)^{p-1} - \dots \right) = 0 \Leftrightarrow \left(\frac{1}{x}\right)^p - \phi_1 \left(\frac{1}{x}\right)^{p-1} - \dots - \phi_p = 0, |\frac{1}{x}| > 1, “\Leftarrow” \text{ 成立}$$

考虑时间序列:  $Y_t = 0.2Y_{t-1} + 0.63Y_{t-2} + e_t + 1.2e_{t-1} + 0.35e_{t-2}$ , 其中  $e_t$  是方差为  $\sigma_e^2$  的白噪声过程

- 如果该模型可以简化, 请写出它的简化形式
- 这个过程可逆吗? 平稳吗?
- 如果平稳, 转换成MA( $\infty$ )形式  $Y_t = e_t + \sum_{j=1}^{\infty} \Psi_j e_{t-j}$ 。写出  $\Psi_j$  的前四项和一般形式。

Proof.

(a). 3) 入滞后算子,  $Y_t = 0.2B Y_t + 0.63 B^2 Y_t + e_t + 1.2 B e_t + 0.35 B^2 e_t$

$$(1 - 0.2B - 0.63B^2) Y_t = (1 + 1.2B + 0.35B^2) e_t$$

则  $Y_t = \phi(B)^{-1} \theta(B) e_t$ , 其中  $\phi(B) = 1 - 0.2B - 0.63B^2$ ,  $\theta(B) = 1 + 1.2B + 0.35B^2$

(b). ▷ MA 特征多项式:  $\theta(x) = 1 + 1.2x + 0.35x^2$ , 且 MA 特征方程  $\theta(x) = 0$  有两根

均小于 -1, 可知其可逆

▷ AR 特征多项式:  $\phi(B) = 1 - 0.2B - 0.63B^2 \Rightarrow AR$  特征方程:  $\phi(x) = 1 - 0.2x - 0.63x^2 = 0$

可知, 特征方程的两个根  $|x_1| > 1, |x_2| > 1$ , 可知其平稳

(C).

法一:

$$Y_t = 0.2Y_{t-1} + 0.63Y_{t-2} + e_t + 1.2e_{t-1} + 0.35e_{t-2}, \text{ 将 } Y_{t-1} \text{ 展开后, 得}$$

$$\begin{aligned} Y_t &= 0.2(0.2Y_{t-2} + 0.63Y_{t-3} + e_{t-1} + 1.2e_{t-2} + 0.35e_{t-3}) + 0.63Y_{t-2} + e_t + 1.2e_{t-1} + 0.35e_{t-2} \\ &= 0.67Y_{t-2} + 0.126Y_{t-3} + e_t + 1.4e_{t-1} + 0.59e_{t-2} + 0.07e_{t-3}, \text{ 再将 } Y_{t-2} \text{ 展开} \\ &= 0.67(0.2Y_{t-3} + 0.63Y_{t-4} + e_{t-2} + 1.2e_{t-3} + 0.35e_{t-4}) + 0.126Y_{t-3} + e_t + 1.4e_{t-1} + 0.59e_{t-2} + 0.07e_{t-3} \\ &= 0.26Y_{t-3} + 0.422Y_{t-4} + e_t + 1.4e_{t-1} + 1.26e_{t-2} + 0.874e_{t-3} + 0.2345e_{t-4} \end{aligned}$$

此时  $\bar{\psi}_1 = 1.4$   $\bar{\psi}_2 = 1.26$ , 而再对  $Y_{t-3}$  和  $Y_{t-4}$  作下阶展开, 可知,

$$\begin{aligned} \bar{\psi}_3 &= 0.26(0.2Y_{t-4} + 0.63Y_{t-5} + e_{t-3} + 1.2e_{t-4} + 0.35e_{t-5}) + 0.422(Y_{t-4} + \dots) + 0.874e_{t-3} \\ &+ 0.2345e_{t-4} = 0.4741Y_{t-4} + (\dots) + 1.134e_{t-3} + 0.5465e_{t-4} \end{aligned}$$

则可知, 其中的参数与  $e_{t-3}$   $e_{t-4}$  无关,  $\bar{\psi}_3 = 1.134$ ,  $\bar{\psi}_4 = 0.5465 + 0.4741 = 1.0206$

综上,  $\bar{\psi}_1 = 1.4$   $\bar{\psi}_2 = 1.26$   $\bar{\psi}_3 = 1.134$   $\bar{\psi}_4 = 1.0206$

法二：试运用泰勒展开，注意到特征方程为  $(1-0.9x)(1+0.7x) \therefore \phi^{-1}(x) = \frac{1}{(1-0.9x)(1+0.7x)}$

$$\text{另有 } \theta(x) = 1 + 1.2x + 0.35x^2 = (1+0.5x)(1+0.7x) \therefore \phi^{-1}(x)\theta(x) = \frac{1+0.5x}{1-0.9x}$$

$$\text{由此知 } \phi^{-1}(x)\theta(x) = -\frac{5}{9} + \frac{14/9}{1-0.9x} = -\frac{5}{9} + \frac{14}{9} \frac{1}{1-0.9x} = f(x)$$

可知  $\phi^{-1}(x)\theta(x)$  的前五项小节为

$$\begin{aligned} ① \quad & \frac{f^{(0)}}{0!}(x-0)^0 = f(0) = 1 \quad ② \quad \frac{f^{(1)}}{1!}(x-0)^1 = \frac{14}{9}(-0.9)(-\frac{1}{(1-0.9x)^2})_{x=0} x^1 = 1.4x \\ ③ \quad & \frac{f^{(2)}}{2!}(x-0)^2 = \frac{1}{2} \frac{14}{9} (-0.9)^2 (2(1-0.9x)^3)_{x=0} x^2 = \frac{1}{2} \cdot \frac{14}{9} \cdot 0.81 \cdot 2 \cdot x^2 = 1.26x^2 \\ ④ \quad & \frac{f^{(3)}(0)}{3!}(x-0)^3 = \frac{1}{6} \cdot \frac{14}{9} \cdot (-0.9)^3 (-6(1-0.9x)^4)_{x=0} x^3 = 1.134x^3 \\ ⑤ \quad & \frac{f^{(4)}(0)}{4!}(x-0)^4 = \frac{1}{24} \cdot \frac{14}{9} (-0.9)^4 (-24(1-0.9x)_{x=0}) x^4 = 1.0206x^4 \end{aligned}$$

$$\text{从而: } \psi_1 = 1.4 \quad \psi_2 = 1.26 \quad \psi_3 = 1.134 \quad \psi_4 = 1.0206$$

另: 由泰勒级数及重叠规则, 我们知, 其一般形式为  $\psi_j = 1.4(0.9)^{j-1}$

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5.7 考虑两个模型: A:  $Y_t = 0.9Y_{t-1} + 0.09Y_{t-2} + e_t$

$$B: Y_t = Y_{t-1} + e_t - 0.1e_{t-1}$$

(a). 分别将每个模型具体的ARIMA形式, 即 p, d, q 分别为多少, 参数值  $\phi$  和  $\theta$  是什么?

(b). 什么情况下两个模型会不同?

(c). 什么情况下两个模型会相似? (比较冲击重和飞权重.)

Sol.

(a). A: 特征方程为  $\phi(x) = 1-0.9x - 0.09x^2$  所有根都落于单位根之外, 因此

其为平稳的 AR(2) 过程,  $p=2, d=0, q=0, \phi_1=0.9, \phi_2=-0.09$

B:  $\phi(x) = 1-x \quad \theta(x) = 1-0.1x$ , 故其为非平稳不可逆的 IMA(1,1) 过程

可写为  $Y_t - Y_{t-1} = \nabla' Y_t = W_t = e_t - 0.1e_{t-1}, p=0, d=1, q=1, \theta_1=0.1$

(b). 如 (a) 中证明的, A 平稳, 而 B 不平稳

(c).

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$\phi$ 权重:

AR(2):

根据(4.3.21)

$$\begin{cases} \psi_0 = 1 \\ \psi_1 - \phi_1\psi_0 = 0 \\ \psi_j - \phi_1\psi_{j-1} - \phi_2\psi_{j-2} = 0 \quad j = 2, 3, \dots \end{cases}$$

$$\rightarrow \psi_0 = 1, \psi_1 = \phi_1, \psi_2 = \phi_1^2 + \phi_2, \dots$$

```
In [11]: phi1 = 0.9
phi2 = 0.09

psi.a = NULL
psi.a[1] = 1
psi.a[2] = phi1

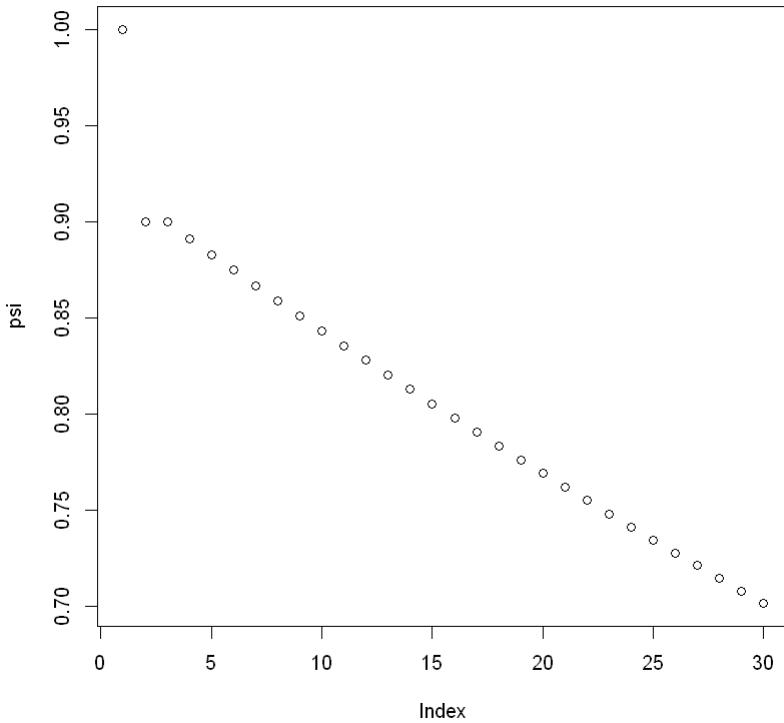
for (i in 3:30){
  psi.a[i] = phi1 * psi.a[i-1] + phi2 * psi.a[i-2]
}

psi
```

1 · 0.9 · 0.9 · 0.891 · 0.8829 · 0.8748 · 0.866781 · 0.8588349 · 0.8509617 · 0.843160671 ·  
0.8354311569 · 0.8277725016 · 0.820184055561 · 0.8126651751489 · 0.8052152226345 ·  
0.797833566134451 · 0.790519579558111 · 0.783272642554401 · 0.776092140459191 ·  
0.768977464243168 · 0.761928010460178 · 0.754943181196045 · 0.748022384017857 ·  
0.741165031923715 · 0.734370543292951 · 0.72763834183679 · 0.720967856549477 ·  
0.71435852165984 · 0.707809776583309 · 0.701321065874364

```
In [10]: plot(psi, main= expression(paste("AR(2)过程", psi, "权重")))
```

### AR(2)过程 $\psi$ 权重



IMA(1,1):

根据(5.2.6):

$$Y_t = e_t + (1 - \theta)e_{t-1} + (1 - \theta)e_{t-2} + \cdots + (1 - \theta)e_{-m} - \theta_{-m-1}$$

据此计算知, IMA(1,1)过程的 $\psi$ 权重为:

$$\psi_0 = 1, \psi_1 = 1 - \theta = 0.9, \psi_2 = 1 - \theta = 0.9, \psi_3 = 0.9, \psi_4 = 0.9 \dots,$$

即

$$\psi_j = \begin{cases} 1, & j = 0, \\ 0.9, & j = 1, 2, \dots \end{cases}$$

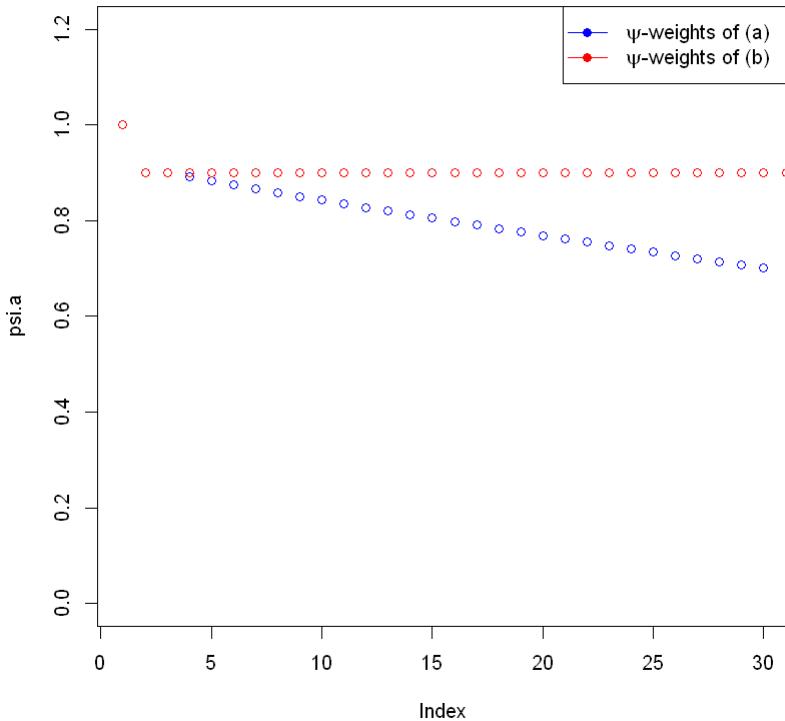
```
In [ ]: psi.b = c(1,rep(0.9,30))

# psi.a , psi.b 画图比较
plot(psi.a,
      col = 'blue',
      main=expression(paste("AR(2)、IMA(1,1)过程",psi,"权重比较")),
      ylim=c(0,1.2))

points(psi.b, col="red")

# 添加图例
legend("topright",
       legend = c( expression(paste(psi,'-weights of (a)')), expression(paste(psi,'-weights of (b)'))),
       col = c("blue", "red"),
       pch = c(16, 16),
       lty = c(1, 1))
```

### AR(2)、IMA(1,1)过程 $\psi$ 权重



可见，在序列的次序数不大时，应该认为两个序列 $\psi$ 权重是有偏差但依然相对接近的。

---

$\pi$ 权重：

**AR(2):**

对于AR(2)过程， $\pi$ 权重为就是原本的系数。

$$\pi_0 = 1, \pi_1 = \phi_1 = 0.9, \pi_2 = \phi_2 = 0.09, \text{而 } \pi_j = 0 \text{ for } j = 3, 4, 5, \dots$$

$$\text{即 } \pi_j = \begin{cases} 0.9, & j = 1 \\ 0.9, & j = 2 \\ 0, & j = 3, 4, 5, \dots \end{cases}$$

**IMA(1,1):**

对于IMA(1,1)过程，可将起逆转为MA( $\infty$ )过程，继而可得 $\pi$ 权重为：

$$\pi_k = (1 - \theta)\theta^{k-1}, k = 1, 2, 3, \dots$$

$$\text{则 } \pi_k = (1 - \theta)\theta^{k-1} = \begin{cases} 0.9, & k = 1 \\ 0.09, & k = 2 \\ 0.009, & k = 3 \\ 0.0009, & k = 4 \\ \vdots & \\ \vdots & \\ \vdots & \end{cases}$$

综上可知，AR(2)和IMA(2,2)前两个 $\pi$ 权重是相同的。

而后边的权重亦相差非常小。

---

综合以上分析，可以认为AR(2)和IMA(1,1)的 $\psi$ 权重和 $\pi$ 权重是**非常接近的**。

换句话说，事实上AR(2)和IMA(1,1)的概率分布及预测性能是十分相似的。

## 5.10

非平稳 ARIMA 序列可以由以下方法模拟：

首先模拟相应的平稳 ARMA 系列，然后对其“求和”（实际上部分求和）。

使用统计软件来模拟具有不同参数值的各种 IMA 的 (1, 1) 和 IMA(2, 2) 序列。

注意在这些模拟序列中出现的任何随机“趋势”。

(Solution).

1

```
In [8]: # 加载必要的库
library(stats)

# 设置随机种子以保证结果可复现
set.seed(123)

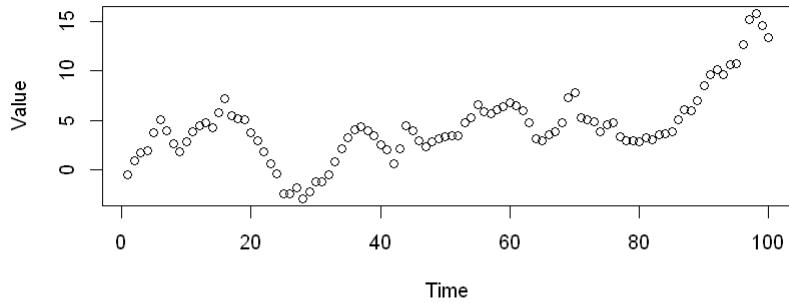
# 模拟IMA(1,1)序列
# 对应的ARMA模型为MA(1)，参数为theta = 0.5
theta_11 <- 0.5
ma11 <- arima.sim(model = list(ma = theta_11), n = 100)
ima11 <- cumsum(ma11) # 对MA序列进行累积求和

# 模拟IMA(2,2)序列
# 对应的ARMA模型为MA(2)，参数为theta = c(0.5, -0.3)
theta_22 <- c(0.5, -0.3)
ma22 <- arima.sim(model = list(ma = theta_22), n = 100)
ima22 <- cumsum(ma22) # 对MA序列进行累积求和

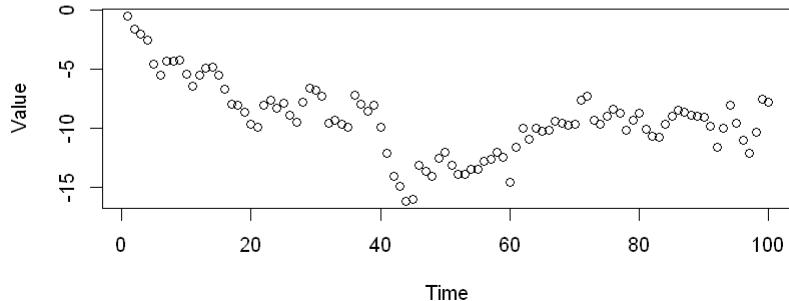
# 绘制模拟的IMA序列图
par(mfrow = c(2, 1)) # 设置绘图区域为2行1列

plot(ima11, main = expression(IMA(1,1)~with~theta == 0.5), ylab = "Value", xlab =
plot(ima22, main = expression(IMA(2,2)~with~theta[1] == 0.5~and~theta[2] == -0.3
```

IMA(1, 1) with  $\theta = 0.5$



IMA(2, 2) with  $\theta_1 = 0.5$  and  $\theta_2 = -0.3$



## 2

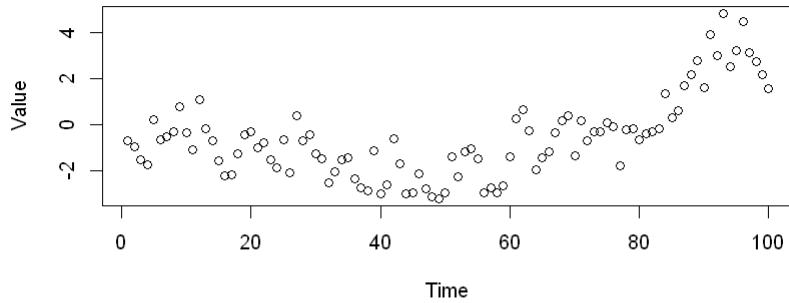
```
In [9]: # 模拟IMA(1,1)序列
# 对应的ARMA模型为MA(1), 参数为theta = -0.5
theta_11 <- -0.5
ma11 <- arima.sim(model = list(ma = theta_11), n = 100)
ima11 <- cumsum(ma11) # 对MA序列进行累积求和

# 模拟IMA(2,2)序列
# 对应的ARMA模型为MA(2), 参数为theta = c(0.5, 0.5)
theta_22 <- c(-0.5, 0.5)
ma22 <- arima.sim(model = list(ma = theta_22), n = 100)
ima22 <- cumsum(ma22) # 对MA序列进行累积求和

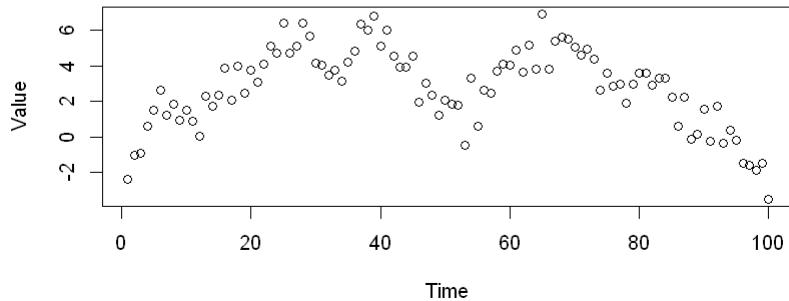
# 绘制模拟的IMA序列图
par(mfrow = c(2, 1)) # 设置绘图区域为2行1列

plot(ima11, main = expression(IMA(1,1)~with~theta == -0.5), ylab = "Value", xlab =
plot(ima22, main = expression(IMA(2,2)~with~theta[1] == -0.5~and~theta[2] == 0.5), ylab = "Value", xlab =
```

IMA(1, 1) with  $\theta = -0.5$



IMA(2, 2) with  $\theta_1 = -0.5$  and  $\theta_2 = 0.5$



### 3

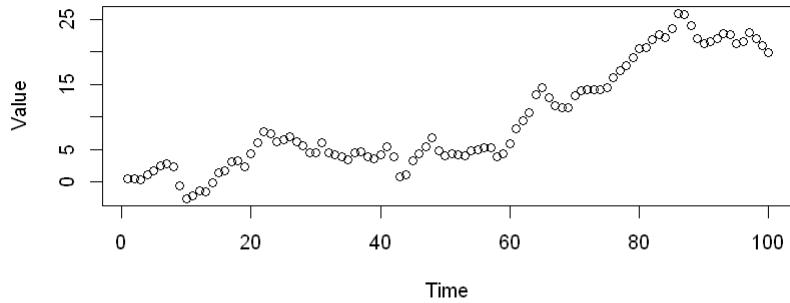
```
In [10]: # 模拟IMA(1,1)序列
# 对应的ARMA模型为MA(1), 参数为theta = 0.7
theta_11 <- 0.7
ma11 <- arima.sim(model = list(ma = theta_11), n = 100)
ima11 <- cumsum(ma11) # 对MA序列进行累积求和

# 模拟IMA(2,2)序列
# 对应的ARMA模型为MA(2), 参数为theta = c(0.7, 1)
theta_22 <- c(0.7, 1)
ma22 <- arima.sim(model = list(ma = theta_22), n = 100)
ima22 <- cumsum(ma22) # 对MA序列进行累积求和

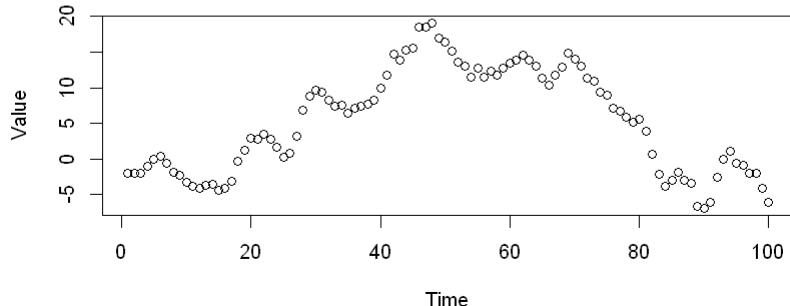
# 绘制模拟的IMA序列图
par(mfrow = c(2, 1)) # 设置绘图区域为2行1列

plot(ima11, main = expression(IMA(1,1)~with~theta == 0.7), ylab = "Value", xlab =
plot(ima22, main = expression(IMA(2,2)~with~theta[1] == 0.7~and~theta[2] == 1),
```

IMA(1, 1) with  $\theta = 0.7$



IMA(2, 2) with  $\theta_1 = 0.7$  and  $\theta_2 = 1$



从三次随机模拟在100步的时间维度上，可以注意到某种趋势，序列似乎确有向上、下的趋势。

注意到，令IMA(1,1)序列的参数 $\theta$ 和IMA(2,2)序列的第一个参数 $\theta_1$ 相等，这两个时间序列在100步的时间中表现出相反的走势。