

P<sub>100</sub> 6.14 对于一个长度为 169 的序列，我们求得  $r_1 = 0.41$ ,  $r_2 = 0.32$ ,  $r_3 = 0.26$ ,  $r_4 = 0.21$ ,  $r_5 = 0.16$ . 什么样的 ARIMA 适合该种自相关模式?

Proof.

$$r_2/r_1 = 0.78 \quad r_3/r_2 = 0.81 \quad r_4/r_3 = 0.81 \quad r_5/r_4 = 0.76$$

可以发现， $r_i$  随着  $i$  的增大而递减且递减的幅度是稳定的。

$$\text{可以设 } r_{t+1} \approx 0.8, \text{ 则有 } r_k = 0.8^{k-1} r_1 = 0.8^k r_1$$

对于 AR(1) 过程： $Y_t = \phi Y_{t-1} + e_t$ . 知  $\gamma_1 = \text{Cov}(Y_{t+1}, Y_t)$

$$\gamma_1 = \text{Cov}(\phi Y_{t-1} + e_{t+1}, Y_t) = \phi \gamma_0$$

$$\gamma_2 = \text{Cov}(Y_{t+2}, Y_t) = \text{Cov}(\phi Y_{t+1} + e_{t+2}, Y_t) = \text{Cov}(\phi(\phi Y_{t-1} + e_{t+1}) + e_{t+2}, Y_t)$$

$$= \text{Cov}(\phi^2 Y_{t-1} + \phi e_{t+1} + \phi + e_{t+2}, Y_t) = \phi^2 \gamma_0$$

$$\text{同理 } \gamma_k = \phi^k \gamma_0, \text{ 且 } p_k = \phi^k = p_1^k$$

$p_k = \phi^k$  这一特征与题给序列并不相符。因为  $\phi = 0.8$  并不能刻画序列数值

(如 ARMA(1,1)).  $Y_t = \phi Y_{t-1} + e_t + \theta e_{t-1}$

$$\begin{aligned} \gamma_1 &= \text{Cov}(\phi Y_{t-1} + e_t + \theta e_{t-1}, Y_{t-1}) = \phi \gamma_0 + \theta \text{Cov}(e_{t-1}, Y_{t-1}) = \phi \gamma_0 + \theta \text{Cov}(e_{t-1}, \phi Y_{t-2} + e_{t-1} + \theta e_{t-2}) \\ &= \phi \gamma_0 + \theta \sigma_e^2 \rightarrow p_1 = \phi + \frac{\theta \sigma_e^2}{\gamma_0} \end{aligned}$$

$$\begin{aligned} \gamma_2 &= \text{Cov}(\phi Y_{t-1} + e_t + \theta e_{t-1}, Y_{t-1}) = \phi \gamma_1 + \theta \text{Cov}(e_{t-1}, Y_{t-1}) = \phi \gamma_1 + \theta \sigma_e^2 = \\ &= \phi^2 \gamma_0 + \phi \frac{\theta \sigma_e^2}{\gamma_0} + \theta \sigma_e^2 = \phi^2 \gamma_0 + \theta(\phi + 1) \sigma_e^2, \quad p_2 = \phi^2 + \frac{\theta(\phi + 1)}{\gamma_0} \sigma_e^2 \end{aligned}$$

$$\begin{aligned} \gamma_3 &= \text{Cov}(\phi Y_{t-1} + e_t + \theta e_{t-1}, Y_{t-1}) = \phi \gamma_2 + \theta \text{Cov}(e_{t-1}, Y_{t-1}) = \phi^3 \gamma_0 + \theta(\phi^2 + \phi + 1) \sigma_e^2 \\ p_3 &= \phi^3 + \frac{\theta(\phi^2 + \phi + 1)}{\gamma_0} \sigma_e^2 \dots \end{aligned}$$

所以对于 ARMA(1,1) 而言， $p_{t+1}/p_t \approx \phi$  取为 0.8，而  $\gamma_1 \neq 0.8$  是合理的，再保证  $\theta \frac{\sigma_e^2}{\gamma_0} \times r_1 - \theta = 0.41 - 0.8 = 0.39$  即可。

综上，ARIMA(1,1) 是合理的，其中假设  $\phi = 0.8$ ,  $\theta = \frac{0.39}{\sigma_e^2}$

6.15 某序列及其一阶差分序列的样本 ACF 列于下表，此次  $n = 100$ .

滞后	1	2	3	4	5	6
$Y_t$ 的 ACF	0.97	0.97	0.93	0.85	0.80	0.71
$\nabla Y_t$ 的 ACF	-0.42	0.18	-0.02	0.07	-0.10	-0.09

只基于这些信息，我们会为该序列考虑什么样的 ARIMA 模型？

Ans,

$Y_t$  的 ACF 在第 6 阶仍未消失，而  $\nabla Y_t$  的 ACF 在第 2 阶就迅速来到 0 附近。

说明  $\nabla Y_t$  有可能为  $MA(5), MA(1)$

首先  $H_{00}: \nabla Y_t \sim MA(0)$ ，此时  $|r_1| > \frac{2}{\sqrt{n}}$ ，所以拒绝  $H_{00}$

另外  $H_{01}: \nabla Y_t \sim MA(1) \Leftrightarrow Y_t \sim IMA(1, 1)$  在  $H_0$  下， $\sqrt{n} r_2 \sim N(0, 1+2p_1^2)$

$$|r_2| = 0.18 < \frac{2}{\sqrt{n}} \sqrt{1+2p_1^2} \approx \frac{2}{\sqrt{n}} \sqrt{1+2r_1^2} = \frac{2}{\sqrt{100}} \sqrt{1+2(-0.42)^2} = 0.2326$$

所未可拒绝原假设，接受  $H_0: Y_t \sim IMA(1, 1)$

6.16 对一个长度为 64 的序列，样本偏自相关函数如下：

滞后	1	2	3	4	5
PACF	0.47	-0.34	0.20	0.02	-0.06

这种情况下，我们应该考虑什么样的模型。

Ans.

$H_{00}: Y_t \sim AR(0)$ ,  $|\hat{p}_1| > 2/\sqrt{n}$ ;  $H_{01}: Y_t \sim AR(1)$ ,  $|\hat{p}_2| > 2/\sqrt{n}$ ;

则对  $H_{00}, H_{01}$  均拒绝

$H_{02}: Y_t \sim AR(2)$ , 而  $|\hat{p}_{33}| = 0.2 < 2/\sqrt{n} = 2/\sqrt{64} = 0.25$

所以接受原假设  $H_0$ ，即认为  $Y_t \sim AR(2)$

## 6.20

模拟  $n=48$ ,  $\phi = 0.7$  的 AR(1) 时间序列。

```
In [5]: set.seed(123) # 设置随机种子, 保证结果可重复  
n <- 48  
phi <- 0.7  
  
x <- arima.sim(model = list(ar = phi), n = n) # 生成 AR(1) 时间序列
```

(a)

计算该模型在 1 阶和 5 阶滞后处的理论自相关系数。

Solution.

```
In [6]: rho1 = 0.7  
rho5 = 0.7^5
```

```
In [7]: rho1
```

0.7

```
In [8]: rho5
```

0.16807

对于 AR(1) 模型, 其自相关函数为:

$$\rho_k = \phi^k$$

因此, 1 阶滞后处的理论自相关系数为:

$$\rho_1 = \phi^1 = 0.7$$

5 阶滞后处的理论自相关系数为:

$$\rho_5 = \phi^5 = 0.7^5 \approx 0.16807$$

(b)

计算 1 阶和 5 阶滞后处样本自相关系数, 并将其与理论自相关值进行比较。用方程 (6.1.5) 和 (6.1.6) 量化这个比较。

Solution.

```
In [9]: acf_values <- acf(x, plot = FALSE) # 计算自相关系数  
acf_values
```

```
Autocorrelations of series 'x', by lag
```

0	1	2	3	4	5	6	7	8	9	10
1.000	0.716	0.529	0.371	0.204	0.119	-0.027	-0.117	-0.221	-0.254	-0.287
11	12	13	14	15	16					
-0.276	-0.285	-0.282	-0.111	-0.066	0.024					

```
In [10]: r1 <- acf_values$acf[2] # 1 阶滞后处的样本自相关系数  
r5 <- acf_values$acf[6] # 5 阶滞后处的样本自相关系数
```

```
In [11]: r1
```

```
0.716129494482611
```

```
In [12]: r1 - rho1
```

```
0.0161294944826107
```

```
In [13]: r5
```

```
0.118618618903509
```

```
In [14]: r5 - rho5
```

```
-0.0494513810964908
```

由此可见，样本自相关系数 $r_1, r_5$ 与理论自相关系数 $\rho_1, \rho_5$ 之间**差异比较小**。

$$Var(r_1) \approx \frac{1 - \phi^2}{n} \quad (6.1.5)$$

$$Var(r_k) \approx \frac{1}{n} \left[ \frac{1 + \phi^2}{1 - \phi^2} \right], \quad \text{对较大的 } k \quad (6.1.6)$$

```
In [15]: Var_r1 = (1 - phi^2) / n  
Var_r1
```

```
0.010625
```

```
In [16]: Var_r5 = 1 / n * ((1 + phi^2) / (1 - phi^2))  
Var_r5
```

```
0.0608660130718954
```

样本自相关系数与真实理论自相关系数值的差异与估计量的方差有关。方差越小，估计的精度越高。

方差  $Var(r_1)$  和  $Var(r_5)$  都非常小，所以样本自相关系数的估计精度还是比较高的。

### (c)

使用新的模拟重复 (b)。描述在相同条件下，估计的精度如何随所选样本的不同而变化。

Solution.

```
In [17]: set.seed(789) # 设置随机种子, 保证结果可重复  
x <- arima.sim(model = list(ar = phi), n = n) # 生成 AR(1) 时间序列
```

```
In [18]: acf_values <- acf(x, plot = FALSE) # 计算自相关系数  
  
r1 <- acf_values$acf[2] # 1 阶滞后处的样本自相关系数  
r5 <- acf_values$acf[6] # 5 阶滞后处的样本自相关系数
```

```
In [19]: r1 - rho1  
  
0.0131175271544546
```

```
In [20]: r5 - rho5  
  
-0.0331016417255391
```

```
In [21]: Var_r1 = (1 - phi^2) / n  
Var_r1  
  
0.010625
```

```
In [22]: Var_r5 = 1 / n * ((1 + phi^2) / (1 - phi^2))  
Var_r5  
  
0.0608660130718954
```

不同的样本会导致样本自相关系数有所不同，但总体上方差大致相同，因此 $r$ 作为 $\rho$ 的估计量应当在特定的理论值周围波动。

## (d)

如果软件允许，重复模拟序列并多次计算  $r_1$  和  $r_5$ ，并且构建  $r_1$  和  $r_5$  的样本分布。

描述估计的精度如何随着相同条件下所选择的样本的不同而变化。

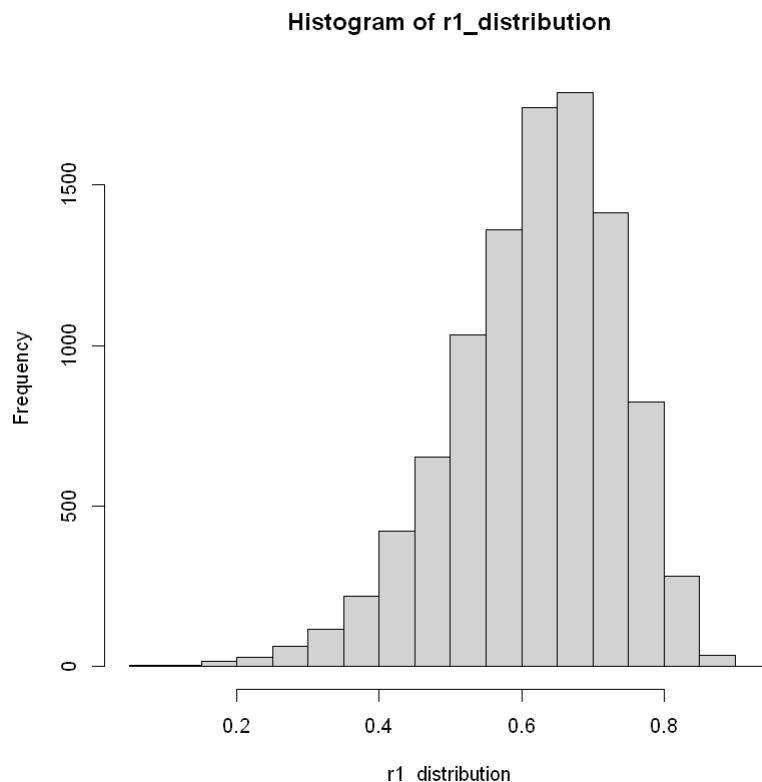
方程 (6.1.5) 给出的大样本方差与你的样本分布的方差接近程度如何？

Solution.

```
In [ ]: # 设置重复次数  
num_simulations <- 10000  
  
# 存储每次模拟的 r1 和 r5  
r1_distribution <- numeric(num_simulations)  
r5_distribution <- numeric(num_simulations)  
  
for (i in 1:num_simulations) {  
  # 模拟 AR(1) 序列  
  set.seed(i+10) # 设置随机种子, 保证结果可重复  
  x_sim = arima.sim(model = list(ar = phi), n = n)  
  
  # 计算样本自相关系数  
  acf_sim <- acf(x_sim, plot = FALSE)  
  r1_distribution[i] <- acf_sim$acf[2]  
  r5_distribution[i] <- acf_sim$acf[6]  
}
```

```
# 计算样本分布的均值  
mean_r1 <- mean(r1_distribution)  
mean_r5 <- mean(r5_distribution)
```

In [73]: `hist(r1_distribution)`

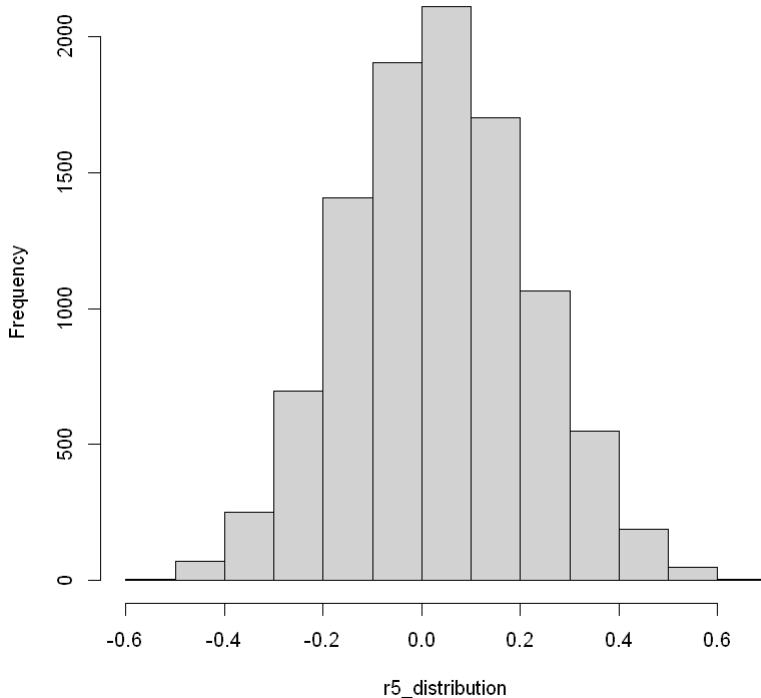


In [74]: `summary(r1_distribution)`

```
Min. 1st Qu. Median Mean 3rd Qu. Max.  
0.05942 0.54793 0.63094 0.61815 0.70134 0.90860
```

In [75]: `hist(r5_distribution)`

Histogram of r5\_distribution



In [76]: `summary(r5_distribution)`

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
-0.55854	-0.09543	0.03152	0.03336	0.15937	0.62919

分布情况展示如上。

$r_1$ 的理论值 $\rho_1 = 0.7$ 大致在3/4分位数附近， $r_5$ 的理论值 $\rho_5 = 0.16807$ 已经超过3/4分位数。

由此可见，参数估计的精度随着样本的不同而变化。但是估计值的方差与方程（6.1.5）给出的方差很接近，具体如下：

In [61]: `variance_r1_sample = var(r1_distribution)`  
`variance_r1`

0.0137251755097358

In [62]: `variance_r5_sample = var(r5_distribution)`  
`variance_r5`

0.0318308478291206

In [63]: `variance_r1_sample - Var_r1`

0.0029426968843183

In [64]: `variance_r5_sample - Var_r5`

-0.0268002088519372

在大样本的渐进性质下，方差的估计是愈发趋向准确的。

## 6.21

模拟  $n=60$ ,  $\theta=0.5$  的 MA(1) 时间序列。

```
In [23]: n = 60  
theta = 0.5  
  
set.seed(61)  
x <- arima.sim(model = list(ma = -theta), n = n) # 生成 MA(1) 时间序列
```

(a)

计算该模型 1 阶滞后处的理论自相关系数。

Solution.

```
In [24]: -theta/(1+theta^2)
```

-0.4

理论自相关系数为-0.4

(b)

计算 1 阶滞后处样本自相关系数，并将其与理论自相关系数进行比较。用图表 6-2 量化这个比较。

Solution.

```
In [25]: acf_values <- acf(x, plot = FALSE) # 计算自相关系数  
r1 = acf_values['1']
```

```
In [26]: r1
```

Autocorrelations of series 'x', by lag

1  
-0.412

从中我们得到，样本自相关系数  $r_1 = -0.412$  与理论自相关系数  $\rho_1 = -0.4$  接近。

```
In [22]: 0.79 / n^0.5
```

0.101988561450129

根据表 (6.2),  $\sqrt{Var(r_1)} = \frac{0.79}{\sqrt{n}} \approx 0.10199$

发现估计值与真实值之间的差异也就是一个大样本标准差。估计的准确度是较高的。

(c)

使用新的模拟重复 (b)。描述在相同条件下，估计的精度如何随所选样本的不同而变化。

Solution.

```
In [29]: set.seed(996)
x <- arima.sim(model = list(ma = -theta), n = n) # 生成 MA(1) 时间序列

acf_values <- acf(x, plot = FALSE) # 计算自相关系数
r1 = acf_values['1']
r1
```

```
Autocorrelations of series 'x', by lag
```

```
1
-0.295
```

这次的估计偏误明显会大一点。不同的样本会导致**样本自相关系数有所不同**，但总体上方差大致相同，因此 $r$ 作为 $\rho$ 的估计量应当在特定的理论值周围波动。

## (d)

如果软件允许，重复模拟序列并多次计算  $r_1$ ，并且构建  $r_1$  的样本分布。描述估计的精度如何随着相同条件下所选择样本的不同而变化。图表 6-2 给出的大样本方差与你的样本分布的方差接近程度如何？

Solution.

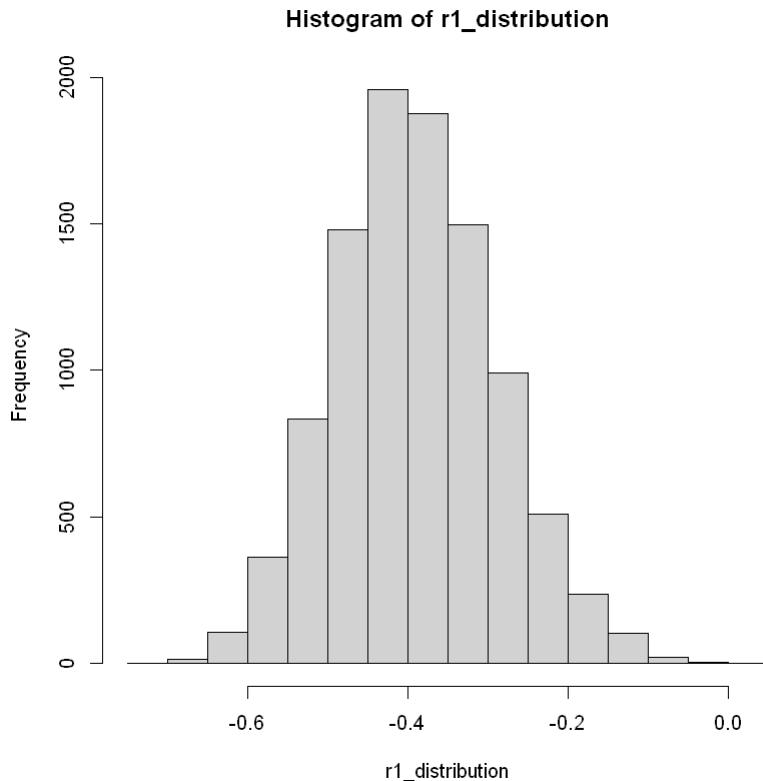
```
In [37]: # 设置重复次数
num_simulations <- 10000

# 存储每次模拟的 r1
r1_distribution <- numeric(num_simulations)

for (i in 1:num_simulations) {
  # 模拟 AR(1) 序列
  set.seed(i+10077) # 设置随机种子，保证结果可重复
  x_sim = arima.sim(model = list(ma = -theta), n = n)

  # 计算样本自相关系数
  acf_sim <- acf(x_sim, plot = FALSE)
  r1_distribution[i] <- acf_sim$acf[2]
}
```

```
In [38]: hist(r1_distribution)
```



```
In [39]: summary(r1_distribution)
```

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
-0.73653	-0.45925	-0.39374	-0.38948	-0.32294	0.01198

```
In [40]: mean(r1_distribution)
```

-0.389483034641693

由此可见，大样本下估计值 $r_1$ 是比较接近理论值 $\rho_1$ 的。

在此次模拟中，1/4分位数和3/4分位数分别位于-0.45和-0.32附近，说明估计值在理论值-0.4附近波动，大部分估计的精度是比较高的。从分布直方图来看，分布并不厚为，偏误较大的估计值是很少的。

```
In [42]: var(r1_distribution)**0.5
```

0.10029475248109

样本方差开根 $\sqrt{Var(r_1)} \approx 0.10029$ ，与表（6.2）给出的0.10199相当接近。

- ① 若AR(1)的模型改为 $Y_t = \phi_0 + \phi_1 Y_{t-1} + e_t$ , 其中 $e_t \sim N(0, \sigma_e^2)$ , 求 $(\phi_0, \phi_1, \sigma_e^2)$ 的无条件似然函数和极大似然估计需要满足的方程。

Sol.

$$L(\phi_0, \phi_1, \sigma_e^2) = f(Y_1, Y_2, \dots, Y_n) = f(Y_2, \dots, Y_n | Y_1) f(Y_1)$$

$$\left\{ \begin{array}{l} Y_2 - \phi_0 = \phi_1 Y_1 + e_2 \\ Y_3 - \phi_0 = \phi_1 Y_2 + e_3 \\ \vdots \\ Y_n - \phi_0 = \phi_1 Y_{n-1} + e_n \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} e_2 = Y_2 - \phi_0 - \phi_1 Y_1 \\ \vdots \\ e_n = Y_n - \phi_0 - \phi_1 Y_{n-1} \end{array} \right.$$

$$\begin{aligned} \therefore f(Y_2, \dots, Y_n | Y_1) &= \Pr(e_2 = Y_2 - \phi_0 - \phi_1 Y_1, \dots, e_n = Y_n - \phi_0 - \phi_1 Y_{n-1}) \\ &= [(2\pi)^{-\frac{n-1}{2}} \sigma_e^{-n+1}] \exp \left\{ -\sum_{i=2}^n \frac{(Y_i - \phi_0 - \phi_1 Y_{i-1})^2}{2\sigma_e^2} \right\} \\ &= (2\pi)^{-\frac{n-1}{2}} \sigma_e^{-n+1} \exp \left\{ -\frac{1}{2\sigma_e^2} \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1})^2 \right\} \end{aligned}$$

另, 若认为该AR(1)过程平稳, 则必有 $Y_i$ 可被 $\{e_i\}$ 线性表示.

$$P(Y_i \text{服从正态分布. } \text{Var}(Y_i) = \frac{\sigma_e^2}{1-\phi_1^2})$$

$$P(Y_1) = (2\pi)^{-\frac{1}{2}} \sigma_e^{-1} \exp \left\{ -\frac{(1-\phi_1^2)}{2\sigma_e^2} (Y_1 - \mu)^2 \right\}$$

$$\begin{aligned} \therefore L(\phi_0, \phi_1, \sigma_e^2) &= P(Y_2, \dots, Y_n | Y_1) P(Y_1) \\ &= (2\pi)^{-\frac{n}{2}} \sigma_e^{-n} \exp \left\{ -\frac{1}{2\sigma_e^2} \left[ \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1})^2 + (1-\phi_1^2)(Y_1 - \mu)^2 \right] \right\} \end{aligned}$$

其中在平稳性假定之下,  $E(Y_t) = \phi_0 + \phi_1 E(Y_{t-1}) + E(e_t) = \phi_0 + \phi_1 \mu \Rightarrow \mu = \frac{\phi_0}{1-\phi_1}$

$$\text{则 } L(\phi_0, \phi_1, \sigma_e^2) = (2\pi)^{-\frac{n}{2}} \sigma_e^{-n} \exp \left\{ -\frac{1}{2\sigma_e^2} \left[ \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1})^2 + (1-\phi_1^2)(Y_1 - \frac{\phi_0}{1-\phi_1})^2 \right] \right\}$$

为无条件似然函数.

$$\textcircled{1} \quad \frac{\partial L}{\partial \phi_0} = 0 \Leftrightarrow \frac{\partial \log L}{\partial \phi_0} = 0 \Leftrightarrow \frac{\partial}{\partial \phi_0} \left[ \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1})^2 + (1-\phi_1^2)(Y_1 - \frac{\phi_0}{1-\phi_1})^2 \right] = 0$$

$$\Leftrightarrow -2 \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1}) - \frac{2}{1-\phi_1} (1-\phi_1^2) (Y_1 - \frac{\phi_0}{1-\phi_1}) = 0$$

$$\Leftrightarrow -2 \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1}) - 2(1+\phi_1) (Y_1 - \frac{\phi_0}{1-\phi_1}) = 0$$

$$\Leftrightarrow \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1}) + (1+\phi_1) (Y_1 - \frac{\phi_0}{1-\phi_1}) = 0$$

$$\begin{aligned} \textcircled{2} \quad \frac{\partial L}{\partial \phi_1} = 0 &\Leftrightarrow \frac{\partial \log L}{\partial \phi_1} \Leftrightarrow \frac{\partial}{\partial \phi_1} \left[ \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1})^2 + (1-\phi_1^2) (Y_1 - \frac{\phi_0}{1-\phi_1})^2 \right] = 0 \\ &\Leftrightarrow \left[ \sum_{i=2}^n (-2Y_{i-1})(Y_i - \phi_0 - \phi_1 Y_{i-1}) - 2\phi_1(Y_1 - \frac{\phi_0}{1-\phi_1})^2 + 2(1-\phi_1^2)(Y_1 - \frac{\phi_0}{1-\phi_1}) \frac{\partial(Y_1 - \frac{\phi_0}{1-\phi_1})}{\partial \phi_1} \right] \\ &= \left[ \sum_{i=2}^n (-2Y_{i-1})(Y_i - \phi_0 - \phi_1 Y_{i-1}) - 2\phi_1(Y_1 - \frac{\phi_0}{1-\phi_1})^2 + 2(1-\phi_1^2)(Y_1 - \frac{\phi_0}{1-\phi_1})(\phi_0(1-\phi_1)^{-2}) \right] \\ &= -2 \left[ \sum_{i=2}^n Y_{i-1}(Y_i - \phi_0 - \phi_1 Y_{i-1}) + \phi_1(Y_1 - \frac{\phi_0}{1-\phi_1})^2 - 2 \frac{1+\phi_1}{1-\phi_1} (Y_1 - \frac{\phi_0}{1-\phi_1}) \right] = 0 \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad \frac{\partial L}{\partial \sigma_e^2} = 0 &\Leftrightarrow \frac{\partial \log L}{\partial \sigma_e^2} \Leftrightarrow \frac{\partial}{\partial \sigma_e^2} (\sigma_e^2)^{-\frac{n}{2}} + \sigma_e^{-n} \frac{\partial}{\partial \sigma_e^2} (-\frac{1}{2}(\sigma_e^2)^{-1}) [\dots] = 0 \\ &\Leftrightarrow \sigma_e^{-n} \left\{ \frac{\partial}{\partial \sigma_e^2} (\sigma_e^2)^{-\frac{n}{2}} + \frac{\partial}{\partial \sigma_e^2} (-\frac{1}{2}(\sigma_e^2)^{-1}) [\dots] \right\} = 0 \\ &\Leftrightarrow -\frac{n}{2}(\sigma_e^2)^{-\frac{n}{2}-1} + (-\frac{1}{2})(-\sigma_e^2)^{-2} \left[ \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1})^2 + (1-\phi_1^2)(Y_1 - \frac{\phi_0}{1-\phi_1})^2 \right] = 0 \\ &\Leftrightarrow -n \sigma_e^{-n+2} + \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1})^2 + (1-\phi_1^2)(Y_1 - \frac{\phi_0}{1-\phi_1})^2 = 0 \end{aligned}$$

综上，应满足方程组

$$\begin{cases} \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1}) + (1+\phi_1)(Y_1 - \frac{\phi_0}{1-\phi_1}) = 0 \\ \sum_{i=2}^n Y_{i-1}(Y_i - \phi_0 - \phi_1 Y_{i-1}) + \phi_1(Y_1 - \frac{\phi_0}{1-\phi_1})^2 - 2 \frac{1+\phi_1}{1-\phi_1} (Y_1 - \frac{\phi_0}{1-\phi_1}) = 0 \\ -n \sigma_e^{-n+2} + \sum_{i=2}^n (Y_i - \phi_0 - \phi_1 Y_{i-1})^2 + (1-\phi_1^2)(Y_1 - \frac{\phi_0}{1-\phi_1})^2 = 0 \end{cases}$$

② 对AR(2)模型：

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + e_t, \quad e_t \sim N(0, \sigma_e^2)$$

► 求给定( $Y_1, Y_2$ )时的条件似然函数  $L(\phi_1, \phi_2, \mu, \sigma_e^2 | Y_1, Y_2)$ ，并求条件似然估计  $\phi_1, \phi_2, \mu, \sigma_e^2$  需要满足的方程

► 求无条件似然函数  $L(\phi_1, \phi_2, \mu, \sigma_e^2)$  (提示:  $(Y_1, Y_2)^\top$  服从二维正态分布)，并求无条件似然估计  $\phi_1, \phi_2, \mu, \sigma_e^2$  需要满足的方程

Sol.

1).

$$\begin{cases} Y_3 + \mu = \phi_1(Y_2 - \mu) + \phi_2(Y_1 - \mu) + e_3 \\ Y_4 + \mu = \phi_1(Y_3 - \mu) + \phi_2(Y_2 - \mu) + e_4 \\ \vdots \\ Y_n + \mu = \phi_1(Y_{n-1} - \mu) + \phi_2(Y_{n-2} - \mu) + e_n \end{cases}$$

$$\therefore L(\phi_1, \phi_2, \mu, \sigma_e^2 | Y_1, Y_2) = \Pr(e_i = Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu), i=3, 4, \dots, n)$$

$$= \prod_{i=3}^n \Pr(e_i = Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu))$$

$$\text{其中 } \Pr_{\phi_1, \phi_2, \mu}(\dots) = (2\pi)^{\frac{n}{2}} \sigma_e^{-n} \exp \left\{ -\frac{1}{2\sigma_e^2} [Y_1 - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2 \right\}$$

$$\therefore L(\phi_1, \phi_2, \mu | \sigma_e^2 | Y_1, Y_2) = \prod_{i=3}^n \Pr_{\phi_1, \phi_2, \mu}(\dots)$$

$$= (2\pi)^{\frac{1-n}{2}} \sigma_e^{2-n} \exp \left\{ -\frac{1}{2\sigma_e^2} \sum_{i=3}^n [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2 \right\}, \text{ 即为条件似然函数.}$$

使  $L$  达到最大值  $\Leftrightarrow \log L$  达到最大值

$$\textcircled{1} \quad \frac{\partial}{\partial \phi_1} \log L = 0 \Leftrightarrow \frac{\partial}{\partial \phi_1} \sum_{i=3}^n [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2 = 0$$

$$\Leftrightarrow 2 \sum_{i=3}^n (-1(Y_{i-1} - \mu)) [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)] = 0$$

$$\Leftrightarrow \sum_{i=3}^n (Y_{i-1} - \mu) [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)] = 0$$

$$\textcircled{2} \quad \frac{\partial}{\partial \phi_2} \log L = 0 \Leftrightarrow \sum_{i=3}^n (Y_{i-2} - \mu) [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)] = 0$$

$$\textcircled{3} \quad \frac{\partial}{\partial \mu} \log L = 0 \Leftrightarrow \frac{\partial}{\partial \mu} \sum_{i=3}^n [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2 = 0$$

$$\Leftrightarrow \sum_{i=3}^n 2[-1 + \phi_1 + \phi_2] [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)] = 0$$

$$\Leftrightarrow \sum_{i=3}^n (Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)) = 0$$

$$\textcircled{4} \quad \frac{\partial}{\partial \sigma_e^2} \log L = \frac{\partial}{\partial \sigma_e^2} \left\{ \left(1 - \frac{n}{2}\right) \log 2\pi + \left(1 - \frac{n}{2}\right) \log \sigma_e^2 - \frac{1}{2} (\sigma_e^2)^{-1} \sum_{i=3}^n [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2 \right\} = 0$$

$$\Leftrightarrow \left(1 - \frac{n}{2}\right) (\sigma_e^2)^{-1} + \frac{1}{2} (\sigma_e^2)^{-2} \sum_{i=3}^n [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2 = 0$$

综上,  $\phi_1, \phi_2, \mu, \sigma_e^2$  需满足

$$\begin{cases} \sum_{i=3}^n (Y_{i-1} - \mu) [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)] = 0 \\ \sum_{i=3}^n (Y_{i-2} - \mu) [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)] = 0 \\ \sum_{i=3}^n (Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)) = 0 \\ (n-2) \sigma_e^2 - \sum_{i=3}^n [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2 = 0 \end{cases}$$

2).

$$\text{Cov}(Y_1 - \mu, Y_2 - \mu) = \text{Cov}(Y_1, \mu + \phi_1(Y_1 - \mu) + \phi_2(Y_2 - \mu) + e_2) = \phi_1 \tau_0 + \phi_2 \tau_1 = \rho_1$$

$$\text{Corr}(Y_1 - \mu, Y_2 - \mu) = \phi_1 + \phi_2 \frac{\tau_1}{\tau_0} = \phi_1 + \phi_2 \frac{(\phi_1 \tau_0)}{(\phi_1 \tau_0 + \phi_2 \tau_1)} = \phi_1 \left(1 + \frac{\phi_2}{\phi_1}\right) = \frac{\phi_1}{1 + \phi_2} = \rho_1 \quad \text{错误}$$

$$\Pr(Y_1 = Y_1, Y_2 = Y_2) = (2\pi)^{-1} (\sigma_e^2)^{-1} (1 - \rho_1^2)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2(1 - \rho_1^2)} \left( \frac{(Y_1 - \mu)^2}{\tau_0} - \frac{2\rho_1 (Y_1 - \mu)(Y_2 - \mu)}{\tau_0} + \frac{(Y_2 - \mu)^2}{\tau_0} \right) \right\}$$

$$L(\phi_1, \phi_2, \mu | \sigma_e^2) = L(\phi_1, \phi_2, \mu | \sigma_e^2 | Y_1, Y_2) P_{\sigma_e^2}(Y_1, Y_2), \text{ 可得 } L(\phi_1, \phi_2, \mu | \sigma_e^2) \text{ 可整理为下式:}$$

$$L(\phi_1, \phi_2, \mu | \sigma_e^2) = (2\pi)^{-\frac{n}{2}} \sigma_e^{-n} \exp \left\{ -\frac{1}{2\sigma_e^2} \sum_{i=3}^n [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2 - \frac{1}{2(1 - \rho_1^2)\tau_0} [(Y_1 - \mu)^2 - 2\rho_1 (Y_1 - \mu)(Y_2 - \mu) + (Y_2 - \mu)^2] \right\}$$

若达到极值,  $\log L = \log L(\phi_1, \phi_2, \mu, \sigma_e^2 | Y_1, Y_2) + \log p_2(Y_1, Y_2)$

$$\begin{aligned} \log L(\phi_1, \phi_2, \mu, \sigma_e^2) &= -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \sigma_e^2 - \frac{1}{2\sigma_e^2} \sum_{i=3}^n [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2 \\ &\quad - \frac{1}{2} \log(1 - p_1^2) - \frac{1}{2(1 - p_1^2)\sigma_0} [(Y_1 - \mu)^2 - 2p_1(Y_1 - \mu)(Y_2 - \mu) + (Y_2 - \mu)^2] \end{aligned}$$

$$\text{其中 } p_1 = \frac{\phi_1}{1 - \phi_2}$$

$$\text{最大化 } \log L(\dots) \text{ 得到 } \frac{\partial \log L}{\partial \sigma_e^2} = 0 \Rightarrow \hat{\sigma}_e^2 = \frac{1}{n} \sum_{i=3}^n [Y_i - \mu - \phi_1(Y_{i-1} - \mu) - \phi_2(Y_{i-2} - \mu)]^2$$

同样地,  $\hat{\phi}_1, \hat{\phi}_2, \hat{\mu}$  满足  $\frac{\partial \log L}{\partial \phi_1} = 0, \frac{\partial \log L}{\partial \phi_2} = 0, \frac{\partial \log L}{\partial \mu} = 0$

- ③ 若AR(2)模型改为  $Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$ , 其中  $e_t \sim N(0, \sigma_e^2)$ , 求无条件似然函数  $L(\phi_0, \phi_1, \phi_2, \sigma_e^2)$  以及极大似然估计需要满足的方程。

Sol.

$$\begin{aligned} f(Y_3, Y_4, \dots, Y_n | Y_1, Y_2) &= L(\phi_0, \phi_1, \phi_2, \sigma_e^2 | Y_1, Y_2) = \prod_{i=3}^n \Pr(e_i = Y_i - \phi_0 - \phi_1 Y_{i-1} - \phi_2 Y_{i-2}) \\ &= \prod_{i=3}^n (2\pi)^{-\frac{1}{2}} \sigma_e^{-1} \exp \left\{ -\frac{1}{2\sigma_e^2} (Y_i - \phi_0 - \phi_1 Y_{i-1} - \phi_2 Y_{i-2})^2 \right\} \\ &= (2\pi)^{\frac{n-1}{2}} \sigma_e^{-(n-2)} \exp \left\{ -\frac{1}{2\sigma_e^2} \sum_{i=3}^n (Y_i - \phi_0 - \phi_1 Y_{i-1} - \phi_2 Y_{i-2})^2 \right\} \end{aligned}$$

$$\text{另有: } f(Y_1, Y_2) = (2\pi)^{-1} (\sigma_e^2)^{-1} (1 - p_1^2)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2(1 - p_1^2)} \left( \frac{(Y_1 - \mu)^2}{\sigma_0^2} - \frac{2p_1(Y_1 - \mu)(Y_2 - \mu)}{\sigma_0} + \frac{(Y_2 - \mu)^2}{\sigma_0^2} \right) \right\}$$

$$\begin{aligned} \Rightarrow L(\phi_0, \phi_1, \phi_2, \sigma_e^2) &= (2\pi)^{\frac{n}{2}} (\sigma_e^2)^{-\frac{n}{2}} (1 - p_1^2)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2(1 - p_1^2)} \left( \frac{(Y_1 - \mu)^2}{\sigma_0^2} - \frac{2p_1(Y_1 - \mu)(Y_2 - \mu)}{\sigma_0} + \frac{(Y_2 - \mu)^2}{\sigma_0^2} \right) \right. \\ &\quad \left. - \frac{1}{2\sigma_e^2} \sum_{i=3}^n (Y_i - \phi_0 - \phi_1 Y_{i-1} - \phi_2 Y_{i-2})^2 \right\} \end{aligned}$$

$$\text{Cov}(Y_t, Y_{t-1}) = \sigma_1 = \text{Cov}(\phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, Y_{t-1}) = \phi_1 \sigma_0 + \phi_2 \sigma_1 \Rightarrow \sigma_1 = \frac{\phi_1 \sigma_0}{1 - \phi_2}$$

$$\Rightarrow p_1 = \frac{\text{Cov}(Y_t, Y_{t-1})}{\sigma_0} = \phi_1 / (1 - \phi_2)$$

$$E(Y_t) = E(\phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t) \Rightarrow \mu = \phi_0 / (1 - \phi_1 - \phi_2)$$

$$\begin{aligned} \log L(\phi_0, \phi_1, \phi_2, \sigma_e^2) &= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \sigma_e^2 - \frac{1}{2} \log(1 - p_1^2) - \frac{1}{2\sigma_e^2} \sum_{i=3}^n (Y_i - \phi_0 - \phi_1 Y_{i-1} - \phi_2 Y_{i-2})^2 \\ &\quad - \frac{1}{2(1 - p_1^2)} \left( \frac{(Y_1 - \mu)^2}{\sigma_0^2} - \frac{2p_1(Y_1 - \mu)(Y_2 - \mu)}{\sigma_0} + \frac{(Y_2 - \mu)^2}{\sigma_0^2} \right) \end{aligned}$$

$$\text{其中, } p_1 = \phi_1 / (1 - \phi_2), \mu = \phi_0 / (1 - \phi_1 - \phi_2)$$

若要令 L 达到最大值， $\partial \log L(\dots) / \partial \phi_e^2 = 0 \Rightarrow \hat{\sigma}_e^2 = \frac{1}{n} \sum_{i=3}^n (Y_i - \phi_0 - \phi_1 Y_{i-1} - \phi_2 Y_{i-2})^2$

另对于  $\hat{\phi}_0, \hat{\phi}_1, \hat{\phi}_2$ ，有  $\partial \log L(\dots) / \partial \phi_0 = 0, \partial \log L(\dots) / \partial \phi_1 = 0, \partial \log L(\dots) / \partial \phi_2 = 0$

P122-123 7.17, 7.24 P135 8.6, 8.11 见 R 代码实现

## 7.17

模拟一个  $\phi = 0.7, \theta = 0.4, n = 72$  的 ARMA(1,1) 序列。

```
In [1]: # 设置参数  
phi = 0.7  
theta = 0.4  
n = 72  
  
# 生成序列  
set.seed(114514)  
series=arima.sim(n=n,list(ar=phi,ma=-theta))
```

(a)

求  $\phi$  和  $\theta$  的矩估计。

Solution.

对于 ARMA(1,1) 模型，其矩估计是通过样本自相关函数来估计模型参数。我们利用样本自相关系数与理论自相关系数之间的关系，解方程得到参数估计值。

对于 ARMA(1,1)，有：

$$\hat{\phi} = r_2/r_1 \quad (1)$$

$$r_1 = \frac{(1 - \hat{\theta}\hat{\phi})(\hat{\phi} - \hat{\theta})}{1 - 2\hat{\theta}\hat{\phi} + \hat{\theta}^2} \quad (2)$$

这里需要先后解这两个方程来估计  $\hat{\phi}$  和  $\hat{\theta}$ ：

```
In [2]: acf(series, lag.max = 5, plot = FALSE)
```

```
Autocorrelations of series 'series', by lag
```

0	1	2	3	4	5
1.000	0.287	0.220	0.213	0.093	0.263

```
In [3]: r1 = acf(series, lag.max = 5, plot = FALSE)$acf[[2]]  
r2 = acf(series, lag.max = 5, plot = FALSE)$acf[[3]]
```

```
In [4]: r1
```

0.287291239592992

基于  $r_1, r_2$ ，我们可以解方程得到  $\hat{\phi}$  和  $\hat{\theta}$  的估计值。

```
In [5]: hat_phi = r2/r1
```

```
In [6]: hat_phi
```

0.767211185178692

$\hat{\phi}$  如上展示,

带入 $r_1$  的公式中, 解方程得到  $\hat{\theta}$  的估计值为复数解, 因此该模型不适合使用矩估计对参数 $\theta$ 作估计。

### (b)

求  $\phi$  和  $\theta$  的条件最小二乘估计, 并与 (a) 比较。

Solution.

```
In [7]: arima(series, order=c(1,0,1), method='CSS')
```

```
Call:  
arima(x = series, order = c(1, 0, 1), method = "CSS")  
  
Coefficients:  
      ar1      ma1  intercept  
      0.7226 -0.4719     0.1295  
s.e.  0.2070  0.2493     0.2376
```

$\sigma^2$  estimated as 1.057: part log likelihood = -104.15

可知,  $\hat{\phi}_{LS} = 0.7226$ ,  $\hat{\theta}_{LS} = 0.4719$

$\hat{\phi}_{LS}$ 与 (a) 中差别不算太大。

### (c)

求  $\phi$  和  $\theta$  的极大似然估计, 并与 (a) 和 (b) 比较。

Solution.

```
In [8]: arima(series, order=c(1,0,1), method='ML')
```

```
Call:  
arima(x = series, order = c(1, 0, 1), method = "ML")  
  
Coefficients:  
      ar1      ma1  intercept  
      0.8262 -0.6124     0.0511  
s.e.  0.1588  0.2092     0.2611
```

$\sigma^2$  estimated as 1.023: log likelihood = -103.09, aic = 214.18

可知,  $\hat{\phi}_{ML} = 0.8262$ ,  $\hat{\theta}_{ML} = 0.6124$

相较而言, 这次模拟的偏误算是较大了

### (d)

在选取相同参数和样本规模的情况下, 使用新的模拟序列重复 (a)、(b) 和 (c), 并将本次与前次的模拟结果进行比较。

Solution.

```
In [24]: # 生成序列  
set.seed(99993)  
series=arima.sim(n=n,list(ar=phi,ma=-theta))
```

```
In [25]: acf(series, lag.max = 5, plot = FALSE)
```

Autocorrelations of series 'series', by lag

lag	0	1	2	3	4	5
1.000	1.000	0.355	0.240	0.084	0.130	-0.044

```
In [26]: r1 = acf(series, lag.max = 5, plot = FALSE)$acf[[2]]  
r2 = acf(series, lag.max = 5, plot = FALSE)$acf[[3]]
```

```
In [27]: hat_phi = r2/r1
```

```
In [29]: # 矩估计  
hat_phi
```

0.677423780427269

```
In [31]: # 最小二乘  
arima(series,order=c(1,0,1),method='CSS')
```

Call:

```
arima(x = series, order = c(1, 0, 1), method = "CSS")
```

Coefficients:

	ar1	ma1	intercept
ar1	0.3432	0.0202	0.2393
s.e.	0.2831	0.2685	0.2052

sigma^2 estimated as 1.252: part log likelihood = -110.25

```
In [33]: # 极大似然估计  
arima(series,order=c(1,0,1),method='ML')
```

Call:

```
arima(x = series, order = c(1, 0, 1), method = "ML")
```

Coefficients:

	ar1	ma1	intercept
ar1	0.5938	-0.2655	0.1959
s.e.	0.2292	0.2736	0.2359

sigma^2 estimated as 1.257: log likelihood = -110.49, aic = 228.97

与前一次模拟相比，这一次通过矩估计得到的 $\hat{\phi}$ 与理论值相差更小，总的来说两次模拟都还算准确。

然而，通过最小二乘和极大似然估计得到的 $\hat{\phi}$ 与理论值并不那么接近，这有点奇怪。

原因可能是多方面的。首先，ARMA(1,1)模型是一个简单的模型，其参数估计可能受到样本规模和噪声的影响。其次，ARMA(1,1)模型的参数估计可能受到模型假设的限制，例如，模型的阶数和参数的约束条件。最后，ARMA(1,1)模型的参数估计可能受到数据集的特性和分布的影响。

## 7.24

模拟一个  $\phi_1 = 1.0$ ,  $\phi_2 = -0.6$ ,  $n = 48$  的 AR(2) 序列, 但误差项从有 8 个自由度的卡方分布中抽样得到。

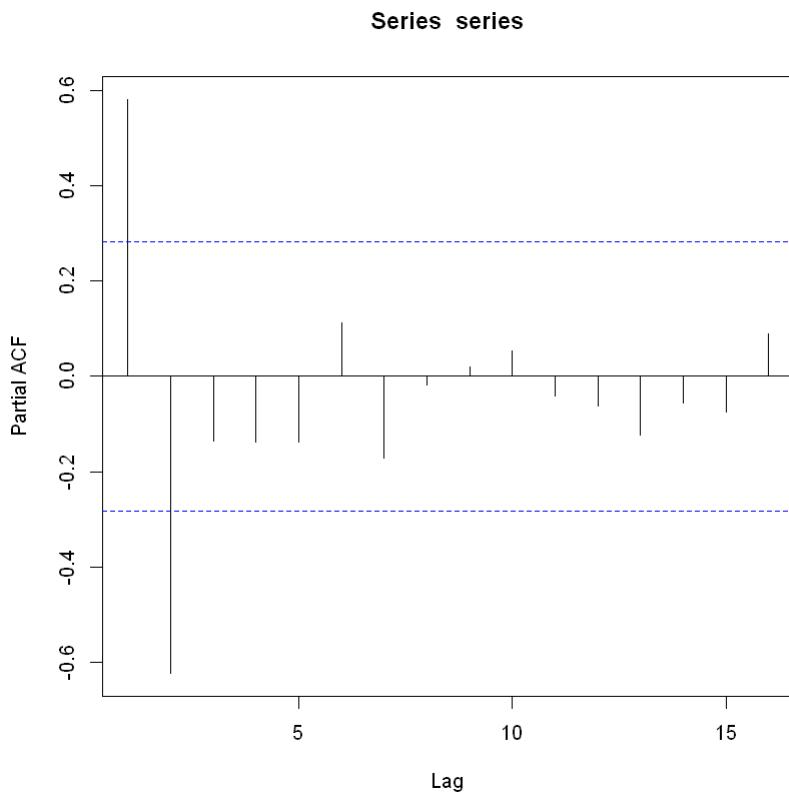
```
In [1]: phi1=1.0  
phi2=-0.6  
n=48  
  
set.seed(114514)  
series=arima.sim(n=n,list(ar=c(phi1,-phi2)),innov=rchisq(n,8))
```

(a)

绘出该序列的样本 PACF, 建议建立 AR(2) 模型吗?

Solution.

```
In [2]: pacf(series)
```



由图可知, PACF 在 2 阶之后截尾, 因此建议建立 AR(2) 模型。

(b)

估计  $\phi_1$  和  $\phi_2$ , 并解释结果。

Solution.

```
In [ ]: arima(series,order=c(2,0,0))
```

```

Call:
arima(x = series, order = c(2, 0, 0))

Coefficients:
      ar1      ar2  intercept
    0.9703 -0.6412   12.9186
  s.e.  0.1091  0.1061   0.8365

sigma^2 estimated as 14.8:  log likelihood = -133.52,  aic = 275.05
 $\hat{\phi}_1 = 0.970$ ,  $\hat{\phi}_2 = -0.641$ .

```

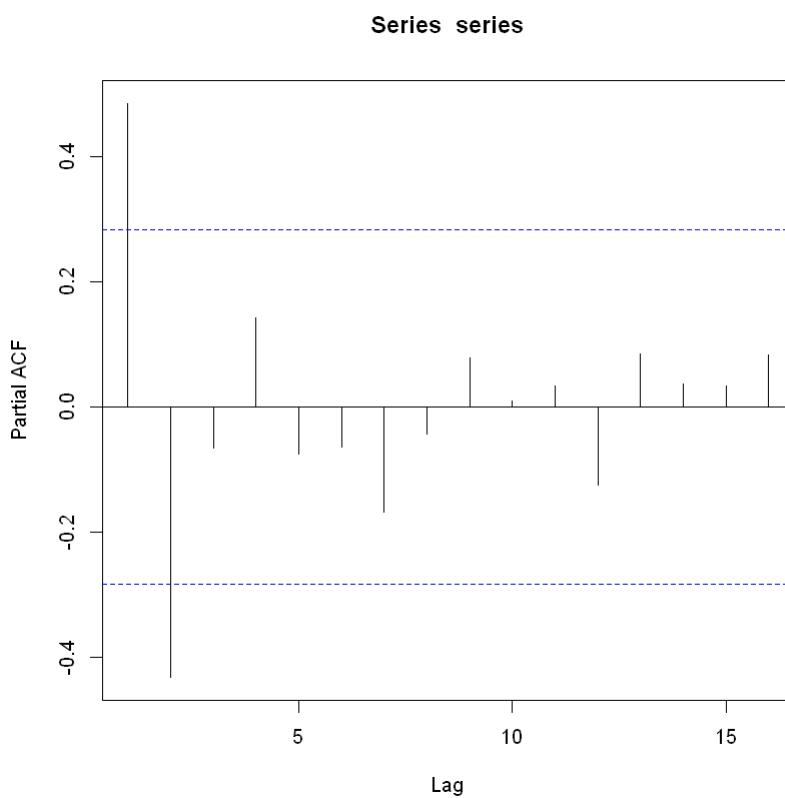
估计值和真实参数较为接近。

(c)

在相同条件下，使用新的模拟序列重复 (a) 和 (b)。

Solution.

```
In [6]: set.seed(10086)
series=arima.sim(n=n,list(ar=c(phi1,-phi2)),innov=rchisq(n,8))
pacf(series)
```



依然符合2阶截尾。

```
In [7]: arima(series,order=c(2,0,0))
```

```
Call:  
arima(x = series, order = c(2, 0, 0))  
  
Coefficients:  
ar1      ar2    intercept  
0.7626  -0.4924   11.7763  
s.e.  0.1302   0.1305   0.6350  
  
sigma^2 estimated as 10.11:  log likelihood = -124.06,  aic = 256.12
```

可见，尽管仍然暗示着AR(2)模型，但估计的参数和真实参数有较大偏差。

## 8.6

模拟  $n = 48$ ,  $\phi_1 = 1.5$ ,  $\phi_2 = -0.75$  的 AR(2) 模型。

```
In [23]: # 设置随机种子以确保结果可重复
set.seed(10086)

# (a) 模拟 AR(2) 模型并拟合
n <- 48
phi1 <- 1.5
phi2 <- -0.75

# 生成 AR(2) 模型的数据
ar2_data <- arima.sim(n = n,
                      model = list(order = c(2, 0, 0),
                                   ar = c(phi1, phi2)))
```

(a)

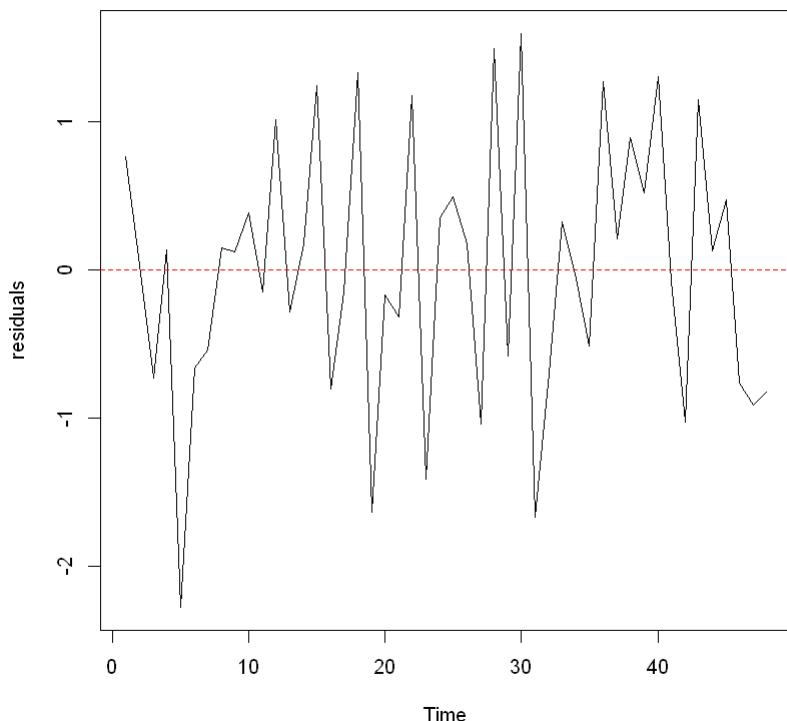
拟合正确识别的 AR(2) 模型，观察残差的时间序列图，该图支持 AR(2) 的识别吗？

Solution.

```
In [24]: # 拟合 AR(2) 模型
ar2_model <- arima(ar2_data,
                     order = c(2, 0, 0),
                     include.mean = TRUE)
residuals <- residuals(ar2_model)
```

```
In [25]: # 绘制残差的时间序列图
plot(residuals, type = "l", main = "残差的时间序列图")
abline(h = 0, col = "red", lty = 2)
```

残差的时间序列图



看不出明显的趋势或周期，这较为支持AR(2)的识别。

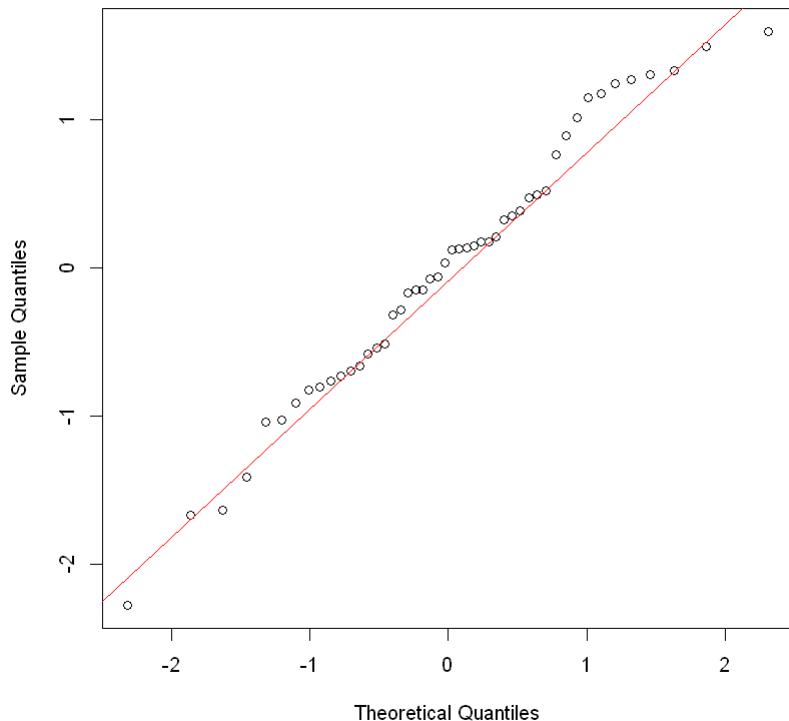
(b)

展示标准残差的正态分位数-分位数图，该图支持 AR(2) 的识别吗？

Solution.

```
In [26]: qqnorm(residuals, main = "残差的正态分位数-分位数图")
qqline(residuals, col = "red")
```

残差的正态分位数-分位数图



可以看出，大部分的残差点都支持正态分布的假设。但存在为数不少的异常点，明显偏离正态分布分位数直线。

总的来讲，还是较为支持AR(2)的识别。

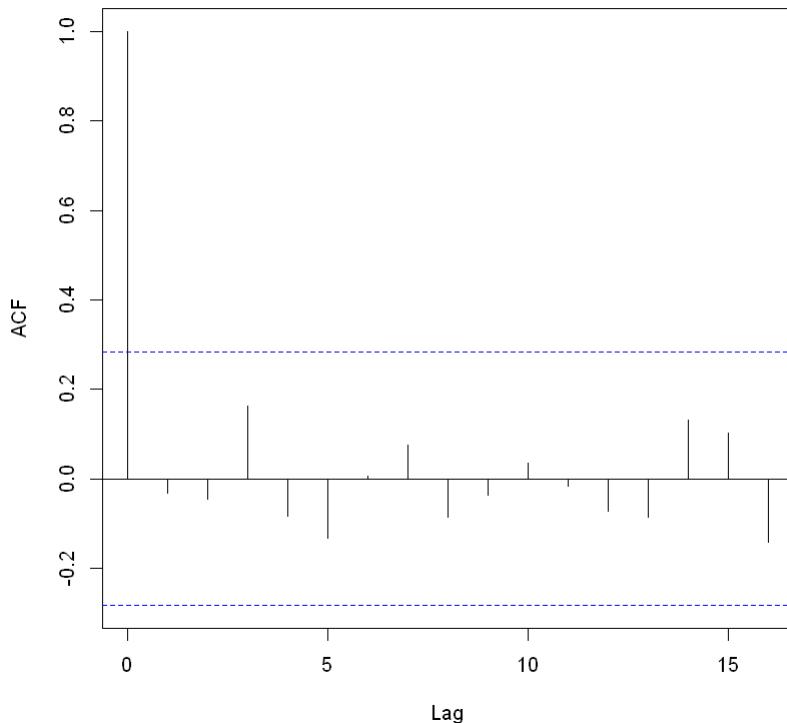
(c)

展示残差的样本 ACF，该图支持 AR(2) 的识别吗？

Solution.

```
In [19]: acf(residuals, main = "残差的样本 ACF")
```

残差的样本 ACF



残差的样本 ACF 在滞后项上都在置信区间内（蓝色阴影区域），说明残差没有显著的自相关性，很支持 AR(2) 的识别。

(d)

计算 Ljung-Box 统计量，求和至  $K = 12$ ，该统计量支持 AR(2) 的识别吗？

Solution.

```
In [27]: library(tseries)
K = 12
ljung_box_test = Box.test(residuals,
                           lag = K,
                           type = "Ljung-Box",
                           fitdf = 2)
ljung_box_test
```

Box-Ljung test

```
data: residuals
X-squared = 16.608, df = 10, p-value = 0.08351
```

Ljung-Box 统计量的值为 **16.608**。

观察到， $p$  值为 0.08351，大于 0.05，无法拒绝原假设，说明残差序列可以被认为是白噪声序列，支持 AR(2) 的识别。

## 8.11

图表 6-31 建议对石油价格的对数差分序列识别 AR(1) 模型或者 AR(4) 模型（文件名为 oil.price）。

```
In [3]: library(TSA)
data("oil.price")

Warning message:
"package 'TSA' was built under R version 4.3.3"

Attaching package: 'TSA'

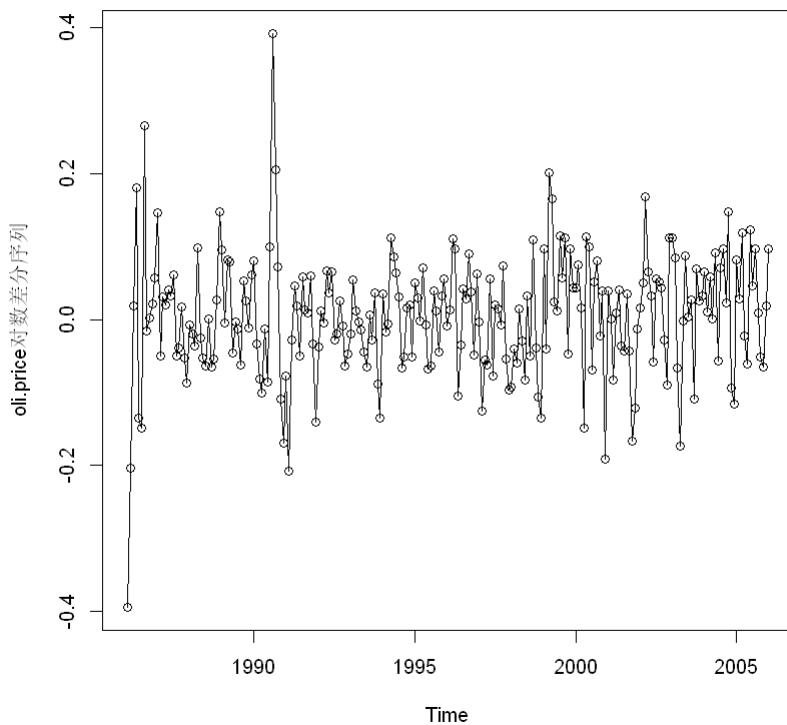
The following objects are masked from 'package:stats':
  acf, arima

The following object is masked from 'package:utils':
  tar
```

```
In [4]: summary(oil.price)

   Min. 1st Qu. Median    Mean 3rd Qu.    Max.
 11.35   18.01   20.25   24.09   27.31   65.59
```

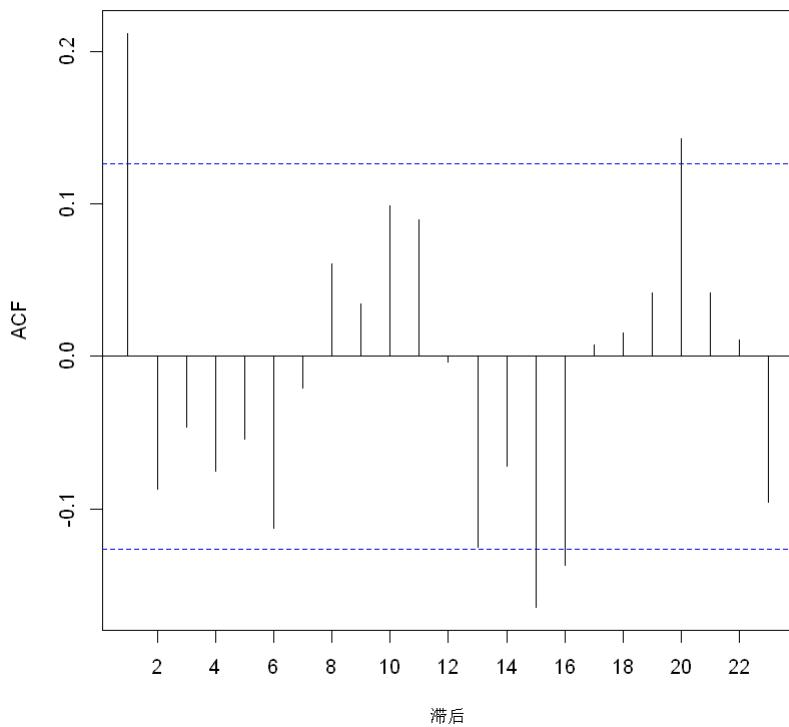
```
In [5]: plot(diff(log(oil.price)),
      type='o',
      ylab='oli.price对数差分序列')
```



图表 6-32

```
In [6]: acf(as.vector(diff(log(oil.price))),
      xaxp=c(0,22,11),
      xlab='滞后',
      main = "石油价格对数差分的样本ACF")
```

石油价格对数差分的样本ACF



(a)

用极大似然估计法估计两个模型，用本章讨论的诊断检验对结果进行比较。

Solution.

```
In [7]: # 拟合模型
model.1 = arima(log(oil.price), order = c(1, 1, 0), method = 'ML')
model.2 = arima(log(oil.price), order = c(4, 1, 0), method = 'ML')
```

```
In [8]: model.1

Call:
arima(x = log(oil.price), order = c(1, 1, 0), method = "ML")

Coefficients:
      ar1
      0.2364
  s.e.  0.0660

sigma^2 estimated as 0.006787:  log likelihood = 258.55,  aic = -515.11
∇¹Y_t = 0.2364Y_{t-1} + e_t
```

```
In [9]: model.2

Call:
arima(x = log(oil.price), order = c(4, 1, 0), method = "ML")

Coefficients:
      ar1      ar2      ar3      ar4
      0.2673  -0.1550  0.0238  -0.0970
  s.e.  0.0669  0.0691  0.0691  0.0681

sigma^2 estimated as 0.006603:  log likelihood = 261.82,  aic = -515.64
∇¹Y_t = 0.2673Y_{t-1} - 0.1550Y_{t-2} + 0.0238Y_{t-3} - 0.0970Y_{t-4} + e_t
```

可见，ARIMA(4,1,0)拟合得到的AIC稍好些。

另外，两个模型得到的系数估计差别并不大。在ar1这一项上差的不大。同时对于ARIMA(4,1,0)来说，ar3、ar4的系数估计都非常小。

(b)

图表 6-32 建议为对数差分序列设定 MA(1) 模型，用极大似然法估计该模型并用本章讨论的检验进行诊断。

Solution.

```
In [10]: model.3 = arima(log(oil.price), order = c(0,1,1), method = 'ML')

model.3
```

```
Call:  
arima(x = log(oil.price), order = c(0, 1, 1), method = "ML")
```

```
Coefficients:  
ma1  
0.2956  
s.e. 0.0693
```

```
sigma^2 estimated as 0.006689: log likelihood = 260.29, aic = -518.58
```

$$\nabla^1 Y_t = e_t - 0.2956e_{t-1}$$

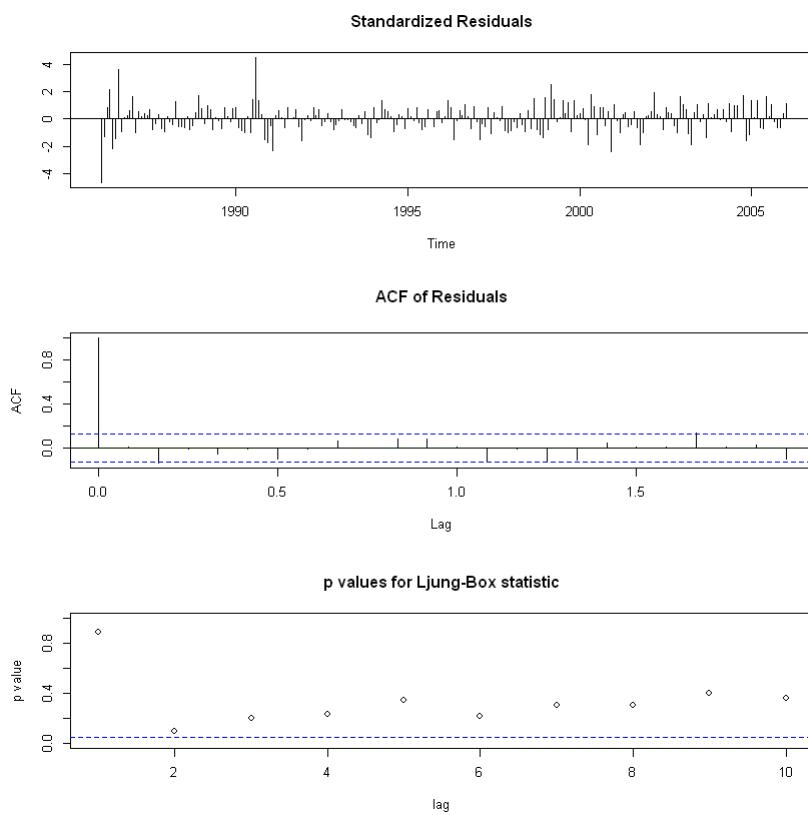
值得注意的是，这个拟合结果的AIC值比前面的都要出色。

(c)

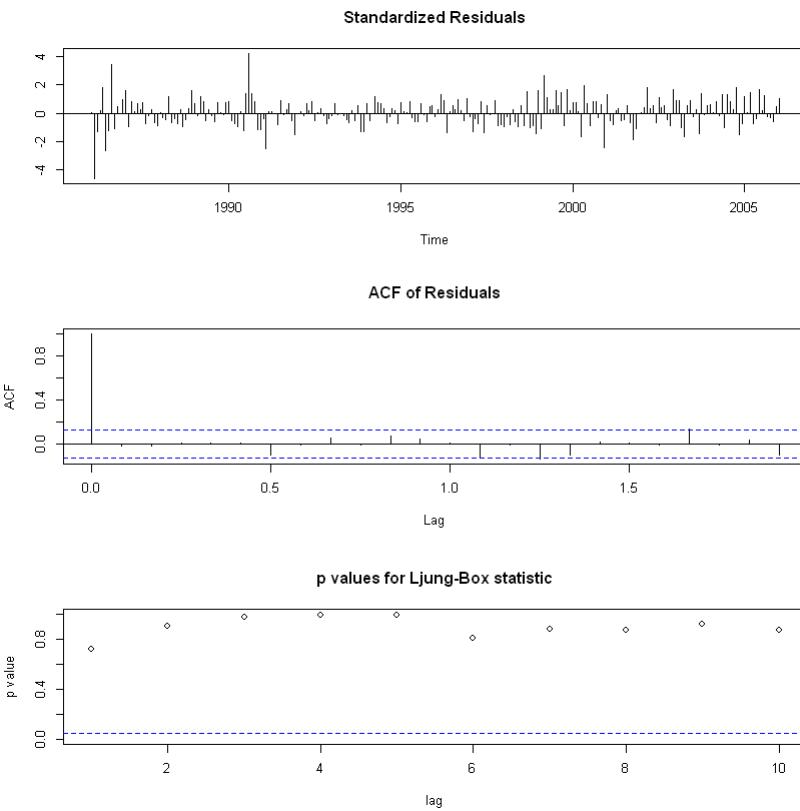
基于 (a)、(b) 部分给出的结果，你更偏好 AR(1)、AR(4) 和 MA(1) 三个模型中的哪一个？

Solution.

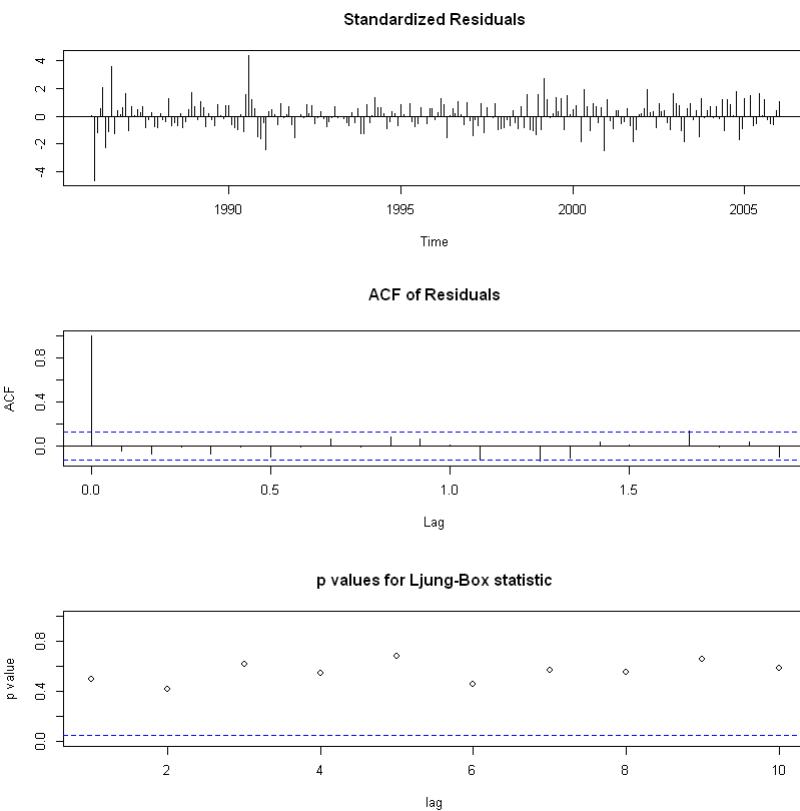
```
In [13]: tsdiag(model.1)
```



```
In [14]: tsdiag(model.2, main='Model 3')
```



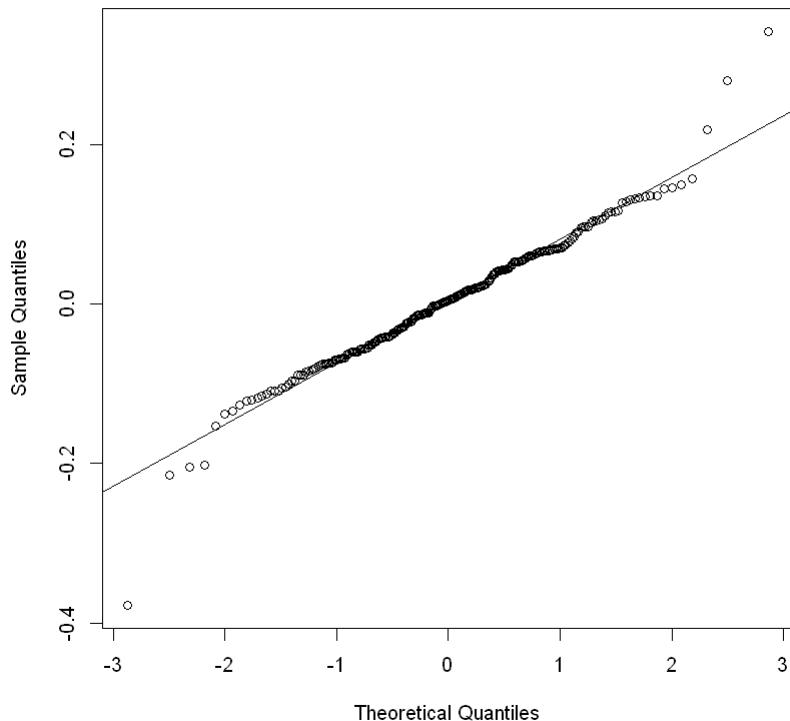
```
In [15]: tsdiag(model.3, main='Model 3')
```



通过诊断可以发现，AR(4)和MA(1)的结果都没有较大的问题。但AR(1)在Ljung-Box检验中表现得比较差。

```
In [22]: qqnorm(residuals(model.2), main='QQnorm for Model.2')
qqline(residuals(model.2))
```

QQnorm for Model.2



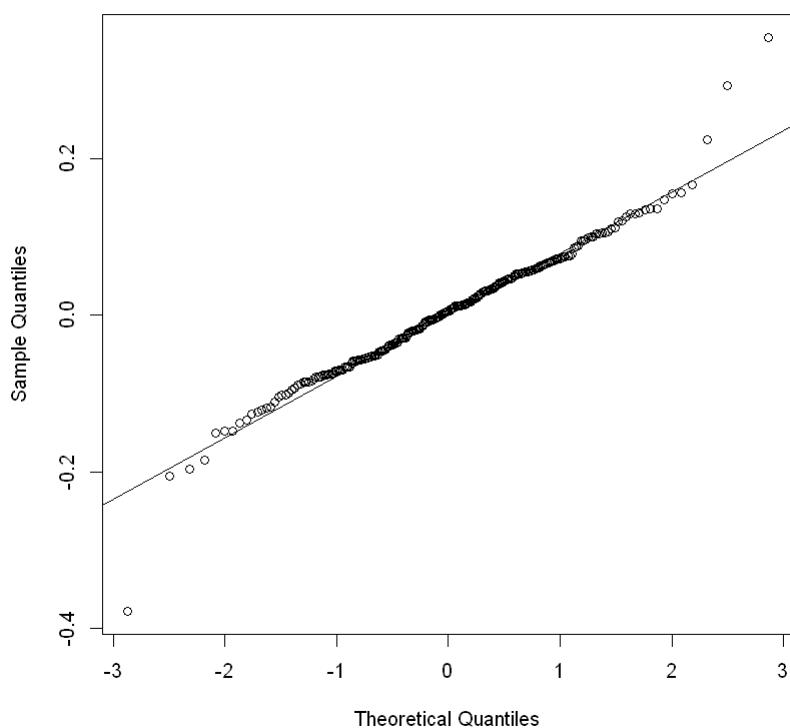
```
In [23]: shapiro.test(residuals(model.2))
```

Shapiro-Wilk normality test

```
data: residuals(model.2)
W = 0.96957, p-value = 4.959e-05
```

```
In [ ]: qqnorm(residuals(model.3), main='QQnorm for Model.3')
qqline(residuals(model.3))
```

QQnorm for Model.3



```
In [21]: shapiro.test(residuals(model.3))
```

```
Shapiro-Wilk normality test
```

```
data: residuals(model.3)  
W = 0.96883, p-value = 3.937e-05
```

进一步确认，AR(4)和MA(1)的残差分布都较为接近正态分布。

在这样的条件下，根据AIC值，我更倾向于用IMA(1,1)去拟合模型。