First order PDEs:

A first order PDE in two independent variables *x*, *y* and the dependent variable *z* can be written in the form

$$F(x, y, z, p, q) = 0. \rightarrow (1)$$

Here $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$.

First order PDEs arise in many applications, such as

• Transport of material in a fluid flow • Propagation of wave-fronts in optics.

Classification of first-order PDEs:

- If (1) is of the form

  $$a(x, y)\frac{\partial z}{\partial x} + b(x, y)\frac{\partial z}{\partial y} = c(x, y)z + d(x, y)$$

  then it is called *linear* first-order PDE.

  Example: $(x^2 - 2y)p + (e^x + 3x)q = x + 3z.$

- If (1) has the form

  $$a(x, y)\frac{\partial z}{\partial x} + b(x, y)\frac{\partial z}{\partial y} = c(x, y, z)$$

  then it is called *semi-linear* because it is linear in the leading (highest-order) terms $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. However, it need not be linear in *z*.

  Example: $(x^2 - 3y^3)p + (\cos x - y)q = x + 3z^2$, or, $= x + \tan z$.

- If (7) has the form

  $$a(x, y, z)\frac{\partial z}{\partial x} + b(x, y, z)\frac{\partial z}{\partial y} = c(x, y, z)$$

  then it is called *quasi-linear* PDE. Here the function *F* in (1) is linear in the derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ with the coefficients *a*, *b* and *c* depending on the independent variables *x* and *y* as well as on the unknown *z*.

  Example: $(3x - y + z^2)\, p + (2x - y + z)q = 6z^3 - 3y + 6.$

- If the function *F* in (1) is not linear in the derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, then (1) is said to be *non-linear*.

  Examples: $z^2p^2 - (2x + y)q^{\frac{1}{2}} = x^2 + z^2 + 2y,$

  $(x + y)\tan p - 2x\, q = z^3 - \log y, \quad pq - z = 2x.$

§*Lagrange's method for solving quasi-linear first order PDE*

Let

$$u\,(x, y, z) \;=\; a \quad \text{and} \quad v\,(x, y, z) \;=\; b$$

(where u and v are two definite function of x, y, z and a, b are arbitrary constants), be two independent solution (integrals) of the ODE's

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\,. \quad \rightarrow \quad (1)$$

Then the general solution of the linear PDE

$$Pp + Qq = R \longrightarrow (2)$$

is given by

$$\varphi(u, v) \;=\; 0. \longrightarrow (3)$$

Here P, Q, R are functions of x, y, z and φ is an arbitrary function. $\left(p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}\right)$

Definition: Equation (2) is known as Lagrange's equation.

Equations (1) are known as auxiliary equation corresponding to Lagrange's eqn. (2)

Proof. We have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = d\,\lambda \quad \rightarrow (1)$$

Then, $\quad dx \;=\; P\,d\lambda, dy \;=\; Q\,d\lambda, dz \;=\; Rd\lambda \rightarrow (4)$

Taking differential on both sides of $u\,(x, y, z) \;=\; a$ we get

$$du = 0 \Rightarrow \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz = 0$$

$$\text{Or,} \quad u_x\,Pd\lambda \;+\; u_y Q d\lambda + u_z R d\lambda \;=\; 0$$

$$\text{Or,} \quad Pu_x \;+\; Qu_y + Ru_z \;=\; 0 \longrightarrow (5)$$

Similarly from $v\,(x, y, z) \;=\; b$ we get

$$v_x\,Pd\lambda \;+\; v_y Q d\lambda + v_z R d\lambda \;=\; 0$$

$$\text{Or,} \quad Pv_x \;+\; Qv_y + Rv_z \;=\; 0 \longrightarrow (6)$$

Differentiating $u\,(x, y, z) \;=\; a$ with respect to x we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\,.\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; u_x + pu_z = 0$$

$$\Rightarrow u_x = -pu_z . \qquad \rightarrow (7)$$

Differentiating $u\,(x, y, z) = a$ with respect to *x* we get

$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \Rightarrow u_y + qu_z = 0$$

$$\Rightarrow u_y = -qu_z . \qquad \rightarrow (8)$$

Substituting (7) and (8) in (5) we obtain

$$P(-pu_z) + Q(-qu_z) + Ru_z = 0$$

Or, $\qquad -Pp - Qq + R = 0$

Or, $\qquad Pp + Qq = R.$

Thus $u(x, y, z) = a$ is a solution of (2) similarly we can show that $v(x, y, z) = b$ is a solution of (2).

From (5) and (6) we have,

$$Pu_x + Qu_y + Ru_z = 0 \rightarrow (5)$$

$$Pv_x + Qv_y + Rv_z = 0 \rightarrow (6)$$

Thus,

$$\frac{P}{u_y v_z - u_z v_y} = \frac{Q}{u_z v_x - u_x v_z} = \frac{R}{u_x v_y - u_y v_x} = \frac{1}{\mu}$$

$$\therefore u_y v_z - u_z v_y = \mu P, \quad u_z v_x - u_x v_z = \mu Q, \quad u_x v_y - u_y v_x = \mu R. \rightarrow (9)$$

We have

$$\varphi(u, v) = 0 . \qquad \rightarrow (3)$$

Differentiating $\varphi = 0$ with respect to *x* and *y* in turn we get,

$$\frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

Or, $\varphi_u \left(\frac{\partial u}{\partial u} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \varphi_v \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$

$$\Rightarrow \varphi_u(u_x + pu_z) + \varphi_v(v_x + pv_z) = 0 \rightarrow (10)$$

and,

$$\varphi_u(u_y + qu_z) + \varphi_v(v_y + qv_z) = 0 \rightarrow (11)$$

Thus, we get some solutions for φ_u and φ_v , if

$$\begin{vmatrix} u_x + pu_z & v_x + pv_z \\ u_y + qu_z & v_y + qv_z \end{vmatrix} = 0$$

Or, $(u_x + pu_z)(v_y + qv_z) - (u_y + qu_z)(v_x + pv_z) = 0$

Or, $(u_x v_y - u_y v_x) + p(u_z v_y - u_y v_z) + q\,(u_x v_z - u_z v_x) = 0.$

By virtue of equations (9), the above equation reduces to

$$\mu R - \mu Pp - \mu Qq = 0$$

Or, $\quad$ Pp + Qq = R.

Methods for solving auxiliary equation:

Method 1:

Example 1:

Solve $xyp - x^2q + yz = 0$

Solution:

Let us write the given PDE as $xyp - x^2q = -yz$.

Comparing the above PDE with the general form $P\,p + Q\,q = R$, we find,

auxiliary equations as

$$\frac{dx}{xy} = \frac{dy}{-x^2} = \frac{dz}{-yz} \qquad \left(\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\right)$$

Taking 1st and 2nd ratios we get,

$$\frac{dy}{dx} = \frac{-x^2}{xy} = -\frac{x}{y}$$

Or, $\quad ydy + xdx = 0$

Or, $\quad d\left(\frac{x^2 + y^2}{2}\right) = 0$

Integrating the above equation we get, $\left(\frac{x^2 + y^2}{2}\right) = \text{const.}$

Or $x^2 + y^2 = a$(say)

Taking 1st and 3rd ratios we get,

$$\frac{dx}{xy} = \frac{dz}{-yz} \Rightarrow \frac{dz}{z} + \frac{dx}{x} = 0$$

Or, $d\,(\log z) + d\,(\log x) = 0 \Rightarrow d\,(\log zx) = 0$

Integrating,

$$\log zx = \text{const.} = c_1 \quad \Rightarrow zx = e^{c_1} = b \text{ (say)}$$

Thus the two solutions are,

$$u(x, y, z) = x^2 + y^2 = a \quad \text{and} \quad v(x, y, z) = zx = b$$

Thus, the general solution to the given PDE is,

$\varphi(u, v) = 0 \qquad$ where φ is an arbitrary function.

Example 2:

Solve $xp + 2yq = (x + y)\,z$.

Solution:

Comparing the given PDE with the general form $P\,p\;+\;Q\,q\;=\;R$, we find,

auxiliary equations as

$$\frac{dx}{x} = \frac{dy}{2y} = \frac{dz}{(x+y)z} \qquad \left(\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\right)$$

Taking 1st and 2nd ratios we get,

$$\frac{dx}{x} = \frac{dy}{2y}$$

Integrating, $\quad \log\left(\frac{x^2}{y}\right) = a$.

Taking 2nd and 3rd ratios we get,

$$\frac{dy}{2y} = \frac{dz}{(x+y)z} \Rightarrow \frac{2dz}{z} = \frac{x+y}{y}\,dy = \frac{x}{y}\,dy + dy$$

Using 1st and 2nd ratios the RHS of the above equation becomes $2dx + dy$. Thus,

$$2\frac{dz}{z} = 2dx + dy$$

Integrating, $\quad 2\log z \;=\; 2x \;+\; y \;+\; a.$

Or, $\log z^2 - 2x - y \;=\; a.$

Thus, the general solution to the given PDE is,

$$\left(\frac{x^2}{y}, \log(z^2) - 2x - y\right) = 0$$ where φ is an arbitrary function.

Method 2:

Solve one differential equation, use that solution in finding the second solution.

Example 1:

Solve $\;xp \;+\; (y \;+\; x^2)\,q \;=\; y \;+\; z$.

Solution:

The auxiliary equations are, $\quad \frac{dx}{x} = \frac{dy}{y+x^2} = \frac{dz}{y+z}$.

Taking 1st and 2nd ratios we get,

$$\frac{dy}{dx} = \frac{y+x^2}{x} = \frac{y}{x} + x$$

Or, $\frac{dy}{dx} - \frac{y}{x} = x$ (linear 1st order eqn)

This is of the form $\quad \frac{dy}{dx} + P(x).y = Q(x)$.

Here the Integrating Factor (I.F.) $= e^{\int P(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x}$.

$$\frac{d}{dx}\left(y\, e^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

Thus, $$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{x}{x} = 1$$

$$\text{Or, } d\left(\frac{x}{y}\right) = dx \Rightarrow \frac{y}{x} = x + a$$

$$\text{Or, } y = x^2 + ax$$

Taking 1st and 3rd ratios we get, $\frac{dx}{x} = \frac{dz}{y+z}$

$$\text{Or, } \frac{dz}{dx} = \frac{y}{x} + \frac{z}{x} = x + a + \frac{z}{x}$$

$$\text{Or, } \frac{dz}{dx} - \frac{z}{x} = x + a$$

Proceeding as above we get, $\frac{d}{dx}\left(\frac{z}{x}\right) = 1 + \frac{a}{x}$

Integrating, $\frac{z}{x} = x + a \log x + L$

$$= x + \left(\frac{y}{x} - x\right)\log x + L$$

$$\text{Or, } \frac{z}{x} - x - \left(\frac{y}{x} - x\right)\log x = L\ .$$

Thus, the general solution to the given PDE is,

$$\varphi\left(\frac{y}{x} - x,\ \frac{z}{x} - x - \left(\frac{y}{x} - x\right)\log x\right) = 0$$ where φ is an arbitrary function.

Example 2:

Solve $xp - yq = xy$.

Solution:

The auxiliary equations are, $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{xy}$.

Taking 1st and 2nd ratios we get, $xy = a$.

Thus from 1st and 3rd ratios we get, $\frac{dx}{x} = \frac{dz}{a}$.

Integrating, $a \log x = z + L \Rightarrow xy \log x - z = L$.

Similarly from 2nd and 3rd ratios we will get, $\log y + z = L$.

Thus, the general solution to the given PDE is,

either $\varphi\,(xy,\ xy\log x - z) = 0$, or, $\varphi\,(xy,\ xy \log y + z) = 0$ where φ is an arbitrary function.

Method 3:

From the auxiliary equations we have,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = d\lambda \text{ (say) .}$$

$$\Rightarrow dx \ = pd\,\lambda, \ dy \ = \lambda d\lambda, \ dz \ = \ Rd\lambda.$$

Let us take a triplet (f, g, h) of suitable numbers or functions of x, y, z. Then,

$$\frac{fdx+gdy+hdz}{fP+gQ+hR} = \frac{f.Pd\lambda+g.Qd\lambda+h.Rd\lambda}{fP + gQ + hR} = d\lambda\ .$$

Thus, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{fdx+gdy+hdz}{fP+gQ+hR}$.

Example 1:

Solve $(y + z)p = (z + x)q = x + y.$

Solution:

The auxiliary equations are, $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$.

Taking $(f, g, h) = (1,1,1)$ we find each ratio $= \frac{1.dx+1.dy+1.dz}{1.(y+z)+1.(z+x)+1.(x+y)} = \frac{dx+dy+ dz}{2\,(x+y+z)}$.

Taking $(f, g, h) = (1, -1,0)$ we find each ratio $= \frac{1.dx+(-1).dy+0.dz}{1.(y+z)+(-1).(z+x)+0.(x+y)}$

$$= \frac{dx-dy}{y+z-z-x} = \frac{dx-dy}{y-x}$$

Taking $(f, g, h) = (1,0, -1)$ we find each ratio $= \frac{1.dx+0.dy+(-1).dz}{1.(y+z)+0.(z+x)+(-1).(x+y)} = \frac{dx-dz}{z-x}$.

Thus we have,

$$\frac{dx+dy+ dz}{2(x+y+z)} = \frac{dx-dy}{y-x} = \frac{dx-dz}{z-x}\ .$$

Taking 1st and 2nd ratios and integrating we get,

$$\frac{1}{2}\ln\,(x + y + z) = -\ln\,(x - y) + a \quad \Rightarrow (x + y + z)(x - y)^2 = a_1\ .$$

Similarly from 1st and 3rd ratios we get,

$$(x + y + z)(z - x)^2 = b_1\ .$$

Thus, the general solution to the given PDE is,

$\varphi\ [(x + y + z)(x - y)^2, (x + y + z)(z - x)^2] = 0$ where φ is an arbitrary function.

Example 2:

Solve $(x-y)p+(x+y)q=\frac{x^2+y^2}{z}$.

(left as exercise).

Method 4:

It is a variety of the 3rd method. Here we choose *(f, g, h)* such that $fP+gQ+hR=\ 0$ and at the same time $fdx+gdy+hdy=du$. *Then,*

$$fdx+gdy+hdy=d\lambda.0=0\ .$$

Then, $du=0$ or, $=a$.

Example 1:

Find the solution of the equation $z(x+y)p+z(x-y)q=x^2+y^2$.

Solution:

The auxiliary equations are, $\frac{dx}{z(x+y)}=\frac{dy}{z(x-y)}=\frac{dz}{x^2+y^2}$

Choose (f,g,h) such that $fP+\ gQ+hR=0.$

i,e $fz(x+y)\ +\ gz(x-y)\ +\ h(x^2+y^2)=0$

If we choose $(f.g,h)$ as $(y,x,-z)$, then

$$fP+gQ+hR=z\ (xy+y^2+x^2-xy)-z(x^2+y^2)=0$$

So, each ratio $=d\lambda=\frac{fdx+gdy+hdz}{fP+gQ+hR}$

Or, $\frac{ydx+xdy-zdz}{0}=d\lambda\ \Rightarrow\ d(xy)-d\left(\frac{z^2}{2}\right)=0\ \Rightarrow\ d\left(xy-\frac{z^2}{2}\right)=0$.

Integrating, $-\frac{z^2}{2}=a$; a arbitrary constant.

Next let us take $(f,g,h)=(-x,y,z)$. Then,

$$fP+gQ+hR=-xz(x+y)+yz(x-y)+(x^2+y^2)$$

$$=z(-x^2-xy+xy-y^2+x^2+y^2)=0.$$

So, each ratio $=d\lambda=\frac{fdx+gdy+hdz}{fP+gQ+hR}$

Or, $\frac{-xdx+ydy+zdz}{0}=d\lambda\ \Rightarrow\ =-xdx+ydy+zdz=0\ \Rightarrow\ d\left(-\frac{1}{2}x^2+\frac{1}{2}y^2+\frac{1}{2}z^2\right)=0$

Integrating, $-\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=$ constant.

Or, $x^2-y^2-z^2=b.$

So, the general solution of the given equation

$$\varphi\left(xy - \frac{z^2}{2}, x^2 - y^2 - z^2\right) = 0 \text{ where } \varphi \text{ is an arbitrary function.}$$

Example 2:

Find the solution of the equation $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.

Solution:

The auxiliary equations are,

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}.$$

Choose (f, g, h) such that $fP + gQ + hR = 0$.

Taking $(f, g, h) = (x, y, z)$ we find, $x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2) = 0$

Also, $xdx + ydx + zdx = 0 \Rightarrow \frac{1}{2}d(x^2 + y^2 + z^2) = 0$.

Integrating, $x^2 + y^2 + z^2 = \text{const} = a$.

Taking $(f, g, h) = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ we find,

$\frac{1}{x}.x(z^2 - y^2) + \frac{1}{y}.y(x^2 - z^2) + \frac{1}{z}.z(y^2 - x^2) = z^2 - y^2 + x^2 - z^2 + y^2 - x^2 = 0$.

Also, $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \Rightarrow d(\log x + \log y + \log y) = 0 \Rightarrow d(\log xyz) = 0$

Integrating, $\log xyz = \text{const} = d \Rightarrow xyz = \text{const} = e^d = b$.

So, the general solution of the given equation

$$\varphi(x^2 + y^2 + z^2,\ xyz) = 0$$

where φ is an arbitrary function.

Linear Partial Differential Equations of Order One with n Independent Variables:

Let $x_1, x_2, x_3, \ldots, x_n$ be n independent variables and z be a dependent function depending on $x_1, x_2, x_3, \ldots, x_n$. Also, let $p_i = \frac{\partial z}{\partial x_i}; i = 1,2, \ldots, n$. Then, the general linear partial differential equation of order one with n independent variables is given by

$$P_1p_1 + P_2p_2 + P_3p_3 + \ldots + P_np_n = R \ldots(1)$$

where $P_1, P_2, P_3, \ldots, P_n$ are the functions of $x_1, x_2, x_3, \ldots, x_n$ and R is a function of $x_1, x_2, x_3, \ldots, x_n$ and z. Thus, $R = R\ (x_1, x_2 \ldots \ldots \ldots x_n, z)$ (not containing any $p_i's$)

The above partial differential equation (1) can be solved by the generalization of Lagrange's method. Therefore, the system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \ldots\ldots\ldots = \frac{dx_n}{P_n} = \frac{dz}{R} \qquad \ldots(2)$$

Let $u_1(x_1, x_2, x_3, \ldots, x_n, z) = c_1$, $u_2(x_1, x_2, x_3, \ldots, x_n, z) = c_2$, $u_3(x_1, x_2, x_3, \ldots, x_n, z) = c_3, \ldots$, $u_n(x_1, x_2, x_3, \ldots, x_n, z) = c_n$ be any n independent integrals of (2).

Then, the general solution of (1) is given by

$$\phi(u_1, u_2, u_3, \ldots\ldots, u_n) = 0 \qquad \ldots(3)$$

SOLVED EXAMPLES

Example 1. Solve $x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 = x_1x_2x_3$.

Solution. The given equation is a linear partial differential equation with three independent variables x_1, x_2 and x_3 and z as a dependent function depending on x_1, x_2 and x_3.

Comparing the given partial differential equation with $P_1p_1 + P_2p_2 + P_3p_3 + \ldots + P_np_n = R$, we have

$$P_1 = x_2x_3, \quad P_2 = x_3x_1, \quad P_3 = x_1x_2 \text{ and } R = x_1x_2x_3$$

∴ The system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{p_1} = \frac{dx_2}{p_2} = \frac{dx_3}{p_3} = \frac{dz}{R} \text{ or } \frac{dx_1}{x_2x_3} = \frac{dx_2}{x_3x_1} = \frac{dx_3}{x_1x_2} = \frac{dz}{x_1x_2x_3} \qquad \ldots(1)$$

Taking the first and the second fractions of (1), we get

$$x_1dx_1 = x_2dx_2 \text{ so that } \frac{x_1^2}{2} = \frac{x_2^2}{2} + \frac{C_1}{2}$$

which gives $x_1^2 - x_2^2 = c_1$ or $u_1 \equiv x_1^2 - x_2^2 = c_1$ ...(2)

Taking the second and the third fractions of (1), we get

$$x_2 dx_2 = x_3 dx_3 \quad \text{so that} \quad \frac{x_2^2}{2} = \frac{x_3^2}{2} + \frac{c_2}{2}$$

which give $\quad x_2^2 - x_3^2 = c_2 \qquad$ or $\qquad u_2 \equiv x_2^2 - x_3^2 = c_2 \qquad \ldots(3)$

Again, taking the third and fourth fractions of (1), we get

$$dz = x_3 dx_3 \quad \text{so that} \quad z = \frac{x_3^2}{2} + \frac{c_3}{2}$$

which gives $\ 2z - x_3^2 = c_3 \qquad$ or $\qquad u_3 \equiv 2z - x_3^2 = c_3 \qquad \ldots(4)$

Finally, from (2), (3) and (4), the general solution of the given partial differential equation is

$$\phi(x_1^2 - x_2^2, x_2^2 - x_3^2, 2z - x_3^2) = 0 \qquad \ldots(5)$$

Example 2. Solve $P_1 p_1 + P_2 p_2 + P_3 p_3 = az + \frac{x_1 x_2}{x_3}$.

Solution: The given equation is a linear partial differential equation with three independent variables x_1, x_2, x_3 and z as a dependent function depending on x_1, x_2 and x_3.

Comparing the given partial differential equation with $P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots = R$, we have

$$P_1 = x_1, P_2 = x_2,\ P_3 = x_3 \text{ and } R = az + \frac{x_1 x_2}{x_3}.$$

∴ The system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \frac{dz}{R} \quad \text{or} \quad \frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dx_3}{x_3} = \frac{dz}{az + \frac{x_1 x_2}{x_3}} \qquad \ldots(1)$$

Taking the first and the second fractions of (1), we have

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} \quad \text{so that} \quad \log x_1 = \log x_2 + \log c_1$$

$$\therefore \quad \frac{x_1}{x_2} = c_1 \quad \text{i.e.} \quad u_1 = \frac{x_1}{x_2} = c_1 \qquad \ldots(2)$$

Taking the second and the third fractions of (1), we have

$$\frac{dx_2}{x_2} = \frac{dx_3}{x_3} \quad \text{so that} \quad \log x_2 = \log x_3 + \log c_2$$

$$\therefore \quad \frac{x_2}{x_3} = c_2 \quad \text{i.e.} \quad u_2 = \frac{x_2}{x_3} = c_2 \qquad \ldots(3)$$

Again, taking the first and fourth fractions of (1), we have

$$\frac{dx_1}{x_1} = \frac{dz}{az+\frac{x_1 x_2}{x_3}} = \frac{dz}{az+c_2 x_1}, \text{ since } \frac{x_2}{x_3} = c_2$$

or $$\frac{az+c_2 x_1}{x_1} = \frac{dz}{dx_1} \quad \text{i.e.,} \quad \frac{dz}{dx_1} - \left(\frac{a}{x_1}\right) z = c_2 \qquad \ldots(4)$$

which is a linear differential equation whose integrating function (I.F.) is given as follows :

$$\text{I.F. of (4)} = e^{-a\int \frac{dx_1}{x_1}} = e^{-a \log x_1} = x_1^{-a}$$

$\therefore$ The solution of the linear differential equation (4) is given by

$$zx_1^{-a} = c_2 \int x_1^{-a}\, dx_1 + c_3 \quad \text{or} \quad zx_1^{-a} = c_2 \left(\frac{x_1^{1-a}}{1-a}\right) + c_3$$

or $$zx_1^{-a} = \frac{x_2}{x_3} . \frac{x_1^{1-a}}{(1-a)} + c_3, \text{ since from (2), } c_2 = \frac{x_2}{x_3}$$

$$\therefore \quad \frac{z}{x_1^a} - \left(\frac{x_1^{1-a}}{1-a}\right)\frac{x_2}{x_3} = c_3 \ \text{ i.e. } \ u_3 = \frac{z}{x_1^a} - \left(\frac{x_1^{1-a}}{1-a}\right)\frac{x_2}{x_3} = c_3 \qquad \ldots(5)$$

Finally, from (2), (3) and (5), the general solution of the given partial differential equation is