(iii) For Markowitz's model we can generate efficient frontier easily provided short selling is allowed (we refer to the *two fund theorem* in this regard). For L_∞-risk model discussed here, generation of efficient frontier becomes easier provided short selling is *not* allowed (we refer to Theorem 6.4.1 in this regard).

(iv) Cai et al. [23] L_∞-risk model can be made more robust if more flexibility is allowed with regard to the parameters μ_i $(i = 1, 2, \ldots, n)$. Deng et al. [36] presented a minimax type model for portfolio optimization where $\mu_i's$ are allowed to be in the interval, $a_i \leq \mu_i \leq b_i$ (i=1,2,...,n). They used the celebrated *minimax theorem* along with the results of Cai et al. [23] to solve the resulting optimization problem.

So far we have considered certain variations of Markowitz's mean-variance model where the portfolio risk is taken different from the standard deviation (e.g. L_1-risk or L_∞-risk) but it is still essentially based on moments of the portfolio return. In the coming sections, we shall discuss few variations of Markowitz's model where portfolio risk is quantile based. In particular, we discuss VaR and CVaR based portfolio optimization problems.

6.5 Value-at-Risk of an Asset

Before defining *value-at-risk* of an asset (denoted by VaR), we consider the below given example for the sake of motivation.

Consider two time instances $t = 0$ and $t = T$. Let $S(0) = 100$ and assume that we buy a share of stock at $t = 0$ to sell it at $t = T$. As S(T) is a random variable, we cannot predict the quantum of profit/loss. Obviously we shall suffer a loss if $S(T) < 100\ e^{rT}$, where r is the risk-free rate under continuous compounding.

A natural question at this stage could be to determine the probability that loss is less than or equal to a specified amount, say Rs 20, i.e. $P[(100\ e^{rT} - S(T)) \leq 20]$. This probability can be easily be computed once the probability distribution of S(T) is known. But it becomes more interesting if this question is reversed. Here we fix the probability, say 95%, and seek the *amount* such that the probability of loss not exceeding *this amount* is more than or equal to 95%. The amount so obtained is essentially called the value-at-risk of the given asset. Thus

$$P[(100\ e^{rT} - S(T)) \leq VaR] \geq 0 \cdot 95\ , \tag{6.37}$$

and VaR represents the predicted maximum loss with specified probability (0.95 in our example) over a certain period of time which is T in our case.

The expression (6.37) suggests that VaR is a measure related to percentiles of loss distributions. In certain sense, it tries to answer the basic question which every investor seems to ask at some point in time, namely, what is the most he/she can lose on his/her investments? Value-at-Risk tries to answer this question within a reasonable bound. VaR and also other downside risk measures are very useful in assessing the risk for securities with asymmetric return distributions, such as call and put options.

VaR has been developed by JP Morgan, and made available through Risk Metrics software in October 1994. We now give the general definition of VaR for a random variable X. This random variable may represent the loss distribution of the asset return with $-X$ being represented as gain.

Definition 6.5.1 (Value at Risk) *Let X be the given random variable and α be the given probability level. Then the VaR of X with confidence level $(1-\alpha)$, $0 < \alpha < 1$, denoted by $VaR_{(1-\alpha)}(X)$ is defined as*

$$VaR_{(1-\alpha)}(X) = Min\{z : \ F_X(z) \geq (1-\alpha)\} = F_X^{-1}(1-\alpha)\ , \tag{6.38}$$

where F_X denotes the cumulative distribution function of the random variable X.

In view of the above definition $VaR_{(1-\alpha)}(X)$ is a lower $(1-\alpha)$ percentile of the random variable X. If we substitute the expression for $F_X(z)$ in (6.38), we obtain

$$VaR_{(1-\alpha)}(X) = \mathrm{Min}\{z : \ P(X \leq z) \geq (1-\alpha)\}. \tag{6.39}$$

Thus $VaR_{(1-\alpha)}(X)$ for an asset is the value z such that the probability that the maximum loss X is at most z, is at least $(1-\alpha)$.

The use of VaR involves two chosen parameters. These are confidence level $(1-\alpha)$ and the holding period T of the asset. The choice of α, and hence$(1-\alpha)$, depends on the purpose to which our risk measure is utilized. In practice α is typically taken as 10%, 5% *and* 1%, so that the typical confidence levels are 90%, 95% *and* 99%. The usual holding periods are one day or one month, but it can be even one quarter or more. Given the confidence level $(1-\alpha)$ and horizon T, VaR is a bound such that the loss over the horizon is less than this bound with probability equal to the confidence coefficient. For example, if horizon is one week, the confidence level is 99% (so $\alpha = 0\cdot 01$) and VaR is Rs 50,000, then there is only a 1% chance of loss exceeding Rs 50,000 over the next week.

From (6.39) we note that $VaR_{(1-\alpha)}(X) = F_X^{-1}(1-\alpha)$ and therefore for a continuous loss distribution $VaR_{(1-\alpha)}(X)$ is simply the loss such that

$$P(X \leq VaR_{(1-\alpha)}(X)) = (1-\alpha).$$

Let us recall that if $X \sim \mathcal{N}(\mu, \sigma^2)$ then the q-percentile of X is $\mu + \sigma\Phi^{-1}(q)$, where Φ is the standard normal density function. Therefore

$$VaR_{(1-\alpha)}(X) = \mu + \sigma\Phi^{-1}(1-\alpha) ,$$

where $(1-\alpha)$-percentile of Φ, namely $\Phi^{-1}(1-\alpha)$, is that value of the standard normal variate for which the area in the left is $(1-\alpha)$.

The above relation shows that for normally distributed random variables, VaR is proportional to the standard deviation.

Example 6.5.1 *Suppose that the stock price is lognormal with mean 12% and standard deviation 30%. Let the interest rate r for the period $t = 0$ to $t = 1$ be 8% and stock price $S(0)$ at time $t=0$ be Rs 100. Determine the VaR for the given stock at 95% confidence level.*

Solution We are given that

$$ln\left(\frac{S(1)}{S(0)}\right) \sim \mathcal{N}(0.12, (0.30)^2) ,$$

and our aim is to find VaR for $(1-\alpha) = 0.95$, i.e. for $\alpha = 0.05$. By definition, loss equals $(100e^{0.08} - S(1))$ and

$$P[(100e^{0.08} - S(1)) \leq VaR] = 0.95. \tag{6.40}$$

Now from the given information

$$\frac{ln\left(\frac{S(1)}{S(0)}\right) - 0.12}{0.30} \sim \mathcal{N}(0,1). \tag{6.41}$$

Therefore (6.40) gives

$$P\left[\frac{S(1)}{S(0)} \geq \frac{(100e^{0.08} - VaR)}{100}\right] = 0.95$$

i.e.

$$P\left[ln\left(\frac{S(1)}{S(0)}\right) \geq ln\frac{(100e^{0.08} - VaR)}{100}\right] = 0.95$$

i.e.

$$P\left[\frac{ln\left(\frac{S(1)}{S(0)}\right)-0.12}{0.30} \geq \frac{ln\left(\frac{(100e^{0.08}-VaR)}{100}\right)-0.12}{0.30}\right] = 0.95. \qquad (6.42)$$

But from (6.41) and (6.42), we get

$$\frac{ln\left(\frac{100e^{0.08}-VaR}{100}\right)-0.12}{0.30} = -1.645,$$

which determines

$$VaR = 100(e^{0.08} - e^{-0.3750}) = Rs\ 39.50.$$

□

Example 6.5.2 *Let an investment A return a gain of* Rs 100 *with probability* 0.96 *and a loss of* Rs 200 *with probability* 0.04. *Obtain VaR of the given investment with* 95% *confidence level.*

Solution We note that our definition of VaR is based on the probability distribution of the random variable X, where X represents the loss of the given investment. Therefore from the given information we have

$$X = \begin{cases} -100, & \text{with probability} \quad 0 \cdot 96 \\ 200, & \text{with probability} \quad 0 \cdot 04. \end{cases}$$

For determining the value of VaR of the given investment we need to use the relation

$$VaR_{(1-\alpha)}(X) = \text{Min}\{z : P(X \leq z) \geq (1-\alpha)\}$$

with $(1-\alpha) = 0 \cdot 95$. Since X is a discrete random variable we have

$$P(X \leq -100) = P(X = -100) = 0 \cdot 96 \geq 0 \cdot 95,$$

and $P(X \leq 200) = 1 \geq 0 \cdot 95$. But -100=Min(-100, 200) and therefore the required VaR is $Rs\ -100$.

□

Example 6.5.3 *Let there be two identical bonds A and B. Each of these defaults with probability 0.04 giving a loss of* Rs 100. *Further there is no loss if default does not occur. Let default occur independently and C be a portfolio consisting of these two bonds A and B. Obtain VaR of bonds A and B; and also of the portfolio C at* 95% *confidence level.*

Solution Following on lines similar to Example 6.5.2, it is simple to get VaR(A)=0 and VaR(B)=0. Since C is the portfolio consisting of bonds A and B, its return R becomes the sum of returns of bonds A and B. Writing returns in terms of the loss function, we obtain the distribution of loss X of investment C as

$$X = \begin{cases} 0, & \text{with probability} \quad (0 \cdot 96)^2 = 0 \cdot 9216 \\ 200, & \text{with probability} \quad (0 \cdot 04)^2 = \cdot 0016 \\ 100, & \text{with probability} \quad (1 - (0 \cdot 96)^2 - (0 \cdot 04)^2) = 0 \cdot 0768. \end{cases}$$

Therefore at 95% confidence level,

$$VaR(C) = Rs\ 100\ ,$$

because P(loss of investment $C \leq 100$) = 0.9216 + 0.0768 = .9984 > 0.95, and P(loss of investment $C \leq 200$) = 1 > 0 · 95, but 100 = Min(100, 200). Here we may note that $P(X \leq 0) = P(X = 0) = 0 \cdot 9216 < 0 \cdot 95$.

□

Some Theoretical and Computational Difficulties with VaR

(i) In Example 6.5.3, at 95% confidence level VaR(C)=100 but VaR(A)+VaR(B)=0. Thus $VaR(C) > VaR(A) + VaR(B)$. But this violates the principle that *diversification reduces risk.* We expect a *good* risk measure to respect this principle. Unfortunately VaR does not do so. Mathematically it means that *VaR is not a subadditive risk measure*. We call a risk measure f to be *subadditive* if for two different investments A and B,

$$f(A + B) \leq f(A) + f(B)\ ,$$

i.e. the total risk of two different investment portfolios does not exceed the sum of individual risks.
In Example 6.5.3, diversification has actually increased the risk if VaR is used as a risk measure.

(ii) VaR does not pay any attention to the magnitude of losses beyond the VaR value. For example it is very unlikely that an investor will take a neutral view for two portfolios with identical expected return and VaR, but return distribution of one portfolio having short left tail and other having a long left tail.

(iii) There is additional difficulty with VaR in its computation and optimization. When VaR is computed by generating scenarios, it turns out to be a non smooth and non convex optimization problem is required to be solved.

The above shortcomings of VaR has motivated researchers to look for other quantile based risk measures and CVaR (conditional value at risk) is one such risk measure.

Before we proceed with the discussion of CVaR, we remark that inspite of the difficulties outlined above, VaR is still very popular in the market. Therefore we need to discuss some non-parametric and parametric methods for its estimation.

Nonparametric Estimation of VaR

Here we describe a procedure for the nonparametric estimation of VaR. Since it is a nonparametric estimation, the loss distribution is not assumed to be in any parametric family of distributions such as normal distributions.

To motivate the procedure, let us consider a simple example. Suppose that we hold a Rs 20,000 position in an index fund, e.g. NIFTY, so that our returns are those of this index. Suppose we require a 24-hour VaR by using 1000 daily returns on NIFTY for the period ending on December 31,2010. These are approximately the last four years of daily returns in the time series of NIFTY.

Let the confidence level $(1-\alpha)$ be 95%, i.e. $\alpha = 5\%$. Since 5% of 1000 is 50 and VaR is a loss which is minus of the revenue, we can estimate the required VaR by taking the 50^{th} smallest daily return. Let this be $r_{(50)}$, the 50^{th} *order statistic* of the sample of historic returns. This means that a daily return of $r_{(50)}$ or less occurred only 5% time in the historic data. Equivalently this means that a daily loss of $-r_{(50)}$ or more occurred only 5% time in the historic data. A return of $r_{(50)}$ on a Rs $20,000$ investment yields a revenue of $20,000 r_{(50)}$ which can be taken as an estimate of VaR at 95% confidence level. Specifically if $r_{(50)} = -0 \cdot 0227$, then $-r_{(50)} = 0 \cdot 0227$ and the estimated VaR is $20,000(0 \cdot 0227)$, i.e. $Rs\,454$.

The above procedure can be generalized easily. We first note that $(1-\alpha)^{th}$ percentile of loss equals the α^{th} percentile of the returns. Suppose that there are n returns $r_1, r_2, \ldots, r_n$ in the historic sample and let k be equal to $(n\alpha)$ rounded to the nearest integer. Then α percentile of the sample of returns is the k^{th} smallest return, that is, the k^{th} order statistic $r_{(k)}$ of the sample of returns. If S is the size of the initial investment then the required estimate of VaR is $-Sr_{(k)}$. Here minus sign converts revenue (return times initial investment) to a loss.

Remark 6.5.1 *Estimation using sample percentiles is only feasible if the sample size is large. If T(holding period) is taken a quarter rather than a day, then in a four years duration we will have only 16 observations. In such a situation, increasing the number of years for historical data would also not increase sample size substantially, and also because of volatility there may be bias in our estimate.*

Parametric Estimation of VaR

For small sample size, VaR can be best estimated by parametric technique. We discuss only the case of normal distribution because it is extremely simple.

Since, in general, the historical data or returns is given, we have

$$VaR_{(1-\alpha)}(loss) = -VaR_{(\alpha)}(return).$$

Therefore

$$VaR_{(1-\alpha)}(X) = -S(\overline{\mu} + \Phi^{-1}(\alpha)\sigma),$$

which can be estimated by $-S(\overline{X} + \Phi^{-1}(\alpha)\hat{\sigma})$. Here $\overline{X}$ and $\hat{\sigma}$ are the mean and standard deviation of sample of returns and S is the size of the initial investment.

Let us assume that from the given historical data, $\overline{X} = -3 \cdot 107 \times 10^{-4}$ and $\hat{\sigma} = 0 \cdot 0151$. Also $\Phi^{-1}(\alpha) = -1 \cdot 645$ for $\alpha = 0 \cdot 05$. Then the estimate of the required VaR is $((-3 \cdot 107 \times 10^{-4}) + (-1 \cdot 645)(0 \cdot 0151)(20,000)) = Rs\ 471$.

In this example, the estimate of the expected return, is negative. This may be because the particular years used include a prolonged bear market. However, we certainly do not expect average future returns to be negative, otherwise we would not have invested Rs 20,000 in the market.

Since in actual practice, normality assumption is not going to hold, we need to apply historical simulation and Monte Carlo strategies to generate a large number of possible scenarios. Also we need to understand VaR for a portfolio of assets which could include options along with certain number of stocks. The estimation of VaR in this situation also requires Monte Carlo simulation. We shall discuss Monte carlo simulation in Chapter 14.

6.6 Conditional Value-at-Risk

One major criticism of VaR is that it pays no attention to the magnitude of losses beyond the VaR value. Also VaR is not subadditive, which not only violates the principle of diversification, but also creates computational difficulty in the resulting portfolio optimization problem. These undesirable features of VaR led to the development of *conditional value-at-risk*, denoted by CVaR, which is obtained by computing the expected loss given that the loss exceeds VaR. Some other names for CVaR are *expected shortfall*, *expected tail loss* and *tail VaR*. CVaR has been introduced by Rockafellar and Uryasev [111] who studied many mathematical properties of CVaR in detail.

To define CVaR we first define the occurrence of a tail event. We say that a *tail event occurs* if the loss exceeds the VaR. Then CVaR is the conditional expectation of loss given that the tail event occurs. We now proceed to define CVaR and discuss its minimization mathematically. To be specific, we define CVaR in context of portfolio optimization.

We consider a portfolio of assets with random returns. Let $f(w,r)$ denote the loss function when we choose the investment w from a set of feasible portfolios and r is the realization of random returns. In the context of our portfolio optimization problem, $f(w,r) = -r^T w$ where $r = (r_1, r_2, \ldots, r_n)^T$ is the vector of random returns and $w = (w_1, w_2, \ldots, w_n)^T$ with $e^T w = 1$, e being the vector $(1, 1, \ldots, 1)^T$. Here in the expression of $f(w,r)$, minus sign has been taken to express return $r^T w$ as loss function. We assume that the return vector r has a probability density function $p(r)$, e.g., the random returns may have a multivariate normal distribution.

For a fixed weight vector w, we define

$$\psi(w,q) = \int_{f(w,r)\le q} p(r)dr, \tag{6.43}$$

which represents the cumulative distribution function of the loss associated with the weight vector w. In fact $\psi(w,q) = P(loss \le q)$. Therefore for a given confidence level $(1-\alpha)$, the $VaR_{(1-\alpha)}(w)$ associated with the portfolio is given by

$$VaR_{(1-\alpha)}(w) = \text{Min}\{q \in \mathbf{R} : \psi(w,q) \ge (1-\alpha)\}.$$

Definition 6.6.1 (Conditional Value-at-Risk) *Let w be a given portfolio and $(1-\alpha)$ be the given confidence level. Then $CVaR_{(1-\alpha)}(w)$ associated with the portfolio w is defined as*

$$CVaR_{(1-\alpha)}(w) = \frac{1}{\alpha}\int_{f(w,r)\ge VaR_{(1-\alpha)}(w)} f(w,r)p(r)dr. \tag{6.44}$$

Remark 6.6.1 *In special case of returns having discrete probability distribution, we have*

$$CVaR_{(1-\alpha)}(w) = \frac{1}{\alpha}\left(\sum_{j\in J_1} p_j f(w, r^{(j)})\right), \tag{6.45}$$

where $J = \{1, 2, \ldots, m\}$, $J_1 = \{j \in J : \ f(w, r^{(j)}) \ge VaR(w)\}$ and the vector $r^{(j)}$ is the j^{th} realization of the return vector r with probability p_j.

Remark 6.6.2 *In certain sense, (6.44) and (6.45) tell that $CVaR_{(1-\alpha)}(w)$ is the average of the outcomes greater than $VaR_{(1-\alpha)}(w)$. This is certainly true for continuous distribution functions; but for general distributions this is not exactly true. There are certain subtle aspects which need to be explained. We shall refer to Sriboonchitta et el [128] in this regard.*

Example 6.6.1 *Let the loss function $f(w,r)$ be given by $f(w,r) = -r$ where $r = 75 - j$, $(j = 0, 1, 2, \ldots, 99)$ with probability 1%. Evaluate $VaR(w)$ and $CVaR(w)$ at 95% confidence level.*

Solution The loss function $f(w,r) = -r$, where $r = 75 - j$, $(j = 0, 1, 2, \ldots, 99)$, takes 100 values given by $(-75, -74, -73, \ldots, 0, 1, \ldots, 19, 20, 21, 22, 23$ and $24)$ with equal probability $p = 0 \cdot 01$. Therefore $VaR(w) = 20$ for $1 - \alpha = 0 \cdot 95$.
We next evaluate $CVaR_{(1-\alpha)}(w)$ for $(1 - \alpha) = 0.95$ by using formula (6.45). This gives

$$CVaR(w) = \frac{1}{0 \cdot 05}(20 + 21 + 22 + 23 + 24)(0 \cdot 01) = 22.$$

Thus at 95% confidence level VaR is 20 and CVaR is 22.

□

In the above example for the same confidence level CVaR is more than VaR. We have the following result in this regard.

Lemma 6.6.1. *For the given confidence level $(1-\alpha)$, $CVaR_{(1-\alpha)}(w) \geq VaR_{(1-\alpha)}(w)$.*

Proof. We have

$$\begin{aligned} CVaR_{(1-\alpha)}(w) &= \frac{1}{\alpha} \int_{f(w,r) \geq VaR_{(1-\alpha)}(w)} f(w,r)p(r)dr \\ &\geq \frac{1}{\alpha} \int_{f(w,r) \geq VaR_{(1-\alpha)}(w)} VaR_{(1-\alpha)}(w)p(r)dr \\ &= \frac{VaR_{(1-\alpha)}(w)}{\alpha} \int_{f(w,r) \geq VaR_{(1-\alpha)}(w)} p(r)dr \\ &= VaR_{(1-\alpha)}(w). \end{aligned}$$

□

Remark 6.6.3 *As CVaR of a portfolio is always more than or equal to VaR for same $(1 - \alpha)$, portfolios with small CVaR will also have small VaR. But this does not mean that the minimization of CVaR is equivalent to the minimization of VaR.*

Minimization of CVaR

We now discuss Rockafellar and Uryasev's [111] procedure to minimize CVaR. Since the definition of CVaR involves the VaR function explicitly, it is not very convenient to optimize CVaR directly. Therefore we introduce the auxiliary function

$$F_{(1-\alpha)}(w,q) = q + \frac{1}{\alpha}\int_{f(w,r)\geq q} (f(w,r) - q)p(r)dr. \tag{6.46}$$

If we denote $a^+ = Max(a,0),\ a \in \mathbf{R}$, then from (6.46) we get

$$F_{(1-\alpha)}(w,q) = q + \frac{1}{\alpha}\int (f(w,r) - q)^+ p(r)dr. \tag{6.47}$$

We refer to Rockafellar and Uryasev [111] for below given results.

Lemma 6.6.2. *The auxiliary function $F_{(1-\alpha)}(w,q)$ is a convex function of q.*

Lemma 6.6.3. *$VaR_{(1-\alpha)}(w)$ is a minimizer of $F_{(1-\alpha)}(w,q)$ over q.*

Lemma 6.6.4. *The minimum value of $F_{(1-\alpha)}(w,q)$ over q is $CVaR_{(1-\alpha)}(w)$.*

In view of the above Lemmas, we have for a given w,

$$CVaR_{(1-\alpha)}(w) = \text{Min}_q(F_{(1-\alpha)}(w,q)) = F_{(1-\alpha)}(w, VaR_{(1-\alpha)}(w)). \tag{6.48}$$

The left equality of (6.48) tells that we can minimize CVaR directly without computing VaR first. Since for portfolios, the loss function $f(w,r) = -r^T w$ is a linear and hence also convex function of w, the auxiliary function $F_{(1-\alpha)}(w,q)$ is a convex function of w. Therefore the problem

$$\begin{aligned} &\text{Min} \quad CVaR_{(1-\alpha)}(w) \\ &\text{subject to} \\ &\qquad e^T w = 1, \end{aligned} \tag{6.49}$$

is equivalent to

$$\begin{aligned} &\underset{w,q}{\text{Min}} \quad F_{(1-\alpha)}(w,q) \\ &\text{subject to} \\ &\qquad e^T w = 1, \end{aligned} \tag{6.50}$$

which is a smooth convex optimization problem. Though problem (6.50) can be solved by using standard convex optimization techniques, the formulation (6.50) still needs the computation/determination of joint density function p(r) of random return vector r. In practice it is not simple or even possible, so we present another approach which is based on generation of scenarios via computer simulation. An added advantage of this approach is that it results in a linear programming formulation.

Suppose we have scenarios $r^{(s)}$, (s=1,2,...,n_s), which may represent historical values of the random vector of returns or obtained via computer simulation. We shall assume that all scenarios have equal probability and define the following empirical distribution of the random returns based on the available scenarios

$$\widetilde{F}_{(1-\alpha)}(w,q) = a + \frac{1}{\alpha\, n_s}\sum_{s=1}^{n_s}(f(w,r^{(s)}) - q)^{+}. \tag{6.51}$$

Using (6.51) to approximate $F_{(1-\alpha)}(w,q)$, we get an approximation to the problem (6.50) as

$$\begin{aligned} &\underset{w,q}{\text{Min}} \quad \widetilde{F}_{(1-\alpha)}(w,q) \\ &\text{subject to} \\ &\qquad\qquad e^T w = 1. \end{aligned} \tag{6.52}$$

Now writing $(f(w,r^{(s)}) - q)^{+}$ as z_s and using the definition of a^{+}, we obtain

$$\begin{aligned} &\underset{w,q,z_s}{\text{Min}} \quad q + \frac{1}{\alpha.n_s}\sum_{s=1}^{n_s} z_s \\ &\text{subject to} \\ &\qquad z_s \geq f(w,r^{(s)}) - q, \qquad s = 1,2,\ldots,n_s \\ &\qquad e^T w = 1 \\ &\qquad z_s \geq 0, \qquad s = 1,2,\ldots,n_s. \end{aligned} \tag{6.53}$$

In the context of portfolio optimization, $f(w,r^{(s)}) = -w^T r^{(s)}$ is linear and so the problem (13.5) becomes a linear programming problem.

Most often we try to optimize a suitable performance measure (e.g, expected return) while making sure that certain risk measures do not exceed a threshold value. It could be variance or absolute deviation as has been discussed earlier. When the risk measure is CVaR, the resulting optimization problem is

$$
\begin{aligned}
&\text{Max} \quad \mu^T w \\
&\text{subject to} \\
&\qquad CVaR_{(1-\alpha)}(w) \leq u_\alpha \\
&\qquad e^T w = 1,
\end{aligned} \tag{6.54}
$$

which can be approximated as

$$
\begin{aligned}
&\underset{w,z,q}{\text{Max}} \quad \mu^T w \\
&\text{subject to} \\
&\qquad u_\alpha \;\; \geq q + \frac{1}{\alpha\, n_s} \sum_{s=1}^{n_s} z_s \\
&\qquad z_s \;\; \geq f(w, r^{(s)}) - q \qquad (s = 1, 2, \ldots, n_s) \\
&\qquad e^T w = 1 \\
&\qquad z_s \;\; \geq 0 \qquad (s = 1, 2, \ldots, n_s).
\end{aligned}
$$

In (6.54), we can have more than one CVaR constraint for different levels α. Also similar to Markowitz's model, we can have a trade off between return and CVaR. We can refer to Mansini et al. [89] for more details in this regard.

6.7 Preference Relation, Utility theory and Decision Making

The mean-variance criterion used in the Markowitz portfolio model can also be explained in terms of the expected utility maximization principle in decision making. To understand this aspect we first wish to motivate the readers to the celebrated *Von Neumann- Morgenstern's* model for decision making. This model tries to explain how a *rational person* makes his/her *optimal choice* in a given situation. This requires some discussion on *preference relation* and related results in utility theory.

Choice under certainty

We first consider the situation when there is no uncertainty present, i.e. the alternatives are certain. Let this set of alternatives be denoted by Z. As the choice is being made under certainty, we shall get the element of our choice for sure.

In order to perform our choice, i.e. the selection procedure, we prescribe our own preferences, and use them to select the desired *optimal* element of Z. These preferences are expressed as a *preference relation*, say $\succ$ on Z. Thus we say that