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$T : \mathbb{R} \to \mathbb{R}$ be a map defined by $T(x) = x + 1$. Using above theorem you can say that T is not linear.

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  Answer: Yes. Let $(x_1, x_2) \in \mathbb{R}^2$. Then $(x_1, x_2) = x_1 e_1 + x_2 e_2$.

Then $T(x_1, x_2) = x_1 T(e_1) + x_2 T(e_2)$

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  Answer: No it is not possible.

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Proof:

Let $x \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} c_i u_i$.

Define $T(x) = \sum_{i=1}^{n} c_i w_i$. It is clear that T is well defined because $x = \sum_{i=1}^{n} c_i u_i$, this expression unique.

We first show that T is a linear transformation. Take $x, y \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} c_i u_i$ and $y = \sum_{i=1}^{n} d_i u_i$.

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Let $\alpha, \beta \in \mathbb{F}$. $T(\alpha x + \beta y) = T(\sum_{i=1}^{n}(\alpha c_i + \beta d_i) u_i)$.

$$
\begin{aligned}
T(\alpha x + \beta y) &= \sum_{i=1}^{n}(\alpha c_i + \beta d_i) w_i. \\
&= \alpha \sum_{i=1}^{n} c_i w_i + \beta \sum_{i=1}^{n} d_i w_i. \\
&= \alpha T(x) + \beta T(y).
\end{aligned}
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Hence T is linear.

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To show that $U = T$. Let $x \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} a_i u_i$. Using definition of T we have $T(x) = T(\sum_{i=1}^{n} a_i u_i) = \sum_{i=1}^{n} a_i w_i$.

$$U(x) = U(\sum_{i=1}^{n} a_i u_i)$$

$$= \sum_{i=1}^{n} a_i U(u_i) \text{ (applying the definition of linear transformation)}$$

$$= \sum_{i=1}^{n} a_i w_i.$$

Then $U(x) = T(x)$ for all $x \in \mathbb{V}$. Hence $U = T$.

- [Example]

Take the basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$. Take $1, 2, 3 \in \mathbb{R}$. Then using previous theorem we have a unique linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}$ such that $T(e_1) = 1$, $T(e_2) = 2$, $T(e_3) = 3$ and $T(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3$.

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The previous theorem gives a technique to construct a linear transformation from a finite dimensional vector space to another dimensional vector space over the same filed $\mathbb{F}$.

- **[Definition:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation.
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  2. $R(T) := \{T(x) : x \in \mathbb{V}\}$. you can easily check that $R(T)$ is a subspace of $\mathbb{W}$.

     The subspaces $R(T)$ is called the **range space** of T.

- **[Example:]** $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a map defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$$

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$\dim(N(T)) = 1.$

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Let $y = (y_1, y_2) \in R(T)$. Then there exists $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $(y_1, y_2) = T(x_1, x_2, x_3)$.

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$\dim(R(T)) = 2$

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  The $\dim(R(T))$ is called the **rank** of T and $\dim(N(T))$ is called the **nullity** of T.

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To show $Ker(T) = \{0\}$. Let $x \in ker(T)$. Then $T(x) = 0$ and we know that $T(0) = 0$. Since T is one-one, then $x = 0$. Hence $Ker(T) = \{0\}$.

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Given that $T(u_1), \ldots, T(u_n)$ are LI. To show $u_1, \ldots, u_n$ are LI.

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Given that $T(u_1), \ldots, T(u_n)$ are LI. To show $u_1, \ldots, u_n$ are LI.

$c_1 u_1 + \cdots + c_n u_n = 0_{\mathbb{V}}$.

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$c_1 = c_2 = \ldots = c_n = 0$ as $T(u_1), \ldots, T(u_n)$ are LI.

Converse is not true in general. That is if $u_1, \ldots, u_n$ are LI, then $T(u_1), \ldots, T(u_n)$ may or may not be LI.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ be a LT defined by $T(x_1, x_2) = (x_1 - x_2, x_2 - x_1)$. Take $u_1 = (1, 0)$ and $u_2 = (1, 1)$. Notice that u_1, u_2 are LI. But $T(u_1) = (1, -1)$ and $T(u_2) = (0, 0)$ are LD.

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Therefore $T(u_{k+1}), \ldots, T(u_n)$ are LI. Hence $\{T(u_{k+1}), \ldots, T(u_n)\}$ is a basis of $R(T)$. Then $\dim(R(T)) = n - k$.

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$T_{\mathbb{S}}$ is a LT from $\mathbb{S}$ to $\mathbb{W}$. Then $Ker(T_{\mathbb{S}}) \subseteq Ker(T)$ and $R(T_{\mathbb{S}}) \subseteq R(T)$.

$nullity(T_{\mathbb{S}}) \leq k$ and $rank(T_{\mathbb{S}}) \leq m$. Then $nullity(T_{\mathbb{S}}) + rank(T_{\mathbb{S}}) \leq k + m$ and $\dim(\mathbb{S}) = m + k + 1$. Then rank nullity theorem is not true on $\mathbb{S}$.

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Applications of Rank-Nullity Theorem: Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the filed $\mathbb{F}$ such that $\dim(\mathbb{V}) = \dim(\mathbb{W})$. Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then the following are equivalent.

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2. T is onto.

- Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the filed $\mathbb{F}$ such that $\dim(\mathbb{V}) < \dim(\mathbb{W})$. Then there is no onto linear transformation from $\mathbb{V}$ to $\mathbb{W}$.

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- **[Definition:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation. Then T is said to be isomorphism if T is bijective (one-one+onto).

- [Example:]

1. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{V}$ be defined by $T(x) = \alpha x$, $\alpha \neq 0$. Then T is an isomorphism.

2. Let $\mathbb{V} = M_{n\times m}(\mathbb{R})$ be the set of $n \times m$ matrices with real entries and let $\mathbb{W} = \mathbb{R}^{mn}$. Define $T : \mathbb{V} \to \mathbb{W}$ by

$T(A) = (a_{11}, \ldots, a_{1m}, a_{21}, \ldots, a_{2m}, \ldots, a_{n1}, \ldots, a_{nm})$. Here $A = (a_{ij}) \in \mathbb{V}$. Then T is an isomorphism.

4. Let $\mathbb{V} = M_{n\times n}(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Define $T : \mathbb{V} \to \mathbb{R}$ by $T(A) = trace(A)$. Then T is not an isomorphism as T is not one-one.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation define by $T(x_1, x_2) = (x_1, x_1 - x_2)$. Then T is an isomorphism.

- **[Definition:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the same field $\mathbb{F}$. Then $\mathbb{V}$ and $\mathbb{W}$ are said to be isomorphic if there is an isomorphism from $\mathbb{V}$ to $\mathbb{W}$.

- **[Example:]**

1. $\mathbb{R}^n$ and $\mathbb{R}^m$ are isomorphic if and only if $m = n$.

2. $\mathbb{R}^{mn}$ are isomorphic to $\mathbb{V} = M_{n\times m}(\mathbb{R})$, the set of $n \times m$ matrices with real entries.

3. $\mathbb{R}^n$ is isomorphic to $\mathbb{P}_n(x, \mathbb{R})$, set of all real polynomials of degree at most n.

- **[Theorem:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two finite dimensional vector spaces over the same field $\mathbb{F}$. Then $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if and only if $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

Proof: We first assume that $\mathbb{V}$ and $\mathbb{W}$ are isomorphic.

- **[Theorem:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two finite dimensional vector spaces over the same field $\mathbb{F}$. Then $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if and only if $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

  Proof: We first assume that $\mathbb{V}$ and $\mathbb{W}$ are isomorphic. Let T be an isomorphism from $\mathbb{V}$ to $\mathbb{W}$.

- **[Theorem:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two finite dimensional vector spaces over the same field $\mathbb{F}$. Then $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if and only if $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

  Proof: We first assume that $\mathbb{V}$ and $\mathbb{W}$ are isomorphic. Let T be an isomorphism from $\mathbb{V}$ to $\mathbb{W}$. Let $\{u_1, \ldots, u_n\}$ be a basis of $\mathbb{V}$.

  Since T is one-one, then $\{T(u_1), \ldots, T(u_n)\}$ is linearly independent.

  $\mathbb{W} = R(T) = \text{ls}(\{T(u_1), \ldots, T(u_n)\})$.

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  $\mathbb{W} = R(T) = \text{ls}(\{T(u_1), \ldots, T(u_n)\})$. Hence $\{T(u_1), \ldots, T(u_n)\}$ is a basis of $\mathbb{W}$.

- **[Theorem:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two finite dimensional vector spaces over the same field $\mathbb{F}$. Then $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if and only if $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

Proof: We first assume that $\mathbb{V}$ and $\mathbb{W}$ are isomorphic. Let T be an isomorphism from $\mathbb{V}$ to $\mathbb{W}$. Let $\{u_1, \ldots, u_n\}$ be a basis of $\mathbb{V}$.

Since T is one-one, then $\{T(u_1), \ldots, T(u_n)\}$ is linearly independent.

$\mathbb{W} = R(T) = \mathrm{ls}(\{T(u_1), \ldots, T(u_n)\})$. Hence $\{T(u_1), \ldots, T(u_n)\}$ is a basis of $\mathbb{W}$. Then $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

Converse: We now assume that $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

- **[Theorem:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two finite dimensional vector spaces over the same field $\mathbb{F}$. Then $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if and only if $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

Proof: We first assume that $\mathbb{V}$ and $\mathbb{W}$ are isomorphic. Let T be an isomorphism from $\mathbb{V}$ to $\mathbb{W}$. Let $\{u_1, \ldots, u_n\}$ be a basis of $\mathbb{V}$.

Since T is one-one, then $\{T(u_1), \ldots, T(u_n)\}$ is linearly independent.

$\mathbb{W} = R(T) = \text{ls}(\{T(u_1), \ldots, T(u_n)\})$. Hence $\{T(u_1), \ldots, T(u_n)\}$ is a basis of $\mathbb{W}$. Then $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

Converse: We now assume that $\dim(\mathbb{V}) = \dim(\mathbb{W})$. Let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be two bases of $\mathbb{V}$ and $\mathbb{W}$, respectively.

We have a linear transformation T such that $T(u_i) = v_i$ for $i = 1, \ldots, n$. We can easily check that T is bijective.

- **[Definition:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the same field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation. Then T is called invertible if T is bijective (one-one+onto).

- **[Example:]**

1. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{V}$ be defined by $T(x) = \alpha x$, $\alpha \neq 0$. Then T is invertible.

2. Let $\mathbb{V} = M_{n\times m}(\mathbb{R})$ be the set of $n \times m$ matrices with real entries and let $\mathbb{W} = \mathbb{R}^{mn}$. Define $T : \mathbb{V} \to \mathbb{W}$ by

$T(A) = (a_{11}, \ldots, a_{1m}, a_{21}, \ldots, a_{2m}, \ldots, a_{n1}, \ldots, a_{nm})$. Here $A = (a_{ij}) \in \mathbb{V}$. Then T is invertible.

4. Let $\mathbb{V} = M_{n\times n}(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Define $T : \mathbb{V} \to \mathbb{R}$ by $T(A) = trace(A)$. Then T is not invertible.

- We use $\mathcal{L}(\mathbb{V}, \mathbb{W})$ to denote set of all linear transformation from $\mathbb{V}$ to $\mathbb{W}$.

- Let $S, T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ and $\alpha \in \mathbb{R}$. Then $S + T$ and αS defined by $(S + T)(x) = S(x) + T(x)$ (vector addition) and $(\alpha S)x = \alpha S(x)$ (scalar multiplication).

- [**Theorem:**] Let $\mathbb{V}$ and $\mathbb{W}$ be two VS over $\mathbb{F}$. Then $\mathcal{L}(\mathbb{V}, \mathbb{W})$ is also a vector space with respect to above two operations over $\mathbb{F}$.

Proof It is trivial.

- **[Definition:]**

1. A linear transformation T from $\mathbb{V}$ to $\mathbb{V}$ is called a **linear operator**.

2. A linear transformation T from $\mathbb{V}$ to $\mathbb{F}$ is called a **linear functional**.

- **[Example]**

1. Let $T : \mathbb{M}_{n\times n} \to \mathbb{R}$ defined as $T(A) = trace(A)$, $A \in \mathbb{M}_{n\times n}$. T is linear functional.

2. Let $T : C[0,1] \to \mathbb{R}$ defined as $T(f) = \int\limits_{0}^{1} f(x)dx$. T is linear functional.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$. T is linear operator.

- **[Definition:]** The space $\mathcal{L}(\mathbb{V}, \mathbb{F})$ is called the **dual space** of $\mathbb{V}$ and it is denoted by $\mathbb{V}^*$. Elements of $\mathbb{V}^*$ are usually denoted by lower case letters f, g, etc.

- **[Theorem:]** Let $\mathbb{V}$ be a finite dimensional space and $B = \{v_1, \ldots, v_n\}$ be an ordered basis of $\mathbb{V}$.

  For each $j \in \{1, \ldots, n\}$, let $f_j : \mathbb{V} \to \mathbb{F}$ be defined by $f_j(x) = \alpha_j$ for $x = \sum_{j=1}^{n} \alpha_j v_j$.

  Then the following are true.

  1. $f_1, \ldots, f_n$ are in $\mathbb{V}^*$ and they satisfy $f_i(v_j) = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$.

  2. $\{f_1, \ldots, f_n\}$ is a basis of $\mathbb{V}^*$.

Proof: We first show that $f_i(v_j) = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$.

$v_j = 0v_1 + \cdots + v_j + \cdots + 0v_n$. Using the definition of f_i, we have

$$f_i(v_j) = \begin{cases} 1 & \text{if } i = j) \\ 0 & \text{if} i \neq j \end{cases}$$

That is $f_i(v_j) = \delta_{ij}$.

We now show that $f_i \in \mathbb{V}^*$, that is f_i is a linear functional for $i = 1, \ldots, n$.

Let $x, y \in \mathbb{V}$. Then $x = a_1 v_1 + \cdots + a_n v_n$ and $y = b_1 v_1 + \cdots + a_n v_n$.

Using definition of f_i we have, $f_i(x) = a_i$ and $f_i(y) = b_i$.

Let $\alpha, \beta \in \mathbb{F}$. Then $\alpha x + \beta y = (\alpha a_1 + \beta b_1)v_1 + \cdots + (\alpha a_n + \beta b_n)v_n$.

Using definition of f_i we have

$f_i(\alpha x + \beta y)$

$= \alpha a_i + \beta b_i$

$= \alpha f_i(x) + \beta f(y)$.

Hence f_i is linear transformation from $\mathbb{V}$ to $\mathbb{F}$ for $i = 1, \ldots, n$. We have proved that $f_i \in \mathbb{V}^*$ for $i = 1, \ldots, n$.

We mow show that $\{f_1, \ldots, f_n\}$ is a basis of $\mathbb{V}^*$. We first show that $\{f_1, \ldots, f_n\}$ is linearly independent.

$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0.$

$(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n) v_1 = 0(v_1).$

$c_1 f_1(v_1) + \cdots + c_n f_n(v_1) = 0.$

$c_1 = 0.$

Similarly you can show that $c_2 = \cdots = c_n = 0$. Hence $\{f_1, \ldots, f_n\}$ is linearly independent.

We now show that $\text{ls}(\{f_1, \ldots, f_n\}) = \mathbb{V}^*$.

Let $f \in \mathbb{V}^*$. Let $f(v_i) = c_i$ for $i = 1, \ldots, n$ where $c_1, \ldots, c_n \in \mathbb{F}$. We have to show that $f = a_1 f_1 + \cdots + a_n f_n$ where $a_, \ldots, a_n \in \mathbb{F}$.

Let $x \in \mathbb{V}$. Then $x = b_1 v_1 + \cdots + b_n v_n$.

$f(x) = f(b_1 v_1 + \cdots + b_n v_n)$

$= b_1 f(v_1) + \cdots + b_n f(v_n)$

$= c_1 b_1 + \cdots + c_n b_n$

$= c_1 f_1(x) + \cdots + c_n f(x)$

$f(x) = (c_1 f_1 + \cdots + c_n f)(x)$ for all $x \in \mathbb{V}$. Therefore $f = c_1 f_1 + \cdots + c_n f$. Hence $\text{ls}(\{f_1, \ldots, f_n\}) = \mathbb{V}^*$.

- **[Definition:]** Let $\mathbb{V}$ be a finite dimensional space and $B = \{v_1, \ldots, v_n\}$ be an order basis of $\mathbb{V}$. A basis $\{f_1, \ldots, f_n\}$ of $\mathbb{V}^*$ such that $f_i(v_j) = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$. Then $\{f_1, \ldots, f_j\}$ is called **dual basis** of $\mathbb{V}^*$.

- **[Example:** How to compute dual basis:]

Let $\mathbb{V} = \mathbb{R}^2$. Let $B = \{(1,0), (0,1)\}$ be a basis of $\mathbb{V}$. Find the dual basis of $\mathbb{V}^*$ corresponding B.

Let $\{f_1, f_1\}$ be the dual basis of $\mathbb{V}^*$ corresponding B.

$f_1(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$ and $f_2 = \beta_1 x_1 + \beta_2 x_2$ where $\alpha_1, \alpha_2 . \beta_1, \beta_2 \in \mathbb{R}$.

$f_1(1,0) = 1 \implies \alpha_1 = 1$ and $f_1(0,1) = 0 \implies \alpha_2 = 0$

Similarly $f_2(1,0) = 0 \implies \beta_1 = 0$ and $f_2(0,1) = 0 \implies \beta_2 = 1$.

$f_1(x_1, x_2) = x_1$ and $f_2(x_1, x_2) = x_2$

- **[Theorem:]** If $\mathbb{V}$ is finite dimensional, then $\mathbb{V}$ and $\mathbb{V}^*$ are isomorphic.

Proof: We have seen that $\dim(\mathbb{V}) = \dim(\mathbb{V}^*)$. Then they are isomorphic.

Let $\mathbb{V}$ and $\mathbb{W}$ be two FDVS over $\mathbb{F}$. Let $B_1 = \{u_1, \ldots, u_n\}$ and $B_2 = \{v_1, \ldots, v_m\}$ be two ordered bases of $\mathbb{V}$ and $\mathbb{W}$, respectively.

Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation.

$T(u_j) \in \mathbb{W}$. Then there exist $a_{ij} \in \mathbb{F}$ for $i = 1, \ldots, m$ such that

$T(u_j) = a_{1j}v_1 + a_{2j}v_2 + \cdots + a_{mj}v_m$ for $j = 1, \ldots, n$.

Let $x \in \mathbb{V}$. There exist $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ such that $x = \sum_{j=1}^{n} \alpha_j u_j$. That is

$$[x]_{B_1} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

$$
\begin{aligned}
T(x) &= T\Big(\sum_{j=1}^{n} \alpha_j u_j\Big) \\
&= \sum_{j=1}^{n} \alpha_j T(u_j) \\
&= \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{m} a_{ij} v_i \\
&= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \alpha_j a_{ij} \Big) v_i.
\end{aligned}
$$

Let $[T(x)]_{B_2} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$.

Then $\beta_i = \sum_{j=1}^{n} \alpha_j a_{ij}$ for $j = 1, \ldots, m$

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

$[T(x)]_{B_2} = A[x]_{B_1}$ where $A = [a_{ij}]_{m \times n}$. That is co-ordinate of $T(x)$ with respect to the basis B_2 is $[T(x)]_{B_2}$ which can be calculated using the co-ordinate of x with respect to basis B_1.

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-z \end{bmatrix}$. Let $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ be bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$.

Let $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$. Then $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The co-ordinate of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with respect to B_1 is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$$[T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}]_{B_2} = A[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}]_{B_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The matrix $A = (a_{ij})$ in the above discussion is called the **matrix representation** of T with respect to the <u>o</u>rdered bases B_1 and B_2 of $\mathbb{V}$ and $\mathbb{W}$, respectively. This matrix is usually denoted by $[T]_{B_1 B_2}$, that is, $[T]_{B_1 B_2} = (a_{ij})$.

Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces over the same field $\mathbb{F}$. Let $\dim(\mathbb{V}) = n$ and $\dim(\mathbb{W}) = m$. Assume that $B = \{u_1, \ldots, u_n\}$ and $B' = \{v_1, \ldots, v_m\}$ are ordered basis of $\mathbb{V}$ and $\mathbb{W}$ respectively.

1. We have seen that for each linear transformation $T : \mathbb{V} \to \mathbb{W}$, we have a matrix $A \in \mathbb{M}_{m\times n}(\mathbb{F})$ such that $[T]_{BB'} = A$.
2. Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$. Then there exists a linear transformation $T : \mathbb{V} \to \mathbb{W}$ such that $A = [T]_{BB'}$ and such linear transformation is $T(u_j) = \sum\limits_{i=1}^{m} a_{ij} v_i$ for $j = 1 \ldots, n$.
3. Let $T, S : \mathbb{V} \to \mathbb{W}$ be two linear transformation. Let B_1 and B_2 be two bases of $\mathbb{V}$ and $\mathbb{W}$, respectively. Then $[T + S]_{B_1B_2} = [T]_{B_1B_2} + [S]_{B_1B_2}$ and $[\alpha T]_{B_1B_2} = \alpha[T]_{B_1B_2}$.

- **[Theorem:]** Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces over the same field $\mathbb{F}$. Let $\dim(\mathbb{V}) = n$ and $\dim(\mathbb{W}) = m$. Then $\mathcal{L}(\mathbb{V}, \mathbb{W})$ is isomorphic to $\mathbb{M}_{m\times n}(\mathbb{F})$.

  Proof: Let $B = \{u_1, \ldots, u_n\}$ and $B' = \{v_1, \ldots, v_m\}$ be bases of $\mathbb{V}$ and $\mathbb{W}$, respectively.

  Define $\zeta : \mathcal{L}(\mathbb{V}, \mathbb{W}) \to \mathbb{M}_{m\times n}(\mathbb{F})$ such that $\zeta(T) = [T]_{BB'}$.

  Using previous remark it is cleared that ζ is linear from $\mathcal{L}(\mathbb{V}, \mathbb{W})$ to $\mathbb{M}_{m\times n}(\mathbb{F})$. We now show that ζ is bijective.

  Let $T \in Ker(\zeta)$. Then $\zeta(T) = 0_{m\times n}$.

  This implies that $[T]_{BB'} = 0_{m\times n}$.

  This implies $[T(x)]_{B'} = 0_{n\times 1}$, co-ordinate of $T(x)$ for each $x \in \mathbb{V}$ with respect to B' is zero. Hence $T(x) = 0_{\mathbb{W}}$ for each $x \in \mathbb{V}$. Then $T = 0$. Therefore $Ker(\zeta) = \{0\}$.

We now show that ζ is onto. Let $A \in \mathbb{M}_{m\times n}$. Define $T(u_j) = \sum_{i=1}^{m} a_{ij} v_i$ for $j = 1 \ldots, n$. It is clear that $[T]_{BB'} = A$. Hence ζ is onto.

Therefore $\mathcal{L}(\mathbb{V}, \mathbb{W})$ is isomorphic to $\mathbb{M}_{m\times n}(\mathbb{F})$.

- **[Theorem:]** Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces over the same field $\mathbb{F}$. Let $\dim(\mathbb{V}) = n$ and $\dim(\mathbb{W}) = m$. Then dimension of $\mathcal{L}(\mathbb{V}, \mathbb{W}) = mn$.

- **[Theorem:]** Let $\mathbb{V}$ be a finite dimensional vector space over the same field $\mathbb{F}$. Let S and T be two linear transformations from $\mathbb{V}$ and to $\mathbb{V}$. Let B be an ordered basis of $\mathbb{V}$. Then $[S \circ T]_{BB} = [S]_{BB}[T]_{B,B}$.

  Proof: Let $B = \{v_1, \ldots, v_n\}$. Let $[T]_{BB} = A$ and $[S]_{BB} = C$.

  Then $T(v_i) = a_{1i}v_1 + a_{2i}v_2 + \cdots + a_{ni}v_n$ for $i = 1, \ldots, n$.

  $S(v_i) = b_{1i}v_1 + b_{2i}v_2 + \cdots + b_{ni}v_n$ for $i = 1, \ldots, n$.

  $(S \circ T)(v_1) = S(T(v_1))$

  $= S(a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n)$

  $= a_{11}S(v_1) + a_{21}S(v_2) + \cdots + a_{n1}S(v_n)$

  $= a_{11}(b_{11}v_1 + b_{21}v_2 + \cdots + b_{n1}v_n) + a_{21}(b_{12}v_1 + b_{22}v_2 + \cdots + b_{n2}v_n) + \cdots + a_{n1}(b_{1n}v_1 + b_{2n}v_2 + \cdots + b_{nn}v_n)$

$$= (a_{11}b_{11} + a_{21}b_{12} + \cdots a_{n1}b_{1n})v_1 + \cdots + (a_{11}b_{n1} + a_{21}b_{n2} + \cdots a_{n1}b_{nn})v_n$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$$

$$\left[(S \circ T)(u_i)\right] = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}$$

$$\left[S \circ T\right]_{BB} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- **[Remark:]** Let $\dim V = n$ and $\dim W = m$. Let $T : \mathbb{V} \to \mathbb{W}$ and $S : \mathbb{W} \to \mathbb{V}$ be two linear transformation. Let B and B' be bases of $\mathbb{V}$ and $\mathbb{W}$, respectively. Then $\left[S \circ T\right]_{BB'} = [S]_{BB'}[T]_{BB'}$.

- **[Theorem:]**Let $\mathbb{V}$ be a finite dimensional vector space over the same field $\mathbb{F}$. Let T be an invertible linear transformation from $\mathbb{V}$ and to $\mathbb{V}$. Let B be an ordered basis of $\mathbb{V}$. Then $[T^{-1}]_{BB} = [T]_{BB}^{-1}$.