Practice Problem Set 4
(Duality Theory and Sensitivity Analysis)

(1) Consider the following problem.

Maximize $Z = -x_1 - 2x_2 - x_3,$

subject to

$$x_1 + x_2 + 2x_3 \le 12$$
$$x_1 + x_2 - x_3 \le 1$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

(a) Construct the dual problem.
(b) Use duality theory to show that the optimal solution for the primal problem has $Z \le 0$.

(2) Consider the following problem.

Maximize $Z = 3x_1 + x_2 + 4x_3,$

subject to

$$6x_1 + 3x_2 + 5x_3 \le 25$$
$$3x_1 + 4x_2 + 5x_3 \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

The corresponding final set of equations yielding the optimal solution is

$$(0) \quad Z + 2x_2 + \frac{1}{5}x_4 + \frac{3}{5}x_5 = 17$$

$$(1) \quad x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3}x_5 = \frac{5}{3}$$

$$(2) \quad x_2 + x_3 - \frac{1}{5}x_4 + \frac{2}{5}x_5 = 3.$$

(a) Identify the optimal solution from this set of equations.
(b) Construct the dual problem.
(c) Identify the optimal solution for the dual problem from the final set of equations. Verify this solution by solving the dual problem graphically.
(d) Suppose that the original problem is changed to

Maximize $Z = 3x_1 + 3x_2 + 4x_3,$

subject to

$$6x_1 + 2x_2 + 5x_3 \le 25$$
$$3x_1 + 3x_2 + 5x_3 \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Use duality theory to determine whether the previous optimal solution is still optimal.
(e) Now suppose that the only change in the original problem is that a new variable x_{new} has been introduced into the model as follows:

Maximize $\quad Z = 3x_1 + x_2 + 4x_3 + 2x_{new}$,

subject to

$$6x_1 + 3x_2 + 5x_3 + 3x_{new} \le 25$$
$$3x_1 + 4x_2 + 5x_3 + 2x_{new} \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_{new} \ge 0.$$

Use duality theory to determine whether the previous optimal solution, along with $x_{new} = 0$, is still optimal.

(3) Consider the following problem.

Maximize $\quad Z = 2x_1 - x_2 + x_3$,

subject to

$$3x_1 + x_2 + x_3 \le 60$$
$$x_1 - x_2 + 2x_3 \le 10$$
$$x_1 + x_2 - x_3 \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	0	0	$\frac{3}{2}$	0	$\frac{3}{2}$	$\frac{1}{2}$	25
x_4	(1)	0	0	0	1	1	-1	-2	10
x_1	(2)	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	15
x_2	(3)	0	0	1	$-\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	5

1. Find the range of value of c_1, c_2 and c_3 for which the current basis (solution) is optimal.
2. Find the range of value of b_1, b_2 and b_3 for which the current basis is feasible.
3. Now you are to conduct sensitivity analysis by *independently* investigating each of the following six changes in the original model. For each change, use the sensitivity analysis procedure to revise this final tableau and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the right-hand sides

$$\text{from} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 70 \\ 20 \\ 10 \end{bmatrix}.$$

(b) Change the coefficients of x_1

$$\text{from} \quad \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

(c) Change the coefficients of x_3

$$\text{from} \quad \begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}.$$

(d) Change the objective function to $Z = 3x_1 - 2x_2 + 3x_3$.
(e) Introduce a new constraint $3x_1 - 2x_2 + x_3 \leq 30$. (Denote its slack variable by x_7.)
(f) Introduce a new variable x_8 with coefficients

$$\begin{bmatrix} c_8 \\ a_{18} \\ a_{28} \\ a_{38} \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 2 \end{bmatrix}.$$