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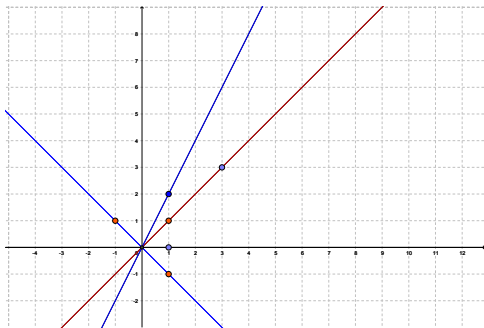
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- **[Geometrically]** An eigenvector, corresponding to a real nonzero eigenvalue, points in a direction in which it is stretched by the matrix and the eigenvalue is the factor by which it is stretched. If the eigenvalue is negative, the direction is reversed.

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There exists a non-zero vector $x \in \mathbb{F}^n$ such that $Ax = \lambda x$.

$$Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0$$

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This says that the system of homogeneous equations $(A - \lambda I)y = 0$ has non-trivial solution. Hence $\text{rank}(A - \lambda I) < n$. Then $\det(A - \lambda I) = 0 = \det(\lambda I - A)$. This implies λ is a root of $\det(xI - A)$.

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We have to calculate the roots of $\det(xI - A)$.

$$\det(xI - A) = \begin{vmatrix} x-1 & -2 & -1 \\ -2 & x-1 & -1 \\ -1 & -1 & x-2 \end{vmatrix} = x^3 - 4x^2 - x + 4.$$

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Then REF of $A - I$ is $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The set $\left\{ k \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} : k \in \mathbb{R} - \{0\} \right\}$ is the set of all eigenvectors.

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Now we are able to answer our question. If $x^2 + 1$ is the characteristic polynomial of a matrix $A \in \mathbb{M}_2(\mathbb{R})$, then A does not have eigenvalues. Here is that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

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Proof: Let A and B be two similar matrices.

Then there exists a nonsingular matrix P such that $P^{-1}AP = B$.

Then $\det(B - xI)$

$$= \det(P^{-1}AP - xI)$$

$$= \det(P^{-1}AP - xP^{-1}P)$$

$$= \det(P^{-1}) \det(A - xI) \det(P) = \det(A - xI).$$

Converse is not true. Consider the following two matrices.

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. These two matrices have the same characteristic polynomial but they are not similar.

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- Let $C = \begin{bmatrix} A_{n \times n} & D_{n \times m} \\ 0_{m \times n} & B_{m \times m} \end{bmatrix}$. Then characteristic polynomial of C , $P_C(x) = P_A(x)P_B(x)$.

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We have $C - xI_{n+m} = \begin{bmatrix} A_{n \times n} - xI_n & D_{n \times m} \\ 0_{m \times n} & B_{m \times m} - xI_m \end{bmatrix}$.

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$$\text{Then } \det(C) = \begin{vmatrix} A_{n \times n} - xI_n & D_{n \times m} \\ 0_{m \times n} & B_{m \times m} - xI_m \end{vmatrix} = \det(A - xI_n) \det(B - xI_m).$$

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The characteristic polynomial of A is $x^2 + 1$ and this polynomial does not have any real root. Hence A does not have any eigenvalues. Then

$$f(A) = A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues of $f(A)$ are $-1, -1$. Then there is no eigenvalue μ in A such that $f(\mu) = -1$.

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That means if (λ, x) is an eigenpair of $f(A)$, then there may not have

$$\bullet \begin{bmatrix} A_{n \times n} & B_{n \times m} \\ C_{m \times n} & D_{m \times m} \end{bmatrix} \begin{bmatrix} E_{n \times n} & F_{n \times m} \\ G_{m \times n} & H_{m \times m} \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

- **[Result]** Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ and $B \in \mathbb{M}_{n \times m}$, where $m \geq n$. Then $P_{AB}(x) = x^{m-n}P_{BA}(x)$.

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We can write
$$\begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0_n \end{bmatrix} \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} 0_m & 0 \\ B & BA \end{bmatrix}$$

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The matrix $\begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}$ is invertible and $\begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix}$ is the inverse.

$$\bullet \begin{bmatrix} A_{n \times n} & B_{n \times m} \\ C_{m \times n} & D_{m \times m} \end{bmatrix} \begin{bmatrix} E_{n \times n} & F_{n \times m} \\ G_{m \times n} & H_{m \times m} \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

• **[Result]** Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ and $B \in \mathbb{M}_{n \times m}$, where $m \geq n$. Then $P_{AB}(x) = x^{m-n} P_{BA}(x)$.

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$$x_{k+1} = c_1 x_1 + c_2 x_2 + \dots + c_k x_k.$$

$$Ax_{k+1} = c_1 Ax_1 + c_2 Ax_2 + \dots + c_k Ax_k.$$

$$\lambda_{k+1} x_{k+1} = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots + c_k \lambda_k x_k.$$

$$\lambda_{k+1}(c_1x_1 + c_2x_2 + \cdots + c_kx_k) = c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_k\lambda_kx_k.$$

$$(\lambda_{k+1} - \lambda_1)c_1x_1 + \cdots + (\lambda_{k+1} - \lambda_k)c_kx_k = 0$$

$$(\lambda_{k+1} - \lambda_1)c_1 = \cdots = (\lambda_{k+1} - \lambda_k)c_k =$$

$c_1 = \cdots = c_k = 0$ as $\lambda_{k+1} - \lambda_i \neq 0$ for $i = 1, \dots, k$.

Hence x_{k+1} is zero a contradiction. Then $k = p$.

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Then we have $\sum_{i=1}^n \lambda_i = \text{trace}(A)$ and $\prod_{i=1}^n \lambda_i = \det(A)$.

Let $A \in \mathbb{M}_n(\mathbb{F})$. The $\det(A)$ and $\text{trace}(A)$ are known to you. But you cannot write $\det(A)$ is the product of the eigenvalues of A and $\text{trace}(A)$ is the sum of the eigenvalues of A . Because A may not have n number of eigenvalues.

For example $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$ in $\mathbb{M}_n(\mathbb{R})$. It has exactly one eigenvalue which is 1. The $\det(A) = 4$ which is not the product of the eigenvalues of A .

If $A \in \mathbb{M}_n(\mathbb{F})$ and the $\det(A)$ and $\text{trace}(A)$ are known to you. Then it is always true $\det(A)$ is the product of the eigenvalues of A and $\text{trace}(A)$ is the sum of the eigenvalues of A .

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 - $E_\lambda(A)$ is a subspace of \mathbb{F}^n .
 - Then $E_\lambda(A)$ is called the **eigenspace** of A corresponding to the eigenvalue of λ .
 - The subspace E_λ is a finite dimensional for each eigenvalue λ of A . The dimension of E_λ is called the **geometric multiplicity** of λ with respect to A .

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Discussions: Let $B \in \mathbb{M}_n(\mathbb{F})$. Then we can write B in the following way

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$$= P^{-1}[\lambda x_1 : \dots : \lambda x_m : Ax_{m+1} : \dots : Ax_n].$$

We can show that $P^{-1}(\lambda x_j) = \lambda P^{-1}x_j = \lambda e_j$ (Here x_j is the j th column of P and P^{-1} is the inverse of P) $j = 1, \dots, m$. Then

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So the algebraic multiplicity of λ with respect to A is at least m and the theorem follows.

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• Suppose we have a non-diagonal matrix A and if we are able to show that A is similar to a diagonal matrix, then we can easily find the above information for A .

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Then you can check that $P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. Hence A is diagonalizable matrix.

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Therefore $AP_i = d_i P_i$ for $i = 1, \dots, n$.

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This implies that d_i is an eigenvalue of A and corresponding eigenvector P_i for $i = 1, \dots, n$

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Hence $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore $P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This implies that $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is not possible.

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Q2. If A is diagonalizable, then how do I calculate such P matrix?

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Consider the following matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ in $\mathbb{M}_n(\mathbb{R})$. Then A has exactly one eigenvalue which is 1 with algebraic multiplicity 1.

So the above argument is not true if we consider some other field \mathbb{F} instead of \mathbb{C} .

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So we have at most $n - 1$ eigenvector. A contradiction that a diagonalizable matrix must have n linearly eigenvectors.

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The eigenspace $E_{\lambda_t}(A)$ gives us at most $m_t - 1$ linearly independent eigenvectors.

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Contradiction because we assume that the geometric multiplicity of λ_t is strictly less than m_t . Hence the geometric multiplicity of λ_i is m_i for $i = 1, \dots, k$.

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We have already proved that $E_{\lambda_i}(A) \cap E_{\lambda_j}(A) = \{0\}$ for $1 \leq i, j \leq k$ and $i \neq j$.

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Proof: We first assume that A is diagonalizable. Then the geometric multiplicity of λ_i is m_i for $i = 1, \dots, k$.

We have already proved that $E_{\lambda_i}(A) \cap E_{\lambda_j}(A) = \{0\}$ for $1 \leq i, j \leq k$ and $i \neq j$.

Then $\dim(E_{\lambda_1}(A) + E_{\lambda_2}(A) + E_{\lambda_3}(A) + \dots + E_{\lambda_k}(A)) = \sum_{i=1}^k \dim(E_{\lambda_i}(A)) = \sum_{i=1}^k m_i = n$.

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It can be proved that $\cup_{i=1}^k B_i$ is a basis of $E_{\lambda_1}(A) \oplus E_{\lambda_2}(A) \oplus E_{\lambda_3}(A) \oplus \cdots \oplus E_{\lambda_k}(A) = \mathbb{C}^n$.

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The cardinality of $\cup_{i=1}^k B_i$ is n . Hence we have n linearly independent vectors. Thus A is diagonalizable.

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We have seen that the algebraic multiplicity equal to the geometric multiplicity for each eigenvalue. Hence A is diagonalizable.

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If the cardinality $|B_i| = m_i$ for $i = 1, \dots, k$ (that is algebraic multiplicity = geometric multiplicity), then A is diagonalizable.

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We extend $\{x\}$ to a basis of \mathbb{C}^n which is $\{x, x_1, \dots, x_{n-1}\}$.

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$$P^*AP = [\lambda e_1 : \cdots : z_n]. \text{ Where } z_i = \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \\ \vdots \\ z_{ni} \end{bmatrix} = P^*Ay_i \text{ for } i = 2, \dots, n$$

$$P^*AP = \begin{bmatrix} \lambda & z_{12} & z_{13} & z_{14} & \cdots & z_{1n} \\ 0 & z_{22} & z_{23} & z_{24} & \cdots & z_{2n} \\ 0 & z_{32} & z_{33} & z_{34} & \cdots & z_{3n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & z_{n2} & z_{n3} & z_{n4} & \cdots & z_{nn} \end{bmatrix}.$$

$$P^*AP = \begin{bmatrix} \lambda & z^t \\ 0 & C \end{bmatrix}, \text{ where } z = \begin{bmatrix} z_{12} \\ z_{13} \\ z_{14} \\ \vdots \\ z_{1n} \end{bmatrix} \text{ and } C = \begin{bmatrix} z_{22} & z_{23} & z_{24} & \cdots & z_{2n} \\ z_{32} & z_{33} & z_{34} & \cdots & z_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ z_{n2} & z_{n3} & z_{n4} & \cdots & z_{nn} \end{bmatrix}$$

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Consider $U = PW$, it is easy to check that U is unitary and U^*AU is upper triangular.

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2. The eigenvectors corresponding to distinct eigenvalues of Hermitian (or real-symmetric) matrix are orthogonal.

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Real part of λ zero. Hence λ is purely imaginary or zero.

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 - The monic polynomial of the least degree which annihilates A is called the **minimal polynomial** of A . We use $m_A(x)$ to denote the minimal polynomial of A .

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- Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The minimal polynomial of A is $x - 1$. Consider the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The minimal polynomial of B is $(x - 1)(x + 1)$.

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Since $m_A(x)$ is a minimal polynomial, we have $m_A(A) = 0_n$. Therefore $m_A(\lambda) = 0$. Hence λ is a root of $m_A(x)$.

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Using above theorem we know that $0, -2, 2$ are the roots of the minimal polynomial of A . The following are the possibilities to be minimal polynomial.

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The first one is the characteristic polynomial of A , hence it annihilates the matrix A .

We can check that second polynomial also annihilates A . Among these two polynomials the second one is minimal. Hence the minimal polynomial is $x(x - 2)(x + 2)$.

Note: If $x(x - 2)(x + 2)$ does not annihilate A then the minimal polynomial is $x^2(x - 2)(x + 2)$.

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- **[Theorem:]** Let $A \in \mathbb{M}_n(\mathbb{C})$. Then A is diagonalizable if and only if the minimal polynomial of A is the product of distinct linear factors.