DEPARTMENT OF MATHEMATICS, IIT KHARAGPUR

Partial Differential Equation Autumn 2022 Mid Semester

Subject Name: Partial Differential Equation FM = 30M Subject Code: MA20203

Attempt all questions below

Symbols: $p \equiv \frac{\partial z}{\partial x}, q \equiv \frac{\partial z}{\partial y}, r \equiv \frac{\partial^2 z}{\partial x^2}, s \equiv \frac{\partial^2 z}{\partial x \partial y}, t \equiv \frac{\partial^2 z}{\partial y^2}$.

$f_x \equiv \frac{\partial f}{\partial x}, f_y \equiv \frac{\partial f}{\partial y}, f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}, f_{xy} = f_{yx} \equiv \frac{\partial^2 f}{\partial x \partial y}, f_{yy} \equiv \frac{\partial^2 f}{\partial y^2}$

Each question carries 5 Marks.

- Attempt all questions. Two Parts, Two pages. Part A contains Q1-Q3, and Part B contains Q4-6.
- Answer each question in both parts sequentially. Start each question from fresh page.
- Leave one blank page after Part A.

Part A

1. **[3+2=5M]**

a) Find the general solution of the 1st order PDE $(x-y)y^2p + (y-x)x^2q = (x^2+y^2)\,z$.

b) If the integral surface of the above PDE passing through the curve $xz = k^3, y = 0$, is given in the form $z^a(x^3+y^3)^b = k^c(x-y)^d$, then find the values of $a+b$ and $c+d$. The k is a fixed constant.

2. **[3+2=5M]**

a) Consider the 1st order PDE $2xz - x^2p - 2xyq + pq = 0$. By using Charpit's method if a complete integral is found in the form $z = ay^m + b(x^n - a)$, where a and b are arbitrary constants, then find m and n.

b) Write the PDE $(p^2+q^2) = 4$ in one of the standard/special types and hence find its complete integral.

3. **[3+2=5M]**

a) Wave propagation for an infinite string is governed by the equation $\frac{\partial^2 u(x,t)}{\partial t^2} = 49\frac{\partial^2 u(x,t)}{\partial x^2}, t \geq 0, x \in (-\infty, +\infty)$ subject to initial conditions $u(x,0) = \cos x$ (argument in radian), $u_t(x,0) = x$. Obtain d'Alembert's solution of the system in the simplest possible form. [No credit without showing the steps of derivation of d'Alembert's solution]

b) Consider the wave propagation of a semi-infinite string governed by the equation $\frac{\partial^2 u(x,t)}{\partial t^2} = 16\frac{\partial^2 u(x,t)}{\partial x^2}, t \geq 0, x \in (0,\infty)$ with the initial condition $u(x,0) = \sin x$ (argument in radian), $u_t(x,0) = x^2$, and the boundary condition $u(0,t) = 0$. Obtain the values of the d'Alembert's solution $u(x,t)$ at i) $(x,t) = (4,1)$, and ii) $(x,t) = (1,4)$. [You may directly write the d'Alembert's solution $u(x,t)$ (derivation need not be shown), and use this as the formula for finding values of $u(4,1)$ and $u(1,4)$.]

*****************END of Part A, Part B on Next Page*****************

Part B

4. **[2+3=5M]**

a) Eliminating arbitrary function f from the following family of surfaces

$$f(zx/y, x^2+y^2) = 0 \ (y \neq 0),$$

derive the PDE, and express it as $Pp + Qq = R$. State whether the resulting PDE is linear, semi-linear, quasi-linear or nonlinear. [Negative answer, e.g. not semi-linear, will not be awarded.]

b) Consider the following 2nd order PDE defined for $xy > 0$:

$$\ln\left(\frac{ax}{3y}\right) r + \sqrt{\ln\left\{\frac{4(a+1)y}{x}\right\}}\, s - \frac{1}{4} t + p^2 - pq = x + 2\sqrt{a}y + az^2 \ (a > 0).$$

Find the value/s of a for which the PDE would be elliptic.

5. **[1+1+3=5M]**

a) The solution of a heat equation $\alpha\psi_t = \psi_{xx}, x \in (0,1), \alpha \neq 0$ is of the form $\psi(x,t) = X(x)T(t)$, where, for all time $t > 0$, the solution vanishes at both ends $x = 0$ and at $x = 1$. Using the separation $X''/\alpha X = T'/T = -\mu^2$, find the range of allowed values of μ.

b) Consider a heat equation $2\partial u/\partial t = \partial^2 u/\partial x^2$ defined on $0 < x < l, t > 0$. Given that for all time $t > 0$, the solution $u(0,t) = 0$ and $u(l,t) = \sin(lt)$. Find a transformation $u \to U$ so that $U(0,t) = U(l,t) = 0$ for all $t > 0$ and for all $l > 0$.

c) Given that $\phi(0,t) = 0 = \phi(2,t) - 1$, where $\phi(x,t)$ satisfies the PDE $\phi_t = \phi_{xx}, x \in (0,2), t > 0$. It is known that $\phi(x,0) = kx, x \in (0,2)$. If $\Phi(0,t) = \Phi(2,t) = \Phi(x,0) - 2x = 0$ for all $t > 0$, and for all $x \in (0,2)$, determine the value of k. The $\Phi(x,t)$ is a suitably transformed dependent variable from the original dependent variable $\phi(x,t)$ that also satisfies the same PDE: $\Phi_t = \Phi_{xx}, x \in (0,2), t > 0$.

6. **[1+4=5M]**

a) The solution of certain heat equation with homogeneous BC (i.e. zero value at both ends) is $\psi(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{2}\right) \exp[-n^2\pi^2 t], x \in (0,2), t \geq 0$. If $\psi(x,0) = \sin^2\left(\frac{\pi x}{2}\right)$ for $x \in (0,1)$ and $\psi(x,0) = 0$ for $x \in (1,2)$, compute C_1, C_2.

b) Consider the IBVP: $\psi_t = \psi_{xx}, 0 < x < 1, t > 0$, BC: $\psi(0,t) = 0, \psi(1,t) = 1, \forall\, t > 0$, IC: $\psi(x,0) = \cos^3(\pi x), x \in (0,1)$. The Fourier-series solution is given below

$$\psi(x,t) = x + \sum_{n=1}^{\infty} C_n \sin(n\pi x) \exp[-n^2\pi^2 t].$$

Compute the Fourier coefficients C_n, and give compact expressions separately for $C_{2k}, k = 1,2, \ldots$ and for $C_{2k+1}, k = 0,1,2, \ldots$.

**************End of Part B and End of Questions**************