0.2 Min Max risk

See class note for details.

$L_\infty-$Risk(or Minmax Risk) of a portfolio $(w(x_1, x_2, ...x_n))$ is defined as

$$\max_i \ E(\mid r_i x_i - \mu_i x_i \mid)$$

Denote $q_i x_i = E(\mid r_i x_i - \mu_i x_i \mid)$.

Min-Max portfolio optimization model is

$$\min_x \max_i \{q_i x_i\} \ Subject \ to \ \mu^T x \geq MR, \ e^T x = M$$

This is same as

$$\min_x \ y \ Subject \ to \ y \geq q_i x_i, \ \mu^T x \geq MR, \ e^T x = M$$

0.2.1 Relation between variance risk measure and MAD risk measure

Discussed in class. See class note.

0.3 Application of Integer Programming in Finance

Integer programming techniques(Branch and Bound method and Gomory cutting plane method are discussed in the class. See class note. You may refer S S Rao book.)

0.3.1 Combinatorial auction

Auction is a process of buying and selling goods by offering them up for bid, taking bid and selling the item to the highest bidder.

- $M = \{1, 2, ..., m\}$ be the set of items the auctioneer has to sell.

- Bid is a pair $B_j = (S_j, p_j)$, where $s_j \subseteq M$ is a nonempty set of items and p_j is the price offer for this bid.
- Suppose the auctioneer has received n bids $B_1, B_2, ...B_n$.
- How should the auctioneer determine the winners in order to maximize his revenues?
- Let $x_j = 1$ if bid B_j wins and 0 if looses.

$$
\begin{aligned}
\max \quad & \sum_j p_j x_j \\
S.To \quad & \sum_{j:i\in S_j} x_j \leq 1, \quad i = 1, 2, ..., m \\
& x_j \in \{0, 1\}, j = 1, 2, ..., n
\end{aligned}
$$

Example Suppose there are 4 items for sale and following bids are received. Formulate the auction model to maximize profit.

$$B_1 = (\{1, 2\}, 6), B_2 = (\{1, 2, 4\}, 7), B_3 = (\{1, 3, 4\}, 8)$$

$$
\begin{aligned}
\max \quad & 6x_1 + 7x_2 + 8x_3 \\
s.to \quad & x_1 + x_2 + x_3 \leq 1 \\
& x_1 + x_2 \leq 1 \\
& x_3 \leq 1 \\
& x_2 + x_3 \leq 1, x_j \in \{0, 1\}
\end{aligned}
$$

0.3.2 Application: Index fund

There are two ways of portfolio management: (i) active portfolio management which uses technical and fundamental analysis as well as forecasting techniques, (ii) passive portfolio management avoids forecasting techniques and relies on diversification. There are two types of passive management strategies: bye and hold, and indexing. We will see how integer programming helps to create an index fund.

Objective is to choose the portfolio that mirrors the movement of a broad market portfolio or market index. Such a portfolio is known as INDEX FUND. Some market indices of S & P BSE SENSEX, S & P BSE 100, S & P BSE MIDCAP etc.

INDEX FUND MODEL:

Suppose a fund manager wants to construct an index fund. Strategies for forming an index fund of choosing a broad market index as a proxy for an entire market. We propose a large scale deterministic model for aggregating a broad market index of stocks into a smaller more manageable index fund. This model clusters the assets into groups of similar assets and selects one representative asset from each group.

ρ_{ij} = Similarity between stock i and stock j. $\rho_{ii} = 1$ and $0 \cancel{\geq} < \rho_{ij} < 1$ for $i \neq j, i, j = 1, 2, ..., n$.

$$y_j = \begin{cases} 1 & if \quad \text{j stock is selected in the index fund} \\ 0 & if \quad \text{j stock is rejected in the index fund} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & if \quad \text{j stock is most similar to stock i} \\ 0 & if \quad \text{j otherwise} \end{cases}$$

Objective of the model is to maximize similarity among all stocks, which is $\max \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} x_{ij}$

Suppose total number of stocks in the index fund is q. Hence $\sum_{j=1}^{n} y_j = q$

Each stock i has exactly one representative stock j in the index fund. Hence $\cancel{\sum_{i=1}^{n}} \sum_{j=1}^{n} x_{ij} = 1$.

Stock i can be represented by stock j in the index fund. Hence $x_{ij} \leq y_j$ for all i, j

The corresponding optimization model is

$$\max \sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}x_{ij}$$
$$subject\ \ to \sum_{j=1}^{n} y_j = q, \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = 1, x_{ij} \leq y_j$$
$$x_{ij}, y_j \in \{0, 1\}$$

0.4 Stochastic Programming Application in Finance: Value at Risk and Conditional Value at risk

For stochastic programming technique see class note. Discussed in detail in the class. You may see S.S.Rao book also.
Application:
Time value of Money: Let $V(t)$ denote the value at money of the investment amount M for time period t,expressed in years, with certain interest rate $r\%$.

- In case of simple interest rate $r\%$ per year, $V(t) = M(1 + rt)$.
- In case of periodic compound interest rate $r\%$ over the period m, which may be annually semi-annually, quarterly etc., $V(t) = M(1 + \frac{r}{m})^{tm}$.
- In case of continuous compounding,

$$\begin{aligned} V(t) &= Lim_{m\to\infty} M(1 + \frac{r}{m})^{tm} \\ &= M.Lim_{m\to\infty}[(1 + \frac{r}{m})^{m/r}]^{rt} \\ &= M e^{rt} \end{aligned}$$

0.4.1 Value at Risk of an asset

Suppose an investor purchases a share of a stock at time $t = 0$ with purchase price $S(0)$ and wants to sell at some time $t = T$. Then $S(T)$ is a random variable. The

investor will suffer loss if $S(T) < V(T)$. A natural question raises here to determine the probability that the loss is less than or equal to some fixed amount say K, that is $P(V(T) - S(T) \leq K)$.
$P(V(T) - S(T) \leq K)$ can be determined when the probability distribution of $S(T)$ is known. We may reverse this logic by fixing the probability in place of fixing the amount as follows.
With fixed probability α, determine a value so that the probability of loss not exceeding that amount is more than equal to α. That is $P(V(T) - S(T) \leq K) \geq \alpha$
VaR represents the predicted maximum loss with specified probability for a certain period of time. Hence VaR is a measure related to percentiles of loss distribution. (VaR concept is developed by JP Morgan related software is Risk Metrics Software available since 1994.)

Definition 0.4.1. *Let X be a random variable and α be the given probability level. Then the VaR of X with confidence level $1-\alpha$, denoted by $VaR_{1-\alpha}(X)$ is defined as*

$$VaR_{1-\alpha}(X) = Min \ \{z : P(X \leq z) \geq 1 - \alpha\}$$

This is a chance constrained programming problem:

$$Min \ z \ subject \ to \ P(X \leq z) \geq 1 - \alpha,$$

which is same as

$$Min \ z \ subject \ to \ z \geq F_X^{-1}(1 - \alpha),$$

where F_X is the cumulative distribution function of X. Minimum takes place at $z* = F_X^{-1}(1-\alpha)$. Hence $VaR_{1-\alpha}(X) = F_X^{-1}(1-\alpha)$
In practice α is considered as $1\%, 5\%$ and 10%

- Find VaR with 95% confidence level if the loss function follows $N(1, 4)$.
- Suppose the stock price follows log normal distribution with mean 12% and standard deviation 30%. The interest rate is 8% for the period $t = 0$ to $t = 1$ and

continuous compound. Stock price at time $t = 0$ is Rs 100. Determine VaR of this stock at 95% confidence level.

Difficulties in VaR:

- VaR is not sub additative.That is for two different investments A and B, $VaR(A \bigcup B) \nleq VaR(A) + VaR(B)$. Hence total risk of two different portfolios investments may exceeds the sum of the individual risks. This violates the principle that diversification reduces risk.
- VaR does not pay attention towards tail risk.
- Difficult to solve if the constraint is of the form $VaR_{1-\alpha}(\sum a_i x_i) = p$ in case the random variables are not identically distributed.

0.4.2 Conditional Value at Risk

Conditional value at risk is obtained by computing the expected loss given that the loss exceeds $VaR_{1-\alpha}$.
Consider a Portfolio of n assets with random return $r = (r_, r_2, ..., r_n)$.
$f(w, r)$= Loss function when investment is w from a set of feasible portfolios. $w = (w_1, w_2, ..., w_n)$
$-f(w, r) = w^T r$ is the return of the portfolio. $p(r)$ is the probability density function of r.
For a given investment w, define

$$\Psi(w, r) = \int_{f(w,r) \leq z} p(r) dr$$

$\Psi(w, r)$ is the cumulative distribution of loss associated with the investment vector w.
Hence

$$VaR_{1-\alpha}(w) = Min\{z : \Psi(w, r) \geq 1 - \alpha\}$$

Definition 0.4.2. *Let w be a given portfolio and $1 - \alpha$ be the given confidence level.*

Then $CVaR_{1-\alpha}(w)$ associated with the portfolio w is defined as

$$CVaR_{1-\alpha}(w) = \frac{1}{\alpha}\int_{f(w,r)\geq VaR_{1-\alpha}(w)} f(w,r)p(r)dr$$

In case the returns having discrete probability distribution,

$$CVaR_{1-\alpha}(w) = \frac{1}{\alpha}\sum_{j:f(w,r^j)\geq VaR_{1-\alpha}(w)} f(w,r^j)p(r^j),$$

r^j is the j^{th} realization vector of r. ($CVaR_{1-\alpha}(w)$ is the average risk greater than $VaR_{1-\alpha}(w)$)

Example 0.4.1. *Loss function of a stock is $f(w,r) = -r$, where $r = 75 - j, j = 0, 1, ...99$ with probability 1%. Find VaR and CVaR at 95% confidence level.*

Ans: Values of $r^j = 75, 74, ...0, -1, -2, ... - 19, -20, -21, -22, -23, -24.$

X denotes the discrete random variable representing loss function.

$X : -r^j = -75, -74, ..., 0, 1, 2, ..., 19, 20, ..., 24,$ $p^j = P(X = r^j) = 0.01\ \forall j$

$P(X = -75) = 0.01; P(X \leq -74) = P(X = -75)+P(X = -74) = 0.02; ....; P(X \leq 19) = 0.94; P(X \leq 20) = 0.95; ..., P(X \leq 24) = 1$

$VaR_(0.95) = Min_j\{-r^j : P(X \leq -r^j) \geq 0.95\} = 20$

$CVaR_{0.95}(w) = \frac{1}{0.05}(20 + 21 + 22 + 23 + 24)(0.01) = 22$

Example 0.4.2. *Suppose price of an asset follows normal distribution with mean 20% and standard deviation 10%. The interest rate is compound with interest rate 2% for one year term. At the start of the year asset price is Rs 50. Find VaR with 90% confidence level.*

Ans: $A(t)$ =Price of the asset. $A(0) = 50$. $V(t) = 50(1 + \frac{0.02}{1})^{t.1} = 50(1.02)^t$

$\frac{A(1)}{A(0)} \sim N(0.2, 0.1^2)$. *That is* $A(1) \sim 50.N(0.2, 0.1^2)$

Loss function is $V(1) - A(1) = 50(1.02) - A(1)$

$P(V(1) - A(1) \leq VaR_{0.99}) = 0.99$

$P(50(1.02) - A(1) \leq VaR_{0.99}) = 0.99$

$P(A(1) \geq 50(1.02) - VaR_{0.99}) = 0.99$

$1 - P(\frac{A(1)-0.2}{0.1} \leq \frac{50(1.02)-VaR_{0.99}-0.2}{0.1}) = 0.99$

$F(\frac{50(1.02)-VaR_{0.99}-0.2}{0.1}) = 0.01$
$VaR_{0.99} = 50(1.02) - 0.2 - (0.1).F^{-1}(0.01)$

Rockafeller Approach:
$r^s = (s = 1, 2, ..., n_s) =$ Historical values of the random vector of returns, obtained via computer simulation.
Assume that all scenarios have equal probability.
Define the following empirical distribution of the random returns based on the available scenarios.

$$\hat{F}_{1-\alpha}(w, q) = q + \frac{1}{\alpha.n_s}\sum_{s=1}^{n_s}(f(w, r^s) - q)^+,$$

where $a^+ = \max\{0, a\}$.
The portfolio optimization model becomes

$$\min_{w,q} \hat{F}_{1-\alpha}(w, q) \;\; s.to \;\; e^T w = 1,$$

which is equivalent to

$$\begin{aligned} \min_{w,q,z_s} \;\; & q + \frac{1}{\alpha.n_s}\sum_{s=1}^{n_s} z_s \\ s.to \;\; & z_s \geq f(w, r^s) - q, \;\; s = 1, 2, ..., n_s \\ & e^T w = 1, z_s \geq 0, \;\; s = 1, 2, ..., n_s \end{aligned}$$

Other CVaR models:
R Manisi, W.Ogryczak and S.Uryasev ”Conditional Value at risk and related linear programming model for portfolio optimization”, Annals of Operations Research, vol 2007.

$$\max \;\; \mu^T w \;\; s.to \;\; CVaR_{1-\alpha}(w) \leq u_\alpha, \;\; e^T w = 1$$

$$\begin{aligned}
\max \quad & \mu^T w \\
s.to \quad & u_\alpha \geq q + \frac{1}{\alpha.n_s}\sum_{s=1}^{n_s} z_s \\
& z_s \geq f(w, r^s) - q, \quad s = 1, 2, ..., n_s \\
& e^T w = 1, z_s \geq 0, \quad s = 1, 2, ..., n_s
\end{aligned}$$

0.5 Coherent risk

A risk measure $\rho(X)$ assigns a numerical value to a random wealth X. ρ is said to be Coherent risk measure (see Artzner et al. (1999)) if this satisfies following properties.

- $\rho(0) = 0$. That is, the risk of holding no assets is zero.
- If X_1, X_2 are two portfolios then $X_1 \leq X_2 \Rightarrow \rho(X_1) \geq \rho(X_2)$. That is, if portfolio X_2 always has better values than portfolio X_2 under almost all scenarios then the risk of X_2 should be less than the risk of X_1 In financial risk management, monotonicity implies a portfolio with greater future returns has less risk.
- (Sub-additivity)If X_1, X_2 are two different portfolios then $\rho(X_1+X_2) \leq \rho(X_1)+\rho(X_2)$. Indeed, the risk of two portfolios together cannot get any worse than adding the two risks separately: this is the diversification principle. In financial risk management, sub-additivity implies diversification is beneficial.
- (Positive homogeneity) For $\alpha \geq 0$, $\rho(\alpha X) = \alpha\rho(X)$. That is, if you double your portfolio then you double your risk. In financial risk management, positive homogeneity implies the risk of a position is proportional to its size.
- (Translation invariance) If A is a deterministic portfolio with guaranteed return a then $\rho(X+A) = \rho(X)-a$. The portfolio A is just adding cash $a$to your portfolio X. In particular, if $\rho(X) = a$ then $\rho(X + A) = 0$. In financial risk management, translation invariance implies that the addition of a sure amount of capital reduces the risk by the same amount.

The notions of Sub-additivity and Positive Homogeneity can be replaced by the notion of convexity: $\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$ for $\lambda \in [0,1]$

Risk measure	Positive Homogeneity	Sub-additivity	Monotonicity	Translation Invariant
Variance	No	Yes	No	No
VaR	Yes	No	Yes	Yes
CVaR	Yes	Yes	Yes	Yes
Semi Variance	No	Yes	No	No
MAD	Yes	Yes	Yes	Yes