Duality Theorems

Theorem 1: The dual problem of the dual is the primal

Proof:

$\boxed{\textbf{D}}$

$$\begin{aligned} &\text{Min } \mathbf{yb} \\ &\text{S.t.} \quad \mathbf{yA} \geq \mathbf{c} \\ &\mathbf{y} \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\text{Max } -\mathbf{yb} \\ &\text{S.t.} \quad -\mathbf{yA} \leq -\mathbf{c} \\ &\mathbf{y} \geq 0 \end{aligned}$$

$\boxed{\textbf{D'}}$ $\qquad\qquad$ $\boxed{\textbf{P}}$

$$\begin{aligned} &\text{Min } -\mathbf{cx} \\ &\text{S.t.} \quad -\mathbf{Ax} \geq -\mathbf{b} \\ &\mathbf{x} \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\text{Max } \mathbf{cx} \\ &\text{S.t.} \quad \mathbf{Ax} \leq \mathbf{b} \\ &\mathbf{x} \geq 0 \end{aligned}$$

Primal [P]

Max Z = **cx** **Dual variable**

S.t. $\mathbf{Ax} \leq \mathbf{b}$ **y**

$\mathbf{x} \geq \mathbf{0}$

Dual [D]

Min W = **yb**

S.t. $\mathbf{y}\mathbf{A} \geq \mathbf{c}$

$\mathbf{y} \geq \mathbf{0}$

Theorem 2: Weak Duality Theorem (WDT)

If **x** is a f.s. to [P] and **y** is a f.s. to [D], then $\mathbf{cx} \leq \mathbf{yb}$.
(i.e. maximum profit is constrained by the value of resources)

Proof: **x** is feasible to [P] => $\mathbf{Ax} \leq \mathbf{b}$ **(1)**

y is feasible to [D] => $\mathbf{yA} \geq \mathbf{c}$ **(2)**

Pre-multiply (1) by **y** and post multiply (2) by **x**, then

$$\mathbf{cx} \leq \mathbf{yAx} \leq \mathbf{yb} \quad \Rightarrow \mathbf{cx} \leq \mathbf{yb}$$

Observations from WDT ($\mathbf{cx} \leq \mathbf{yb}$)

Note: Primal refers to the maximization problem and dual refers to the corresponding minimization problem

1) The Objective function value of the primal for its any feasible solution is a lower bound to the minimum value of the dual problem
2) Similarly, the Objective function value of the dual for its any feasible solution is an upper bound to the maximum value of the primal problem
3) If the primal is unbounded (i.e., Max Z->+∞), then the dual problem is infeasible.
4) If the dual is unbounded (i.e., Min W->-∞), then the primal is infeasible.

Theorem 3: Strong Duality Theorem (SDT)

If **x*** and **y*** are feasible to [P] and [D] respectively and **cx* = y*b**, then **x*** and **y*** are optimal to [P] and [D], respectively.

Proof: From WDT, **cx** ≤ **yb** for any feasible **x** and **y**.
Then **cx** ≤ **y*b** for any feasible **x** and **y***.
But **cx* = y*b (Given)**
Then **cx ≤ cx*** for any feasible **x**
=> x* is optimal to [P] {maximization problem}

Similarly, y* is optimal to [D]

Simplex Tableau for Primal at any iteration

BV	$\mathbf{x}^T$	$\mathbf{x}_s^T$	RHS
$\mathbf{x}_B$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$
Z	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$

Proposition: If **B** is the optimal primal basis, then optimal solution of the dual problem is $\mathbf{y^* = c_B B^{-1}}$

Proof:

From SDT, $\mathbf{cx^*=y^*b}$

$\Rightarrow \mathbf{c_B x_B = y^*b}$

$\Rightarrow \mathbf{c_B B^{-1} b = y^*b}$

$\Rightarrow \mathbf{y^* = c_B B^{-1}}$

Economic interpretation of Duality (dual variable)

If **B** is the optimal primal basis

At optimality, $\mathbf{X_B} = \mathbf{B^{-1}b}$ and So $Z = \mathbf{C_B B^{-1} b} = \mathbf{C_B B^{-1}} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_i \\ \vdots \\ \mathbf{b}_m \end{bmatrix}$

Suppose optimal basis **B** does not change by changing the ith resource from b_i to $b_i + \Delta b_i$.

$$\boldsymbol{b}' = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_i + \Delta \boldsymbol{b}_i \\ \vdots \\ \boldsymbol{b}_m \end{bmatrix}$$

So $Z' = \mathbf{C_B B^{-1} b'}$

The Change in objective function,

$$\Delta Z = C_B B^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Delta b_i \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} y_1^* & y_2^* & \cdots & y_i^* & \cdots & y_m^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Delta b_i \\ \vdots \\ 0 \end{bmatrix}_{m \times 1} = y_i^* \Delta b_i$$

$\Rightarrow \Delta Z / \Delta b_i = \mathbf{y_i}^* =$ Shadow Price or Dual Price

Complementary Slackness Theorem (CST)

- Suppose

  Primal [P] Max $\mathbf{Z}=\mathbf{cx}$ s.t. $\mathbf{Ax}\leq\mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$

  Dual [D] Min $\mathbf{W}=\mathbf{yb}$ s.t. $\mathbf{yA} \geq \mathbf{c}$, $\mathbf{y}\geq 0$

- If $\mathbf{x}$ and $\mathbf{y}$ are feasible solution to P and D, respectively. Then $\mathbf{x}$ and $\mathbf{y}$ are optimal to their respective problems iff

$$(\mathbf{yA}-\mathbf{c})\mathbf{x} + \mathbf{y}(\mathbf{b}-\mathbf{Ax}) = 0$$

Complementary Slackness Theorem (CST)

Proof: $\mathbf{Ax} + \mathbf{u} = \mathbf{b}$ (i), $\mathbf{yA} - \mathbf{v} = \mathbf{c}$ (ii)

Where $\mathbf{u}, \mathbf{v} \geq 0$

$$\mathbf{u}=\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_m \end{bmatrix} \qquad \mathbf{v}=\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

Pre-multiply (i) by **y** and post-multiply (ii) by **x,** we get

$$\mathbf{yAx} + \mathbf{yu} = \mathbf{yb} \quad \text{(iii)} \quad \text{and} \quad \mathbf{yAx} - \mathbf{vx} = \mathbf{cx} \quad \text{(iv)}$$

Now, (iii) – (iv),

$$\mathbf{yu} + \mathbf{vx} = \mathbf{yb} - \mathbf{cx} \Rightarrow \mathbf{y(b\text{-}Ax)+(yA\text{-}c)x} = \mathbf{yb} - \mathbf{cx}$$

and from the given condition $\mathbf{y(b\text{-}Ax)+(yA\text{-}c)x = 0}$

$\Rightarrow \mathbf{yb} - \mathbf{cx} = \mathbf{0} \Rightarrow \mathbf{yb} = \mathbf{cx}$, which implies optimality of **x** and **y** (from SDT)

The proof is complete when the other side of statement is also proved, i.e. if x and y are optimal (=> $\mathbf{yb} = \mathbf{cx}$) it must be $\mathbf{yu} + \mathbf{vx} = 0$, which is straightforward.

Implications of CST

(i) Complementary slackness conditions

Since $\mathbf{x}$, $\mathbf{u}$, $\mathbf{y}$, $\mathbf{v} \geq 0$, if the sum of nonnegative terms equals zero, then each term is zero.

$$\mathbf{yu} = 0 \quad \Rightarrow \quad \mathbf{y}(\mathbf{b} - \mathbf{Ax}) = 0$$

and

$$\mathbf{vx} = 0 \quad \Rightarrow \quad (\mathbf{yA} - \mathbf{c})\mathbf{x} = 0$$

(ii) Using duality interpretations and the condition

If $y_i > 0 \Rightarrow u_i = 0$

Thus, if another party is willing to +ve prices, then it must be the case that the resource must have been utilized fully. (Valuable resource)

If $u_i > 0 \Rightarrow y_i = 0$ (resource not used fully)

(iii) CST conditions can be used to find an optimal primal solution from an optimal dual solution, and vice-versa

Example

[P]

		Dual variable	Surplus variable
	Min $W = 2y_1 + 3y_2+5y_3+2y_4+3y_5$		
Subject to	$y_1 + y_2 + 2y_3 + y_4 + 3y_5 \geq 4$	x_1	v_1
	$2y_1 - 2y_2 + 3y_3 + y_4 + y_5 \geq 3$	x_2	v_2
	and $y_1, y_2, y_3, y_4, y_5 \geq 0$		

[D]

			Slack variable
	Max $Z = 4x_1+3x_2$		
Subject to	$x_1+2x_2 \leq 2$	(1)	u_1
	$x_1-2x_2 \leq 3$	(2)	u_2
	$2x_1+3x_2 \leq 5$	(3)	u_3
	$x_1+x_2 \leq 2$	(4)	u_4
	$3x_1+x_2 \leq 3$	(5)	u_5
	$x_1, x_2 \geq 0$		

Optimal Solution:
$x_1 = 4/5$, $x_2 = 3/5$ and $Z = 5$

For which constraints, slack > 0 => $u_2, u_3, u_4 > 0$. From CST, primal variable y's corresponding to these constraints = 0 => $y_2= y_3= y_4 = 0$

Also, from CST, since $x_1, x_2 > 0$ => primal constraint should be satisfied as equality, i.e. $v_1= v_2 = 0$

Now, [P] reduces to $y_1 + 3y_5 = 4$ and $2y_1 + y_5 = 3$ => $y_1 = 1$, $y_5 = 1$ and $W = 5$

Dual Simplex Method

- Primal Problem:

$$\text{Maximize } Z = \mathbf{cx}$$
$$\text{Subject to } \mathbf{Ax} \leq \mathbf{b}$$
$$\mathbf{x} \geq 0$$

- Basis $\mathbf{B}$ is primal feasible if and only if $\mathbf{x_B} = \mathbf{B^{-1}b} \geq 0$
- Suppose this basis $\mathbf{B}$ is also feasible for the dual problem (called dual feasible), then

$$\mathbf{y} = \mathbf{c_B B^{-1}} \geq 0$$

and $\mathbf{yA} \geq \mathbf{c}$

$$\Rightarrow \mathbf{c_B B^{-1} A} - \mathbf{c} \geq 0$$

$$\Rightarrow z_j - c_j \geq 0, \forall j$$

which are optimality conditions for the primal problem

- Thus, if a Basis $\mathbf{B}$ is dual feasible, then it is also primal optimal
- Since, this is primal optimal, using SDT ($\mathbf{cx} = \mathbf{yb}$ **as** $\mathbf{x_B} = \mathbf{B^{-1}b}$ and $\mathbf{y} = \mathbf{c_B B^{-1}}$) $\Rightarrow$ it is also dual optimal
- Thus, the main crux in the LP is to find the basis $\mathbf{B}$ which is both primal and dual feasible $\rightarrow$ Optimality

- **(Primal) simplex method:** Moves from one primal feasible basis to another till it achieves dual feasibility, i.e.

  $z_j - c_j \geq 0 \implies$ optimality

- **Dual simplex method:** Starts with a dual feasible Basis and move towards primal feasibility

- Difference from primal simplex
  - Entering variable / leaving rule
  - Optimality

Algorithm: Dual Simplex

1) The problem should maximization type with $\leq$ constraints

2) Updated objective function coefficients are $\geq 0 \Rightarrow$ Dual feasibility

3) At least one RHS is negative $\Rightarrow$ Primal infeasibility

4) Leaving variable: The basic variable with most (-ve) RHS

5) Entering variable: Maximum ratio rule, that x_j for which the ratio

   $\left\{\frac{z_j - c_j}{a_{jk}} : a_{jk} < 0\right\}$ is maximum, where x_k is leaving variable

6) Perform elementary row operations as usual

Example: Dual simplex

- Min $Z = 3x_1 + 2x_2$

Subject to $3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

- Convert into maximization type with all the constraints $\leq$ type

$Max\ Z' = -Z = -3x_1 - 2x_2$ max $z' + 3x_1 + 2x_2 = 0$

Subject to

$-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

- Augmented form

$Max\ Z' = -Z = -3x_1 - 2x_2$

Subject to

$-3x_1 - x_2 + x_3 = -3$

$-4x_1 - 3x_2 + x_4 = -6$

$x_1 + x_2 + x_5 = 3$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Solving using Dual simplex

	Basis	x_1	x_2	x_3	x_4	x_5	RHS
Iteration 0	x_3	-3	-1	1	0	0	-3
	x_4	-4	-3	0	1	0	-6
	x_5	1	1	0	0	1	3
	Z'	3	2	0	0	0	0
	Ratio	-3/4	**-2/3**	-	-	-	
Iteration 1	x_3	-5/3	0	1	-1/3	0	-1
	x_2	4/3	1	0	-1/3	0	2
	x_5	-1/3	0	0	1/3	1	1
	Z'	1/3	0	0	2/3	0	-4
	Ratio	**-1/5**	-	-	-2	--	
Iteration 2	x_1	1	0	-3/5	1/5	0	**3/5**
	x_2	0	1	4/5	-3/5	0	**6/5**
	x_5	0	0	-1/5	2/5	1	**6/5**
	Z'	0	0	1/5	3/5	0	**-21/5**

Dual feasible and optimal: $x_1 = {}^{3}/_{5}, x_2 = {}^{6}/_{5}, Z = -Z' = {}^{21}/_{5}$

Graphical Interpretation