- **[Definition:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the **same field** $\mathbb{F}$.

- **[Definition:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the **same field** $\mathbb{F}$.

  A mapping $T : \mathbb{V} \to \mathbb{W}$ is said to be a linear transformation if

  $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in \mathbb{V}$ and for all $\alpha, \beta \in \mathbb{F}$.

- **[Definition:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the **same field** $\mathbb{F}$.

  A mapping $T : \mathbb{V} \to \mathbb{W}$ is said to be a linear transformation if

  $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in \mathbb{V}$ and for all $\alpha, \beta \in \mathbb{F}$.

- **[Example:]**
  1. Let $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = x$. Then T is a linear transformation.

- **[Definition:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the **same field** $\mathbb{F}$.

A mapping $T : \mathbb{V} \to \mathbb{W}$ is said to be a linear transformation if

$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in \mathbb{V}$ and for all $\alpha, \beta \in \mathbb{F}$.

- **[Example:]**
  1. Let $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = x$. Then T is a linear transformation.
  2. Let $A \in \mathbb{M}_{m\times n}(R)$, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $T(x) = Ax$, $x \in \mathbb{R}^n$. Then T is a linear transformation.

- **[Definition:]** Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the **same field** $\mathbb{F}$.

  A mapping $T : \mathbb{V} \to \mathbb{W}$ is said to be a linear transformation if

  $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in \mathbb{V}$ and for all $\alpha, \beta \in \mathbb{F}$.

- **[Example:]**
  1. Let $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = x$. Then T is a linear transformation.
  2. Let $A \in \mathbb{M}_{m\times n}(R)$, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $T(x) = Ax$, $x \in \mathbb{R}^n$. Then T is a linear transformation.
  3. Let $T : C[a, b] \to \mathbb{R}$ be defined by $T(f) = \int\limits_a^b f(x)dx$. Then T is a linear transformation.

1. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over the same field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{W}$

1. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over the same field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{W}$ be defined by $T(x) = 0$, $x \in \mathbb{V}$. This transformation is called **Zero transformation**.

1. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over the same field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{W}$ be defined by $T(x) = 0$, $x \in \mathbb{V}$. This transformation is called **Zero transformation**.

2. Let $\mathbb{V}$ be vector space over the field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{V}$ be defined by

1. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over the same field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{W}$ be defined by $T(x) = 0$, $x \in \mathbb{V}$. This transformation is called **Zero transformation**.

2. Let $\mathbb{V}$ be vector space over the field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{V}$ be defined by $T(x) = x$, $x \in \mathbb{V}$. This transformation is called **identity transformation**.

1. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over the same field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{W}$ be defined by $T(x) = 0$, $x \in \mathbb{V}$. This transformation is called **Zero transformation**.
2. Let $\mathbb{V}$ be vector space over the field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{V}$ be defined by $T(x) = x$, $x \in \mathbb{V}$. This transformation is called **identity transformation**.
3. Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$ and let $\lambda \in \mathbb{F}$. Let

1. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over the same field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{W}$ be defined by $T(x) = 0$, $x \in \mathbb{V}$. This transformation is called **Zero transformation**.
2. Let $\mathbb{V}$ be vector space over the field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{V}$ be defined by $T(x) = x$, $x \in \mathbb{V}$. This transformation is called **identity transformation**.
3. Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$ and let $\lambda \in \mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{V}$ be defined by $T(x) = \lambda x$, $x \in \mathbb{V}$. This transformation is called **scalar transformation**.

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT.

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

Proof: We know that $0_{\mathbb{V}} + 0_{\mathbb{V}} = 0_{\mathbb{V}}$.

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

Proof: We know that $0_{\mathbb{V}} + 0_{\mathbb{V}} = 0_{\mathbb{V}}$.

$T(0_{\mathbb{V}} + 0_{\mathbb{V}}) = T(0_{\mathbb{V}})$

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

Proof: We know that $0_{\mathbb{V}} + 0_{\mathbb{V}} = 0_{\mathbb{V}}$.

$T(0_{\mathbb{V}} + 0_{\mathbb{V}}) = T(0_{\mathbb{V}})$

$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) = T(0_{\mathbb{V}})$

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

Proof: We know that $0_{\mathbb{V}} + 0_{\mathbb{V}} = 0_{\mathbb{V}}$.

$T(0_{\mathbb{V}} + 0_{\mathbb{V}}) = T(0_{\mathbb{V}})$

$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) = T(0_{\mathbb{V}})$

$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}})) = T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}}))$

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

Proof: We know that $0_{\mathbb{V}} + 0_{\mathbb{V}} = 0_{\mathbb{V}}$.

$$T(0_{\mathbb{V}} + 0_{\mathbb{V}}) = T(0_{\mathbb{V}})$$

$$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) = T(0_{\mathbb{V}})$$

$$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}})) = T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}}))$$

$$\implies T(0_{\mathbb{V}}) + 0_{\mathbb{W}} = 0_{\mathbb{W}}$$

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

Proof: We know that $0_{\mathbb{V}} + 0_{\mathbb{V}} = 0_{\mathbb{V}}$.

$$T(0_{\mathbb{V}} + 0_{\mathbb{V}}) = T(0_{\mathbb{V}})$$

$$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) = T(0_{\mathbb{V}})$$

$$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}})) = T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}}))$$

$$\implies T(0_{\mathbb{V}}) + 0_{\mathbb{W}} = 0_{\mathbb{W}}$$

$$\implies T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$$

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

Proof: We know that $0_{\mathbb{V}} + 0_{\mathbb{V}} = 0_{\mathbb{V}}$.

$T(0_{\mathbb{V}} + 0_{\mathbb{V}}) = T(0_{\mathbb{V}})$

$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) = T(0_{\mathbb{V}})$

$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}})) = T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}}))$

$\implies T(0_{\mathbb{V}}) + 0_{\mathbb{W}} = 0_{\mathbb{W}}$

$\implies T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

$T : \mathbb{R} \to \mathbb{R}$ be a map defined by $T(x) = x + 1$.

- **[Theorem:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a LT. Then $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$

Proof: We know that $0_{\mathbb{V}} + 0_{\mathbb{V}} = 0_{\mathbb{V}}$.

$$T(0_{\mathbb{V}} + 0_{\mathbb{V}}) = T(0_{\mathbb{V}})$$

$$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) = T(0_{\mathbb{V}})$$

$$\implies T(0_{\mathbb{V}}) + T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}})) = T(0_{\mathbb{V}}) + (-T(0_{\mathbb{V}}))$$

$$\implies T(0_{\mathbb{V}}) + 0_{\mathbb{W}} = 0_{\mathbb{W}}$$

$$\implies T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$$

$T : \mathbb{R} \to \mathbb{R}$ be a map defined by $T(x) = x + 1$. Using above theorem you can say that T is not linear.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}$ be a LT. Then T has the following form.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}$ be a LT. Then T has the following form.

  $T(x_1, \ldots, x_n) = \sum_{i=1}^{k} \alpha_i x_i$ for some $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}$ be a LT. Then T has the following form.

  $T(x_1, \ldots, x_n) = \sum_{i=1}^{k} \alpha_i x_i$ for some $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

  Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}$ be a LT. Then T has the following form.

$T(x_1, \ldots, x_n) = \sum_{i=1}^{k} \alpha_i x_i$ for some $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

Let $T(e_i) = \alpha_i$ for $i = 1, \ldots, n$.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}$ be a LT. Then T has the following form.

  $T(x_1, \ldots, x_n) = \sum_{i=1}^{k} \alpha_i x_i$ for some $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

  Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

  Let $T(e_i) = \alpha_i$ for $i = 1, \ldots, n$.

  Let $x = [x_1, x_2, \ldots, x_n] \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} x_i e_i$.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}$ be a LT. Then T has the following form.

  $T(x_1, \ldots, x_n) = \sum_{i=1}^{k} \alpha_i x_i$ for some $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

  Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

  Let $T(e_i) = \alpha_i$ for $i = 1, \ldots, n$.

  Let $x = [x_1, x_2, \ldots, x_n] \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} x_i e_i$.

  $T(x) = T(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i T(e_i)$

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}$ be a LT. Then T has the following form.

$T(x_1, \ldots, x_n) = \sum_{i=1}^{k} \alpha_i x_i$ for some $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

Let $T(e_i) = \alpha_i$ for $i = 1, \ldots, n$.

Let $x = [x_1, x_2, \ldots, x_n] \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} x_i e_i$.

$$T(x) = T(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i T(e_i)$$

$$T(x) = \sum_{i=1}^{n} \alpha_i x_i$$

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

  Then there exist linear transformations $T_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$

  such that $T(x) = (T_1(x), \ldots, T_m(x))$ for all $x \in \mathbb{R}^n$.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Then there exist linear transformations $T_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$

such that $T(x) = (T_1(x), \ldots, T_m(x))$ for all $x \in \mathbb{R}^n$.

Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

  Then there exist linear transformations $T_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$

  such that $T(x) = (T_1(x), \ldots, T_m(x))$ for all $x \in \mathbb{R}^n$.

  Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

  Let $T(e_i) = (\alpha_1^i, \ldots, \alpha_m^i)$ for $i = 1, \ldots, n$.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

  Then there exist linear transformations $T_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$

  such that $T(x) = (T_1(x), \ldots, T_m(x))$ for all $x \in \mathbb{R}^n$.

  Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

  Let $T(e_i) = (\alpha_1^i, \ldots, \alpha_m^i)$ for $i = 1, \ldots, n$.

  Let $x = [x_1, x_2, \ldots, x_n] \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} x_i e_i$.

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

  Then there exist linear transformations $T_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$

  such that $T(x) = (T_1(x), \ldots, T_m(x))$ for all $x \in \mathbb{R}^n$.

  Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

  Let $T(e_i) = (\alpha_1^i, \ldots, \alpha_m^i)$ for $i = 1, \ldots, n$.

  Let $x = [x_1, x_2, \ldots, x_n] \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} x_i e_i$.

  $$T(x) = T(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i T(e_i)$$

- **[Theorem:]** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

  Then there exist linear transformations $T_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$

  such that $T(x) = (T_1(x), \ldots, T_m(x))$ for all $x \in \mathbb{R}^n$.

  Proof: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$.

  Let $T(e_i) = (\alpha_1^i, \ldots, \alpha_m^i)$ for $i = 1, \ldots, n$.

  Let $x = [x_1, x_2, \ldots, x_n] \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} x_i e_i$.

  $$T(x) = T(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i T(e_i)$$

  $$T(x) = \sum_{i=1}^{n} (\alpha_1^i, \ldots, \alpha_m^i) x_i$$

$$T(x) = (\sum_{i=1}^{n} x_i \alpha_i^1, \ldots, \sum_{i=1}^{n} x_i \alpha_i^m)$$

$$T(x) = (\sum_{i=1}^{n} x_i \alpha_i^1, \ldots, \sum_{i=1}^{n} x_i \alpha_i^m)$$

$T(x) = (T_1(x), \ldots, T_n(x))$, where $T_i(x) = \sum_{i=1}^{n} x_i \alpha_i^i$

$$T(x) = (\sum_{i=1}^{n} x_i\alpha_i^1, \ldots, \sum_{i=1}^{n} x_i\alpha_i^m)$$

$$T(x) = (T_1(x), \ldots, T_n(x)), \text{ where } T_i(x) = \sum_{i=1}^{n} x_i\alpha_i^i$$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by $T(x_1, x_2) = (x_2 - x_1, x_1^2, x_2)$.

$$T(x) = (\sum_{i=1}^{n} x_i \alpha_i^1, \ldots, \sum_{i=1}^{n} x_i \alpha_i^m)$$

$$T(x) = (T_1(x), \ldots, T_n(x)), \text{ where } T_i(x) = \sum_{i=1}^{n} x_i \alpha_i^i$$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by $T(x_1, x_2) = (x_2 - x_1, x_1^2, x_2)$.

  T is linear or not?

$$T(x) = (\sum_{i=1}^{n} x_i \alpha_i^1, \ldots, \sum_{i=1}^{n} x_i \alpha_i^m)$$

$$T(x) = (T_1(x), \ldots, T_n(x)), \text{ where } T_i(x) = \sum_{i=1}^{n} x_i \alpha_i^i$$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by $T(x_1, x_2) = (x_2 - x_1, x_1^2, x_2)$.

  T is linear or not?

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(e_1) = (1, 1)$ and $T(e_2) = (-1, 1)$.

$$T(x) = (\sum_{i=1}^{n} x_i \alpha_i^1, \ldots, \sum_{i=1}^{n} x_i \alpha_i^m)$$

$$T(x) = (T_1(x), \ldots, T_n(x)), \text{ where } T_i(x) = \sum_{i=1}^{n} x_i \alpha_i^i$$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by $T(x_1, x_2) = (x_2 - x_1, x_1^2, x_2)$.

  T is linear or not?

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(e_1) = (1, 1)$ and $T(e_2) = (-1, 1)$.

  Could it be possible to get the linear map explicitly?

$$T(x) = (\sum_{i=1}^{n} x_i \alpha_i^1, \ldots, \sum_{i=1}^{n} x_i \alpha_i^m)$$

$$T(x) = (T_1(x), \ldots, T_n(x)), \text{ where } T_i(x) = \sum_{i=1}^{n} x_i \alpha_i^i$$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by $T(x_1, x_2) = (x_2 - x_1, x_1^2, x_2)$.

  T is linear or not?

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(e_1) = (1, 1)$ and $T(e_2) = (-1, 1)$.

  Could it be possible to get the linear map explicitly?

  Answer: Yes. Let $(x_1, x_2) \in \mathbb{R}^2$. Then $(x_1, x_2) = x_1 e_1 + x_2 e_2$.

Then
$$\begin{aligned} T(x_1, x_2) &= x_1 T(e_1) + x_2 T(e_2) \\ &= x_1(1,1) + x_2(-1,1) \\ &= (x_1 - x_2, x_1 + x_2) \end{aligned}$$

Then
$$\begin{aligned} T(x_1, x_2) &= x_1 T(e_1) + x_2 T(e_2) \\ &= x_1(1,1) + x_2(-1,1) \\ &= (x_1 - x_2, x_1 + x_2) \end{aligned}$$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(1,0) = (1,1)$.

Then
$$\begin{aligned} T(x_1, x_2) &= x_1 T(e_1) + x_2 T(e_2) \\ &= x_1(1,1) + x_2(-1,1) \\ &= (x_1 - x_2, x_1 + x_2) \end{aligned}$$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(1,0) = (1,1)$.

Could it be possible to get the linear map explicitly?

Then $T(x_1, x_2) = x_1 T(e_1) + x_2 T(e_2)$

$$= x_1(1,1) + x_2(-1,1)$$

$$= (x_1 - x_2, x_1 + x_2)$$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(1,0) = (1,1)$.

Could it be possible to get the linear map explicitly?

Answer: No it is not possible.

- **[Theorem:]** Let $\mathbb{V}$ be a finite-dimensional vector space over the field $\mathbb{F}$ and let $\{u_1, \ldots, u_n\}$ be an **ordered basis** for $\mathbb{V}$.

- **[Theorem:]** Let $\mathbb{V}$ be a finite-dimensional vector space over the field $\mathbb{F}$ and let $\{u_1, \ldots, u_n\}$ be an **ordered basis** for $\mathbb{V}$. Let $\mathbb{W}$ be a vector space over the same field $\mathbb{F}$ and let $w_1, \ldots, w_n$ be any vectors in $\mathbb{W}$.

- **[Theorem:]** Let $\mathbb{V}$ be a finite-dimensional vector space over the field $\mathbb{F}$ and let $\{u_1, \ldots, u_n\}$ be an **ordered basis** for $\mathbb{V}$. Let $\mathbb{W}$ be a vector space over the same field $\mathbb{F}$ and let $w_1, \ldots, w_n$ be any vectors in $\mathbb{W}$. Then there is precisely one linear transformation T from $\mathbb{V}$ into $\mathbb{W}$ such that $T(u_j) = w_j$, for $j = 1, \ldots, n$.

- **[Theorem:]** Let $\mathbb{V}$ be a finite-dimensional vector space over the field $\mathbb{F}$ and let $\{u_1, \ldots, u_n\}$ be an **ordered basis** for $\mathbb{V}$. Let $\mathbb{W}$ be a vector space over the same field $\mathbb{F}$ and let $w_1, \ldots, w_n$ be any vectors in $\mathbb{W}$. Then there is precisely one linear transformation T from $\mathbb{V}$ into $\mathbb{W}$ such that $T(u_j) = w_j$, for $j = 1, \ldots, n$.

  Proof:

- **[Theorem:]** Let $\mathbb{V}$ be a finite-dimensional vector space over the field $\mathbb{F}$ and let $\{u_1, \ldots, u_n\}$ be an **ordered basis** for $\mathbb{V}$. Let $\mathbb{W}$ be a vector space over the same field $\mathbb{F}$ and let $w_1, \ldots, w_n$ be any vectors in $\mathbb{W}$. Then there is precisely one linear transformation T from $\mathbb{V}$ into $\mathbb{W}$ such that $T(u_j) = w_j$, for $j = 1, \ldots, n$.

Proof:

Let $x \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} c_i u_i$.

- **[Theorem:]** Let $\mathbb{V}$ be a finite-dimensional vector space over the field $\mathbb{F}$ and let $\{u_1, \ldots, u_n\}$ be an **ordered basis** for $\mathbb{V}$. Let $\mathbb{W}$ be a vector space over the same field $\mathbb{F}$ and let $w_1, \ldots, w_n$ be any vectors in $\mathbb{W}$. Then there is precisely one linear transformation T from $\mathbb{V}$ into $\mathbb{W}$ such that $T(u_j) = w_j$, for $j = 1, \ldots, n$.

Proof:

Let $x \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} c_i u_i$.

Define $T(x) = \sum_{i=1}^{n} c_i w_i$. It is clear that T is well defined because $x = \sum_{i=1}^{n} c_i u_i$, this expression unique.

We first show that T is a linear transformation. Take $x, y \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} c_i u_i$ and $y = \sum_{i=1}^{n} d_i u_i$.

We first show that T is a linear transformation. Take $x, y \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} c_i u_i$ and $y = \sum_{i=1}^{n} d_i u_i$.

Let $\alpha, \beta \in \mathbb{F}$. $T(\alpha x + \beta y) = T(\sum_{i=1}^{n}(\alpha c_i + \beta d_i)u_i)$.

$$
\begin{aligned}
T(\alpha x + \beta y) &= \sum_{i=1}^{n}(\alpha c_i + \beta d_i)w_i. \\
&= \alpha \sum_{i=1}^{n} c_i w_i + \beta \sum_{i=1}^{n} d_i w_i. \\
&= \alpha T(x) + \beta T(y).
\end{aligned}
$$

Hence T is linear.

Uniqueness: Suppose that there is another linear transformation U such that $U(u_i) = w_i$.

Uniqueness: Suppose that there is another linear transformation U such that $U(u_i) = w_i$.

To show that $U = T$. Let $x \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} a_i u_i$. Using definition of T we have $T(x) = T(\sum_{i=1}^{n} a_i u_i) = \sum_{i=1}^{n} a_i w_i$.

$$U(x) = U(\sum_{i=1}^{n} a_i u_i)$$

$$= \sum_{i=1}^{n} a_i U(u_i) \text{ (applying the definition of linear transformation)}$$

$$= \sum_{i=1}^{n} a_i w_i.$$

Then $U(x) = T(x)$ for all $x \in \mathbb{V}$. Hence $U = T$.

- [Example]

Take the basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^n$. Take $1, 2, 3 \in \mathbb{R}$. Then using previous theorem we have a unique linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}$ such that $T(e_1) = 1$, $T(e_2) = 2$, $T(e_3) = 3$ and $T(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3$.

Take the basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^n$. Take $1, 2, 3 \in \mathbb{R}$. Then using previous theorem we have a unique linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}$ such that $T(e_1) = 2$, $T(e_2) = 1$, $T(e_3) = 3$ and $T(x_1, x_2, x_3) = 2x_1 + x_2 + 3x_3$. This transformation is different between the previous transformation.

The previous theorem gives a technique to construct a linear transformation from a finite dimensional vector space to another dimensional vector space over the same filed $\mathbb{F}$.