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- How do we understand the geometry of an arbitrary vector space?

- Inner product can be used to understand the geometry of abstract vector spaces.

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An inner product is a mapping $\langle ., . \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{K}$ which satisfies the following conditions.

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Ans: This is not an inner product. Take $x = (i, i, \ldots, i)$. Then $\langle x, x \rangle = \sum_{i=1}^{n} i \cdot i = -n$ which is negative. Therefore fails to satisfy property 1.

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Then $\langle f + g, h\rangle = \int_a^b \big(f(x) + g(x)\big)\, h(x) dx$

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$$= \langle f, h\rangle + \langle g, h\rangle.$$

- Let $f : [a, b]$ be integrable and $f(x) \geq 0$. Let f is continuous at $x = c$ and $f(c) > 0$ (resp. $f(c) < 0$). Then $\int_a^b f(x)dx > 0$ (resp. $\int_a^b f(x)dx < 0$).

Sol: Case I. Let $a < c < b$.

f is continuous at $x = c$. Take $\epsilon = \frac{f(c)}{2}$ then there exists $\delta > 0$ such that

for each $x \in (c - \delta, c + \delta) \implies |f(x) - f(c)| < \frac{(f(c))}{2}$.

Then $-\frac{(f(c))}{2} < f(x) - f(c) < \frac{(f(c))}{2}$ for all $x \in (c - \delta, c + \delta)$.

Therefore $f(x) > \frac{(f(c))}{2} > 0$ for all $x \in (c - \delta, c + \delta)$.

$\int_a^b f^2(x)dx = \int_a^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^{b} f(x)dx > 0$
as $\int_{c-\delta}^{c+\delta} f^2(x)dx > 0$.

Case II. $c = a$

When $c = a$, we have a neighborhood $(a, a+\delta)$ where f is positive.

$\int_a^b f^2(x)dx = \int_a^{a+\delta} f(x)dx + \int_b^{a+\delta} f(x)dx > 0$

Case-III. $c = b$. Same as Case II.

To show $\langle f, f\rangle = \int_a^b f(x)f(x)dx = 0 \implies f \equiv 0$.

Here f^2 is nonnegative function and continuous. If f is not identically zero, then there exists $c \in [a, b]$ such that $f(c) \neq 0$. Therefore $f^2(c) > 0$. Using above result we have $\int_a^b f(x)f(x)dx > 0$, a contradiction. Hence $f \equiv 0$.

- $\mathbb{V} = \mathbb{C}(\mathbb{R})$ and $\mathbb{K} = \mathbb{R}$. Let $\langle f, g \rangle = \int_a^b f(x)g(x)dx$ for all $f, g \in \mathbb{C}(\mathbb{R})$ where a and b are fixed real numbers.

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$$f(x) = \begin{cases} a - x & \text{if } x \leq a \\ 0 & \text{if } x \in [a, b] \\ x - b & \text{if } x \geq b \end{cases}$$

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Hence $\langle f, g \rangle = \int_a^b f(x)g(x)dx$ is not an inner product on $C(\mathbb{R})$.

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Let $x, y \in \mathbb{V}$. Then $x = \sum_{i=1}^{k} a_i u_i$ and $y = \sum_{i=1}^{k} b_i u_i$ for some $a_i, b_i \in \mathbb{K}$ for $i = 1, \ldots, k$.

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We now show that $\langle , \rangle$ is an IP on $\mathbb{V}$.

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Then $x + y = \sum_{i=1}^{k} (a_i + b_i) u_i$. Therefore,

$$\begin{aligned}
\langle x + y, z \rangle &= \sum_{i=1}^{k} (a_i + b_i) \overline{c_i} \\
&= \sum_{i=1}^{k} (a_i \overline{c_i} + b_i \overline{c_i}) \\
&= \langle x, z \rangle + \langle y, z \rangle.
\end{aligned}$$

Case II. $\mathbb{V}$ is infinite dimensional.

Exercise.