Partial Differential Equation (PDE) Part C

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In this last Part of course, we will study three celebrated 2nd order linear PDEs subject to different types of Boundary Conditions (BC) and Initial Conditions (IC):

1. Wave Equation: (1D) $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$, [$x$ is position and t is time]
2. Heat Equation (or Diffusion Equation): (1D) $\frac{\partial\psi}{\partial t} = \alpha\frac{\partial^2\psi}{\partial x^2}$
3. Laplace Equation: $\nabla^2\psi = 0$, [∇^2 is called Laplacian Operator] where in 3D Cartesian coordinates, $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

We already know above three PDEs are respectively Hyperbolic, Parabolic and Elliptic [See classification scheme discussed in Part B]. Before going into detailed study of these PDEs, we need some conception about Fourier series.

- Introduction to Fourier Series

Historically it was Great Mathematician Joseph Fourier who in his research paper in 1807 showed that any arbitrary piecewise continuous function $f(x)$ may be represented by an infinite trigonometric series of sine and cosine functions. Fourier's motivation was to express the solution of "Heat Equation" over a metal plate analytically. Hence, the name of the trigonometric series is given as "Fourier series".

The study of the Theory of Fourier Series involves seeking the important question of whether Fourier series converges always to the function $f(x)$ which is being represented by the series, what the principles are to determine Fourier coefficients etc. In this section, I'll not deal with this study, my intention is to quote standard results without proof, as these will be required in later stage when I'll discuss solution of some well-known second order PDEs, e.g. Finding Fourier series solution of Wave Equation needs the following results.

At the outset Fourier series is a way of approximating a periodic function like in Taylor's series expansion for smooth functions. One major concept is that even non-periodic function defined on a finite interval may be represented by Fourier series by making its periodic extensions on full real line, where, of course, our interest will be the value of Fourier series in the finite interval only.

We will consider only real-valued function $f(x)$ of single real variable x.

❖ Suppose $f(x)$ is a periodic function of period 2π defined on $\mathbb{R}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]. \tag{1.1}$$

Fourier coefficients a_n, b_n can be computed by using following orthogonality property of sine and cosine functions:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \begin{cases} 0\,, m \neq n \\ \pi\,, m = n \end{cases}, \quad \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \begin{cases} 0\,, m \neq n \\ \pi\,, m = n \end{cases} \tag{1.2a}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0, \forall\, m, n\,. \qquad [m, n \text{ are non-negative integers }] \tag{1.2b}$$

Assuming that the infinite series is term by term integrable, we see that if we multiply (1.1) by $\sin(mx)\,,\cos(mx)$ and integrate over the interval $[-\pi,\pi]$, then due to the orthogonal relations (1.2), all but one term in RHS will disappear, thereby getting Fourier coefficients as follows:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\ , b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\ ,\quad n = 0,1,2, \dots, \tag{1.3}$$

where for the case of finding a_0, we just integrate (1.1), and use the relation

$$\int_{-\pi}^{\pi} \sin(mx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos(mx)\mathrm{d}x = 0, m, n \neq 0. \tag{1.4}$$

Note that in formula (1.3), I do not separately put a_0, and do not exclude $n = 0$, as in most literature, because for $n = 0$, formula gives correct expression for a_0, and the extra term b_0 is, in fact, identically zero by the formula. Clearly, formula (1.3) yield coefficients if the function $f(x)$ is integrable in the interval. Some points worth mentioning:

- Instead of the interval $[-\pi,\pi]$, we may take any interval of length 2π on $\mathbb{R}$, in formula (1.3), for instance, we may choose $[0,2\pi]$, if periodicity of $f(x)$ is exhibited in $[0,2\pi]$. It is easy to verify that orthogonal relations hold true for any interval of length 2π.
- If instead of $\mathbb{R}$, $f(x)$ is defined only on $[-\pi,\pi]$, that means given $f(x)$ is no longer periodic. In this circumstance, we make a periodic extension of $f(x)$ on $\mathbb{R}$ by $F(x)$, where $F(x) = f(x), x \in [-\pi,\pi]$ and $F(x) = F(x + 2k\pi)$ outside for non-zero integer k, i.e we make a copy of shape of $f(x)$ in all other interval of length 2π. Note that $F(x)$, thus constructed, is periodic so

that above Fourier series representation is possible for it, but our interest lies only in $[-\pi, \pi]$.

- If given $f(x)$ is even or odd function, then Fourier series (1.1) reduces to Fourier-cosine or Fourier-sine series as follows:

$$f(-x) = f(x): b_n = 0 \Longrightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \tag{1.5a}$$

$$f(-x) = -f(x): a_n = 0 \Longrightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin(nx). \tag{1.5b}$$

- If $f(x)$ is defined on $[0, \pi]$, then there are two ways of periodic extension: Even extension: Define $f(-x) = f(x), x \in [-\pi, 0]$, and then make identical copies in all intervals of length 2π, by constructing $F(x)$, discussed above for second instance. In this case we have half-range Fourier-cosine series (1.5a), wherein a_ns are given by

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, \mathrm{d}x. \tag{1.6a}$$

Odd Extension: Define $f(-x) = -f(x), x \in [-\pi, 0]$, and then make identical copies in all intervals of length 2π, by constructing $F(x)$, discussed above for second instance. In this case we have half-range Fourier-sine series (1.5b), wherein b_ns are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, \mathrm{d}x. \tag{1.6b}$$

- Suppose $f(x)$ is a periodic function of period $2L$ defined on $\mathbb{R}$

Simply make a change of variables $x \to x' = \pi x/L$ so that we will have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right], \tag{1.7}$$

where the coefficients are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \;, b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) , \quad n = 0,1,2, \ldots \tag{1.8}$$

Note that all of the arguments for above instances apply for period $2L$, so I'm not repeating those.

Fourier series converges to $f(x)$ at all points in the interval except for discontinuities. At point of discontinuity and at end-points it converges to average of right-hand and left-hand limits of $f(x)$, e.g. At the point $x = -L, L$, value of Fourier series converges to $A = \frac{1}{2}\left[\lim_{x \to -L+0} f(x) + \lim_{x \to L-0} f(x)\right]$.

❖ Generalized Fourier Series

We have seen that Fourier Series is an expansion of $f(x)$ in terms of elements of set S of periodic functions $\{1, \sin(nx), \cos(nx)\}_{n=1}^{\infty}$. Now property of the set S is that it is a complete* set of orthogonal functions. There exists many other such set, e.g. $\{P_n(x)\}_{n=0}^{\infty}$ is also a complete set of orthogonal functions, where $P_n(x)$ is Legendre polynomial, defined on $[-1,1]$, and is given by

$$P_n(x) = \sum_{m=0}^{N}(-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m}, N = n/2 \text{ or } (n-1)/2 \quad (1.9a)$$

according as n is even or odd. First three members are $P_0(x) = 1, P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$. Orthogonality relation is

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = \frac{2}{2n+1}\delta_{mn}. \quad (1.9b)$$

Thus a function $f(x)$ can be expressed as an infinite series of Legendre polynomials, known as <u>Fourier-Legendre Series</u>

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\mathrm{d}x. \quad (1.9c)$$

- Fourier series solution of Wave Equation using “Method of Separation”

In Part A, I’ve derived d’Alembert’s solution of 1D Wave Equation using Characteristic directions in the context of infinite string and semi-infinite string. Here, I’ll solve 1D Wave Equation for finite string, and use Method of Separation of Variables. The precise problem, to be studied, is given as follows

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}, x \in [0, l], t \geq 0, \quad [\text{c} > 0 \text{ is fixed constant}] \quad (2.1a)$$

where solution $\psi(x,t)$ satisfies following Boundary and Initial conditions:

BC : $\psi(0,t) = \psi(l,t) = 0$, IC: $\psi(x,0) = f(x), \psi_t(x,0) = g(x)$ (2.1b)

This type of Problem is known as Initial Boundary Value Problem (IBVP). Note that two independent variables x, t respectively denote position and time. The zero BC on both ends, considered in (2.1b), are called <u>Homogeneous BC</u>. There exists Problem with <u>Non-homogeneous BC</u>, wherein zero value is for one end, and at other end solution has non-zero value [This non-zero term may be a non-zero constant or may be a time-dependent function].

Given BC in Problem (2.1) implies that we are considering a string, tied at both ends

*For the conception of “Complete Set”, see books on Mathematical Analysis at $x = 0$ and $x = l$. The transverse vibration of string will start from a disturbance given at equilibrium, and $\psi(x,t)$ denotes the profile of this transverse vibration. IC means that at $t = 0$, the profile ψ of wave and the velocity ψ_t of wave are prescribed by $f(x)$ and $g(x)$. The constant c is the speed of wave generated by vibration.

We will use Method of separation. Its concept is to assume solution in the following separable form:

$$\psi(x,t) = \mathrm{X(x)T(t)}. \tag{2.2}$$

We are to find X(x) and T(t). Substituting into Wave equation (2.1a),

$$X''T = \frac{1}{c^2}T''X \Longrightarrow \frac{X''}{X} = \frac{T''}{c^2T} = -k^2. \tag{2.3}$$

where prime denotes ordinary differentiation of factor functions X, T w.r.t. their single arguments. Point is that if the assumed solution would really be a solution, then each ratio in (2.3) must be independent of x, t, since x, t are independent variables. So, we introduce a separation constant $-k^2$, negative sign & square form is just for convenience. Actually, at this point, we do not know whether k is real or complex. However, it can be seen easily that for $k = 0$, we would end up with a zero solution, and hence is rejected:

$$k = 0 \Longrightarrow X(x) = AX + B, \text{BC } A = B = 0.$$

So from now on we will consider $k \neq 0$. Separation (2.3) gives following two ODEs:

$$X'' = -k^2X,\ T'' = -c^2k^2T\ \ (k \neq 0). \tag{2.4}$$

A word of Caution. Do not consider these two as just two simple ODEs for some fixed k, rather consider them as eigenvalue equation for the differential operators$^{\top}$ $\partial^2/\partial x^2$ and $\partial^2/\partial t^2$ with sets of eigenvalues $\{-k^2\}, \{-c^2k^2\}$. That is, I mean to say that, all possible values of k have to be obtained, so that complete solution will be given by the sum of all solutions (eigenvectors) X_k, T_k by the Principle of Linear Superposition. Here the suffix means the eigenvector corresponding to an eigenvalue (a particular value of k). It is easy to write both eigenvectors formally:

$$X_k(x) = C_{1,k}\cos(kx) + C_{2,k}\sin(kx), T_k(t) = D_{1,k}\cos(ckt) + D_{2,k}\sin(ckt), \tag{2.5}$$

where $\{C_{j,k}, D_{j,k}\}$ are four families of arbitrary constants, suffixes just means dependence on k.

$^{\perp}$Conception of eigenvalue equation for matrix and differential operator are identical.

Using BC in (2.1b)

$$X_k(0) = X_k(l) = 0 \Longrightarrow C_{1,k} = 0,\ \sin(kl) = 0 \Longrightarrow k = k_n = \frac{n\pi}{l}\ , n = 1,2,3 \ldots \quad (2.6)$$

Equation (2.6) gives all eigenvalues of both equations of (2.4). We see now that the separation constant k is real number, and takes values for all natural numbers.

Substituting for $X(x), T(t)$ into assumed solution (2.2), we get

$$\psi(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[A_n \cos\left(\frac{n\pi c}{l}t\right) + B_n \sin\left(\frac{n\pi c}{l}t\right)\right], \quad (2.7)$$

where for convenience we have identified $A_n = C_{2,k}D_{1,k}|_{k=k_n}, B_n = C_{2,k}D_{2,k}|_{k=k_n}$, so that we are to determine two families of arbitrary constants $\{A_n, B_n\}$.

Using 1st IC in (2.1b),

$$\psi(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right). \quad (2.8)$$

Equation (2.8) is Fourier-sine series [see Sec 1]. Thus we get the constants $\{A_n\}$:

$$A_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x\ , n = 1,2,3, \ldots.. \quad (2.9)$$

From (2.7), differentiating partially w.r.t.

$$\psi_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right)\left[-A_n \sin\left(\frac{n\pi c}{l}t\right) + B_n \cos\left(\frac{n\pi c}{l}t\right)\right]. \quad (2.10)$$

Using 2nd IC in (2.1b),

$$\psi_t(x,0) = g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{l} B_n\right) \sin\left(\frac{n\pi x}{l}\right). \quad (2.11)$$

Equation (2.11) is also Fourier-sine series. So , we get, $\{B_n\}$

$$\frac{n\pi c}{l} B_n = \frac{2}{l}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x\ , n = 1,2,3, \ldots. \quad (2.12a)$$

Hence, final solution of IBVP of (2.1) of 1D Wave Equation for finite string is

$$\psi(x,t) = \frac{2}{l}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[\cos\left(\frac{n\pi c}{l}t\right)\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \quad + \right.$$
$$\left. \frac{l}{n\pi c} \sin\left(\frac{n\pi c}{l}t\right)\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x\right]. \quad (2.12b)$$

Example: Suppose a string is tightly tied at both ends at $x = 0,2$. If the string is released from rest with a initial profile $f(x) = \sin^3(\pi x/2)$, find the solution $\psi(x,t)$

of transverse vibration at any point $x(0 \leq x \leq l)$ at any instant $t(\geq 0)$. Assume 2 m/s is the speed of wave, and all quantities are in ms unit.

Solution: Transverse vibration of a finite string is described by Wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{4}\frac{\partial^2 \psi}{\partial t^2}, \qquad x \in [0,2], t \geq 0$$

Note that $\psi_t(x,t)$ denotes velocity of wave at any position x. In question, string is released from rest, so that $\psi_t(x,0) = 0$. Also, $\psi(x,t)$ denotes profile of wave at any position x at any instant t, so that according to given data about initial profile, $\psi(x,0) = \sin^3(\pi x/2)$. Thus IC are

$$\psi(x,0) = \sin^3(\pi x/2), \psi_t(x,0) = 0$$

Regarding BC, since string is tied at both ends, we have, as discussed above,

$$\psi(0,t) = \psi(2,t) = 0 \,, \forall\, t \geq 0$$

Hence, to solve this IBVP, we will use Method of Separation of Variables. I'll not repeat the steps [see discussion above]. Final solution is

$$\psi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)\cos(n\pi t)\,, A_n = \int_0^2 \sin^3\left(\frac{\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)\mathrm{d}x$$

It is trivial exercise to compute A_n for $n = 1,2,3, ...$

Check that $A_1 = \frac{3}{4}, A_3 = -\frac{1}{4}$ and all other $A_n = 0$. Hence final solution is

$$\psi(x,t) = \frac{3}{4}\sin\left(\frac{\pi x}{2}\right)\cos(\pi t) - \frac{1}{4}\sin\left(\frac{3\pi x}{2}\right)\cos(3\pi t)$$

Interpretation of Fourier Series Solution of finite string as "Standing Wave Profile"

Recall that in Part B, I gave interpretation of D′Alembert's solution (without external force) for infinite string as progressive wave, which was travelling in both forward & backward direction. This is the reason why that solution is also called "Travelling Wave Solution".

In contrast, for the case of finite string, we will see that the Fourier series solution represents "Stationary Wave". Note that solution (2.7) can be split into two standing wave profiles:

$$\psi(x,t) = \sum_{n=1}^{\infty}\left[\frac{A_n}{2}\,\psi_n^{(1)} + \frac{B_n}{2}\psi_n^{(2)}\right], \qquad (2.13a)$$

$$\psi_n^{(1)}(x,t) = \sin\left(\frac{n\pi}{l}(x+ct)\right) + \sin\left(\frac{n\pi}{l}(x-ct)\right), \qquad (2.13b)$$

$$\psi_n^{(2)}(x,t) = \cos\left(\frac{n\pi}{l}(x-ct)\right) - \cos\left(\frac{n\pi}{l}(x+ct)\right) \qquad (2.13c)$$

Each term in (2.13a) represents sinusoidal waves of amplitudes $A_n/2, B_n/2$ moving towards left or right with speed c. Also, note that sine term goes to cos term by shifting the argument a by $a+\pi/2$. In the language of wave, this shift is called "Phase Shift", i.e. phase of $\psi_n^{(2)}$ differs from that of $\psi_n^{(1)}$ by $\pi/2$.

Now, let us examine for each $n = 1,2,3, ...$, how many times n-th wave crosses equilibrium position (horizontal axis). The point at which wave crosses horizontal position is called node of that wave. Since $\psi_n^{(1)}(x_m,t) = 0$ for $x_m = \frac{m}{n}l$, the number of nodes of n-th mode wave is $n+1$, since m varies from 0 to n. Note that string is tied at both ends, and so wave can't go beyond the positions $x = 0$ and $x = l$, and hence m is restricted to take only $(n+1)$ values $0,1,2, \dots n$.

So, we understand that in the mix of waves given by n-th mode $\psi_n^{(1)}$, there will be exactly n full waves of $(n+1)$ nodes for each $n = 1,2,3, \dots$. Similar interpretation will be for other component $\psi_n^{(2)}$, which gives also n full waves of $(n+1)$ nodes for each n, but the phase of each wave will differ from corresponding wave of 1st component by $\pi/2$.

Number of waves per unit distance is measured by "Wave Number" $k = n\pi/l$, and the length of full cycle is $\lambda = 2\pi/k$. Frequency of the wave is $\omega = ck$. The periodic time is $T = 2\pi/\omega$, which is time lapse between two same phases.

Note that $\lambda/c = T$, so that after time T wave reaches same phase. This can be checked straightforwardly as

$$j = 1,2: \psi_n{}^{(j)}\left(x, t+\frac{\lambda}{c}\right) = \psi_n{}^{(j)}(x,t), \text{ since } \frac{n\pi c}{l}\left(t+\frac{\lambda}{c}\right) = \frac{n\pi ct}{l} + 2\pi, n = 1,2, \dots .$$

Thus if an observer is stationed at a fixed point x, then exactly one full cycle passes her after time T.

This completes description of the interpretation of solution of finite-string wave problem as standing wave profile.

- 1D Heat Equation

Heat equation in 1D (i.e. for one position variable x) with time variable t is a parabolic PDE of following form

$$\frac{\partial\psi}{\partial t} = \alpha\frac{\partial^2\psi}{\partial x^2}, 0 \le x \le l, t \ge 0. \quad [\alpha > 0] \tag{3.1}$$

This equation is also called "Heat Conduction Equation" or "Diffusion Equation". The meaning of these names is that this equation describes rate of change of temperature ψ over time on a thin rod of length l. The rod is sufficiently thin so that heat-distribution is equal over the cross-section at any time t. The surface of the rod is insulated in order to prevent heat-loss through the boundary. One physical requirement is that the solution must tend to zero or (more generally) must remain finite after a long time. BC may be Homogeneous or Non-homogeneous. Below I'll discuss Heat Equation with both BC.

Problem with Homogeneous BC

Zero values at both ends

Consider following Problem where both ends are maintained at zero temperature for all time:

$$\psi_t = \alpha\psi_{xx}, 0 \le x \le l, t \ge 0, \quad [\alpha > 0] \tag{3.2a}$$

BC: $\psi(0,t) = \psi(l,t) = 0, \forall\, t \ge 0,$ IC: $\psi(x,0) = f(x), x \in [0,l]$ (3.2b)

Solution: We use Method of Separation of Variables. Assume following solution:

$$\psi(x,t) = X(x)T(t). \tag{3.3}$$

Substituting (3.3) into (3.2a), solution is separated as follows

$$\frac{X''}{X} = \frac{T'}{\alpha T} = -\mu^2, \tag{3.4}$$

where prime denotes ordinary differentiation of X, T w.r.t their single arguments, and μ is called separation constant. Note that the negative sign and square form of separation constant is just for convenience. At this point we do not know whether μ is real or complex.

Equation for $T(t)$:

$$T' = -\alpha\mu^2 T \Longrightarrow T_\mu(t) = A_\mu \exp(-\alpha\mu^2 t). \tag{3.5}$$

Since solution must be bounded, we understand μ must be real:

$$T(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty \Longrightarrow \mu \in \mathbb{R}. \tag{3.6}$$

Due to square form, without loss of generality, we can set $\mu \geq 0$. For $\mu = 0$, the corresponding eigenvector $T_{\mu=0}(t) = A_{\mu=0}$ is bounded, and hence, up to now, μ is any non-negative real number including zero.

Equation for $X(x)$:

$$X'' + \mu^2 X = 0 \Longrightarrow X_\mu(x) = B_\mu \cos(\mu x) + C_\mu \sin(\mu x). \tag{3.7}$$

Using BC in (3.2b),

$$X_\mu(0) = 0 \Longrightarrow B_\mu = 0, \tag{3.8a}$$

$$X_\mu(l) = 0 \Longrightarrow \sin(\mu l) = 0 \Longrightarrow \mu_n = n\pi/l, n = 1,2,3 \dots . \tag{3.8b}$$

Note that since $X_\mu(x)$ must be non-zero, restrictions (3.8a) and (3.8b) implies that we are to exclude $n = 0$ value (i.e. $\mu = 0$ considered before is now rejected). Equation (3.8) constitutes complete set of all eigenvalues, and so by the principle of linear superposition, full solution will be given by the sum of all eigenvectors over these eigenvalues as follows:

$$\psi(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \exp\left[-\left(\frac{n\pi}{l}\right)^2 \alpha t\right], \tag{3.9}$$

where, I've clubbed two arbitrary constants into a single constant by the defining C_n as $C_n \equiv A_{\mu_n} C_{\mu_n}$. The family $\{C_n\}_{n=1}^{\infty}$ will be obtained by using IC in (3.2b):

$$\psi(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right). \tag{3.10}$$

The series (3.10) is Fourier-sine series [see Sec 1]. Hence, Fourier coefficients are

$$C_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x . \tag{3.11}$$

Substituting coefficients from (3.11) into (3.9), final solution of 1D Heat Equation with homogeneous BC is as follows

$$\psi(x,t) = \frac{2}{l}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \exp\left[-\left(\frac{n\pi}{l}\right)^2 \alpha t\right] \int_0^l f(x') \sin\left(\frac{n\pi x'}{l}\right) \mathrm{d}x' . \tag{3.12}$$

In above equation, we change in the integral of (3.12) the variable x to x' to indicate that it is not a position variable, it is just an integral variable.

Example: Find the solution of following Heat equation with given IC and homogeneous BC

$$\psi_t = 2\psi_{xx}, 0 \le x \le \pi, t \ge 0,$$

BC: $\psi(0,t) = \psi(\pi,t) = 0, \forall\, t \ge 0,$ and $\psi(x,t) \to$ finite as $x \to \infty$

$$\text{IC: } \psi(x,0) = \begin{cases} x, 0 \le x \le \pi/2 \\ \pi - x, \pi/2 \le x \le \pi \end{cases}$$

Solution: Note that second BC is obvious for all problems, and so sometimes this may not be mentioned explicitly in some problem-statements. Proceeding along the steps described above, we get

$\psi(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx)\ \exp(-2n^2 t),$

$$C_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x' \sin(nx')\, dx' + \int_{\pi/2}^{\pi} (\pi - x')\ \sin(nx')\, dx'\right] = \frac{4\sin(n\pi/2)}{n^2\pi}$$

Hence, final solution is

$$\psi(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)\ \sin(nx)}{n^2} \exp(-2n^2 t)$$

Problem with Non-homogeneous BC

Non-zero value at one end

Consider following Problem where for all time, one end is maintained at zero temperature, and other end is held non-zero constant temperature:

$$\psi_t = \alpha\psi_{xx}, 0 \le x \le l, t \ge 0, \quad [\alpha > 0] \tag{3.13a}$$

BC: $\psi(0,t) = 0, \psi(l,t) = k, \forall\, t \ge 0, k \ne 0,$ IC: $\psi(x,0) = f(x), x \in [0,l]$ (3.13b)

Solution: Note that in contrast to previous Problem (3.2), here solution vanishes at one end, but at other end it acquires a non-zero value. This BC is called "Non-homogeneous BC". To reduce the problem in previous form, we will make following change of dependent variable to reduce BC in homogeneous form:

$$\psi(x,t) \rightarrow \Psi(\mathrm{x},\mathrm{t}) = \psi(x,t) - \frac{kx}{l}. \quad (3.14)$$

Equation (3.14) implies that $\Psi_t = \psi_t, \Psi_{xx} = \psi_{xx}$, and BC reduces to $\Psi(0,t) = 0, \Psi(l,t) = 0$. Hence, Problem (3.13) becomes in the previous form:

$$\Psi = \alpha \Psi_{xx}, 0 \le x \le l, t \ge 0, \quad [\alpha > 0] \quad (3.13a)$$

BC: $\Psi(0,t) = 0 = \Psi(l,t), \forall\, t \ge 0$, IC: $\Psi(x,0) = F(x), x \in [0,l]$, (3.13b)

where $F(x) = f(x) - kx/l$. Clearly Problem (3.13) is same as Problem (3.2), since BC now becomes homogeneous. I'll not repeat the steps. Final solution is

$$\Psi(x,t) = \psi(x,t) - \frac{kx}{l} = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \exp\left[-\left(\frac{n\pi}{l}\right)^2 \alpha t\right], \quad (3.14a)$$

$$C_n = \frac{2}{l}\int_0^l F(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x = \frac{2}{l}\int_0^l \left[f(x) - \frac{kx}{l}\right] \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x. \quad (3.14b)$$

After a simple integration, equation (3.14b) gives

$$C_n = (-1)^n \frac{2k}{n\pi} + \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x. \quad (3.14c)$$

Plugging (3.14c) into (3.14a), final solution is in the following form:

$$\psi(x,t) = \frac{kx}{l} + \sum_{n=1}^{\infty} \left[\sin\left(\frac{n\pi x}{l}\right) \exp\left\{-\left(\frac{n\pi}{l}\right)^2 \alpha t\right\} \left\{(-1)^n \frac{2k}{n\pi} + \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \right\}\right]. \quad (3.15)$$

Problem with Non-homogeneous BC Contributed by Prof. R Gayen

Non-zero value at both ends

Consider a thin homogeneous bar of length l. Both ends are maintained at constant temperatures T_1 and T_2. Initial temperature throughout the bar in the cross-section at x is $f(x)$.

The IBVP modelling this setting is as follows

$$\psi_t = \alpha\, \psi_{xx}, 0 \le x \le l, t \ge 0, \quad [\alpha > 0] \quad (3.15.1)$$

BC: $\psi(0,t) = T_1, \psi(l,t) = T_2, \forall\, t \ge 0, k \ne 0$, IC: $\psi(x,0) = f(x), x \in [0,l]$ (3.15.2)

Solution: To convert the problem having homogeneous BC, here following change of dependent variable is useful:

$$\psi(x,t) \to \Psi(\mathrm{x},\mathrm{t}) = \psi(x,t) - \left[T_1 + (T_2 - T_1)\frac{x}{l}\right] \equiv \psi(x,t) - v(x) \quad (3.15.3)$$

This ensures homogeneous BC: $\Psi(0,\mathrm{t}) = 0, \Psi(\mathrm{l},\mathrm{t}) = 0$. And we get converted problem as follows

$$\Psi_t = \alpha\Psi_{xx}, 0 \leq x \leq l, t \geq 0, \quad [\alpha > 0] \quad (3.15.3a)$$

BC: $\Psi(0,t) = 0, \Psi(l,t) = 0, \forall\, t \geq 0, k \neq 0$, IC: $\Psi(x,0) = F(x), x \in [0,l]$ (3.15.3b)

where $F(x) = f(x) - v(x)$.

Problem (3.15.3) is same as Problem (3.13). Hence, without repeating the steps, the solution is written below:

$$\Psi(x,t) = \frac{2}{l}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \exp\left[-\left(\frac{n\pi}{l}\right)^2 \alpha t\right] \int_0^l F(x') \sin\left(\frac{n\pi x'}{l}\right) \mathrm{d}x' \quad (3.15.4)$$

Physical interpretation of above solution Contributed by Prof. R Gayen

Looking into the change of dependent variable (3.15.3), it is easy to understand that the solution has two parts as follows

$$\psi(x,t) = \Psi(\mathrm{x},\mathrm{t}) + v(x) \quad (3.15.4a)$$

where $v(x,t)$ satisfies following steady-state problem

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0\,, v(0) = T_1, v(l) = T_2 \quad (3.15.4b)$$

Solving (3.15.4b), you will get the form of $v(x)$, as given in (3.15.3). Note that differential equation satisfied by $v(x)$ is time-independent, and hence it gives steady-state temperature distribution.

Thus solution $\psi(x,t)$ may be regarded as a decomposition of the temperature distribution into a transient part $\Psi(\mathrm{x},\mathrm{t})$ and a steady-state part $v(x)$. Transient part decays and approaches zero after a long time, since $\lim_{t\to\infty} \Psi(\mathrm{x},\mathrm{t}) = 0$. Steady part $v(x)$ is independent of time which represents the limiting value which temperature approaches after a long time. Note that in this case, steady-state temperature is always a linear function of x.

Exercise: Solve above problem for $T_1 = 1, T_2 = 2, f(x) = 3/2,\ x \in [0,l]$.

Ans: $\Psi(\mathrm{x,t}) = 1 + \frac{x}{l} + \sum_{n=1}^{\infty}\left[\left\{\frac{1+(-1)^n}{n\pi}\right\}\exp\left(-\frac{n^2\pi^2\alpha^2}{l^2}t\right)\sin\left(\frac{n\pi x}{l}\right)\right]$

Problem with Non-homogeneous BC

Time-dependent function at one end

Above Problem may be generalized by taking a time-dependent function at the end at $x = l$. Specifically, let us formulate Problem as follows:

$$\psi_t = \alpha\psi_{xx}, 0 \le x \le l, t \ge 0, \quad [\alpha > 0] \tag{3.16a}$$

BC: $\psi(0,t) = 0, \psi(l,t) = g(t), \forall\, t \ge 0,$ IC: $\psi(x,0) = f(x), x \in [0,l]$ (3.16b)

Solution: With the same spirit in the previous Problem, we made following change of dependent variable

$$\psi(x,t) \to \Psi(\mathrm{x,t}) = \psi(x,t) - \frac{xg(t)}{l}, \tag{3.17}$$

which will make BC homogeneous, but in contrast with above problem, here we will end up to a Heat Equation with a non-homogeneous term. Physically, this non-homogeneous term is interpreted as a source or sink of heat. Thus, Problem (3.16) with transformation (3.17) is formulated as follows:

$$\Psi_t = \alpha\Psi_{xx} + H(x,t), 0 \le x \le l, t \ge 0, \quad \left[\alpha > 0, H(x,t) = \frac{x}{l}g'(t)\right] \tag{3.18a}$$

BC: $\Psi(0,t) = 0 = \Psi(l,t), \forall\, t \ge 0,$ IC: $\Psi(x,0) = F(x), x \in [0,l]$, (3.18b)

where $F(x) = f(x) - xg(0)/l$.

The difficult-level of a non-homogeneous Heat equation is higher than its homogeneous counterpart. In Ref.2, non-homogeneous Wave Equation has been discussed, but non-homogeneous Heat Equation was not discussed. I followed the same pathways, as shown for Wave equation in Ref 2, to solve current non-homogeneous Heat Equation (3.18a), i.e. Below I am proposing following Eigen function expansions for both of solution and the non-homogeneous term in the basis of complete set of orthogonal functions $\{\sin(n\pi x/l)\}_{n=1}^{\infty}$:

$$\Psi(x,t) = \sum_{n=1}^{\infty}\left[u_n(t)\sin\left(\frac{n\pi x}{l}\right)\right], H(x,t) = \sum_{n=1}^{\infty}\left[H_n(t)\sin\left(\frac{n\pi x}{l}\right)\right]. \tag{3.19}$$

The advantage of the representation of the solution $\Psi(x,t)$ in (3.19) is that it automatically satisfies BC (3.18b). Also note that 2nd equation of (3.19) is Fourier-sine series, and hence coefficient functions $H_n(t)$ is known by following formula:

$$H_n(t) = \frac{2}{l}\int_0^l H(x',t)\sin\left(\frac{n\pi x'}{l}\right)\mathrm{d}x'. \tag{3.20}$$

Substituting (3.19) into non-homogeneous Heat Equation (3.18a), and integrating w.r.t x in $[0,l]$ after multiplying by $\sin(m\pi x/l)$, we het following 1st order linear ODE:

$$u_n'(t) + \lambda_n^2 u_n(t) = H_n(t), \quad [\lambda_n = \frac{n\pi\sqrt{\alpha}}{l}], \tag{3.20.1}$$

the solution of which is

$$u_n(t) = \exp(-\lambda_n^2 t)\left[u_n(0) + \int_0^t H_n(\tau)\exp(\lambda_n^2\tau)\,\mathrm{d}\tau\right]. \tag{3.20.2}$$

Note that $u_n(t)$ in (3.20.2) tend to zero after long time, consistent with the physical requirement.

Let us now use IC in (3.18b). We get

$$\Psi(x,0) = F(x) = \sum_{n=1}^{\infty}\left[u_n(0)\sin\left(\frac{n\pi x}{l}\right)\right], \tag{3.20.3}$$

Equation (3.20.33) is Fourier-sine series, hence the numbers $u_n(0)$ in (3.20.2) are computed by the formula:

$$u_n(0) = \frac{2}{l}\int_0^l F(x')\sin\left(\frac{n\pi x'}{l}\right)\mathrm{d}x' = (-1)^{n+1}\frac{2l}{n\pi} + \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x'}{l}\right)\mathrm{d}x', \tag{3.21}$$

where we use the definition $F(x) = f(x) - xg(0)/l$, given in (3.18).

Also, using the definition $(x,t) = xg'(t)/l$, given in (3.18), we get from (3.19)

$$H_n(t) = \frac{2}{l}\int_0^l \frac{x\prime g\prime(t)}{l}\sin\left(\frac{n\pi x'}{l}\right)\mathrm{d}x' = (-1)^{n+1}\frac{2g\prime(t)}{n\pi}. \tag{3.22}$$

Plugging (3.20.2),(3.21) into 1st equation of (3.19), we get solution of transformed problem (3.18) as follows

$$\Psi(x,t) = \sum_{n=1}^{\infty}\left\{\exp(-\lambda_n^2 t)\sin\left(\frac{n\pi x}{l}\right)\left[(-1)^{n+1}\frac{2l}{n\pi} + \frac{2}{l}\int_0^l f(x')\sin\left(\frac{n\pi x'}{l}\right)\mathrm{d}x' + 2(-1)^{n+1}\int_0^\tau \frac{g\prime(\tau)}{n\pi}\exp(\lambda_n^2\tau)\,\mathrm{d}\tau\right]\right\}, \tag{3.23}$$

where $\lambda_n^2 = (n\pi/l)^2$.

Hence, the solution $\psi(x,t)$ of original Problem (3.16) in final form is as follows

$$\psi(x,t) = \frac{xg(t)}{l} + \Psi(x,t). \tag{3.24}$$

Problem with Non-homogeneous BC Contributed by Prof. R Gayen

Time-dependent function at both ends

Above Problem may be further generalized by taking time-dependent functions at the both end at $x = 0, l$. Specifically, let us formulate Problem as follows:

$$\psi_t = \alpha\psi_{xx}, 0 \le x \le l, t \ge 0, \quad [\alpha > 0] \tag{3.25a}$$

BC: $\psi(0,t) = h(t), \psi(l,t) = g(t), \forall\, t \ge 0$, IC: $\psi(x,0) = f(x), x \in [0,l]$ (3.25b)

Solution: $\psi(x,t) \to \Psi(\mathrm{x},\mathrm{t}) = \psi(x,t) - v(x,t)$ (3.26)

Substitute (3.26) into (3.25a).

$$[\Psi_t - \alpha\Psi_{xx}] + [v_t - \alpha v_{xx}] = 0 \tag{3.27}$$

Our target is homogeneous BC for Ψ.

Set

$$v_{xx} = 0 \Longrightarrow v_x = xK_1(t) + K_2(t) \tag{3.28}$$

To find $K_1(t), K_2(t)$, use BC

$$\Psi(0,\mathrm{t}) + v(0,t) = h(t), \Psi(\mathrm{l},\mathrm{t}) + v(l,t) = g(t) \tag{3.29}$$

To get $\Psi(0,\mathrm{t}) = \Psi(\mathrm{l},\mathrm{t}) = 0$, we must have

$$v(0,t) = h(t), v(l,t) = g(t) \tag{3.30}$$

From (3.30) and ((3.28), we have

$$K_2(t) = h(t), K_1(t) = \frac{g(t)-h(t)}{l} \Longrightarrow v(x,t) = \frac{(l-x)h(t)+xg(t)}{l} \tag{3.31}$$

Now, Problem (3.25) converted into following Problem with homogeneous BC:

$$\Psi_t = \alpha\,\Psi_{xx} + H(x,t), 0 \le x \le l, t \ge 0, \quad [\alpha > 0, H(x,t) \equiv -v_t(x,t)] \tag{3.32a}$$

BC: $\Psi(0,t) = 0, \Psi(l,t) = 0, \forall\, t \ge 0$, IC: $\Psi(x,0) = F(x), x \in [0,l]$ (3.32b)

where $F(x) = f(x) - v(x,0)$.

Notice that Problem (3.32) is exactly same as Problem (3.18). Hence, without repeating steps, I quote below the solution

$$\left.\begin{array}{c} \Psi(x,t) = \sum_{n=1}^{\infty}\left[u_n(t)\sin\left(\frac{n\pi x}{l}\right)\right], H(x,t) = \sum_{n=1}^{\infty}\left[H_n(t)\sin\left(\frac{n\pi x}{l}\right)\right] \\ u_n(t) = \exp(-\lambda_n^2 t)\left[\frac{2}{l}\int_0^l F(x')\sin\left(\frac{n\pi x'}{l}\right)\mathrm{d}x' + \int_0^t H_n(\tau)\exp(\lambda_n^2\tau)\,\mathrm{d}\tau\right] \\ H_n(t) = \frac{2}{l}\int_0^l H(x',t)\sin\left(\frac{n\pi x'}{l}\right)\mathrm{d}x' \end{array}\right\} \quad (3.33)$$

Equation (3.33) constitutes solution of heat equation with time-dependent functions at both ends.

Physical interpretation of above solutions

We see that when time-dependent function/s is/are attributed at one/both end/s, solution, as in the case for nonzero constant/s at one/both end/s, is decomposed into transient part $\Psi(\mathrm{x},\mathrm{t})$ decaying as $t \to \infty$, and steady-state part $v(x,t)$. But difference is that in the above two cases steady-state temperature distribution will not approach constant after long time unless the given time-dependent functions $h(t), g(t)$ have the restrictions $\lim_{t\to\infty} h(t) = h_0, \lim_{t\to\infty} g(t) = g_0$.

Before going to next topic, note following remarks

- There may be mixed-type BC like $s_1\psi(0,t) + s_2\psi_t(0,t) = s_3$ at left-end.
- Eigenvalues may not be obtained always in analytical form, like you may get eigenvalue equation like $\tan(\lambda l) = K$, which has infinite number of positive solutions $\lambda = \lambda_n, n = 1,2,3, \dots$.

This completes discussion about the solution of 1D Heat equation.

- Laplace Equation: $\nabla^2\psi = 0$

First of all, note that the Laplacian operator ∇^2 depends on positions, and so Laplace equation doesn't contain time variable t, in contrast to other two PDEs studied previously. Hence, Laplace equation is called "Steady State" equation. Secondly, note that the exact form of the Laplacian operator ∇^2 will depend on dimension of space and the chosen frame of coordinates.

Remark: Laplace equation is homogeneous (w.r.t operators), i.e. $\exists$ no function on RHS. If there is function in RHS, then that non-homogeneous PDE is named as Poisson Equation.

Let me write Laplace equation in 2D and 3D space using Cartesian and polar coordinates.

2D Laplace Equation

Cartesian Coordinates (x, y): $\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, x, y \in (-\infty, +\infty)$.

Polar Coordinates (r, ϕ): $\nabla^2\psi \equiv r^2\frac{\partial^2\psi}{\partial r^2} + r\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial\phi^2} = 0,\ r \in [0, \infty), \phi \in [0,2\pi]$.

3D Laplace Equation

Cartesian Coordinates (x, y, z): $\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0, x, y, z \in (-\infty, +\infty)$.

Spherical Polar Coordinates (r, θ, ϕ): $r \in [0, \infty), \theta \in [0, \pi], \phi \in [0,2\pi]$

$$\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0.$$

Note that using the well-known transformation relations between coordinates in two different frames, it is easy to derive, applying chain rule, one form from other form of Laplace equation, calculations are bit lengthy though. Transformation rules between Cartesian and Polar frame are listed below:

2D: $x = r\cos\phi, y = r\sin\phi$

3D: $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

When we solve Laplace equation on a domain, we choose Cartesian Frame or Polar Frame according to the geometric nature of domain. For instance, over circular or spherical domain, we use polar frame in 2D or 3D, whereas for rectangular or parallelepiped, we use Cartesian frame in 2D or 3D. Furthermore, according to the boundary of the domain, the true ranges* of values of coordinates will be fixed, i.e. to say that the ranges written above is in general sense. For instance, over a full sphere, the ranges of r, θ, ϕ are same as above. But for hemisphere, range of θ will be $[0, \pi/2]$ instead of $[0, \pi]$.

In many different Mathematical modeling, Laplace equation arises. For instance, consider a metal body is kept at

*For understanding range, see books on Geometry.

constant (w.r.t. time) temperature by keeping a source of heat. Then the distribution of heat at each point on the body is given by the solution ψ of Laplace equation that satisfy appropriate boundary conditions at the

boundary of the body. Further, if the body is closed body, say a circle or sphere or a rectangular plate, solution of Laplace equation will depend on whether we need solution at the interior or exterior of the body.

- Different Boundary Conditions related to Laplace Equation

Consider a finite region V of surface S enclosed by a boundary curve Γ.

- ❖ Dirichlet Boundary Condition
  - Interior Problem: To find solution ψ such that $\nabla^2\psi = 0$ <u>within</u> V, and the solution satisfies BC: $\psi = f$ on Γ, f is prescribed.
  - Exterior Problem: To find solution ψ such that $\nabla^2\psi = 0$ <u>outside</u> V, and the solution satisfies BC: $\psi = f$ on Γ, f is prescribed.

<u>Remark</u>: In general, solutions for interior and exterior problems are different. For most of the Exterior Problems, to get bounded solutions we need to impose additional restriction on ψ, like for instance $\psi(r,\theta,\phi) \to 0$ as $r \to \infty, \forall\, \theta, \phi$. Further for both problems, for uniqueness, we need the periodicity condition $\psi(r,\theta,\phi + 2\pi) = \psi(r,\theta,\phi)$. Note that in most Problem-statement, these conditions are not explicitly mentioned, but you need to adopt those for the existence of unique bounded solution.

- ❖ Neumann Boundary Condition
  - Interior Problem: To find solution ψ such that $\nabla^2\psi = 0$ <u>within</u> V, and the solution satisfies BC: $\frac{\partial\psi}{\partial n} = g$ on Γ, g is prescribed, i.e. normal derivative of ψ on boundary curve is prescribed [n is outward drawn normal to the surface of the region].
  - Exterior Problem: To find solution ψ such that $\nabla^2\psi = 0$ <u>outside</u> V, and the solution satisfies BC: $\frac{\partial\psi}{\partial n} = g$ on Γ, g is prescribed.
- ❖ Robin Boundary Condition
  - Interior Problem: To find solution ψ such that $\nabla^2\psi = 0$ <u>within</u> V, and the solution satisfies BC: $\psi = f$ on Γ_1, $\frac{\partial\psi}{\partial n} = g$ on Γ_2 , where $\Gamma = \Gamma_1 \cup \Gamma_2$, f, g are prescribed functions.

- Exterior Problem: To find solution ψ such that $\nabla^2\psi = 0$ outside V, and the solution satisfies BC: $\psi = f$ on $\Gamma_1, : \frac{\partial\psi}{\partial n} = g$ on Γ_2 , where $\Gamma = \Gamma_1 \cup \Gamma_2$, f, g are prescribed functions.

  Remark: In Robin BC, the solution is prescribed on one part of boundary curve, and its normal derivative is prescribed on rest part of boundary. Apart from that, there exists a mixed BC, where certain linear combination of the solution and its normal derivative is prescribed on boundary.

- Solution of Dirichlet Problem using Method of Separation

1. 2D Laplace Equation on Rectangular Domain

Interior Dirichlet Problem

Let us consider following interior problem with Dirichlet BC

$$\nabla^2\psi = 0,\ \ 0 \le x \le a, 0 \le y \le b, \tag{5.1a}$$

BC: $$\psi(x, b) = \psi(a, y) = \psi(0, y) = 0, \psi(x, 0) = f(x) \tag{5.1b}$$

Solution: Since the domain is rectangular, obvious choice is Cartesian Frame, so that equation (5.1a) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0. \tag{5.1c}$$

For interior Problem, we are to find solution $\psi(x, y)$ inside the rectangle, i.e. for $x \in [0, a], y \in [0, b]$. We will use Method of Separation, i.e. assume the solution:

$$\psi(x, y) = X(x)Y(y). \tag{5.2}$$

Substituting (5.2) into (5.1c),

$$YX'' + XY'' = 0 \Longrightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2. \tag{5.3}$$

At this point, we do not know whether separation constant λ is real or complex. Formally we can write the eigenvectors as

$$X_\lambda(x) = A_\lambda \exp(i\lambda x) + B_\lambda \exp(-i\lambda x)\,, Y_\lambda(y) = C_\lambda \exp(\lambda y) + D_\lambda \exp(-\lambda y), \tag{5.4}$$

for $\lambda \neq 0$. Note that $\lambda = 0$ case is rejected, because then given BC implies that solution will be identically zero. Now solution can be written as a superposition of all eigenvectors in the following form:

$$\psi(x,y) = \sum_{\lambda}[\{A_{\lambda}\exp(i\lambda x) + B_{\lambda}\exp(-i\lambda x)\}\{C_{\lambda}\exp(\lambda y) + D_{\lambda}\exp(-\lambda y)\}]. \quad (5.5)$$

Using BC $\psi(x,b) = \psi(a,y) = \psi(0,y) = 0$, we get

$$A_{\lambda} + B_{\lambda} = 0, C_{\lambda}\mathrm{e}^{\lambda b} + D_{\lambda}\mathrm{e}^{-\lambda b} = 0, A_{\lambda}\mathrm{e}^{i\lambda a} + B_{\lambda}\mathrm{e}^{-i\lambda a} = 0\,. \quad (5.6)$$

From 1st and last of (5.6), $B_{\lambda} = -A_{\lambda} \Longrightarrow \exp 2i\lambda a = 1$. Thus we get eigenvalues as

$$\lambda = \lambda_n = \frac{n\pi}{a}\,, n = 1,2,3,\ldots \quad (5.7)$$

2nd relation of (5.6) then gives

$$D_{\lambda_n} = -C_{\lambda_n}\mathrm{e}^{2n\pi b/a}$$

Then solution (5.5) becomes

$$\psi(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left[\frac{n\pi(y-b)}{a}\right], \quad (5.8a)$$

where, for convenience we set $a_n \equiv 4A_{\lambda_n}C_{\lambda_n}\exp\left(\frac{n\pi b}{a}\right)$. Note that the last factor in RHS is hyperbolic function. To get a_n, we use last BC:

$$\psi(x,0) = f(x) = \sum_{n=1}^{\infty}\left[-a_n \sinh\left(\frac{n\pi b}{a}\right)\right]\sin\left(\frac{n\pi x}{a}\right)$$

$$\Longrightarrow a_n = -\frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x \quad (5.8b)$$

Hence, final solution of interior Dirichlet Problem (5.1) as follows:

$$\psi(x,y) = -\frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\sinh\left[\frac{n\pi(y-b)}{a}\right]\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x. \quad (5.9)$$

2. 2D Laplace Equation on Circular Domain
   a. Interior Problem

Let us consider following interior problem with Dirichlet BC

$$\nabla^2\psi = 0,\ \ 0 \le r \le a, 0 \le \phi \le 2\pi, \quad (5.10a)$$

BC: $$\psi(a,\phi) = f(\phi),\ \ 0 \le \phi \le 2\pi \quad (5.10b)$$

Solution: As discussed before, for circular domain, we choose Polar Coordinates. As a result Laplace equation (5.10a) becomes

$$r^2 \frac{\partial^2\psi}{\partial r^2} + r\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial\phi^2} = 0,\ \ r \in [0, a], \phi \in [0,2\pi]. \tag{5.10c}$$

We use Method of Separation. Assume solution in the following form:

$$\psi(r, \phi) = R(r)\Phi(\phi). \tag{5.11}$$

Since r, ϕ are radius vector and azimuthal angle, $R(r)$ is called "Radial Part" and $\Phi(\phi)$ is called "Azimuthal Part". Substituting solution (5.11) into equation (5.10c):

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Phi''}{\Phi} = \lambda^2. \tag{5.12}$$

First let us consider the case $\lambda = 0$

$$\Phi'' = 0 \Longrightarrow \Phi_{\lambda=0}(\phi) = (A_{\lambda=0})\phi + B_{\lambda=0}, \quad rR'' + R' = 0$$

$$\Longrightarrow R_{\lambda=0}(r) = (C_{\lambda=0})\ln r + D_{\lambda=0},$$

so that for $\lambda = 0$, the corresponding eigenvector in the solution is

$$R_{\lambda=0}(r)\Phi_{\lambda=0}(\phi) = (h_{\lambda=0})\,\phi\ln r + (k_{\lambda=0})\ln r + (l_{\lambda=0})\,\phi + (m_{\lambda=0})$$

where $h_{\lambda=0} \equiv (A_{\lambda=0})(C_{\lambda=0}), k_{\lambda=0} \equiv (B_{\lambda=0})(C_{\lambda=0}), l_{\lambda=0} = (A_{\lambda=0})(D_{\lambda=0})$ and $m_{\lambda=0} = (B_{\lambda=0})(D_{\lambda=0})$. Since solution must be bounded at centre (0,0), and since $\lim_{r\to 0}\ln r = \infty$, we have to set $k_{\lambda=0} = 0, h_{\lambda=0} = 0$. Further solution must be single valued, i.e. $\Phi_{\lambda=0}(\phi + 2\pi) = \Phi_{\lambda=0}(\phi)$ must hold, which means $l_{\lambda=0} = 0$ keeping only constant term $m_{\lambda=0} \equiv a_0/2$. Thus $\lambda = 0$ is legitimate eigenvalue whose corresponding contribution to the solution is

$$R_{\lambda=0}(r)\Phi_{\lambda=0}(\phi) = \frac{a_0}{2}. \tag{5.13}$$

Now we will find non-zero eigenvalues, and note that without loss of generality, we can assume $\lambda > 0$. From (5.12), we see that for $\lambda \neq 0$ case, Radial part satisfies Euler equation

$$r^2R'' + rR' - \lambda^2 R = 0,\ \ \lambda > 0, \tag{5.14}$$

which by the change of variable $r \longrightarrow \tilde{r} = \ln r$, reduces to

$$\frac{\mathrm{d}^2R}{\mathrm{d}\tilde{r}^2} - \lambda^2 R = 0 \Longrightarrow R_\lambda(r) = A_\lambda r^\lambda + B_\lambda r^{-\lambda}. \tag{5.15}$$

Note that solution must be bounded at the centre of the circle, so that we have

$$B_\lambda = 0 \Longrightarrow R_\lambda(r) = A_\lambda r^\lambda \text{ (since } \lambda > 0). \tag{5.16}$$

Azimuthal part satisfies following equation:

$$\Phi'' + \lambda^2 \Phi = 0 \Longrightarrow \Phi_\lambda(\phi) = C_\lambda \cos(\lambda\phi) + D_\lambda \sin(\lambda\phi). \tag{5.17}$$

Note that for uniqueness of solution, it must be single-valued function of ϕ:

$$\Phi_\lambda(\phi + 2\pi) = \Phi_\lambda(\phi). \tag{5.18}$$

(5.17) and (5.18) implies following pair of trigonometric equations:

$$\cos\{\lambda(\phi + 2\pi)\} - \cos(\lambda\phi) = 0, \quad \sin\{\lambda(\phi + 2\pi)\} - \sin(\lambda\phi) = 0.$$

Both equations have common factor $\sin(\pi\lambda)$. Hence, we must set

$$\sin(\pi\lambda) = 0 \Longrightarrow \lambda = \lambda_n = n \,, n = 0,1,2,3, \dots , \tag{5.19}$$

which constitute set of all eigenvalues. Note that we include $n = 0$ because previously we show that zero is also eigenvalue.

Substituting (5.13), (5.15), (5.17) into solution (5.11), we have

$$\psi(r, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [h_n \cos(n\phi) + k_n \sin(n\phi)], \tag{5.20}$$

where $h_n \equiv A_\lambda C_\lambda, k_n \equiv A_\lambda D_\lambda$. Using now given BC $\psi(a, \phi) = f(\phi)$, we have from (5.20)

$$f(\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\phi) + b_n \sin(n\phi)], \tag{5.21}$$

where $a_n \equiv a^n h_n, b_n \equiv a^n k_n$. The series (5.21) is full-range Fourier series. Hence coefficients are given by

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \cos(n\phi) \, d\phi \,, b_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \sin(n\phi) \, d\phi \,, n = 0,1,2, \dots \tag{5.22}$$

Hence, the solution (5.20) of interior Dirichlet Problem over circle is

$$\psi(r, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{a^n} [a_n \cos(n\phi) + b_n \sin(n\phi)] \tag{5.23}$$

Substituting now the coefficients from (5.22) into (5.23),

$$\psi(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi + \sum_{n=1}^{\infty} \frac{r^n}{\pi \, a^n} \Big[\cos(n\phi) \int_0^{2\pi} f(\phi) \cos(n\phi) \, d\phi \; + \sin(n\phi) \int_0^{2\pi} f(\phi) \sin(n\phi) \, d\phi \Big] \tag{5.24}$$

This completes solving interior Dirichlet Problem over a circle.

Remark: During steps, I time to time redefine arbitrary constants in order to make the coefficients of the resulting Fourier series looking same as in the series introduced in Sec 1, and hence these changes are just for convenience. You may skip those kind of changes. However, if you have product of arbitrary constants like AB, then it is good idea to absorb those in a single constant by defining $AB \equiv C$.

b. <u>Exterior Problem</u>

Let us consider following interior problem with Dirichlet BC

$$\nabla^2\psi = 0, \;\; a \le r < \infty, 0 \le \phi \le 2\pi, \tag{5.25a}$$

BC: $$\psi(a,\phi) = f(\phi), \;\; 0 \le \phi \le 2\pi \tag{5.25b}$$

Solution: Steps are identical mostly with the interior problem discussed above. However, problem is different because of different boundary conditions on radial part. Below I'll skip repeating steps to that point where procedure will be different, because readers may understand full steps.

We see that for exterior problem, solution must be bounded as $r \to \infty$, and also must be single-valued as before, so that $\Phi(\phi + \pi) = \Phi(\phi)$.

Proceed identically as above with Method of Separation. Check that here also $\lambda = 0$ is legitimate eigenvalue whose corresponding contribution is also same, i.e. a constant $a_0/2$, because $\lim\limits_{r\to\infty,} \ln r = \infty$.

For $\lambda \neq 0$, proceed similarly as before by assuming without loss of generality $\lambda > 0$ as before. Remember for exterior problem, r^{λ} has to be rejected in contrast to interior problem, where we rejected $r^{-\lambda}$ and keep r^{λ} term. All others will be same up to the eigenvalues $\lambda = \lambda_n = n\,, n = 0,1,2,\ldots$, so that we will finally obtain

$$\psi(r,\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}[h_n \cos(n\phi) + k_n \sin(n\phi)]. \tag{5.26}$$

Using BC $\psi(a,\phi) = f(\phi)$, as before, we get full-range Fourier series:

$$f(\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\phi) + b_n \sin(n\phi)], \tag{5.27}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos(n\phi)\,\mathrm{d}\phi\,, b_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin(n\phi)\,\mathrm{d}\phi\,, n = 0,1,2,\ldots, \tag{5.28}$$

where, in contrast to interior problem, $a_n \equiv a^{-n}h_n, b_n \equiv a^{-n}k_n$ [note the negative sign before n in the power of a].

Hence, the solution is

$$\psi(r,\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a^n}{r^n}\left[a_n \cos(n\phi) + b_n \sin(n\phi)\right] \qquad (5.29)$$

Substituting for the Fourier coefficients, final form of solution for the exterior problem is given by

$$\psi(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\mathrm{d}\phi + \sum_{n=1}^{\infty} \frac{a^n}{\pi\, r^n}\left[\cos(n\phi)\int_0^{2\pi} f(\phi)\cos(n\phi)\,\mathrm{d}\phi + \sin(n\phi)\int_0^{2\pi} f(\phi)\sin(n\phi)\,\mathrm{d}\phi\right] \qquad (5.30)$$

This completes solution of exterior Dirichlet problem of 2D Laplace equation over a circle.

Note:

I. Compare interior solution (5.24) with exterior solution (5.30) to see the difference.
II. Students are advised to produce full steps for exterior problem.

- Dirichlet Problem in Spherical Coordinates

<u>3D Laplace Equation over a sphere</u>

a) Interior Problem

Consider following problem inside a solid sphere of radius a

Laplace Equation: $\nabla^2\psi = 0\,,\ \ 0 \le r \le a, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$ (6.1a)

BC: $\psi(a,\theta,\phi) = f(\theta,\phi)$, (6.1b)

Solution: As discussed before, for a spherical domain, we will choose spherical polar coordinates. Thus Laplace equation (6.1a) in (r,θ,ϕ) becomes

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0. \qquad (6.1c)$$

We use Method of Separation. Assume solution in the following separable form:

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi). \qquad (6.2)$$

Note that out of three variables, at the 1st step we separate radial part, and in the 2nd step, we will separate other two variables. Thus, in this 3D problem, two separation constants will arise. The part $Y(\theta,\phi)$ is called spherical harmonic.

Substituting assumed solution (6.12) into Laplace equation (6.1c):

$$\frac{1}{R}\left[r^2R'' + 2rR'\right] = -\frac{1}{Y}\left[\frac{\partial^2 Y}{\partial\theta^2} + \cot\theta\,\frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = l(l+1), \tag{6.3}$$

where, we have taken the separation constant in a special form because that will be useful to express final solution in a convenient form. However, this is again a convenience only, and as usual at this point we do not know whether l is real or complex.

Radial Equation for $R(r)$

We get Euler equation, what we obtained before in another problem:

$$r^2R'' + 2rR' - l(l+1)R = 0 \Longrightarrow R_l(r) = A_l r^l + B_l r^{-(l+1)}. \tag{6.4}$$

Angular Equation for $Y(\theta,\phi)$

From (6.3), the equation for spherical harmonics:

$$\frac{\partial^2 Y}{\partial\theta^2} + \cot\theta\,\frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + l(l+1)Y = 0. \tag{6.5}$$

We now make 2nd separation by assuming following solution for (6.5):

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi). \tag{6.6}$$

Substituting (6.6) into (6.5)

$$\frac{\sin^2\theta}{\Theta}\left[\Theta'' + (\cot\theta)\Theta'\right] + l(l+1)\sin^2\theta = -\frac{1}{\Phi}\Phi'' = m^2, \tag{6.7}$$

where m^2 is 2nd separation constant. This means that the eigenvalues will be labeled by two indices l, m. Note again, at this point we do not know whether m is real or complex.

Equation for Φ

$$\Phi'' + m^2\Phi = 0. \tag{6.8}$$

First check whether $m = 0$ would give legitimate solution.

$$m = 0: \Phi(\phi) = A\phi + B$$

Since solution must be single-valued, i.e. $\Phi(\phi + 2\pi) = \Phi(\phi) \Longrightarrow A = 0$, so that contribution for $m = 0$ is

$$\Phi_{m=0}(\phi) = B. \tag{6.9}$$

Next consider $m \neq 0$ in (6.8) for the equation in Φ. But note that in spite of square form (m^2), here we will not restrict m as strictly positive as we did in previous problems, because unlike previous problems, situation here is a bit complicated due to the fact that there may an inter-dependency between m and 1st separation constant l. Hence, we will proceed with $m \neq 0$ only. The solution for Φ for $m \neq 0$ is

$$\Phi_m(\phi) = C_m \mathrm{e}^{im\phi} + D_m \mathrm{e}^{-im\phi}. \tag{6.10}$$

For single-valuedness, $\Phi_m(\phi + 2\pi) = \Phi_m(\phi)$ for each m, which gives its range:

$$\exp(2im\pi) = 1 \Longrightarrow m = 0, \pm 1, \pm 2, \pm 3, \dots, \tag{6.11}$$

Note that we include $m = 0$, since we show previously that $m = 0$ is allowed, which contributes constant term in solution. However, the range of m, obtained in (6.11) is not finalized, because final range of m may be subset of it due to inter-dependency with 1st separation constant l, i.e. until we find allowed range of l, range of m is not finalized. For now, for all m given in (6.11), eigenvector is given by (6.10).

Equation for Θ

$$\sin^2\theta[\Theta'' + (\cot\theta)\Theta'] + [l(l+1)\sin^2\theta - m^2]\Theta = 0. \tag{6.12}$$

Equation (6.12) is actually "Associated Ligendre Equation"$^{\perp}$ in trigonometric form. It may be reduced to well-known algebraic form by the following change of variable:

$$\theta \to \xi = \cos\theta, -1 \leq \xi \leq 1. \tag{6.13}$$

Change of variable governed by (6.13) reduces (6.12) in the following form:

$$(1-\xi^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}\xi^2} - 2\xi\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} + \left[l(l+1) - \frac{m^2}{1-\xi^2}\right]\Theta = 0. \tag{6.14}$$

Note that $\xi = \pm 1$(i.e. $\theta = 0, \pi$) are singular points. Hence, we need a power series solution around the ordinary point ξ =0 (i.e. around the symmetrical position $\theta = \pi/2$), and it is known that $\exists$ one such convergent infinite series of unit radius of convergence. Further, to have bounded solution, we need a finite solution in θ.

It is well-known from the theory of special functions that Associated Legendre equation (6.14) has a finite solution, if and only if the separation constants l, m have following ranges and inter-dependency:

$$l = 0,1,2, \dots, \quad m = 0, \pm 1, \pm 2, \dots \pm l. \tag{6.15}$$

The finite solution of Associated Legendre equation (6.14) is denoted by $P_l^m(\xi)$, which is $(1-\xi^2)^{|m|/2}$ times a finite polynomial of degree $l-|m|$. This solution is conventionally called "Associated Legendre Polynomial", although it is not a polynomial unless m is even.

Note that range of m is now finalized by inter-dependency as $|m| \leq l$, which is

$\perp$For special functions, see books of Mathematical Methods. Text book and Ref 1 also have discussions.

subset of range given by (6.11) as it should be. Now, since equation (6.15) constitute complete set of eigenvalues, we have to take sum of all terms over the allowed ranges of l, m, by the Principle of Superposition. The eigenvector $\Theta_l^m(\theta)$ corresponding to each pair (l, m) is given by

$$\Theta_l^m(\theta) = A_l^m \; P_l^m(\cos\theta), \tag{6.16}$$

where Associated Legendre Polynomial $P_l^m(\xi)$ is as follows

$$P_l^m(\xi) = (1-\xi^2)^{|m|/2} \frac{\mathrm{d}^{|m|} P_l(\xi)}{\mathrm{d}x^{|m|}}, \tag{6.17}$$

$P_l(\xi)$ being the Legendre Polynomial of degree l, introduced in Sec 1.

Further, we can now safely drop one term from the eigenvector $\Phi_m(\phi)$ in (6.10);

$$\Phi_m(\phi) = C_m \mathrm{e}^{im\phi}. \tag{6.18}$$

We drop $\mathrm{e}^{-im\phi}$ term because it contains in (6.18) for negative values of m from (6.15).

Thus from (6.4), (6.16), (6.18), we have full solution from (6.2) as follows

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[\left(A_l r^l + B_l r^{-(l+1)} \right) A_l^m P_l^m(\cos\theta) C_m \mathrm{e}^{im\phi} \right]. \tag{6.19}$$

Note that for interior problem, solution ψ must be bounded at centre $(r=0)$, and hence $r^{-(l+1)}$ has to be dropped, that means that we are to take $B_l = 0$. Hence, absorbing redundant arbitrary constants, solution (6.19) reduces as follows

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty} r^l \sum_{m=-l}^{l} \left(A_l^m P_l^m(\cos\theta) \mathrm{e}^{im\phi} \right). \tag{6.10}$$

Using BC $\psi(a,\theta,\phi) = f(\theta,\phi)$,

$$f(\theta,\phi) = \sum_{l=0}^{\infty} a^l \sum_{m=-l}^{l} \left(A_l^m P_l^m(\cos\theta) \mathrm{e}^{im\phi} \right). \tag{6.11}$$

To find A_l^m, we need orthogonal relations of P_l^m and $\mathrm{e}^{im\phi}$:

$$\int_{-1}^{1} P_l^m(\xi)\, P_{l'}^m(\xi) \mathrm{d}\xi = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}\ , \ \int_0^{2\pi} \mathrm{e}^{i(m-m')\phi} \mathrm{d}\phi = 2\pi\delta_{mm'} \qquad (6.12)$$

Multiply (6.11) by $\sin\theta\, P_{l'}^m(\cos\theta)\mathrm{e}^{im'\phi}$ and integrating w.r.t. $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$, we get, using orthogonal relation:

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta\ f(\theta,\phi) P_l^m(\cos\theta)\mathrm{e}^{im\phi} \mathrm{d}\theta\mathrm{d}\phi\ = \frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!} a^l A_l^m$$

$$\Longrightarrow A_l^m = \frac{2l+1}{4\pi a^l}\frac{(l-m)!}{(l+m)!} \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta\ f(\theta,\phi)P_l^m(\cos\theta)\mathrm{e}^{im\phi}\mathrm{d}\theta\mathrm{d}\phi\ , \qquad (6.13)$$

where in above equations, prime in l, m has been dropped. Hence, final solution of interior Dirichlet Problem over a sphere with general BC on circumference is given by (6.10). Plugging in (6.13), we may write as follows:

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi}\frac{r^l}{a^l}\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}\left(P_l^m(\cos\theta)\mathrm{e}^{im\phi}\right)$$

$$\times\left[\int_{\phi'=0}^{2\pi}\int_{\theta'=0}^{\pi} \sin\theta'\ f(\theta',\phi')P_l^m(\cos\theta')\mathrm{e}^{im\phi}\mathrm{d}\theta'\mathrm{d}\phi'\right] \qquad (6.14)$$

Note that the integral variables θ, ϕ in (6.13) have been replaced by θ', ϕ' in (6.14) to indicate that (r, θ, ϕ) are polar coordinates of any point inside and on the sphere.

<u>Special BC: Axisymmetric Solution</u>

Consider prescribed function $f(\theta,\phi)$ on circumference is independent of azimuthal angle ϕ, i.e.

$$f(\theta,\phi) = g(\theta), \forall\theta \Longrightarrow \text{BC: } \psi(a,\theta,\phi) = f(\theta). \qquad (6.15)$$

The reason of discussing this separately is that for this kind of BC, solution (6.10) becomes substantially simplified, because in this case solution must be independent of ϕ, i.e. $\psi(r,\theta,\phi) \equiv \psi(r,\theta)$.

This kind of solution is called "Axisymmetric Solution", since it is symmetric about z-axis. So for this particular kind of problem (i.e. with such BC), we need to take $m = 0$ only. Hence, solution (6.10) reduces to the following form:

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty} A_l\, r^l\, P_l(\cos\theta). \qquad (6.16)$$

Using BC $\psi(a,\theta,\phi) = g(\theta)$,

$$g(\theta) = \sum_{l=0}^{\infty} (a^l A_l) P_l(\cos\theta), \tag{6.17}$$

where the coefficients are given by

$$A_l = \frac{2l+1}{2a^l} \int_{-1}^{1} g(\theta) P_l(\cos\theta) \, \mathrm{d}(\cos\theta). \tag{6.18}$$

Plugging (6.18) into (6.16), final axisymmetric solution of interior Dirichlet Problem over a sphere is given by

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty} \frac{2l+1}{2} \left(\frac{r}{a}\right)^l P_l(\cos\theta) \int_{-1}^{1} g(\theta') P_l(\cos\theta') \, \mathrm{d}(\cos\theta'). \tag{6.19}$$

Note that axisymmetric solution is contained in non-axisymmetric solution (6.14) for $m = 0$.

This completes discussion about solution of interior Dirichlet Problem over a sphere of radius a.

Example: Solve Laplace equation inside a sphere of radius 2m with following BC

$$\psi(2,\theta,\phi) = \begin{cases} 4\cos\theta, & 0 \le \theta \le \pi/2 \\ \\ 0, & -\pi/2 \le \theta \le 0 \end{cases} \tag{6.20}$$

Calculate first four non-zero terms of the solution $\psi(r,\theta,\phi), 0 \le r \le 2$.

Solution: I'll not repeat the steps. Note that given BC implies that solution is axisymmetric. Using definitions of $\psi(2,\theta,\phi)$ in (6.20), (6.18) becomes

$$A_l = \frac{2l+1}{2^{l+1}} \int_0^1 4\xi P_l(\xi) \mathrm{d}\xi \quad . \quad [\text{ Note: } \xi = \cos\theta] \tag{6.14}$$

Previously, I gave expression of Legendre polynomial with first few members. Using those, let me show computation of 1st four nonzero coefficients, as asked in question, from (6.14).

$$A_0 = \frac{1}{2}\int_0^1 4\xi P_0(\xi)\mathrm{d}\xi = \frac{1}{2}\int_0^1 4\xi \mathrm{d}\xi = 1, A_1 = \frac{3}{4}\int_0^1 4\xi P_1(\xi)\mathrm{d}\xi = \frac{3}{4}\int_0^1 4\xi^2 \mathrm{d}\xi = 1$$

$$A_2 = \frac{5}{8}\int_0^1 4\xi P_2(\xi)\mathrm{d}\xi = \frac{5}{8}\frac{1}{2}\int_0^1 4\xi(3\xi^2 - 1)\mathrm{d}\xi = \frac{5}{16}$$

$$A_3 = \frac{7}{16}\int_0^1 4\xi P_3(\xi)\mathrm{d}\xi = \frac{7}{4}\frac{1}{2}\int_0^1 \xi(5\xi^3 - 3\xi)\mathrm{d}\xi = 0$$

$$A_4 = \frac{9}{32}\int_0^1 4\xi P_4(\xi)\mathrm{d}\xi = \frac{9}{8}\frac{1}{8}\int_0^1 \xi(35\xi^4 - 30\xi^2 + 3)\mathrm{d}\xi = -\frac{3}{128}$$

Thus solution is

$$\psi(r,\theta,\phi) = 1 + r\cos\theta + \frac{5}{32}r^2(3\cos^2\theta - 1)$$

$$-\frac{3}{1024}r^4\,(35\cos^4\theta - 30\cos^2\theta + 3) + \cdots \qquad (6.15)$$

b) Exterior Problem

You already know that for exterior problem we seek solution $\psi(r,\theta,\phi)$ outside a sphere, i.e for $r \geq a$, a is radius of sphere. As before, here we have to drop r^l term and keep $r^{-(l+1)}$ term, since exterior solution must tend to zero as $r \to \infty$ giving bounded solution. Hence, we have general solution in the following form

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\left(\frac{a}{r}\right)^{l+1}\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}\left(P_l^m(\cos\theta)\mathrm{e}^{im\phi}\right)$$

$$\times\left[\int_{\phi'=0}^{2\pi}\int_{\theta'=0}^{\pi}\sin\theta'\ f(\theta',\phi')P_l^m(\cos\theta')\mathrm{e}^{im\phi}\mathrm{d}\theta'\mathrm{d}\phi'\right], \quad (6.16a)$$

for general BC $\psi(a,\theta,\phi) = f(\theta,\phi)$, and axisymmetric solution for ϕ-independent BC $\psi(a,\theta,\phi) = g(\theta)$ on circumference is

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\frac{2l+1}{2}\left(\frac{a}{r}\right)^{l+1}P_l(\cos\theta)\int_0^{\pi}g(\theta')P_l(\cos\theta')\mathrm{d}(\cos\theta'). \quad (6.16b)$$

Example: Find the bounded solution $\psi(r,\theta,\phi)$ of Laplace equation outside a unit sphere with centre at origin that reduces to $\cos(2\theta)$ on circumference.

Solution: BC implies that solution is axisymmetric. Skipping all steps, final form is given by (6.16b). Let me show computation of coefficients.

Note that $\cos(2\theta) = 2\cos^2\theta - 1 = 2\xi^2 - 1 = \frac{1}{3}[4P_2(\xi) - P_0(\xi)]$, and coefficients A_l is given by

$$A_l = \frac{2l+1}{2}\,\frac{1}{3}\int_{-1}^{1}[4P_2(\xi) - P_0(\xi)]P_l(\xi)\mathrm{d}\xi. \qquad (6.17)$$

Watch the trick! We notice that given function contains only even-degree terms in ξ , which is the characteristic of Legendre polynomial $P_l(\xi)$ Precisely $P_l(\xi)$ is a polynomial in ξ of degree l containing only even-degree or odd-degree terms according as l is even or odd. Then the advantage of writing the integrand in (6.17) in terms of Legendre polynomials is that we can now use orthogonality property of Legendre polynomials:

$$\int_{-1}^{1} P_l(\xi)P_m(\xi)\mathrm{d}\xi = \frac{2}{2l+1}\delta_{lm}, \tag{6.18}$$

where δ_{lm}, known as "Kronecker delta", is defined as $\delta_{lm} = 2/(2l+1)$ for $l = m$, and zero otherwise.

Hence, final take-away is simply by inspection, we understand that only two coefficients A_0 and A_2 will be non-zero.

$$A_0 = -\frac{1}{3}, A_2 = \frac{10}{3}\frac{2}{5} = \frac{4}{3}$$

Exterior solution thus contain only two terms:

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\left(\frac{1}{r}\right)^{l+1} A_l P_l(\cos\theta) = \frac{1}{3}\left[-\frac{1}{r} + \frac{2}{3r^3}(3\cos^2\theta - 1)\right]$$

Note that the above solution is defined for $r \geq 1$, since it is exterior problem. Laplace equation may also be studied with other boundary conditions, stated in Sec5. Interested readers may see Ref 2.

*******************END of Part C*******************