# Duality Theory

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- Every linear programming problem has associated with it another linear programming problem called the **Dual.** The original problem is called **Primal**.

**The watch maker's Problem**

Smith Family: Father (F) and Son (S)

| **Decision Variable** | **Watch Type** | **Profit Per Unit ($)** | **Working hours required of** | |
|---|---|---|---|---|
| | | | **F** | **S** |
| $x_1$ | 1 | 60 | 2 | 3 |
| $x_2$ | 2 | 40 | 1 | 4 |
| $x_3$ | 3 | 80 | 4 | 2 |
| **Maximum Working Hours available per week** | | | 50 | 60 |

Production plan to maximize total profit from 3 type of watches

Max $W = 60\,x_1 + 40\,x_2 + 80\,x_3$

S.t. $2\,x_1 + x_2 + 4\,x_3 \le 50$

$3\,x_1 + 4\,x_2 + 2\,x_3 \le 60$

$x_1, x_2, x_3 \ge 0$

## John Blake

Neighbor and owner of similar product line company (competitor)

Wants to hire Father and Son full time

**How to convince them (Son and Father) to join him (John Blake)?**

Comparable wages with respect to before : “at least” as good as before

Hire them at the minimum possible cost: “just enough” to effect the switch over

F’s wages – \$ $y_1$ per hour

S’s wages – \$ $y_2$ per hour

$$
\begin{aligned}
&\text{Min } B = 50\, y_1 + 60\, y_2 \\
&\text{S.t.} \quad 2\, y_1 + 3\, y_2 \geq 60 \\
&\qquad\quad\; y_1 + 4\, y_2 \geq 40 \\
&\qquad\quad\; 4\, y_1 + 2\, y_2 \geq 80 \\
&\qquad\quad\; y_1, y_2 \geq 0
\end{aligned}
$$

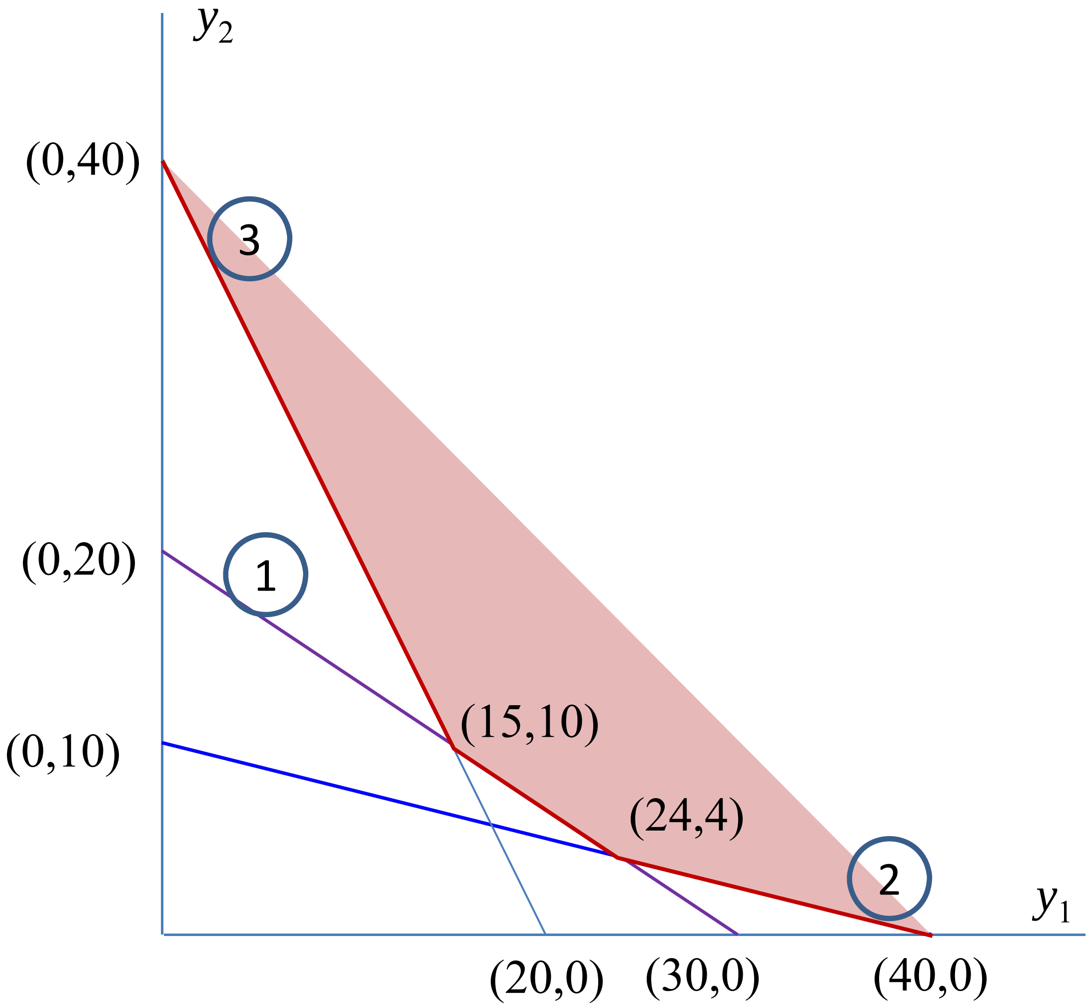
Solution: $y_1 = 15$, $y_2 = 10$, Objective function $B$ = \$1350 per week

# Solution of John Blake Problem

- Obj B is min at

  $y_1 = 15, y_2 = 10$
- Objective function

  $B = \$1350$ per week



**Will the solution of John Blake help to solve Watch maker's problem?**

From watch maker's constraint

$$2x_1 + x_2 + 4x_3 \le 50 \qquad (1)$$

$$3x_1 + 4x_2 + 2x_3 \le 60 \qquad (2)$$

Now, (1)x15 + (2) x 10

$$60x_1 + 55x_2 + 80x_3 \le 1350$$

Also, $60x_1 + 55x_2 + 80x_3 \le 1350 \le 50y_1 + 60y_2$ (for any feasible $y_1$ and $y_2$)

$60x_1 + 40x_2 + 80x_3$ : Watch maker's (Smith Family) objective function

Hence, the best Smith Family could do is \$1350 per week.

How can they make it?

Only by chance if $x_2 = 0$

$$2x_1 + 4x_3 = 50$$

$$3x_1 + 2x_3 = 60 \qquad \Rightarrow x_1 = 70/4 \text{ and } x_3 = 15/4$$

Objective function value W = 60 x 70/4 + 80 x 15/4 = \$1350

**John Blake Problem -> Watch maker's problem & vice-versa**

# Another Example: Diet Problem

$$\text{Min } Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{S.t. } \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad \forall i = 1,2,\ldots,m$$

$$x_j \geq 0, \qquad \forall j = 1,2,\ldots,n$$

**Where,**
$c_j$ : cost per unit of food $j$
$a_{ij}$: amount of nutrient $i$ available in per unit of food $j$
$b_i$: minimum nutritional requirement for nutrient $i$
$x_j$: amount of food to be included in the diet.

**Let us consider a hypothetical dual problem to the diet problem**

**Salesman:** sales pure nutrient pills (i.e. only iron or only vitamins etc.)

- wants to sell pills to the dietician in order to switch completely from foods to pills
- Price pills subject to

  (1) A switch takes place – competitive prices in terms of cost of foods

  (2) Total revenue to salesman is maximized if minimum requirements are sold

## Dual Problem

$$\textbf{Max } W = \sum_{i=1}^{m} b_i y_i$$

**Where,** $y_i$: price of $i$th nutrient pill containing one unit of nutrient $i$

$$\textbf{S.t. } \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \qquad \forall j = 1,2,\ldots,n$$

**(Competitive pricing w.r.t. foods)**

$$y_i \geq 0, \qquad \forall i = 1,2,\ldots,m$$

# How to get Dual (D) from Primal (P)?

**P**

**Max** $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ **Dual Variable**

**S.t.** $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}\, x_n \leq b_1$ $y_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}\, x_n \leq b_2$ $y_2$

**(Optimal Allocation of Resources)**

$\cdot$ $\cdot$

$\cdot$ $\cdot$

$\cdot$ $\cdot$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}\, x_n \leq b_m$ $y_m$

**and** $x_j \geq 0$, for $j = 1, 2, \ldots, n$

**D**

**Min** $W = b_1y_1 + b_2y_2 + \ldots + b_my_m$

**S.t.** $a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}\, y_m \geq c_1$

$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}\, y_m \geq c_2$

**(Optimal Pricing of Resources)**

$\cdot$

$\cdot$

$\cdot$

$a_{1n}y_1 + a_{m2}y_2 + \ldots + a_{mn}\, y_n \geq c_n$

**and** $y_i \geq 0$, for $i = 1, 2, \ldots, m$

# Matrix Form of Primal and Dual Problems

**P** Max $Z = \mathbf{cx}$ **Dual variable**

S.t. $\mathbf{Ax} \le \mathbf{b}$ $\mathbf{y}$

$\mathbf{x} \ge \mathbf{0}$

**D** Min $W = \mathbf{yb}$

S.t. $\mathbf{yA} \ge \mathbf{c}$

$\mathbf{y} \ge \mathbf{0}$ $\mathbf{y} = [y_1\ y_2\ \ldots\ y_m]$

# Summary of P-D Relations (Rules for constructing the dual problem)

| Primal Problem | ↔ | Dual Problem |
|---|---|---|
| Objective: Maximization | ↔ | Objective: Minimization |
| Cost coefficient | ↔ | RHS |
| RHS | ↔ | Cost coefficient |
| Constraints<br>≤<br>=<br>≥ | ↔ | Variables<br>≥ 0<br>Unrestricted in sign<br>≤ 0 |
| Variables<br>≥ 0<br>Unrestricted in sign<br>≤ 0 | ↔ | Constraints<br>≥<br>=<br>≤ |

# Demonstration of Primal-Dual Rule: Numerical Examples

Example 1: **Maximization Type**

Max $Z = x_1 + 4x_2 + 3x_3$

| | Dual Variable |
|---|---|
| S.t. $2x_1 + 3x_2 - 5x_3 \leq 2$ | $y_1$ |
| $3x_1 - x_2 + 6x_3 \geq 1$ | $y_2$ |
| $x_1 + x_2 + x_3 = 4$ | $y_3$ |

$x_1 \geq 0,\ x_2 \leq 0,\ x_3$ unrestricted

Dual

Min $W = 2y_1 + y_2 + 4y_3$

S.t. $2y_1 + 3y_2 + y_3 \geq 1$

$3y_1 - y_2 + y_3 \leq 4$

$-5y_1 + 6y_2 + y_3 = 3$

$y_1 \geq 0,\ y_2 \leq 0,\ y_3$ unrestricted

Example 2: **Minimization Type**

Min $Z = 2x_1 + x_2 - x_3$

| | Dual Variable |
|---|---|
| S.t. $x_1 + x_2 - x_3 = 1$ | $y_1$ |
| $x_1 - x_2 + x_3 \geq 2$ | $y_2$ |
| $x_2 + x_3 \leq 4$ | $y_3$ |

$x_1 \geq 0,\ x_2 \leq 0,\ x_3$ unrestricted

Dual

Max $W = y_1 + 2y_2 + 4y_3$

S.t. $y_1 + y_2 \leq 2$

$y_1 - y_2 + y_3 \geq 1$

$-y_1 + y_2 + y_3 = -1$

$y_1$ unrestricted, $y_2 \geq 0,\ y_3 \leq 0$