Linear algebra ~~for~~ for AI & ML

(November - 3)

Low Rank approximations.

Let $A \in \mathbb{R}^{n \times m}$

$X = \begin{bmatrix} \quad \end{bmatrix}_{n \times r}$ $\quad Y = \begin{bmatrix} \quad \end{bmatrix}_{r \times m}$

Compute "nearest" rank r approximation to A.

Golub

Using SVD,

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_m u_m v_m^T$$

(m : rank of A)

$$\tilde{A} = \underset{\sim}{\sigma_1} \underset{\sim}{u_1} \underset{\sim}{v_1}^T + \cdots + \underset{\sim}{\sigma_r} \underset{\sim}{u_r} \underset{\sim}{v_r}^T$$

$\underset{\sim}{A^T A}$, $\underset{\sim}{A A^T}$

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma^T U^T U \Sigma V^T = V \underbrace{\Sigma^T \Sigma}_{D} V^T$$

$A \in \mathbb{R}^{n\times m}$

$$A \approx \underbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix}_{n\times r}}_{X} \begin{bmatrix} \quad & Y & \quad \end{bmatrix}_{r\times m}$$

$A \approx XY$ ← will have rank 'r'.

<u>Formulation:</u>

$$\min_{X,Y} \|A - XY\|_2 \quad \leftarrow \text{subject } X \geq 0,\ Y \geq 0$$

How do you solve this optimization problem??

- no. of decision variables is $r(m+n)$

$$= \underbrace{rm}_{Y} + \underbrace{rn}_{X}$$

A ← given

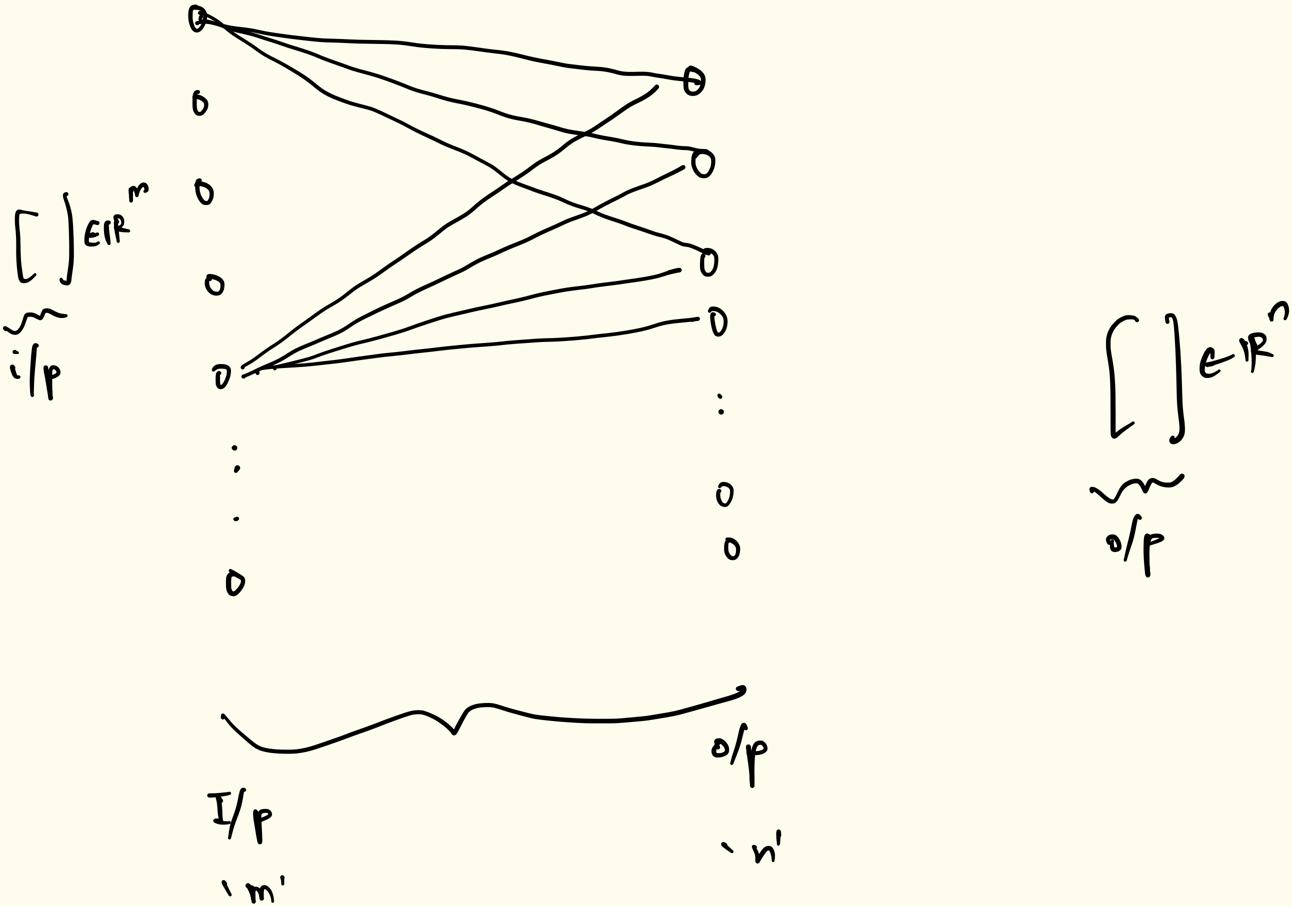
find $\hat{x}$, $\hat{y}$ such that

$$(\hat{x}, \hat{y}) = \underset{x, y}{\arg\min} \| A - xy \|_2$$

Suppose $\underbrace{x = x_0}$ ← given

$$\hat{y}_0 = \underset{y}{\arg\min} \| A - x_0 y \|_2$$

$$\hat{x}_1 = \underset{x}{\arg\min} \| A - x \hat{y}_0 \|_2$$



$[\ ] \in \mathbb{R}^m$
i/p
$[\ ] \in \mathbb{R}^n$
o/p
I/p
'm'
o/p
'n'