Assignment Problem

Introduction

- This is a special type of transportation problem in which each source should have the capacity to fulfill the demand of any one of the destinations.
- In other words, any operator would be able to perform any job regardless of his skill, although the cost(or the time taken) will be more if the job does not match with operator's skill.

- **Examples of assignment problem**

Row	**Column**	**Cell entry**
Jobs	Machines	Processing time/cost
Programmer	Program	Coding time
Operators	Assignments	Processing time/cost
Drivers	Routes	Travel time
Teachers	Subjects	Students pass percentage

General format

n : number of jobs = number of machines

c_{ij} : processing cost (or time) of job i by machine j.

Objective: to assign the jobs to the machines such that the total processing cost (or time) is minimized.

Constraints: Each job is assigned to exactly one machine and Each machine performs exactly one job

Machine

Job		1	2	…	j	…	n	
	1	x_{11} c_{11}	x_{12} c_{12}	x_{13} …	c_{1j}	…	c_{1n}	u_1
	2							u_2
	.							
	i	c_{i1}			c_{ij}		c_{in}	u_i
	.							
	n	c_{n1}	c_{n2}		c_{nj}		c_{nn}	u_n
		v_1	v_2		v_j		v_n	

$\sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij} x_{ij}$

s.t.

Same sol. of. [illegible]

Square cost matrix *n* x *n*

$c'_{ij} = c_{ij} - u_i - v_j$

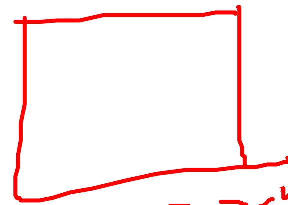
General LP formulation

Decision variables

$$x_{ij} = \begin{cases} 1, & \text{If job } i \text{ is assigned to machine } j \\ 0, & \text{Otherwise} \end{cases}$$

Formulation

$$\text{Minimize } Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}$$



n

$n+n-1 = (2n-1) - n = (n-1) = 0$

No. of variables = n^2

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1, \forall i = 1, 2, \ldots, n$$ **Each job is assigned to exactly one machine**

No. of basic variables = n

(i.e number of variables will take value as 1)

$$\sum_{i=1}^{n} x_{ij} = 1, \forall j = 1, 2, \ldots, n$$ **Each machine performs exactly one job**

if this is solved using simplex

The No. of BV = $n+n-1 = 2n-1$

i.e (n-1) No. of Basic variables will take value = 0

Highly degenerate

$$x_{ij} \geq 0, \forall ij$$

Solving Assignment Problem: Hungarian Algorithm

(Balanced Assignment Prob.)
No. of assignments = No. of assignees.

Theorem: Optimal solution of the assignment problem remains the same if a constant is added/ subtracted to any row/column of cost matrix

Proof:

$$c_{ij} \rightarrow \overline{c_{ij}} = c_{ij} - u_i - v_j$$

$$\overline{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_{ij}}\, x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} (c_{ij} - u_i - v_j)\, x_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} v_j \sum_{i=1}^{n} x_{ij}$$

$$\overline{Z} = Z - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j$$

$$\overline{Z} = Z - \text{constant}$$

$$\text{Min } \overline{Z} = \text{Min } Z$$

$\sum_{i=1}^{n}\sum_{j=1}^{n} u_i x_{ij}$

$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}$

$\sum_{j=1}^{n}\sum_{i=1}^{n} x_{ij}$

u_i, v_j

$\overline{Z}$ and Z

$u_i = \min_{\forall j} c_{ij}$

From constraints

$$\left(\sum_{j=1}^{n} x_{ij} = 1, \forall i\right)$$

$$\left(\sum_{i=1}^{n} x_{ij} = 1, \forall j\right)$$

Can prepare a new matrix with zero cost element for some cells

Hungarian Algorithm

Step 1: Subtract the smallest element of each row from all the elements of that row. (Row reduction)

Step 2: Subtract the smallest element of each column from all the elements of that column. (Column reduction)

(As a result, there would be at least one zero in each row and column of the reduced matrix)

Step 3: Determine an optimal assignment as follows:

(i) Starting with the first row of the reduced matrix, examine each row one by one until a row with exactly one zero is found. Make assignment in the cell with zero element and cross out all other zeros in the corresponding column.

(ii) Repeat the procedure for columns.

Repeat (i) and (ii) until all zeros are either assigned or crossed out.

Step 4: If the number of assigned cells equals the number of rows (columns), then optimal assignment is found and Stop. Otherwise, go to Step 5

Hungarian Algorithm Contd…

Step 5: Cover all the zeros using minimum number of horizontal and/or vertical lines as follows:

(i) Tick all unassigned rows

(ii) If a ticked row has an unassigned zero, then tick the corresponding column (not already ticked)

(iii) If a ticked column has an assignment, then tick the corresponding row (not already ticked)

(iv) Repeat (ii) and (iii) till no more ticking is possible.

(v) Draw lines through unticked rows and ticked columns. The number of lines represents the maximum number of assignments possible.

Step 6: Select the smallest uncovered element. This element is subtracted from every uncovered element and added to every element at the intersection of two lines.

Step 7: Go to Step 3 and repeat the procedure till an optimal solution is obtained.

Example 1: 4 Jobs and 4 persons

Person

Job		P1	P2	P3	P4
	J1	5	9	**3**	6
	J2	8	7	8	**2**
	J3	**6**	10	12	7
	J4	**3**	10	8	6

Row Reduction operation

Person

Job		P1	P2	P3	P4
	J1	2	6	0	3
	J2	6	5	6	0
	J3	0	**4**	6	1
	J4	0	7	5	3

Column Reduction operation

Person

Job		P1	P2	P3	P4
	J1	2	2	$\boxed{0}$	3
	J2	6	1	6	$\boxed{0}$
	J3	$\cancel{0}$	$\boxed{0}$	6	1
	J4	$\boxed{0}$	3	5	3

Assignment

Job	Person
J1	P3
J2	P4
J3	P2
J4	P1

Total Cost = 3+2+10+3 = 18

Example 2: 5 jobs 5 Machines

		Machine				
		M1	M2	M3	M4	M5
Job	J1	11	**7**	10	17	10
	J2	13	21	**7**	11	13
	J3	**13**	13	15	13	14
	J4	18	**10**	13	16	14
	J5	12	**8**	16	19	10

Example: Contd...

Row Reduction Operation

		Machine				
		M1	M2	M3	M4	M5
Job	J1	4	0	3	10	3
	J2	6	14	0	4	6
	J3	0	0	2	0	**1**
	J4	8	0	3	6	4
	J5	4	0	8	11	2

Step 1: Subtract the smallest element of each row from all the elements of that row. (Row reduction)

Example: Contd…

Column Reduction Operation

Machine

Job		M1	M2	M3	M4	M5
	J1	4	0	3	10	2
	J2	6	14	0	4	5
	J3	0	0	2	0	0
	J4	8	0	3	6	3
	J5	4	0	8	11	1

Step 2: Subtract the smallest element of each column from all the elements of that column. (Column reduction)

Example: Contd...

Step 3: Determine an optimal assignment as follows:

(i) Starting with first row of the reduced matrix, examine each row one by one until a row with exactly one zero is found. Make assignment in the cell with zero element and cross out all other zeros in the corresponding column.

(ii) Repeat the procedure for the columns.

Repeat (i) and (ii) until all zeros are either assigned or crossed out.

Machine

Job		M1	M2	M3	M4	M5
	J1	4	$\boxed{0}$	3	10	2
	J2	6	14	$\boxed{0}$	4	5
	J3	$\boxed{0}$	$\cancel{0}$	2	$\cancel{0}$	$\cancel{0}$
	J4	8	$\cancel{0}$	3	6	3
	J5	4	$\cancel{0}$	8	11	1

Number of assignments made is less than number of rows/columns

Example: Contd...

Step 4: If the number of assigned cells equals the number of rows (and columns), then optimal assignment is found and Stop. Otherwise, go to Step 5

Machine

Job		M1	M2	M3	M4	M5
	J1	4	$\boxed{0}$	3	10	2
	J2	6	14	$\boxed{0}$	4	5
	J3	$\boxed{0}$	~~0~~	2	~~0~~	~~0~~
	J4	8	~~0~~	3	6	3
	J5	4	~~0~~	8	11	1

Example: Contd…

Step 5: Cover all the zeros with minimum number of horizontal and/or vertical lines as follows:

(i) Tick all unassigned rows

(ii) If a ticked row has an unassigned zero, then tick the corresponding column (not already ticked)

(iii) If a ticked column has an assignment, then tick the corresponding row (not already ticked)

(iv) Repeat (ii) and (iii) till no more ticking is possible.

(v) Draw lines through unticked rows and ticked columns. The number of lines represents the maximum number of assignments possible.

	M1	M2	M3	M4	M5	
J1	4	$\boxed{0}$	3	10	2	√
J2	6	14	$\boxed{0}$	4	5	
J3	$\boxed{0}$	~~0~~	2	~~0~~	~~0~~	
J4	8	~~0~~	3	6	3	√
J5	4	~~0~~	8	11	**1**	√
		√				

Example: Contd…

Step 6: Select the smallest uncovered element. This element is subtracted from every uncovered element and added to every element at the intersection of two lines.

Step 7: Go to Step 3 and repeat the procedure till an optimal solution is obtained.

	M1	M2	M3	M4	M5	
P1	3	$\boxed{0}$	2	9	**1**	√
P2	6	15	$\boxed{0}$	4	5	
P3	$\boxed{0}$	1	2	~~0~~	~~0~~	
P4	7	~~0~~	2	5	2	√
P5	3	~~0~~	7	10	$\boxed{0}$	
		√				

Example: contd….

	M1	M2	M3	M4	M5	
P1	2	~~0~~	**1**	8	~~0~~	√
P2	6	16	$\boxed{0}$	4	5	
P3	$\boxed{0}$	2	2	~~0~~	~~0~~	
P4	6	$\boxed{0}$	1	4	1	√
P5	3	1	7	10	$\boxed{0}$	√
		√			√	

	M1	M2	M3	M4	M5	
P1	**1**	~~0~~	~~0~~	7	~~0~~	√
P2	6	17	$\boxed{0}$	4	6	√
P3	$\boxed{0}$	3	2	~~0~~	1	
P4	5	$\boxed{0}$	~~0~~	3	1	√
P5	2	1	6	9	$\boxed{0}$	√
		√	√		√	

	M1	M2	M3	M4	M5
P1	$\boxed{0}$	~~0~~	~~0~~	6	~~0~~
P2	5	17	$\boxed{0}$	3	6
P3	~~0~~	4	3	$\boxed{0}$	2
P4	4	$\boxed{0}$	~~0~~	2	1
P5	1	1	6	8	$\boxed{0}$

Assignment:

P1-> M1

P2-> M3

P3-> M4

P4-> M2

P5-> M5

Total Cost = 11+7+13+10+10 = 51