Assignment 5 on Part C
Partial Differential Equations (MA20103)

For convenience of the students, I have listed the superposition principle, wave equation and elliptic equation at the end of the below problems to avoid the confusion related to notations.

1. (a) Find the periodicity of the function $f(x) = x - [x]$.

   (b) Verify whether the superposition principle for

   $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

   is satisfied or not? If not, then specify the reason.

   (c) The ends of a stretched string of length $L = 1$ are fixed at $x = 0$ and $x = 1$. The string is set to vibrate from rest by releasing it from an initial triangular shape modelled by the function

   $$\begin{aligned} f(x) &= \frac{3x}{10} \qquad \text{if } 0 \le x \le \frac{1}{3}, \\ &= \frac{3(1-x)}{20} \qquad \text{if } \frac{1}{3} \le x \le 1. \end{aligned}$$

   Determine the subsequent motion of the string, given that $c = \frac{1}{\pi}$.

2. (a) Show that if a string with initial shape $f(x) = \sin \frac{m\pi x}{L}$ for $0 < x < L$ is set to vibrate from rest, then its vibrations are given by the m^{th} mode.

   (b) Use d'Alembert's principle to verify whether the solution is same as question 2(a).

3. (a) A thin bar of length π units is placed in boiling water (temperature 100°C). After reaching 100°C throughout, the bar is removed from the boiling water. With the lateral sides kept insulated, suddenly, at time $t = 0$, the ends are immeresed in a medium with constant freezing temperature 0°C. Find the temperature distribution $u(x, t)$ for $t > 0$.

   (b) Solve the boundary value problems

   $$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad & \text{in } 0 < x < \pi, t > 0 \\ u(0,t) = 0 \text{ and } u(\pi, t) = 0 \quad & \text{for all } t > 0 \\ u(x, 0) = f(x) \quad & \text{for } 0 < x < \pi, \end{aligned}$$

   where

   $$\begin{aligned} f(x) &= 33x \quad \text{if } 0 < x \le \frac{\pi}{2}, \\ &= 33(\pi - x) \quad \text{if } \frac{\pi}{2} \le x < \pi. \end{aligned}$$

Appendix

Superposition principle: If u_1 and u_2 are the solutions of a linear homogeneous PDEs, then any linear combination $u = c_1u_1 + c_2u_2$, where c_1 and c_2 are constants, is also a solution. If in addition, u_1 and u_2 satisfy a linear homogeneous boundary condition, then so will $u = c_1u_1 + c_2u_2$.

Wave equations: The solution of the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{for } 0 < x < L,\ t > 0$$

with boundary conditions

$$u(0,t) = 0 \text{ and } u(L,t) = 0 \text{ for all } t > 0$$

and initial conditions

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 < x < L$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(b_n \cos\lambda_n t + b_n^* \sin\lambda_n t\right), \tag{1}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad b_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \tag{2}$$

and

$$\lambda_n = c\frac{n\pi}{L},\ n = 1, 2, \ldots \tag{3}$$

Heat equations: The solution of the one dimensional heat equation (boundary value problem)

$$\begin{aligned} \frac{\partial u}{\partial t} - c^2\frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{in } 0 < x < L, t > 0, \\ u(0,t) = 0 \text{ and } u(L,t) &= 0 \quad \text{for all } t > 0, \\ u(x,0) &= f(x) \quad \text{for } 0 < x < L \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t}\sin\frac{n\pi}{L}x, \tag{4}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, \quad \lambda_n = c\frac{n\pi}{L}, \quad n = 1, 2, \ldots \tag{5}$$