# Computational Statistics

Transformation of random vectors

$x \mapsto Ax$ ← linear transformation.

Assume $A$ is invertible.

$Z = Ax$

$$f_Z(z) = \frac{f_X(A^{-1}z)}{|A|} \qquad z \in \mathbb{R}^n$$

where $|A|$ = determinant of $A$

General transformation.

$x \in \mathbb{R}^n$ ; $x \mapsto g(x)$ (vector valued function.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{pmatrix} = g(x)$$

$g_i : \mathbb{R}^n \to \mathbb{R}$
for $i = 1, 2, \dots, n$

Let $X$ be an $n$-dimensional r.v.
$Z = g(X)$ be the transformed r.v.

Q: Find the density of $g(X)$ given the density of $X$.

Assumption: Let $g$ be invertible.

$$x = g^{-1}(z) \quad \forall z \in \mathbb{R}^n$$

$$J_x(g) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix} \text{ : Jacobian matrix}$$

Result:

$$\boxed{f_Z(z) = f_X(g^{-1}(z)) \left| J_z(g^{-1}) \right|} \quad \forall z \in \mathbb{R}^n$$

✓

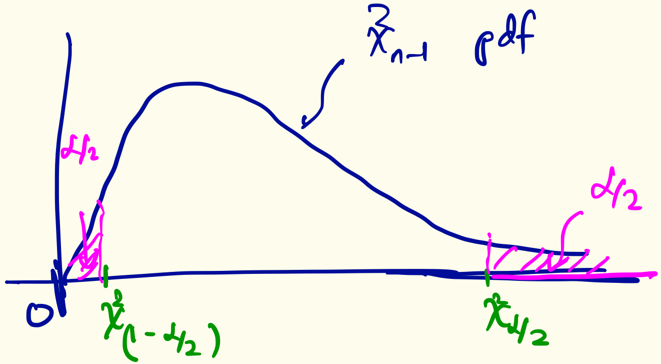
## Digression:

Let $x_1, x_2, \ldots, x_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$s^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2$$ is an unbiased estimator of $\sigma^2$.

$$Y = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$100(1-\alpha)\%$




$$P\left(\chi^2_{(1-\alpha/2)} \leq Y \leq \chi^2_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P\left(\chi^2_{(1-\alpha/2)} \leq \frac{(n-1)s^2}{\sigma^2} \leq \chi^2_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P\left(\underset{\uparrow}{L} \leq \sigma^2 \leq \underset{\uparrow}{U}\right) = 1-\alpha$$

**Ex:** $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(0,1)$

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2}$$

$$-\infty < x_i < \infty$$

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}$$

$$f_X(x) = (2\pi)^{-n/2} e^{-\frac{1}{2}x^T x}$$

$Z = \mu + BX$ where $\mu \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$ and $B$ is invertible.

$$Y = Z - \mu = BX$$

$$f_Y(y) = \frac{1}{|B|(2\pi)^{n/2}} e^{-\frac{1}{2}(B^{-1}y)^T(B^{-1}y)}$$

$$f_Y(y) = \frac{1}{|B|\sqrt{(2\pi)^n}} e^{-\frac{1}{2}y^T (B^{-1})^T B^{-1} y}$$

$$\begin{aligned} &(B^{-1})^T B^{-1} \\ &= (B^T)^{-1} B^{-1} \\ &= (BB^T)^{-1} \\ &= (\Sigma)^{-1} \end{aligned}$$

$$|B| = \sqrt{|\Sigma|}$$

$$f_Y(y) = \frac{1}{\sqrt{|\Sigma|(2\pi)^n}} e^{-\frac{1}{2}y^T \Sigma^{-1} y}$$

$$Z = Y + \mu \Rightarrow Y = Z - \mu$$

$$f_Z(z) = \frac{1}{\sqrt{|\Sigma|(2\pi)^n}} e^{-\frac{1}{2}(z-\mu)^T \Sigma^{-1}(z-\mu)} \qquad z \in \mathbb{R}^n$$

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<u>Ex:</u> $X \sim U[0,1]$

$$\boxed{Z = -\lambda \log(1-X)}$$

$$F_Z(z) = \text{Prob}(Z \leq z)$$

$$= \text{Prob}(-\log(1-x) \leq z)$$

$$= F_X(1-e^{-z}) = 1-e^{-z} \Rightarrow \boxed{f_Z(z) = \frac{1}{\lambda} e^{-z/\lambda} \quad z > 0}$$

Problem: Let $U_1, U_2 \overset{iid}{\sim} U[0,1]$

Define $X_1 = \sqrt{-2\log U_1} \cos(2\pi U_2)$

$X_2 = \sqrt{-2\log U_1} \sin(2\pi U_2)$

Prove that $X_1$ and $X_2$ are iid $N(0,1)$

Soln:

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \qquad X = g(U) = \begin{pmatrix} g_1(U_1, U_2) \\ g_2(U_1, U_2) \end{pmatrix}$$

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$g : U \longmapsto X$

$g^{-1} : X \longmapsto U$

$f_{U_1, U_2}(u_1, u_1) =$

$$\frac{dg_1^{-1}}{dx_1} = -x_1 e^{-\frac{(x_1^2+x_2^2)}{2}} \; ; \; \frac{dg_1^{-1}}{dx_2} = -x_2 e^{-\frac{(x_1^2+x_2^2)}{2}}$$

$$\frac{dg_2^{-1}}{dx_1} = \frac{-1}{2\pi}\left(\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\right)\frac{x_2}{x_1^2} \; ; \; \frac{dg_2^{-1}}{dx_2} = \frac{1}{2\pi}\left(\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\right)\frac{1}{x_1}$$

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}}$$