Some operators:

(1) Back shift operator: (B)

$$B^h X_t = X_{t-h} \qquad h \in \mathbb{Z}$$

eg: $B^2 X_{10} = X_8$

(2) Difference operator: (∇)

$\underline{\nabla \equiv I - B}$ where I stands for identity operator.

$$\nabla X_t \equiv (I - B) X_t = X_t - X_{t-1}$$

$$\Rightarrow \nabla^h X_t \equiv (I - B)^h X_t \equiv \sum_{k=0}^{h} \binom{h}{k} (-1)^{h-k} (B)^{h-k} X_t.$$

$X_1 \; X_2 \; X_3 \; X_4 \; X_5 \; X_6.$

$\nabla \rightarrow \quad \nabla X_2 \; \nabla X_3 \; \nabla X_4 \; \nabla_5 \; \nabla X_6$

$\nabla^2 \rightarrow \quad \nabla^2 X_3 \; \nabla^2 X_4 \; \nabla^2 X_5 \; \nabla^2 X_6.$

$$\underline{\nabla^2 X_3} = \nabla X_3 - \nabla X_2 = X_3 - X_2 - X_2 + X_1 = X_3 - 2X_2 + X_1$$

$$= (I - 2B + B^2) X_3 = \underline{(I - B)^2 X_3.}$$

(3) **Seasonal difference : (∇_S)**

$$\nabla_S = (I - B^S) \neq (I - B)^S = \nabla^S \quad \text{in general.}$$

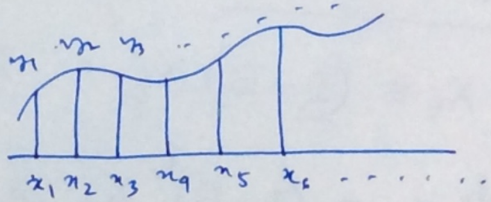
$$X_1\ X_2\ X_3\ X_4 \,\vdots\, X_5\ X_6\ X_7\ X_8 \,\vdots\, X_9\ X_{10}\ X_{11}\ X_{12} \,\vdots\, X_{13}\ X_{14}\ X_{15}\ X_{16} \,\vdots\, \cdots$$

$$\nabla_4 X_6 = (I - B^4) X_6 = X_6 - X_2.$$

application: is to remove seasonal effect from the data. or identify the seasonality

(4) $x_1\ x_2\ x_3\ x_4\ x_5 \ \cdots$

$f(x) \to y_1\ y_2\ y_3\ y_4\ y_5 \ \cdots$



devided difference: $\dfrac{\nabla y_2}{\nabla x_2}, \dfrac{\nabla y_3}{\nabla x_3}, \dfrac{\nabla y_4}{\nabla x_4} \cdots\cdots$

if $\nabla x_i = x_i - x_{i-1} = h$ common differences are same.

$$\frac{\nabla y_i}{\nabla x_i} = \frac{y_i - y_{i-1}}{h} = \frac{f(x_i) - f(x_{i-1})}{h} = \frac{f(x_{i-1} + h) - f(x_{i-1})}{h}.$$

let $h \to 0$. and $h > 0$ it will approximate right side derivative.

Linear process: A time series is said to be a linear process if it has following representation.

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

$\forall t \in \mathbb{Z}$, $Z_t \sim WN(0, \sigma^2)$, $\{\psi_j\}$ are absolutely summable.

i.e. $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

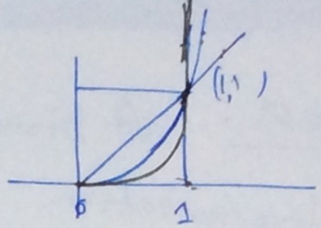
$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j B^j Z_t = \mu + \left(\sum_{j=-\infty}^{\infty} \psi_j B^j\right) Z_t = \mu + \Psi(B) Z_t$$

where $\Psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$

Here $\{X_t\}$ is a linear process.

why do we need $\{\psi_j\}$ ~~to~~ to be absolutely summable?

If we do not have an absolutely summable series then different rearrangements of the series may lead to different limits with sometimes even may not exist also.



$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots\cdots$$

but $\left(\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \cdots\cdots}_{\downarrow\ \infty}\right) - \left(\underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\cdots}_{\downarrow\ \infty}\right)$. limit does not exists.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \cdots\cdots \longrightarrow \infty.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Let $\{X_t\}$ be a linear process. Find its expectation if exists. ?

$Z_t \sim WN$.
$E(Z_t) = 0$
$V(Z_t) = \sigma^2$
$\Rightarrow E(Z_t^2) = \sigma^2$
$\Rightarrow E(|Z_t|^2) = \sigma^2 < \infty$
$\Rightarrow E|Z_t| < \infty$

$$E|X_t| = E\left|\mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}\right|$$

$$\leq E\left(|\mu| + \left|\sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}\right|\right) \qquad \text{triangular inequity}$$

$$\leq |\mu| + E\left(\sum_{j=-\infty}^{\infty} |\psi_j|\,|Z_{t-j}|\right) \qquad \text{triangular inequity}$$

$$= |\mu| + \sum_{j=-\infty}^{\infty} |\psi_j| \left(E|Z_{t-j}|\right) \qquad \text{sum law of expectation.}$$

$$\leq |\mu| + k \sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

$$E(X_t) = \mu + \left(E \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}\right) = \mu. < \infty$$

If $E(|X|^r) < \infty$

then $E(|X|^s) < \infty$ when $0 < s \leq r$

HW