Partial Differential Equation (PDE) Part A

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- Definition of PDE, Its occurrence in geometry & physics, Order & Degree

A PDE arises in geometry and physics when there will be more than one independent variable so that the derivatives of any dependent variable will be partial derivatives instead of ordinary derivatives.

For example, suppose we want to study flow of heat along a 2D body. If we denote the temperature at any point P(x,y) of the solid by z, then naturally z will be function of two independent variables x,y, as temperature may change from point to point. Thus when laws of Physics will be applied, we may get a relation involving x,y,z and partial derivatives of z of order one or higher yielding a PDE.

For simplicity, let us consider two independent variables x,y, and one dependent variable $z(x,y)$, although the terminologies and definitions, introduced below, may be generalized immediately for more than two independent variables. We will use notations $p \equiv \partial z/\partial x, q \equiv \partial z/\partial y$. Also time to time we will use suffixes x,y of dependent variable z to denote partial derivatives, i.e. $z_x \equiv \frac{\partial z}{\partial x}, z_y \equiv \frac{\partial z}{\partial y}, z_{xx} \equiv \frac{\partial^2 z}{\partial x^2}, z_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv z_{yx}, z_{yy} \equiv \frac{\partial^2 z}{\partial y^2}$ etc. Now a general form of PDE will be

$$F\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial x \partial y},\frac{\partial^2 z}{\partial y^2},\ldots\right)=0 \qquad (1.1)$$

Order & Degree of PDE

The order of highest order partial derivative term of (1.1) is the order of PDE (1.1). The exponent of highest order partial derivative term is called the degree of PDE (1.1).

For example, $px+qy=0$ is of 1st order and of 1st degree. The PDE $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ is of 2nd order & 1st degree. The PDEs $p^2x+qy=0, p^2x+qy+x\frac{\partial^2 z}{\partial x^2}=0$ are respectively of 1st order, 2nd degree and 2nd order, 1st degree.

Linear & Nonlinear PDE

PDE (1.1) will be linear if $\exists$ no terms with power higher than one w.r.t. dependent variable z and its partial derivatives. Otherwise PDE will be called nonlinear.

For example, $px+qy=z^2$ is non-linear, $\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2=z$ is also nonlinear. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ is linear.

Linear, Semi-linear, Quasi-linear, Non-linear : Non-linear PDE, defined above, may be further sub-categorized into three types:

- ✓ Semi-linear: Non-linear terms present w.r.t. z, e.g. z^2, z^3 etc, but $\exists$ no non-linear terms like zp, pq, p^2 etc.

  e.g. $x\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial z}{\partial y}-z^3x=0$ is semi-linear.
- ✓ Quasi-linear: Non-linear terms present w.r.t. z and product of z and its partial derivatives, e.g. $z^2, z^3, zp, z^2q, z\partial^2 z/\partial x^2$ etc, but $\exists$ no non-linear terms like $\cancel{zp}, pq, p^2, (\partial^2 z/\partial y^2)^3$ etc.

  e.g. $z\frac{\partial^2 z}{\partial x^2}+z^2\frac{\partial z}{\partial y}-z^3y=0$ is quasi-linear.
- ✓ Non-linear: All non-linear terms present, e.g. $z^3, pq, (\partial^2 z/\partial y^2)^3, z^2q$ etc

  e.g. $p^2x+pqy=1$ is non-linear.
- ✓ Linear: No non-linear terms present.

  e.g $\frac{\partial \psi}{\partial t}=\alpha\frac{\partial^2 z}{\partial x^2}$ is linear.

Remark: In this note, most often I use the term "Linear PDE" to mean Linear/Semi-linear/Quasi-linear PDE when there is no context of classification.

- Formation of PDE by eliminating arbitrary constants/functions

➢ First order PDE

Consider a surface of revolution with z-axis as axes of symmetry. It is well-known from geometry that equation to such surfaces is

$$z=f(x^2+y^2) \quad [f \text{ is arbitrary function}] \tag{2.1}$$

Let us now eliminate arbitrary function f. Denoting $u=x^2+y^2$, using chain rule, we differentiate z in (2.1) partially w.r.t. x & y, and get $p=2x\frac{\mathrm{d}f}{\mathrm{d}u}, q=2y\frac{\mathrm{d}f}{\mathrm{d}u}$. Eliminating f from two equations, we get

$$py-qx=0, \tag{2.2}$$

which is a 1st order 1st degree linear PDE.

In particular, for $f = c \pm \sqrt{x^2 + y^2 - a^2}$, where a, c are arbitrary constants, equation (2.1) becomes

$$x^2 + y^2 + (z - c)^2 = a^2, \qquad (2.3)$$

which represents a system of sphere of radius a with centre $(0,0,c)$ on z-axis.

Let us eliminate a, c from (2.3). Treating z as dependent variable, we differentiate z w.r.t. x & y, and get $p(z-c) + x = 0$, $q(z-c) + y = 0$, so that one arbitrary constant a has been automatically eliminated. Next eliminating c from two resulting equations, we get same PDE $py - qx = 0$, as it should be according to (2.1) and (2.2). But note that in the first case, we eliminate one arbitrary function, whereas in second case, we eliminate two arbitrary constants, i.e. contexts were different.

Hence, given a non-differential relation between $x, y, z(x, y)$ through an arbitrary function f or through two arbitrary constants a, c, we will obtain PDE by eliminating f (or a, c) from given relation and two relations obtained by partial differentiations of given relation w.r.t. x, y. Essentially, this means that given non-differential relation, which represents a family of surfaces in space, is a solution of resulting PDE. This solution surface is called integral surface of PDE.

Note that the PDE thus obtained may be linear or nonlinear. Consider a general relation of the form

$$F(x, y, z, a, b) = 0, \ \text{[a, b are arbitrary constants, F is known function]} \qquad (2.4)$$

To eliminate a, b from (2.4), we partially differentiate (2.4) w.r.t. x, y, and get

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0, \ \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0. \qquad (2.5)$$

From these two and given relation, we can always eliminate a, b to obtain 1st order PDE of the following general form

$$f(x, y, z, p, q) = 0. \quad \text{[f is a known function]} \qquad (2.6)$$

Note that PDE (2.6) may not be linear, in general. See the example below. At this point, note that the surface (2.4) is a solution of PDE (2.6), and hence is an integral surface of (2.6). Later we will see that this solution belongs to a class of integrals of 1st order PDEs.

Example: Eliminate a, b from $F(x, y, z, a, b) \equiv z - (x + a)(y + b) = 0$

Solution: $p = y + b, \ q = x + a$

Substituting for a, b into given relation, resulting PDE is $z = pq$, which is a 1^{st} order but nonlinear PDE.

Example: Find the PDE of a system of spheres of radius 5 with centre in xy-plane.

Solution: Let $C(a, b, 0)$ be centre. Then given equation is

$$(x-a)^2 + (y-b)^2 + z^2 = 25$$

To eliminate two arbitrary constants a, b, differentiate above equation partially w.r.t. x, y (treating $z(x, y)$ as dependent variable):

$$(x-a) + zp = 0, (y-b) + zq = 0$$

Eliminating a, b, from above relation and given equation, we finally have following nonlinear 1^{st} order PDE:

$$z^2(1 + p^2 + q^2) = 25$$

Let me now come back again to the context of elimination of arbitrary functions. Previously, I have shown that elimination of one arbitrary function $f(u)$ from given relation $z = f(u)$, where u is a known function of x, y, leads us to a linear 1^{st} order PDE $py - qx = 0$. Now, let us consider a slight generalization of given relation as

$$f(u, v) = 0 \quad [\text{u}, v \text{ are known functions of } x, y, z] \tag{2.7}$$

Note that $z(x, y)$ is dependent variable. Now, to eliminate arbitrary function f, we differentiate (2.7) partially w.r.t. x, y:

$$(u_x + u_z p)\frac{\partial f}{\partial u} + (v_x + v_z p)\frac{\partial f}{\partial v} = 0\,, (u_y + u_z q)\frac{\partial f}{\partial u} + (v_y + v_z q)\frac{\partial f}{\partial v} = 0. \tag{2.8}$$

It is straightforward to eliminate $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ from two linear algebraic homogeneous equations (2.8) by noticing that for non-zero values of them, coefficient determinant must vanish. Hence, we get

$$\begin{vmatrix} u_x + u_z p & v_x + v_z p \\ u_y + u_z q & v_y + v_z q \end{vmatrix} = 0$$

Quasi-linear

Expanding and collecting terms for p, q, we get a 1^{st} order ~~linear~~ PDE:

$$(u_z v_y - u_y v_z)p + (u_x v_z - u_z v_x)q + (u_x v_y - u_y v_x) = 0. \tag{2.9}$$

Note at this point that PDE (2.9) is the most general form of quasi-linear (Definitions are given above for PDE of arbitrary order, and will be given later separately for 1st order PDE) 1st order PDE of the following form:

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z). \tag{2.10}$$

Note also that the family of surfaces given by (2.7) is a solution of PDE (2.10), and hence defines Integral Surface. Later we will see that this type of solution containing one arbitrary function belongs to a class of integrals of 1st order PDEs.

Example: Eliminate arbitrary function ϕ from $\phi(x+y+z, x^2+y^2-z^2) = 0$

Solution: Denote $u = x+y+z\,, v = x^2+y^2-z^2$. Differentiate given relation $\phi(u,v) = 0$ partially w.r.t. x, y:

$$(1+p)\frac{\partial\phi}{\partial u} + 2(x-zp)\frac{\partial\phi}{\partial v} = 0\,, (1+q)\frac{\partial\phi}{\partial u} + 2(y-zq)\frac{\partial\phi}{\partial v} = 0$$

Eliminating $\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}$, we get

$$\begin{vmatrix} 1+p & 2(x-zp) \\ 1+q & 2(y-zq) \end{vmatrix} = 0 \Longrightarrow (y+z)p - (z+x)q = x-y$$

- Formation of higher order PDE by eliminating arbitrary functions/constants

Consider following relation involving two arbitrary functions f, g

$$z = f(x-cy) + g(x+cy) \quad [c > 0 \text{ is known constant}] \tag{2.10}$$

Let us denote $u = x-cy\,, v = x+cy$. Here, merely differentiating once w.r.t. x, y, we can't eliminate f, g, as evident from two resulting relations:

$$p = \frac{\mathrm{d}f}{\mathrm{d}u} + \frac{\mathrm{d}g}{\mathrm{d}v}\,,\ q = -c\frac{\mathrm{d}f}{\mathrm{d}u} + c\frac{\mathrm{d}g}{\mathrm{d}v}, \tag{2.11}$$

If we now differentiate 1st one of (2.11) w.r.t. x, and 2nd one w.r.t. y, we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}u^2} + \frac{\mathrm{d}^2 g}{\mathrm{d}v^2}\,,\ \frac{\partial^2 z}{\partial y^2} = c^2\frac{\mathrm{d}^2 f}{\mathrm{d}u^2} + c^2\frac{\mathrm{d}^2 g}{\mathrm{d}v^2}. \tag{2.12}$$

We now can eliminate $\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}, \frac{\mathrm{d}^2 g}{\mathrm{d}v^2}$ from two relations (2.12), and get following 2nd order linear PDE:

$$\frac{\partial^2 z}{\partial y^2} = c^2\frac{\partial^2 z}{\partial x^2}. \tag{2.13}$$

Remark: 2^{nd} order linear PDE (2.13) is a well-known equation, known as “Wave Equation”. We will study it at later stage of the course. But, for now, we can definitely say that given relation (2.10) is a solution of Wave Equation (2.13).

Consider now a slight generalization of (2.10) as follows:

$$z = f(u) + g(v) + w\,. \quad [u, v, w \text{ are known functions of } x, y] \tag{2.14}$$

Let us use the notations $r \equiv \frac{\partial^2 z}{\partial x^2}, s \equiv \frac{\partial^2 z}{\partial x \partial y}, t \equiv \frac{\partial^2 z}{\partial x^2}$ for 2^{nd} order partial derivatives. We will also use the symbol f', f'', g', g'' to denote ordinary derivatives of f, g w.r.t. their single arguments. Now we will compute p, q, r, s, t by partially differentiating (2.14) w.r.t. x, y twice:

$$\left.\begin{aligned} p = u_x f' + v_x g' + w_x, \qquad q = u_y f' + v_y g' + w_y \\ r = {u_x}^2 f'' + u_{xx} f' + {v_x}^2 g'' + v_{xx} g' + w_{xx} \\ s = u_x u_y f'' + u_{xy} f' + v_x v_y g'' + v_{xy} g' + w_{xy} \\ t = {u_y}^2 f'' + u_{yy} f' + {v_y}^2 g'' + v_{yy} g' + w_{yy} \end{aligned}\right\} \tag{2.15}$$

So, we have a system of five non-homogeneous linear algebraic equations for four unknowns f', g', f'', g'', and hence eliminating these, we will get a relation, which will be coefficient determinant equal to zero:

$$\begin{vmatrix} p - w_x & u_x & v_x & 0 & 0 \\ q - w_y & u_y & v_y & 0 & 0 \\ r - w_{xx} & u_{xx} & v_{xx} & {u_x}^2 & {v_x}^2 \\ s - w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t - w_{yy} & u_{yy} & v_{yy} & {u_y}^2 & {v_y}^2 \end{vmatrix} = 0\,. \tag{2.16}$$

A close inspection of the determinant in lhs of (2.16) reveals two things:- 1) second order derivative appears so that PDE (2.16) is of second order, 2) Derivative terms appear only in 1^{st} column, and all derivative terms in 1^{st} column are of power 1, so that PDE (2.16) is linear. In fact, expanding the determinant in terms of the elements of 1^{st} column, PDE (2.16) will be of the following form:

$$Rr + Ss + Tt + Pp + Qq = W, \tag{2.17}$$

where P, Q, R, S, T, W are functions of x, y only, and hence it is linear 2^{nd} order PDE.

Note that there are several occasions when we will end up with nonlinear higher order PDE by elimination. For instance, in the model (2.14), if we let the known functions u, v, w to depend on dependent variable z also, resulting PDE after elimination of arbitrary functions f, g will be 2nd order nonlinear PDE. Without going into lengthy calculation, we may infer that by noticing that in the equations (2.15) & (2.16), we will have to replace $u_x, v_x, u_y, v_y, w_x, w_y$ by $(u_x + u_z p), (v_x + v_z p), (u_y + u_z q), (v_y + v_z q), (w_x + w_z p), (w_y + w_z q)$ respectively, so that u_{xx} will be replaced by $(u_x + u_z p)_x$ etc. Thus derivative terms appear all over in the determinant (2.16) unlike the previous case wherein these appear only in 1st column. So it is inevitable that after expansion of the determinant we will get a nonlinear 2nd order PDE.

<u>Formation of PDE by elimination for more than two independent variables</u>

We will just show one examples, as there is no extra theoretical constraint for more than two variables.

Example: Eliminate a, b, c from $\psi(x, y, t) = a(x + y) + b(x - y) + abt + c$

Solution: $\frac{\partial\psi}{\partial x} = a + b$, $\frac{\partial\psi}{\partial y} = a - b$, $\frac{\partial\psi}{\partial t} = ab \Longrightarrow \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = 4\frac{\partial\psi}{\partial t}$

Note that resulting PDE is 1st order nonlinear.

- Classification of 1st order PDE as linear/quasi-linear/semi-linear/nonlinear

From this topic onward, we will consider two independent variables x, y, and one dependent variable $z(x, y)$. A general form of linear 1st order PDE is

$$P(x, y)p + Q(x, y)q = R(x, y)\overset{z}{\cancel{z}} + F(x, y). \tag{3.1}$$

Note that RHS is called non-homogeneous term in the sense that there is no partial derivative term there so that its degree is 0 w.r.t p, q, whereas in LHS degree of each term is one.

In Sec 1, a general definition was given to classify a PDE of arbitrary order as Linear/Semi-linear/Quasi-linear/Non-linear. For the case of 1st order PDE, I've decided to provide separately the definitions regarding same classifications.

<u>Semi-linear</u>: RHS depends arbitrarily on dependent variable z. General form:

$$P(x, y)p + Q(x, y)q = R(x, y, z). \tag{3.2}$$

Note that here RHS may contain nonlinear term like z^2, but LHS doesn't contain terms which are non-linear in p, q, i.e. $\exists$ no term like zp, zq, z^2p etc. Such equation (3.2) is classified as 1st order Semi-linear PDE.

<u>Quasi-linear</u>: Coefficient functions in LHS and RHS depend arbitrarily on dependent variable z. General form:

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z). \tag{3.3}$$

Note here in both of RHS & LHS, there may be non-linear term like zp, zq, z^2p etc in LHS and z^2, z^3 etc in RHS, but $\exists$ no term which is non-linear in derivatives, eg. Terms like p^2, pq, q^2 etc are absent. Such equation (3.3) is classified as Quasi-linear PDE.

Remark: Note that in spite of presence of certain non-linear terms in semi-linear/quasi-linear PDE, these two can be clubbed with linear PDE in the sense that these three categories can be solved by Lagrange's method, to be discussed in following section. This is the reason why sometimes I only use the term "Linear" to refer Linear/Semi-linear/Quasi-linear for brevity when there is no context of classification.

Classification of Integrals (Solutions) of a 1st order PDE

Consider 1st order PDE in its most general form as follows

$$f(x, y, z, p, q) = 0. \tag{3.4}$$

1. Complete Integral (CI): A two-parameter family of solutions of PDE (3.4) of the form

$$F(x, y, z(x, y), a, b) = 0, \qquad [a, b \text{ arbitrary constants}] \tag{3.5}$$

is called Complete Solution (or Integral) of PDE. Recall the discussion in Sec 2 [see equations (2.4)-(2.6)], wherein I've shown that eliminating a, b, one obtains (3.4), which is now understood as CI. Hence, CI of 1st order PDE of two independent variables will contain two arbitrary constants.

2. General Integral (GI): A solution of PDE (3.4) involving one arbitrary function g of the form

$$g(u, v) = 0, \tag{3.6}$$

where u, v are known functions of x, y, z, is called General Solution (or Integral) of PDE (3.4). Recall the discussion in Sec 2 [see equations (2.7)-(2.10)], wherein I've shown that eliminating g, one obtains a 1st order quasi-linear PDE (3.3)which is now understood as GI.

The Author of Textbook commented that the name GI is actually 'illusory' in the sense that it is possible to derive from CI. However, the procedure of derivation of GI from CI is well-known [see Refs 1,2] since long ago. Let me discuss below the procedure without proof [for proofs, interested reader may see Text Book and Ref 1].

To derive GI from CI (3.5), let $b = \phi(a)$, so that (3.5) reduces to following one-parameter family, which is sub-family of surfaces given by CI (3.5):

$$F(x, y, z, a, \phi(a)) = 0. \tag{3.7}$$

We then consider the envelope of one-parameter family of surface (3.7), which can be obtained by partially differentiating (3.7) w.r.t. a:

$$F_a + F_b\, \phi'(a) = 0. \tag{3.8}$$

From equation (3.8), we may solve for a:

$$a = a(x, y, z). \tag{3.9}$$

Substituting for a from (3.9) into (3.7), we have GI of PDE (3.4) in the following form

$$F(x, y, z, a(x, y, z), \phi(a(x, y, z))) = 0. \tag{3.10}$$

Note that GI (3.10) contains one arbitrary function ϕ, which is the property of GI of 1st order PDE (3.4).

3. Particular Integral (PI): If $b = \phi(a)$ is prescribed satisfying some given condition, we will get from (3.10) a solution containing no arbitrary constant or function. This solution is known as Particular Solution (or Integral). Indeed ∃ infinitely many such particular solutions. However, in a physical problem, may be only one of them is acceptable solution.

4. Singular Integral (SI): This is such a solution of PDE (3.4),which, as in the case for 1st order ODE, can't be obtained from GI. This is called Singular Integral (or Solution). Geometrically, SI, if exists, represents envelope of two-parameter family of surfaces, given by CI (3.5). Equation for envelope of (3.5) may be obtained by partially differentiating (3.5) w.r.t. a, b:
$$F_a(x, y, z, a, b) = 0\ , F_b(x, y, z, a, b) = 0 \tag{3.11}$$
If envelope exists, then it will be possible to eliminate a, b from three equations (3.5) and (3.11), and the resulting relation will be SI of PDE (3.4). Note that SI may or may not exist.

5. Special Solution: For some 1st order PDEs, apart from the class of solutions discussed above, ∃ special solution, mentioned in Textbook. See the examples below, taken from Textbook.

Example: Show that $F(x + y, x - \sqrt{z})$ is GI of the PDE $p - q = 2\sqrt{z}$, where F is an arbitrary function. Also check whether $\sqrt{z} = \frac{ax+by}{a-b} + b$ is CI of given PDE, whether a, b $(a \neq b)$ are two arbitrary constants. Finally verify $z = 0$ is yet another solution which can't be obtained from GI, and hence $z = 0$ is a solution.

Solution: Denoting $u = x + y$, $v = x - \sqrt{z}$, eliminate arbitrary function F from the relation $F(u, v) = 0$ by differentiating partially w.r.t. x, y:

$$F_u + \left(1 - \frac{p}{2\sqrt{z}}\right) F_v = 0\,, F_u - \frac{q}{2\sqrt{z}} F_v = 0$$

So, eliminating F_u, F_v from above two equations,

$$\frac{q}{2\sqrt{z}} + \left(1 - \frac{p}{2\sqrt{z}}\right) = 0 \Longrightarrow p - q = 2\sqrt{z}$$

Hence, $F(x + y, x - \sqrt{z})$ is GI of the PDE $p - q = 2\sqrt{z}$.

Next, from $\sqrt{z} = \frac{ax+by}{a-b} + b$, eliminate arbitrary constants a, b by differentiating partially w.r.t. x, y:

$$\frac{p}{2\sqrt{z}} = \frac{a}{a - b}\,, \frac{q}{2\sqrt{z}} = \frac{b}{a - b} \Longrightarrow p - q = 2\sqrt{z}$$

Hence, $\sqrt{z} = \frac{ax+by}{a-b} + b$ is CI of given PDE

Last Part: We see that $z = 0$ can't be obtained from GI or CI, but for $z = 0$, we have $p = 0, q = 0$, and hence given PDE is satisfied by $z = 0$. Note that it is not a SI, since in the process of finding envelope of two-parameter family CI, one has to consider the case $z \neq 0$ to eliminate a, b. Hence, $z = 0$ is a special solution.

- Lagrange's method for solving 1st order linear PDE

Here, for convenience, in the term "Linear", I've clubbed linear, semi-linear and quasi-linear PDEs, due to the fact that all of three categories can be solved by Lagrange's method theoretically (i.e. subject to separability and analytical integrability). Hence, let us consider following 1st order PDE

$$Pp + Qq = R, \tag{4.1}$$

where P, Q, R are functions of x, y, z in general.

Lagrange's Method: General Solution of PDE (4.1) is given by an arbitrary function ϕ of two known functions $u(x, y, z), v(x, y, z)$ as follows

$$\phi(u, v) = 0\,, \tag{4.2}$$

where

$$u(x, y, z) = c_1\,, v(x, y, z) = c_2 \tag{4.3}$$

are solutions of following Differential Equations (DE), known as Lagrange's Auxiliary Equation (AE), given by

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}, \tag{4.4}$$

c_1, c_2 being two arbitrary constants.

The result (4.2)-(4.4) can be proved analytically (by using knowledge of elimination) or geometrically (using geometric properties of surface generated by curves in space).

<u>Analytical Proof</u> I'll not show detail steps as reader can do that using discussions in previous section [see discussions in Sec 2]. I just mention that eliminating arbitrary function ϕ from GS (4.2) by partial differentiation w.r.t. x, y, we will obtain PDE (4.1), where

$$P = u_z v_y - u_y v_z, Q = u_x v_z - u_z v_x, R = u_y v_x - u_x v_y. \tag{4.5}$$

Next we will eliminate c_1, c_2 from solutions $u(x,y,z) = c_1, v(x,y,z) = c_2$ of DE (4.4). Taking differentials of $\mathrm{d}u = 0$, $\mathrm{d}v = 0$, we conclude that following two equations must be compatible:

$$u_x \mathrm{d}x + u_y \mathrm{d}y + u_z \mathrm{d}z = 0, \frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$$

i.e. we must have

$$P u_x + Q u_y + R u_z = 0. \tag{4.6}$$

Similarly for the other solution, we have

$$P v_x + Q v_y + R v_z = 0. \tag{4.7}$$

From (4.6) & (4.7), we get P, Q, R given by (4.5). Hence proved.

<u>Geometrical Proof</u>: In the first step, we will show that all integral surfaces (i.e. solutions) of PDE (4.1) are generated by the integral curves (i.e. solutions) of DE (4.4). In the second step, we will show that all surfaces generated by integral curves of DE (4.4) are integral surfaces of PDE (4.1).

Step 1. Equations (4.2) and (4.3) implies that any integral surface of PDE (4.1) is given by $z = f(x, y)$, f is arbitrary function. Then it is well-known from differential calculus that d.ratios (direction ratios) of normal to this surface is $(F_x, F_y, F_z) = (p, q, -1)$, where $F(x,y,z) = f(x,y) - z$. Now, the PDE (4.1) is the geometric statement of the fact that the normal to the surface is

perpendicular to the direction defined by direction ratios(P, Q, R). This means that the direction (P, Q, R) is tangential to the surface.

On the other hand, any integral curve of DE (4.4) has tangential directions at each point $\mathrm{M}(x, y, z)$ as $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = (P, Q, R)$, which means that the integral curve lies entirely on the surface.

Hence, if we start from an arbitrary point $\mathrm{M}(x, y, z)$ on the surface and move along the direction (P, Q, R), we will trace out an integral curve of DE (4.4), which lies entirely on the surface. Furthermore, given P, Q, R, integral curve through a point M is unique. Thus it is proved that integral surface of PDE (1.1) is generated by integral curves of AE (4.4).

Step 2: Suppose any integral curve of AE (4.4) generates the surface $z = f(x, y)$, then that curve must be perpendicular to the tangent to the surface. Clearly, direction ratios of the integral curve of AE (4.4) are $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = (P, Q, R)$, and tangential directions of the surface $z = f(x, y)$ are $(p, q, -1)$. Hence, geometrical property of perpendicularity is expressed by the following relation

$$Pp + Qq = R,$$

which is given PDE. Hence generated surface $z = f(x, y)$ is integral surface of PDE (4.1).

We are yet to prove that any surface generated by the integral curves of AE (4.4) has an equation of the form (4.2). Let any curve on the surface which is not a particular member of the system (4.3) have equations

$$\chi(x, y, z) = 0\,, \psi(x, y, z) = 0. \tag{4.8}$$

If the curve (4.3) is a generating curve of the surface, it must intersect the curve (4.8). The condition of this intersection will be obtained by eliminating x, y, z from four equations (4.3) & (4.8). This relation will be a relation of the form

$$F(c_1, c_2) = 0\,. \tag{4.9}$$

Thus the surface is generated by curve (4.3) which obey condition (4.9), and hence have equation of the form (4.2).

Conversely, any surface of the form (4.2) is generated by integral curve (4.3) of AE (4.4), because it is that surface generated by those curves which satisfy relation (4.9).

This completes the proof.

Geometrical Interpretation of Lagrange's Equation, Method of Characteristics

The 1st order linear PDE (4.1), also called Lagrange's equation, and the DE (4.4) in Lagrange's method is called Lagrange's auxiliary equation or simply Auxiliary Equation (AE). Now, geometrical interpretation of Lagrange's method is already given in the geometrical proof of the method given above. Let me point out the take away from above proof:

- At any point $\mathrm{M}(x, y, z)$ on the solution surface (known as Integral Surface) $F(x, y, z) \equiv f(x, y) - z = 0$, the direction $(p, q, -1)$ is normal to the solution surface.
- Given PDE $Pp + Qq = R$ implies that the direction (P, Q, R) is perpendicular to the normal at each point $\mathrm{M}(x, y, z)$, and hence the direction (P, Q, R) is tangential to the solution surface. This triad (P, Q, R) is called Characteristic direction.
- Integral curve of AE (4.4) is such a curve in space such that the tangent to this curve has d.ratios (P, Q, R). These curves $u(x, y, z) = c_1$, $v(x, y, z) = c_2$ are called Characteristic Curves.
- Integral surface of PDE is generated by Integral Curve of AE and conversely Integral Curve of AE generates Integral surface of PDE.

The method of solving 1st order linear PDE, due to Lagrange, described above, is also called method of characteristic. Before showing some examples on Lagrange's method, let me mention that the GI $\phi(u, v) = 0$ may also be expressed equivalently as $u = g(v)$ or $v = h(v)$, because of the arbitrariness of the functions.

Example: Find GI of PDE $y^2 p - xyq = x(z - 2y)$ by the method of characteristics

Solution: $P = y^2, Q = -xy, R = x(z - zy)$ are characteristic directions. Characteristic curves are integrals of the characteristic equations

$$\frac{\mathrm{d}x}{y^2} = \frac{\mathrm{d}y}{-xy} = \frac{\mathrm{d}z}{x(z - 2y)}$$

From 1st two fractions, $x\mathrm{d}x + y\mathrm{d}y = 0 \Longrightarrow x^2 + y^2 = c_1$, a family of circles with centre at origin and of radius $\sqrt{c_1}$, one family of characteristic curves.

From last two, we get 1st order linear ODE:

$\frac{\mathrm{d}z}{\mathrm{d}y} + \left(\frac{1}{y}\right) z = 2 \Longrightarrow$IF$= y \Longrightarrow yz - y^2 = c_2$, 2nd family of characteristic curves

Hence, GS is in terms of an arbitrary function: $(x^2 + y^2, yz - y^2) = 0$.

- Integral Surfaces through a given Curve: Cauchy Problem

Given a 1st order PDE $f(x, y, z, p, q) = 0$ and a curve Γ: $x = x_0(t), y = y_0(t), z = z_0(t), t \in [a, b]$, the Cauchy Problem is to find a solution $z = z(x, y)$ of the PDE such that on curve Γ, $z = z_0(t) = z\big(x_0(t), y_0(t)\big), \forall\, t \in [a, b]$, i.e. we are to find an integral surface of PDE which contains the given curve Γ.

Consider 1st order linear PDE in its general form:

$$P(x, y, z(x, y))p + Q(x, y, z(x, y)q = R(x, y, z(x, y). \qquad (5.1)$$

Cauchy Data: Given curve Γ: $x = x_0(t), y = y_0(t), z = z_0(t), t \in [a, b]$, t is just a parameter.

Our problem is to find the integral surface

$G\Big(u\big(x(t), y(t), z(x(t), y(t))\big), v\big(x(t), y(t), z(x(t), y(t))\big)\Big) = 0$, which contains the curve Γ, i.e. the surface passes through the point $\big(x_0(t), y_0(t), z_0(t)\big)$ for all $t \in [a, b]$. Note that this means that G is not an arbitrary function, it will be of specified form, which we are to find out.

Now, from the theory of 1st order linear PDE, discussed before, we know that GI of PDE (5.1) is of the form

$$F(u(x, y, z), v(x, y, z)) = 0, \quad [F \text{ is arbitrary function}] \qquad (5.2a)$$

where

$$u(x, y, z) = c_1, v(x, y, z) = c_2, \qquad (5.2b)$$

are solutions of Lagrange's AE

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}. \qquad (5.2c)$$

Equations (5.2a) & (5.2b) implies that we have a relation between two arbitrary constants c_1, c_2 of the form

$$F(c_1, c_2) = 0. \qquad (5.3)$$

Now, if the integral surface pass through given curve Γ, characterized by single parameter t in the following form:

$$\Gamma: x = x_0(t), y = y_0(t), z = z_0(t), t \in [a, b], \qquad (5.4)$$

then the integral curves $u(x, y, z), v(x, y, z)$ must satisfy following two equations:

$$u\big(x_0(t), y_0(t), z_0(t)\big) = c_1\,, v\big(x_0(t), y_0(t), z_0(t)\big) = c_2,\ \forall t \in [a, b]. \quad (5.5)$$

From two equations in (5.5), we can eliminate single parameter t, to obtain a relation of the type

$$G(c_1, c_2) = 0. \quad (5.6)$$

Note that GI (5.3) contains (5.6) for $F \equiv G$, because F is an arbitrary function. Hence, substituting for arbitrary constants c_1, c_2 from equations (5.2b) into (5.6), we obtain our desired integral surface as follows

$$G\big(u(x, y, z), v(x, y, z)\big) = 0. \quad (5.7)$$

Note that the surface (5.7) contains given curve because of (5.5). Further, this must be integral surface of PDE, as this is derived from GI (5.2a). In fact, it is a PI of PDE.

Example: Find the integral surface of the PDE $(x-y)p + (y-x-z)q = z$ through the circle $x^2 + y^2 = 1, z = 1$.

Solution: Lagrange's AE: $\frac{\mathrm{d}x}{x-y} = \frac{\mathrm{d}y}{y-x-z} = \frac{\mathrm{d}z}{z} \Longrightarrow$ each ratio$= \frac{\mathrm{d}x+\mathrm{d}y+\mathrm{d}z}{0}$

Hence, we get $\mathrm{d}(x+y+z) = 0 \Longrightarrow x+y+z = c_1$

Next, considering 2nd & 3rd fraction, and using above integral curve, we have

$$\frac{\mathrm{d}y}{y-x-z} = \frac{\mathrm{d}z}{z} \Longrightarrow \frac{\mathrm{d}y}{2y-c_1} = \frac{\mathrm{d}z}{z} \Longrightarrow \frac{2y-c_1}{z^2} = c_2$$

Substituting for c_1 from 1st integral curve, we get 2nd integral curve

$$\frac{y-x-z}{z^2} = c_2$$

Now, from above discussion, it is clear that for Cauchy problem, we are to find a fixed relation between c_1, c_2 using Cauchy data.

We see that given curve : $x^2 + y^2 = 1$. Putting $z = 1$ in two integral curves,

$$2y = c_1 + c_2\,, y - x = c_2 + 1 \Longrightarrow 2x = c_1 - c_2 - 2$$

Substituting for x, y from above into $x^2 + y^2 = 1$, the fixed relation is

$$(c_1 - c_2 - 2)^2 + (c_1 + c_2)^2 = 4 \Longrightarrow {c_1}^2 + {c_2}^2 - 2(c_1 + c_2) = 0$$

Finally, we substitute for c_1, c_2 from the integral curves into above relation:

$$z^4(x+y+z)^2 + (y-x-z)^2 - 2z^2[z^2(x+y+z) - (y-x-z)] = 0$$

Remark: In above example, the curve is given in non-parametric form. See the example below, where curve is given in parametric form. Parametric form was taken above for theoretical discussion. Furthermore, non-parametric form of curve in above problem may be put in parametric form also as follows:

$$\Gamma: x(t) = \cos t\,, y(t) = \sin t\,, z(t) = 1, 0 \le t \le 2\pi$$

Thus, from the two integral curves, we must have

$$c_1 = 1 + \sin t + \cos t\,, c_2 = \sin t - \cos t - 1$$

$$\Longrightarrow 2\sin t = (c_1 + c_2), 2\cos t = c_1 - c_2 - 2$$

Eliminating t from above equations, we get

$(c_1 + c_2)^2 + (c_1 - c_2 - 2)^2 = 4$, which is the same fixed relation between c_1, c_2, and so substituting for them from integral curves, we get same surface.

Example: Solve the PDE $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ with initial data : given straight line $x_0(s) = 1, y_0(s) = 0, z_0(s) = s$

Solution: AE: $\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)}$

$\Longrightarrow$ Each ratio$= \frac{zdx+dy+xdz}{0} = \frac{2xdx+2ydy+dz}{0}$

Thus, we get $d(xz + y) = 0\,, d(x^2 + y^2 + z) = 0$

$$\Longrightarrow xz + y = c_1, x^2 + y^2 + z = c_2$$

Hence, we may take GI in the form

$$x^2 + y^2 + z = \phi(xz + y)$$

We are to find specified form of ϕ such that above integral surface passes through the line $x_0(s) = 1, y_0(s) = 0, z_0(s) = s$. Substituting for x, y, z from line to above surface, we get

$$1 + 0 + s = \phi(1.s + 0) \Longrightarrow \phi(s) = 1 + s$$

Thus, desired solution is $x^2 + y^2 + z = 1 + (xz + y)$

Before going to next section, I would like to show an example of finding particular solution for given functional relation between the parameters from CI, and also of finding GI and SI.

Example: Recall the example, discussed in Sec 2 [see 2nd example below (2.6)], where eliminating a, b from $(x - a)^2 + (y - b)^2 + z^2 = 25$, we get PDE $z^2(1 + p^2 + q^2) = 25$, so that former is CI of PDE.

Now, write down GI by taking $b = \phi(a)$. Then choosing $\phi(a) = a$, write down particular solution. Finally find SI, if exists.

Solution: Given CI may be expressed as

$$F(x, y, z, a, b) \equiv (x-a)^2 + (y-b)^2 + z^2 - 25 = 0$$

Taking $b = \phi(a)$, ϕ is arbitrary function, we have one parameter family of surface:

$$F\big(x, y, z, a, \phi(a)\big) \equiv (x-a)^2 + \big(y-\phi(a)\big)^2 + z^2 - 25 = 0$$

The GI is the envelope of above surface, which is a-eliminant of above equation and the following equation obtained by partial differentiation of F w.r.t. a

$$F_a \equiv x - a + \big(y - \phi(a)\big)\phi'(a) = 0$$

Next choosing $\phi(a) = a$, we get two equations from above two equations:

$$(x-a)^2 + (y-a)^2 + z^2 - 25 = 0, x - a + (y-a) = 0 \Longrightarrow 2a = x + y$$

Hence, particular solution of PDE for $\phi(a) = a$: $(x-y)^2 + 2z^2 = 50$

Next, to see SI, if exists, we differentiate CI w.r.t. a, b:

$$(x-a) = 0, (y-b) = 0$$

To eliminate a, b from given CI, we put $a = x, b = y$: $z = \pm 5$

This is required SI. It is easy to verify that this is a solution of PDE.

- Orthogonal Surface to a given System of Surfaces

This is an application of the theory of 1st order linear PDE to Geometry. Here we will determine the system of surfaces orthogonal to a given system of surfaces. Suppose given system is a one-parameter family of surfaces

$$f(x, y, z) = c. \tag{6.1}$$

A system of surfaces will be called orthogonal to given system (6.1), if former cuts each of given surfaces at right angles. Suppose the orthogonal surfaces are given by

$$G(x, y, z) \equiv g(x, y) - z = 0. \tag{6.2}$$

From the geometric interpretation of Lagrange's method for 1st order linear PDE, we know that normal to given surfaces (6.1) at any point M(x, y, z) on the surface is the direction given by $(P, Q, R) = (f_x, f_y, f_z)$, and the normal at common point M on orthogonal surfaces (6.2) is the direction given by $(p, q, -1)$. Now since two systems (6.1) and (6.2) are orthogonal, two normal must be perpendicular to each other, which transpires in mathematical language:

$$Pp + Qq = R. \tag{6.3}$$

Note that $P = f_x, Q = f_y, R = f_z$ are known, since f is given function. Now, PDE (6.3) is 1st order linear PDE, which can be solved by Lagrange's method.

Conversely, any solution of linear PDE (6.3) must be orthogonal to every surface characterized by given system (6.1), since relation (6.3) simply means that normal to any solution of (6.3) is perpendicular to the normal to that member of the given system (6.1) which passes through the same point.

Hence, orthogonal surface to given system (6.1) is determined by general solution of linear PDE (6.3), so that the surfaces orthogonal to the system (6.1) are the surfaces generated by the integral curves of characteristic equation of PDE (6.3):

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R} \Longrightarrow \frac{\mathrm{d}x}{f_x} = \frac{\mathrm{d}y}{f_y} = \frac{\mathrm{d}z}{f_z}$$

Example: Find the system of surfaces orthogonal to given system of surfaces given by $z(x+y) = c(3z+1)$, c is a free parameter.

Solution: Write $f(x, y, z) \equiv \frac{z(x+y)}{3z+1} = c$, we compute $f_x = \frac{z}{3z+1}, f_y = \frac{z}{3z+1}$, $f_z = \frac{(x+y)}{(3z+1)^2}$, so that orthogonal surfaces are general solutions of following PDE:

$$f_x p + f_y q = f_z \Longrightarrow \frac{z}{3z+1} p + \frac{z}{3z+1} q = \frac{(x+y)}{(3z+1)^2}$$

AE is $\frac{\mathrm{d}x}{z(3z+1)} = \frac{\mathrm{d}y}{z(3z+1)} = \frac{\mathrm{d}z}{(x+y)} \Longrightarrow$each ratio$= \frac{\mathrm{d}x - \mathrm{d}y}{0} \Longrightarrow u \equiv x - y = c_1$ is one integral curve. Also, writing AE as

$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{1} = \frac{z(3z+1)\mathrm{d}z}{(x+y)} \Longrightarrow$each ratio$= \frac{x\mathrm{d}x + y\mathrm{d}y - z(3z+1)\mathrm{d}z}{0}$

$\Longrightarrow v \equiv x^2 + y^2 - 2z^3 - z^2 = c_2$, which is our 2nd integral curve.

Hence, orthogonal surfaces are given by the following system

$x^2 + y^2 - 2z^3 - z^2 = \phi(x-y)$, ϕ is arbitrary function.

**************End of Part A****************