Setting up simplex tableau in matrix form

Initial set of equations in matrix form:

$$\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x_s} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} \qquad \textbf{(1)}$$

After any iteration:

$$\begin{bmatrix} Z \\ \mathbf{X}_B \end{bmatrix} = \begin{bmatrix} \mathbf{C}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

Premultiply the initial set of equation (1) by $\begin{bmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix}$, we have

$$\begin{bmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x_s} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

Thus, the set of equations after any iteration in matrix form:

$$\begin{bmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{C}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x_s} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix} \qquad \textbf{(2)}$$

Simplex tableau in matrix form (contd…)

Simplex Tableau in matrix form

Iteration		Coefficient of		RHS
		Original variables (x)	**Slack variables (x_s)**	
0	$\mathbf{x}_B$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
	Z	$\mathbf{-c}$	$\mathbf{0}$	0
Any	$\mathbf{x}_B$	$\mathbf{B^{-1}A}$	$\mathbf{B^{-1}}$	$\mathbf{B^{-1}b}$
	Z	$\mathbf{c}_B\mathbf{B^{-1}A - c}$	$\mathbf{c}_B\mathbf{B^{-1}}$	$\mathbf{c}_B\mathbf{B^{-1}b}$

How to remember the table:

- Basic Formula, $x_B = B^{-1}b$, $Z = c_B B^{-1} b$
- For any iteration, obtain
  - x_B raw by pre-multiplying with $\mathbf{B^{-1}}$ in the initial row of x_B
  - Z raw by pre-multiplying with $c_B\mathbf{B^{-1}}$ in the initial row of x_B and adding it to the initial Z raw

Revised Simplex Method

1. *Initialization:* Same as for the original simplex method.

2. *Iteration:*

Step 1: Determine the entering basic variable: Same as for the original simplex method.

Step 2: Determine the leaving basic variable: Same as for the original simplex method, except calculate only the numbers required to do this [the coefficients of the entering basic variable in every equation but Eq. (0), and then, for each strictly positive coefficient, the right-hand side of that equation].

Step 3: Determine the new BF solution: Derive $\mathbf{B}^{-1}$ and set $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$.

3. *Optimality test:* Same as for the original simplex method, except calculate only the numbers required to do this test, i.e., the coefficients of the *nonbasic variables* in Eq. (0).

In Augment form

$\max Z = 50x_1 + 40x_2$

s.t. $3x_1 + 5x_2 + x_3 = 150$

$x_2 + x_4 = 20$

$8x_1 + 5x_2 + x_5 = 300$

$x_1, x_2, \ldots, x_5 \geq 0$

TechEdge Co. example

$$\underset{\sim}{c} = [50,\ 40]$$

$$[A, I] = \begin{bmatrix} 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 8 & 5 & 0 & 0 & 1 \end{bmatrix} \quad \underset{\sim}{b} = \begin{bmatrix} 150 \\ 20 \\ 300 \end{bmatrix} \quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \underset{\sim}{x_s} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Iteration 0

$$\underset{\sim}{x_B} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B^{-1} \Rightarrow \underset{\sim}{x_B} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \overset{B^{-1}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \overset{b}{\begin{bmatrix} 150 \\ 20 \\ 300 \end{bmatrix}} = \begin{bmatrix} 150 \\ 20 \\ 300 \end{bmatrix}$$

$$\underset{\sim}{c_B} = [0, 0, 0] \Rightarrow Z = \underset{\sim}{c_B} B^{-1} \underset{\sim}{b} = [0, 0, 0] \begin{pmatrix} 150 \\ 20 \\ 300 \end{pmatrix} = 0$$

Iteration 1

Simplex Iteration

	original variables		slack variables			b
Basis	X_1	X_2	X_3	X_4	X_5	RHS
X_3	0	25/8	1	0	-3/8	75/2
X_4	0	1	0	1	0	20
X_1	1	5/8	0	0	1/8	75/2
z	0	-70/8	0	0	50/8	1875

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_1 \end{pmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} 1 & 0 & -3/8 \\ 0 & 1 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}$$

$\underset{\sim}{x}_B = \underset{\sim}{B}^{-1} \underset{\sim}{b}$

$$\Rightarrow \begin{pmatrix} x_3 \\ x_4 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3/8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 150 \\ 20 \\ 300 \end{pmatrix} = \begin{pmatrix} 75/2 \\ 20 \\ 75/2 \end{pmatrix}$$

$$c_B = [0, 0, 50] \Rightarrow Z = [0, 0, 50] \begin{pmatrix} 75/2 \\ 20 \\ 75/2 \end{pmatrix} = 1875$$

$$\underset{\sim}{B}^{-1}\underset{\sim}{A} = \begin{bmatrix} 1 & 0 & -3/8 \\ 0 & 1 & 0 \\ 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 25/8 \\ 0 & 1 \\ 1 & 5/8 \end{bmatrix}$$

$$\underset{\sim}{c}_B \underset{\sim}{B}^{-1} \underset{\sim}{A} - \underset{\sim}{c} = [0 \quad 0 \quad 50] \begin{bmatrix} 0 & 25/8 \\ 0 & 1 \\ 1 & 5/8 \end{bmatrix} - [50 \quad 40] = [0 \quad -70/8]$$

$$\underset{\sim}{c}_B \underset{\sim}{B}^{-1} = [0 \quad 0 \quad 50] \begin{bmatrix} 1 & 0 & -3/8 \\ 0 & 1 & 0 \\ 0 & 0 & 1/8 \end{bmatrix} = [0 \quad 0 \quad \tfrac{50}{8}]$$

Iteration 2

Basis	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	$8/25$	0	$-3/25$	12
x_4	0	0	$-8/25$	1	$3/25$	8
x_1	1	0	$-1/5$	0	$1/5$	30
z	0	0	$14/5$	0	$26/5$	1980

$$\underset{\sim}{x_B} = \begin{bmatrix} x_2 \\ x_4 \\ x_1 \end{bmatrix} \qquad \underset{\sim}{B} = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 1 & 0 \\ 5 & 0 & 8 \end{bmatrix} \Rightarrow \underset{\sim}{B}^{-1} = \begin{bmatrix} \frac{8}{25} & 0 & -3/25 \\ -8/25 & 1 & 3/25 \\ -1/5 & 0 & 1/5 \end{bmatrix}$$

$$\underset{\sim}{x_B} = \underset{\sim}{B}^{-1}\underset{\sim}{b}$$

$$\begin{bmatrix} x_2 \\ x_4 \\ x_1 \end{bmatrix} = \begin{bmatrix} \frac{8}{25} & 0 & -\frac{3}{25} \\ -8/25 & 1 & 3/25 \\ -1/5 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 150 \\ 20 \\ 300 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 30 \end{bmatrix}$$

$$\underset{\sim}{c_B} = [40 \quad 0 \quad 50]$$

$$z = \underset{\sim}{c_B}\underset{\sim}{x_B} = [40 \quad 0 \quad 50] \begin{bmatrix} 12 \\ 8 \\ 30 \end{bmatrix} = 1980$$

$$\underset{\sim}{B}^{-1}\underset{\sim}{A} = \begin{bmatrix} 8/25 & 0 & -3/25 \\ -8/25 & 1 & 3/25 \\ -1/5 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\underset{\sim}{c_B}\underset{\sim}{B}^{-1}\underset{\sim}{A} - \underset{\sim}{c} = [40 \quad 0 \quad 50] \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} - [50 \quad 40] = [0 \quad 0]$$

$$\underset{\sim}{c_B}\underset{\sim}{B}^{-1} = [40 \quad 0 \quad 50] \begin{bmatrix} 8/25 & 0 & -3/25 \\ -8/25 & 1 & 3/25 \\ -1/5 & 0 & 1/5 \end{bmatrix} = \left[\frac{14}{5} \quad 0 \quad \frac{26}{5}\right]$$