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If $x + y \in \mathbb{W}$, then $(x + y) - y = x \in \mathbb{W}$ which is not possible. Therefore either $\mathbb{U} \subset \mathbb{W}$ or $\mathbb{W} \subset \mathbb{U}$.

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  Therefore $\alpha x + \beta y \in \mathbb{U} + \mathbb{W}$. Hence $\mathbb{U} + \mathbb{W}$ is subspace of $\mathbb{V}$.

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Hence $\mathbb{U} + \mathbb{W}$ is the smallest subspace of $\mathbb{V}$ which contains both $\mathbb{U}$ and $\mathbb{W}$.

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Then $\{v_1, \ldots, v_k, u_1, \ldots, u_m, w_1, \ldots, w_p, \})$ is basis of $\mathbb{U} + \mathbb{W}$. Therefore $\dim(\mathbb{U} + \mathbb{W}) = k + m + P + k - k = \dim(\mathbb{U}) + \dim(\mathbb{W}) - \dim(\mathbb{U} \cap \mathbb{W})$.

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Then $x_1 + x_2 = y_1 + y_2$ this implies $x_1 - y_1 = y_2 - x_2$.

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This is possible only when $x = 0$ otherwise the hypothesis is wrong. Then $\mathbb{U} \cap \mathbb{W} = \{0\}$. Hence $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$.