Tutorial Problems set-II

Note: All these problems can be solved using the results of Chapter-2.

[0.0.1] ***Exercise*** Find a necessary and sufficient condition for $\langle x, y\rangle = \sum\limits_{i=1}^{n} \alpha_i x_i y_i$ to be an inner product on $\mathbb{R}^n$.

[0.0.2] ***Exercise*** Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 matrix with real entries. Let $f_A : \mathbb{R}^2 \to \mathbb{R}$ be a map defined by $f_A(x, y) = y^t Ax$, where $x, y \in \mathbb{R}^2$. Show that f_A is an inner product on $\mathbb{R}^2$ if and only if $A = A^t$, $a_{11} > 0$, $a_{22} > 0$ and $det(A) > 0$.

[0.0.3] ***Exercise*** Let $\mathbb{V}$ be a finite-dimensional vector space and let $B = \{u_1, \ldots, u_n\}$ be a basis for $\mathbb{V}$. Let $\langle x, y\rangle$ be an inner product on $\mathbb{V}$. If $c_1, \ldots, c_n$ are any n scalars, show that there is exactly one vector x in $\mathbb{V}$ such that $\langle x, u_1\rangle = c_i$ for $i = 1, \ldots, n$.

[0.0.4] ***Exercise*** Let $(\mathbb{V}, \langle,\rangle)$ be an inner product space. Show that $\langle x, y\rangle = 0$ for all $y \in \mathbb{V}$, then $x = 0$.

[0.0.5] ***Exercise*** Show that $\langle x, y\rangle = \sum\limits_{i=1}^{n} \overline{x_i} y_i$ is not an inner product on $\mathbb{C}^n$.

[0.0.6] ***Exercise*** Let $(\mathbb{V}, \langle,\rangle)$ be a finite inner product space. Prove that for $v \in \mathbb{V} - \{0\}$, the set $W = \{w \in \mathbb{V} : \langle w, v\rangle = 0\}$ is a subspace of $\mathbb{V}$ of dimension $\textsc{dim}\,\mathbb{V} - 1$.

[0.0.7] ***Exercise*** Decide which of the following functions define an inner product $\mathbb{C}^2$. For $x = (x_1, y_1)$, $y = (y_1, y_2)$.

1. $\langle x, y\rangle = x_1\overline{y_2}$
2. $\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2}$
3. $\langle x, y\rangle = x_1y_1 + x_2y_2$
4. $\langle x, y\rangle = 2x_1\overline{y_1} + i(x_2\overline{y_1} - x_1\overline{y_2}) + 2x_2\overline{y_2}$

[0.0.8] ***Exercise*** Let $\mathbb{V}\mathbb{P}_3(x)$ be a subspace of real polynomials of degree at most 3. Equip $\mathbb{V}$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$$

.

1. Find the orthogonal complement of the subspace of scalar polynomials.
2. Apply the Gram Schmidt process to the basis $\{1, x, x^2, x^3\}$.

[0.0.9] ***Exercise*** Find an inner product on $\mathbb{R}^2$ such that $\langle e_1, e_2\rangle = 2$.

[0.0.10] ***Exercise*** Let $\mathbb{V}$ be the space of all $n \times n$ over $\mathbb{R}$ with the inner product $\langle A, B\rangle = trace(AB^t)$. Find the orthogonal complement of the subspaces of diagonal matrices.

[0.0.11] ***Exercise*** Let $(\mathbb{V}, \langle, \rangle)$ be an IPS. Let $\alpha, \beta \in \mathbb{V}$. Then show that $\alpha = \beta$ if and only if $\langle \alpha, \gamma\rangle = \langle \beta, \gamma\rangle$ for all $\gamma \in \mathbb{V}$.

[0.0.12] ***Exercise*** Apply Gram Schmidt process to the vectors $u_1 = (1, 0, 1)$, $u_2 = (1, 0, -1)$ and $u_3 = (0, 3, 4)$ to obtain an orthonormal basis for $\mathbb{R}^2$ with the standard inner product.

[0.0.13] ***Exercise*** Consider the inner product $\langle x, y\rangle = y^t AX$ on $\mathbb{R}^2$ where $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$. Find an orthonormal basis B of $S := \{(x_1, x_2, x_3) : \ x_1 + x_2 + x_3 = 0\}$ and then extend it to an orthonormal basis of $\mathbb{R}^3$.

[0.0.14] ***Exercise*** Let $(\mathbb{V}, \langle, \rangle)$ be an IPS. Let $||u|| = \sqrt{\langle u, u\rangle}$ for all $u \in \mathbb{V}$ be the norm induced by $\langle, \rangle$. Then prove that $||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$.

[0.0.15] ***Exercise*** Let $(\mathbb{V}, \langle, \rangle)$ be a finite dimensional IPS. Let $B = \{u_1, u_2, \ldots, u_n\}$ be a basis of $\mathbb{V}$. Then prove that $\langle u, v\rangle = \bar{y}^t Ax$ for all $u, v \in \mathbb{V}$ where $x = (x_1, \ldots, x_n)^t$,$y = (y_1, \ldots, y_n)^t$ are coordinates of u and v with respect to basis B and $a_{ij} = \langle u_i, u_j\rangle$.