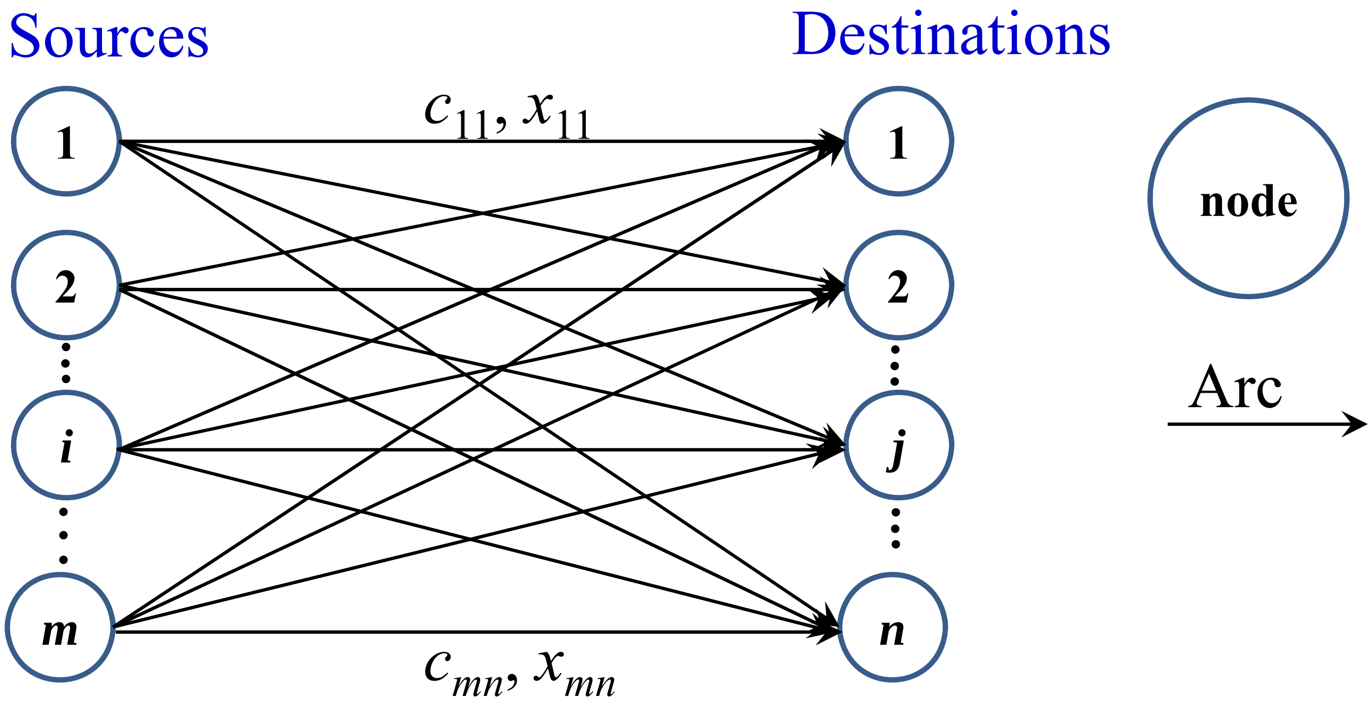
Transportation Problem

- A special class of LP dealing with transporting a commodity from sources (e.g., factories) to destinations (e.g., warehouses, markets).



Network Representation of the transportation model

- **Decision:** To determine number of units of item to be shipped from each source to each destination.
- **Objective:** To minimize the total shipping cost while satisfying supply and demand requirements.

Notations

Parameters

m : number of sources

n : number of destinations

a_i : supply available at source i, $i = 1, 2, \ldots, m$

b_j : demand at destination j, $j = 1, 2, \ldots, n$

c_{ij} : cost of transporting unit good from source i to destination j

Decision

x_{ij} : number of units of item to be shipped from source i to destination j, for all i, j

Number of decision variables? -> mn

General Formulation

$$\text{Minimize Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i, \forall i = 1, 2, \ldots, m \quad \text{(Supply constraints)}$$

$$\sum_{i=1}^{m} x_{ij} \ge b_j, \forall j = 1, 2, \ldots, n \quad \text{(Demand constraints)}$$

$$x_{ij} \ge 0, \forall i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

Formulation of Balanced Transportation Problem

- For balanced Transportation problem:

  Total supply =Total demand, i.e. $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

$$\text{Minimize Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \forall i = 1, 2, \ldots, m \quad \text{(Supply constraints)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \forall j = 1, 2, \ldots, n \quad \text{(Demand constraints)}$$

$$x_{ij} \geq 0, \forall i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

Example

Automobile company

Plants – NDL, LKO, BLR

Distribution Centres – JPR, HBD

Transportation Costs

	JPR (1)	HBD (2)	Plant Capacity
NDL (1)	50	300	1000
LKO (2)	100	250	1500
BLR (3)	500	125	1200
Demands	2300	1400	

Minimize

$$Z = 50\,x_{11} + 300\,x_{12} + 100\,x_{21} + 250\,x_{22} + 500\,x_{31} + 125\,x_{32}$$

Subject to

$$\begin{array}{llllllll}
x_{11} + x_{12} & & & & & & & = 1000 \\
& & x_{21} + x_{22} & & & & & = 1500 \\
& & & & x_{31} + x_{32} & & & = 1200 \\
x_{11} + & & x_{21} + & & x_{31} & & & = 2300 \\
& x_{12} + & & x_{22} & & + x_{32} & & = 1400 \\
& x_{ij} \geq 0,\ \forall\, i, j & & & & & &
\end{array}$$

- **Special structure**

A =

$$\begin{array}{cl}
\left[\begin{array}{ccccccccc}
1 & 1 & 1 & & & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & & & & & 1 & 1 & 1 \\ \hline
1 & & & 1 & & & 1 & & \\
 & 1 & & & 1 & & & 1 & \\
 & & 1 & & & 1 & & & 1
\end{array}\right]
&
\begin{array}{l}
\textbf{Supply} \\ \textbf{Constraints} \\ \\ \textbf{Demand} \\ \textbf{Constraints} \\ \\
\end{array}
\end{array}$$

➔ **Integer BFS**

Number of basic variables

- Ordinarily, there is one basic variable for each functional constraint in a linear programming problem. For transportation problems with m sources and n destinations, the number of functional constraints is m+n. However, the number of basic variables is $(m + n – 1)$.
- The reason is that the functional constraints are equality constraints, and this set of m+n equations has one extra (or redundant) equation that can be deleted without changing the feasible region, i.e. any one of the constraints is automatically satisfied whenever the other $(m + n – 1)$ constraints are satisfied.
- This fact can be verified by showing that any supply constraint exactly equals the sum of the demand constraints minus the sum of the other supply constraints, and that any demand equation also can be reproduced by summing the supply equations and subtracting the other demand constraints.

Transportation Simplex method

For feasibility: transportation problem will have feasible solutions if and only if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Transportation Simplex

Step 1: Find initial Basic feasible solution

Step 2: Is the solution Optimal? If yes, STOP, otherwise go to Step 3

Step 3: Determine entering variable (from non-basic variables).

Step 4: Determine leaving variable (from current basic variables).
Find the new basic solution, and go to Step 2

Initial Basic feasible solution

- **Integer solutions property:** If a_i and b_j (for all i, j) have integer value, then all the basic variables will have *integer value.*

- **General Rule for finding initial BFS:**

Step 1: Select a cell (i, j) for allocation according to some criteria (Northwest corner rule, Least cost rule, Vogel's approximation method)

Step 2: Allocate the amount in the selected cell (i, j) as minimum of (a_i, b_j), and adjust the associated amounts of supply and demand.

Step 3: Cross out the row or column with zero supply or demand. If both row and column are exhausted simultaneously, cross out either row or column, and leave a zero supply (demand) in the uncrossed row (column).

Step 4: If exactly one row or column is left uncrossed, allocate the remaining amount to its cell according to their supply/demand requirements, and stop. Otherwise, go to Step 1

Example Problem

Source \ Destination	1	2	3	4	Supply
1	10	0	20	11	15
2	12	7	9	20	25
3	0	14	16	18	5
Demand	5	15	15	10	45

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 45$$

Northwest Corner Rule

1. Begin by selecting cell (1,1) (northwest corner of the table)
2. If the last selected cell is (i, j) and source i has any supply remaining, then select the next cell as $(i, j+1)$ (One column to right), else select cell $(i+1, j)$ (one row down)

	1	2	3	4	**Supply**
1	10 5	0 10	20	11	15/~~10~~
2	12	7 5	9 15	20 5	25/20/~~5~~
3	0	14	16	18 5	~~5~~
Demand	~~5~~	15/~~5~~	~~15~~	10/~~5~~	

Total cost, $Z = 10\times5+0\times10+7\times5+9\times15+20\times5+18\times5 = 410$

Least Cost Method

- **Allocate as much as possible to the cell with the smallest unit cost (Break tie arbitrarily)**

	1	2	3	4	**Supply**
1	10	0 15	20	11 0	15/~~0~~
2	12	7	9 15	20 10	25/~~10~~
3	0 5	14	16	18 0	5/~~0~~
Demand	~~5~~	~~15~~	~~15~~	~~10~~	

Total cost, Z = 0x15+11x0+9x15+20x10+0x5+18x0 = 335

VOGEL'S APPROXIMATION METHOD

- For each row and column under consideration, calculate penalty as absolute difference between the two smallest unit cost c_{ij} still remaining in that row or column.
- If two cells tie for the minimum cost, the penalty is set at zero.
- Identify **row or column** with the **maximum penalty** (break ties arbitrarily) and select the **cell** with **least cost** in the selected row or column.
- Finally, the last row/column is made satisfied according to Least cost method

	1	2	3	4	Supply	Penalty		
1	10	0 15	20	11 0	15/~~0~~	10	11	9
2	12	7	9 15	20 10	25/~~10~~	2	2	11
3	0 5	14	16	18 0	5/~~0~~	14	2	2
Demand	~~5~~	~~15~~	~~15~~	~~10~~				
Penalty	10	7	7	7				

Total cost, Z = 0x15+11x0+9x15+20x10+0x5+18x0 = 335

Summary of NWC Rule and VAM

- NWC: quick and easy
  - Far from optimality because no attention to costs
- VAM: popular and gives better initial BFS
  - Penalty has intuitive appeal: minimum extra unit cost for failing to make an allocation to cell having smallest unit cost

Dual of Balanced Transportation Problem

Let u_i : dual variable associated with the supply constraint of source i

v_j : dual variable associated with the demand constraint of destination j

$$\max \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j$$

$$s.t.$$

$$u_i + v_j \le c_{ij}, \forall i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n$$

u_i and v_j unrestricted in sign

- **Economic interpretation**

A shipping company proposes to producer (facing the primal problem) to pick an item from source i at the price u_i per unit and deliver it at destination j at the price v_j per unit.

Dual constraint ensures that the prices are **competitive**

Therefore, if shipping company knows all a_i and b_j, then shipping company will find all u_i and v_j to maximize the total return.

Optimality Test in Transportation Simplex

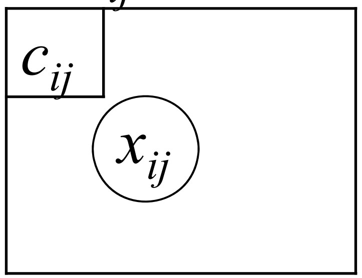
- Equivalent to computing the z_{ij}-c_{ij} of simplex tableau
- For each basic variable $x_{ij} \geq 0$: set $u_i + v_j - c_{ij} = 0$

  (This is equivalent to $\mathbf{C_B B^{-1} A - c} = 0$, i.e. $\mathbf{yA - c} = 0$)
- For each non-basic variable $x_{ij} = 0$, Calculate $u_i + v_j - c_{ij}$
- Since Transportation problem has minimization type objective function, therefore

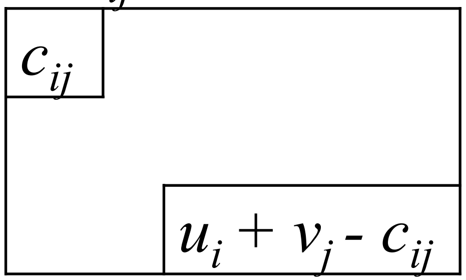
  If $u_i + v_j - c_{ij} > 0$, then corresponding x_{ij} is the candidate for entering variable

  If $u_i + v_j - c_{ij} \leq 0$, $\forall i, j$, then the current solution is optimal

If x_{ij} is a BV



If x_{ij} is a NBV



Step1: Initial BFS (Using NWC rule)

	1	2	3	4	**Dual variable**
1	10 (5)	0 (10)	20	11	u_1
2	12	7 (5)	9 (15)	20 (5)	u_2
3	0	14	16	18 (5)	u_3
Dual Variable	v_1	v_2	v_3	v_4	

Total cost, $Z = 10\times5+0\times10+7\times5+9\times15+20\times5+18\times5 = 410$

Iteration 1

Step 2: Optimality Test (minimization problem)

BFS is optimal if and only if $u_i + v_j - c_{ij} \le 0 \quad \forall i, j$ such that x_{ij} is NBV

If x_{ij} is a BV, then $u_i + v_j = c_{ij}$

How to calculate u_i and v_j?

Number of basic variables $= m + n - 1 =$ number of equations in u_i and v_j

Number of unknowns $= m + n$

Set one of the u_i or v_j to 0 and solve $m + n - 1$ equations to calculate remaining u_i and v_j

Now, for each BV x_{ij}, $u_i + v_j = c_{ij}$

$x_{11} : u_1 + v_1 = 10$

$x_{12} : u_1 + v_2 = 0$

$x_{22} : u_2 + v_2 = 7$

$x_{23} : u_2 + v_3 = 9$

$x_{24} : u_2 + v_4 = 20$

$x_{34} : u_3 + v_4 = 18$

Set $u_1 = 0$, and so $v_1 = 10, v_2 = 0, u_2 = 7, v_3 = 2, v_4 = 13, u_3 = 5$

For each NBV, calculate $u_i + v_j - c_{ij}$ (updated cost coefficients)

$$z_{13} - c_{13} = u_1 + v_3 - c_{13} = 0 + 2 - 20 = -18$$

$$z_{14} - c_{14} = u_1 + v_4 - c_{14} = 0 + 13 - 11 = 2$$

$$z_{21} - c_{21} = u_2 + v_1 - c_{21} = 7 + 10 - 12 = 5$$

$$z_{31} - c_{31} = u_3 + v_1 - c_{31} = 5 + 10 - 0 = 15$$

$$z_{32} - c_{32} = u_3 + v_2 - c_{32} = 5 + 0 - 14 = -9$$

$$z_{33} - c_{33} = u_3 + v_3 - c_{33} = 5 + 2 - 16 = -9$$

					u_i
	10 5	0 10	20 -18	11 2	0
	12 5	7 5	9 15	20 5	7
	0 15	14 -9	16 -9	18 5	5
v_j	10	0	2	13	

The current solution is not optimal

Step 3: Entering Variable

The non-basic variable corresponding to most positive $u_i + v_j - c_{ij}$

⇨ x_{31} is entering variable

Step 4: Leaving variable

- Equivalent to feasibility condition in Simplex.
- No need to calculate explicit ratio as constraint coefficients are 1
- **Equivalent to minimum ratio rule:** Construct a "closed loop" using vertical and horizontal segments, which starts and terminates at the entering variable cell and other corners of the loop lie in basic variable cells.
- If entering variable (x_{31}) is increased by Θ unit, then decrease/increase Θ from various basic variables at the corner of the loop to maintain feasibility. Θ= *minimum of* x_{ij} value from the -Θ cells
- Select leaving variable of smallest value from (-Θ) cell, i.e., (x_{34}) (break tie arbitrarily)

					u_i
	10 5 $-\Theta$	0 10 $+\Theta$	20 -18	11 2	0
	12 5	7 5 $-\Theta$	9 15	20 5 $+\Theta$	7
	0 $+\Theta$ 15	14 -9	16 -9	18 5 $-\Theta$	5
v_j	10	0	2	13	

Iteration 2

					u_i
	10 0 $-\theta$	0 15 $+\theta$	20 -18	11 2	0
	12 $+\theta$ 5	7 0 $-\theta$	9 15	20 10	7
	0 5	14 -24	16 -24	18 -15	-10
v_j	10	0	2	13	

Degenerate Solution

Total cost = 335

⇨ x_{21} is entering variable

⇨ x_{11} is leaving variable

Iteration 3

				u_i
10 -5	0 15 $-\theta$	20 -18	11 $+\theta$ 2	0
12 0	7 0 $+\theta$	9 15	20 10 $-\theta$	7
0 5	14 -19	16 -19	18 -10	-5
v_j: 5	0	2	13	

Degenerate Solution

Total cost = 335

⇨ x_{14} is entering variable

x_{24} is entering variable

Iteration 4

					u_i
	10 -5	0 5	20 -18	11 10	0
	12 0	7 10	9 15	20 -2	7
	0 5	14 -19	16 -19	18 -12	-5
v_j	5	0	2	11	

Optimal solution since $u_i + v_j - c_{ij} \leq 0$ for all non-basic variables

Total cost = 315

Special Cases of Transportation Problems

1. Maximization type:

Transportation-type problems that concern profits or revenues rather than costs with the objective to maximize profits rather than to minimize costs.

- Reverse the optimality and entering variable rule:

i.e., if $u_i + v_j - c_{ij} \geq 0$ for all non-basic variables, then the current BFS is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis

2. Unacceptable Routes:

- Certain origin-destination combinations may be unacceptable due to weather factors, equipment breakdowns, labor problems, or skill requirements that either prohibit, or make undesirable, certain combinations (routes).
- If shipment from source i to destination j is not desirable Assign c_{ij} =M (a Large positive number)

3. Multiple optimal solution:

- If $u_i + v_j - c_{ij}$=0 for at least one non-basic variable cell

Special Cases of Transportation Problems

4. Unbalance Transportation Problem

- **Convert into a Balanced Transportation Problem and apply the procedure**

Case (i): If total supply > total demand

- Add a dummy destination
- Set demand of dummy destination = total supply – total demand
- Set cost of shipping unit item to the dummy destination from each supply node as zero

Case (ii): If total supply < total demand

- Add a dummy supply node
- Set supply of dummy supply node = total demand – total supply
- Set cost of shipping unit item from the dummy supply node to each destination as zero