Assignment 6 on Part C
Partial Differential Equations (MA20103)

For convenience of the students, I have listed the superposition principle, wave equation and elliptic equation at the end of the below problems to avoid the confusion related to notations.

1. (a) Determine the steady state temperature distribution in a 1×1 square plate where one side is held at 100° and other sides are held at 0°. In particular, find the steady state temperature at the center of the plate.

   (b) Obtain the steady state temperature distribution in a rectangular material body, $0 \le x \le 2$ and $0 \le y \le 1$ with boundary conditions

$$f_1(x) = 100, f_2(x) = g_1(y) = 0 \text{ and } g_2(y) = 100(1-y).$$

2. (a) Solve the nonhomogeneous boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{in } 0 < x < 1, t > 0, \\ u(0,t) = 100 \text{ and } u(1,t) &= 100 \quad \text{for all } t > 0, \\ u(x,0) = f(x) &= 50x(1-x) \quad \text{for } 0 < x < 1. \end{aligned}$$

   (b) Find the solution of the following boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < 1,\ t > 0$$

   with boundary conditions

$$u(0,t) = 0 \text{ and } u(1,t) = 0 \text{ for all } t > 0$$

   and initial conditions

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = 0 \text{ for } 0 < x < 1,$$

   where

$$\begin{aligned} f(x) &= 4x \quad \text{for } 0 \le x \le \frac{1}{4}, \\ &= 4\left(\frac{1}{2} - x\right) \quad \text{for } \frac{1}{4} \le x \le \frac{3}{4}, \\ &= 4\,(x-1) \quad \text{for } \frac{3}{4} \le x \le 1. \end{aligned}$$

Appendix

Superposition principle: If u_1 and u_2 are the solutions of a linear homogeneous PDEs, then any linear combination $u = c_1u_1 + c_2u_2$, where c_1 and c_2 are constants, is also a solution. If in addition, u_1 and u_2 satisfy a linear homogeneous boundary condition, then so will $u = c_1u_1 + c_2u_2$.

Elliptic equations: Let $R : 0 \le x \le a, 0 \le y \le b$ be a two dimensional rectangle/region. An elliptic equation/Laplace equation on R together with boundary conditions can be given by

$$\nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with

$$u(x,0) = f_1(x) \quad \text{and} \quad u(x,b) = f_2(x) \text{ for } 0 \le x \le a,$$

$$u(0,y) = g_1(y) \quad \text{and} \quad u(a,y) = g_2(y) \text{ for } 0 \le y \le b.$$

Wave equations: The solution of the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < L,\ t > 0$$

with boundary conditions

$$u(0,t) = 0 \text{ and } u(L,t) = 0 \text{ for all } t > 0$$

and initial conditions

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 < x < L$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(b_n \cos \lambda_n t + b_n^* \sin \lambda_n t \right), \tag{1}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx \tag{2}$$

and

$$\lambda_n = c\frac{n\pi}{L},\ n = 1, 2, ... \tag{3}$$

Heat equations: The solution of the one dimensional heat equation (boundary value problem)

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } 0 < x < L, t > 0,$$
$$u(0,t) = 0 \text{ and } u(L,t) = 0 \quad \text{for all } t > 0,$$
$$u(x,0) = f(x) \quad \text{for } 0 < x < L$$

is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x, \tag{4}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx, \quad \lambda_n = c\frac{n\pi}{L}, \quad n = 1, 2, ... \tag{5}$$