Contents

Chapter 1

Solution of General Nonlinear Programming Problem

General form of an unconstrained optimization problem is

$$(UP): \ \min_{x \in R^n} f(x), \ \ f: R^n \to R$$

General form of a constrained optimization problem is

$$\begin{aligned} (CP): \ \ \min \ \ & f(x) \\ subject \ to \ \ & g_i(x) \leq 0, \ i = 1, 2, ..., m \\ & h_j(x) = 0, \ j = 1, 2, ..., k \\ & x \in R^n, f, g_i, h_j : R^n \to R \end{aligned}$$

If all f, g_i and h_j are linear functions then (CP) is linear programming problem. If at least one of these is a nonlinear function then (CP) is a nonlinear programming problem. If all f, g_i and h_j are convex functions then (CP) is a convex programming problem.

Note 1.1. *Definiteness of a symmetric real matrix $A = (a_{ij})_{n \times n}$ is determined as follows. Suppose rank of $A = r$, signature of $A = s$. λ_i are Eigen values. Definiteness of the quadratic form $x^T Ax$ is same as definiteness of the matrix A.*

1. *A is positive definite* $\equiv \{x^T Ax > 0, \forall x \neq 0\} \equiv \{s = n\} \equiv \{\lambda_i > 0, \forall i\}$
   A is positive definite $\Rightarrow$ all principal minors are > 0
   Leading principal minors are $> 0 \Rightarrow A$ is positive definite.

2. *A is positive semi-definite* $\equiv \{x^T Ax \geq 0, \forall x \in R^n\} \equiv \{s = r\} \equiv$ *all principal minors are* $\geq 0 \equiv \{\lambda_i \geq 0, \forall i\}$.

3. *A is negative definite* $\equiv \{x^T Ax < 0, \forall x \neq 0\} \equiv \{s = -n\} \equiv$ *All principal minors of even order are > 0 and all principal minors of odd order are < 0.* $\equiv \{\lambda_i < 0 \forall i\}$

4. *A is negative semi-definite* $\equiv \{x^T Ax \leq 0, \forall x \in R^n\} \equiv \{s = -r\} \equiv$ *All principal minors of even order are ≥ 0 and all principal minors of odd order are ≤ 0* $\equiv \{\lambda_i \geq 0\} \equiv \{\lambda_i \leq 0\}$.

5. *A is in-definite if neither of above holds* $\equiv \{\mid s \mid < r\}$.

6. *A is positive definite iff A is positive semidefinite and nonsingular.*

Note 1.2. *A twice differentiable function $f : R^n \to R$ is said to be strictly convex on a set $S \subseteq R^n$ iff $\nabla_x^2 f(x) \succ 0,\ \forall\, x \in S$ and convex iff $\nabla_x^2 f(x)$ is positive semidefinite.*
f is said to be strictly concave iff $\nabla_x^2 f(x) \prec 0,\ \forall\, x \in S$

Theorem 1.0.1. *x is a local minimum of (UP) iff $\nabla_x f(x) = 0, \nabla_x^2 f(x) \succ 0$.*

1.0.1 Solution of CP

Example 1: Find the maximum volume of a rectangular parallelopiped whose surface area is at most 10 and at least 6 units units.
Solution of this problem can be found by solving the optimization problem:

$$\max\ \ xyz\ \ subject\ \ to\ \ 3 \leq xy + yz + zx \leq 5,\ \ x, y, z > 0$$

This is a constrained non linear programming problem.

Example 2: Shortest path problem

Find the minimum distance from $(1, 2)$ to the curve $x^2 + x - y = 1$.

Solution of this problem is the solution of the optimization problem :

$$\min\ (x-1)^2 + (y-2)^2,\ subject\ \ to\ \ x^2 + x - y = 1$$

This is a constrained non linear programming problem.

Example 3: Quadratic Programming Problem:

A general quadratic programming problem is:

$$\min\ \ \frac{1}{2}x^T Qx\ \ s.to\ \ Ax = b$$

where $x \in R^n$, Q is a positive definite matrix of order n, A is a matrix of order $m \times n$, b is a vector of order m. Example 4: Least Mean Square Problem

Consider a system of linear equations $Ax = b$, $A = (a_{ij})_{m\times n}$ is a matrix of order $m \times n$, $Rank(A) = m$, $x \in R^n$, $b \in R^m$. That is, find $(x_1, x_2, ..., x_n)$ so that

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1j}x_j + ... + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2j}x_j + ... + a_{2n}x_n &= b_2 \\
&.... \\
a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n &= b_i \\
&.... \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mj}x_j + ... + a_{mn}x_n &= b_m
\end{aligned}$$

Solution of this problem can be found analytically by solving the optimization problem:

$$\min_{x\in R^n} \parallel Ax - b \parallel_2^2$$

This is equivalent to

$$\min_{x\in R^n} \sum_{i=1}^{m}(\sum_{j=1}^{n} a_{ij}x_j - b_i)^2$$

This is an unconstrained quadratic Programming problem.

General structure of a constrained optimization problem is

$$\begin{aligned} \min \quad & f(x) \\ subject\ to \quad & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \\ & \ldots\ldots \\ & g_m(x) \leq 0 \\ & h_1(x) = 0 \\ & h_2(x) = 0 \\ & \ldots\ldots. \\ & h_k(x) = 0 \end{aligned}$$

$f, g_i, h_j : R^n \rightarrow R$, $i = 1, 2, ..., m; j = 1, 2, ..., k$ Construct the Lagrange function for (CP) with dual vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^T$ and $\mu = (\mu_1, \mu_2, ..., \mu_k)^T$, as

$$\begin{aligned} L(x, \lambda, \mu) &= f(x) + \lambda^T g(x) + \mu^T h(x) \\ &= f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{k} \mu_j h_j(x) \end{aligned} \tag{1.0.1}$$

$\nabla_x L(x, \lambda, \mu) = \nabla_x f(x) + \sum_{i=1}^{m} \lambda_i \nabla_x g_i(x) + \sum_{j=1}^{k} \mu_j \nabla_x h_j(x)$

Theorem 1.0.2. *x is a local minimum of (CP) iff f, g_i, h_j are convex functions at x, $\{\nabla_x g_i(x), \nabla_x h_j(x)\}$ is linearly independent, and x satisfies KKT(Karush Kuhn Tucker)* *optimality conditions, which are:*

$$\nabla_x L(x, \lambda, \mu) = 0$$
$$g_i(x) \leq 0, \ i = 1, 2, ..., m$$
$$h_j(x) = 0, \ j = 1, 2, ..., k$$
$$\lambda_i . g_i(x) = 0 \ \forall \, i$$
$$\lambda_i \geq 0, \mu_j \in R, \ \forall \, i, j, (\lambda, \mu) \neq 0$$

1.1 Example

Example 1. Write all necessary and sufficient conditions for the existence of a local optimal solution of the following problem at $(1, 1)$ and verify if these are satisfied or not.

$$\min \ \ x^3y^5 - 3x^2 + 2y \ \ s.to \ \ 3x + 2y^2 \leq 6, x^2 + y \leq 2, 3x - 2y = 1$$

Here $f(x, y) = x^3y^5 - 3x^2 + 2y$, $g_1(x, y) = 3x + 2y^2 - 6$, $g_2(x, y) = x^2 + y - 2$, $h(x, y) = 3x - 2y - 1$

Lagrange function is

$L(x, y; \lambda_1, \lambda_2; \mu) = x^3y^5 - 3x^2 + 2y + \lambda_1(3x + 2y^2 - 6) + \lambda_2(x^2 + y - 2) + \mu(3x - 2y - 1)$.

Optimality Conditions:

1. Feasibility condition: $(1, 1)$ satisfies feasibility conditions $g_1(x, y) \leq 0, g_2(x, y) \leq 0, h(x, y) = 0$.

2. Convexity condition:

   - $\nabla^2 f(1, 1)$ is not a positive definite matrix so f is not a convex function in the nbd of $(1, 1)$.
   - $\nabla^2 g_1(1, 1)$ is a positive semidefinite matrix, hence convex.
   - $\nabla^2 g_2(1, 1)$ is a positive semidefinite matrix, hence convex.
   - h is a linear function, hence this is a convex function.

   Hence this is not a convex programming problem.

3. Dual restriction: $\lambda_1 \geq 0, \lambda_2 \geq 0, \mu \in R$ and $(\lambda_1.\lambda_2, \mu) \neq (0, 0, 0)$

4. Complementary conditions: $\lambda_1 g_1(1, 1) = 0$ means λ_1 may not be zero. $\lambda_2 g_2(1, 1) = 0$ means $\lambda_2 = 0$.

5. Regularity condition: $\{\nabla g_1(1, 1), \nabla g_2(1, 1), \nabla h(1, 1)\}$ is a linearly dependent set. So regularity condition is not satisfied.

6. Normal condition: $\nabla L(1, 1; \lambda_1, \lambda_2; \mu) = 0$, which is

$$7 + 3\lambda_1 + \lambda_2 + 3\mu = 0$$

$$4 - 2\lambda_1 + 3\lambda_2 + \mu = 0$$

Since $\lambda_2 = 0$ so solution of this system is $\lambda_1 = \frac{5}{9}, \mu = \frac{-26}{9}$. Hence dual restriction is satisfied.

Since some optimality conditions are not satisfied so $(1, 1)$ is not a solution.

1.2 Exercise

1. Consider the following two non linear optimization problems.
(i)Verify both necessary and sufficient optimality conditions at (1,1,1) for $(P_1)$and at (1,-1,0)for (P_2) respectively.
(ii)Verify if (P_3) is a convex quadratic programming problem or not.

$$\begin{aligned}(P_1) : Minimize\ 3x_1^2 - 2x_1x_2x_3 + x_2^3x_3 \\ Subject\ to\ 3x_1^2 + x_2x_3 \geq 4 \\ 2x_2 - 3x_3^2 \leq 6 \\ -3x_1 + 2x_2x_3^2 = -1 \\ 2x_1 - 3x_2^2 + 4x_1x_3 = 3\end{aligned}$$

$$
\begin{aligned}
(P_2):\ Maximize\ \ & 3x_1^2 - 2x_1x_2x_3 + x_2^3x_3 \\
Subject\ \ to\ \ & 3x_1^2 + x_2x_3 \geq 3 \\
& 2x_2 - 3x_3^2 \leq 6, x_1 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
(P_3): Minimize\ \ & x_1^2 + x_1x_2 + 6x_2^2 - 2x_2 + 8x_2 \\
Subject\ \ to\ \ & x_1 + 2x_2 \leq 4 \\
& 2x_1 + x_2 \leq 5 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

2. Derive KKT optimality conditions for

$$
\begin{aligned}
Minimize\ \ & 7x_1 - 6x_2 + 4x_3 \\
Subject\ \ to\ \ & 3x_1^2 + x_2x_3 \geq 4 \\
& x_1^2 + 2x_2 + 3x_3^2 = 1 \\
& x_1 + 5x_2 - 3x_3 = 6
\end{aligned}
$$