Part B

Note that in Part A, I show some examples of how to verify for a given solution to be CI of given PDE. In this Part, we will discuss method of finding CI by Charpit's method for a given PDE. Before going to details of this method, I'll quote two results, without proof, as these two results will be used in the discussion of Charpit's method.

Result 1: Lagrange's Theorem for 1st order linear PDE of $n \geq 2$ independent variables x_j and one dependent variable $z(x_1, x_2, \dots, x_n)$:

For convenience, for this result using the symbols $p_j \equiv \partial z/\partial x_j$, we write the PDE as follows

$$P_1 p_1 + P_2 p_2 + \cdots + P_n p_n = R, \tag{A.1}$$

where P_js, R are functions of $x_1, x_2, \dots, x_n, z$. Then general integral is given by

$$\phi(u_1, u_2, \dots u_n) = 0, \tag{A.2}$$

where $u_j(x_1, x_2, \dots, x_n, z) = c_j, j = 1,2, \dots n$ are n independent solutions of following AE

$$\frac{\mathrm{d}x_1}{P_1} = \frac{\mathrm{d}x_2}{P_2} = \cdots = \frac{\mathrm{d}x_n}{P_n}. \tag{A.3}$$

Result 2: Pfaffian Differential equation

$$P\mathrm{dx} + \mathrm{Qdy} + \mathrm{Rdz} = 0, \tag{A.4}$$

is said to be integrable if $\exists$ a relation

$$\phi(x, y, z; c) = 0, \tag{A.5}$$

which satisfy (A.4). (A.5) is one-parameter family of surface in space.

Condition of Integrability of (A.4):

$$P(Q_z - R_y) + Q(R_x - P_z) + R(P_y - Q_x) = 0.$$

- Compatible Systems of 1st order PDEs

We will consider two following PDEs (linear or nonlinear) of two independent variables x, y and one dependent variable (solution) $z(x, y)$, by denoting partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ by p, q respectively:

$$f(x, y, z, p, q) = 0, \;\; g(x, y, z, p, q) = 0, \tag{1.1}$$

where f, g are assumed to be functionally independent, i.e. to say, $\exists$ no relation between f, g which is independent of p, q. This assumption is natural because otherwise two PDEs would be automatically "Compatible".

By "Compatible", we mean every solution $z = z(x, y)$ of one PDE will be solution of other. The assumption made above mathematically transpires to following relation

$$f_p g_q - f_q g_p \neq 0\ , \qquad (1.2)$$

where suffix denote the variable w.r.t. which partial differentiation taken.

Let us now derive the condition of compatibility of two PDEs in (1.1). We begin by assuming the existence of same solution $z = z(x, y)$ for both PDEs. So with the substitution of solution z in both PDEs, we can view them as two non-differential relations for 2 variables p, q, so that under assumption (1.2), these two relations can be solved for p, q:

$$p = \phi(x, y, z), q = \psi(x, y, z) \qquad (1.3)$$

Note $p \equiv \frac{\partial z}{\partial x}, q \equiv \frac{\partial z}{\partial y}$, so that from well-known identity, we can write

$$p\mathrm{d}x + q\mathrm{d}y = \mathrm{d}z \Longrightarrow \phi\mathrm{d}x + \psi\mathrm{d}y - \mathrm{d}z = 0. \qquad (1.4)$$

(1.4) is Pfaffian differential equation (DE), which must be integrable so that solution of (1.4) must exist as $z = z(x, y)$, which is solution of both PDEs, we assumed at beginning. Let me quote (without proof) the formula for a general Paffian DE $P(x, y, z)\mathrm{d}x + Q(x, y, z)\mathrm{d}y + R(x, y, z)\mathrm{d}z = 0$ to be integrable:

$$P(Q_z - R_y) + Q(R_x - P_z) + R(P_y - Q_x) = 0$$

In the present case of DE (1.4), $P = \phi, Q = \psi, R = -1$, so that condition of compatibility of two PDEs in (1.1) is

$$\phi\psi_z - \psi\phi_z - (\phi_y - \psi_x) = 0 \Longrightarrow \psi_x + \phi\psi_z = \phi_y + \psi\phi_z. \qquad (1.5)$$

Our work is not finished yet ! Because compatibility condition will be a differential relation between 7 quantities x, y, z, p, q, f, g, so that we will have to eliminate unknown functions ϕ, ψ using given two relations in (1.1) with assumed solution $z = z(x, y)$ and p, q from (1.3) are substituted into (1.1), i.e. f, g in (1.1) now becomes functions of 5 variables x, y, z, p, q, where p, q are functions of x, y, z by (1.3), and x, y, z will now be considered independent. Apparently, reader may worry at this point that how z may be independent of x, y! The answer to this riddle is as follows: Note that we are here not solving PDEs, in which case z is of course some function of x, y. But here at the beginning we assume that some solution $z = z(x, y)$ exist for both PDEs, that means that we substitute z in (1.1) making z independent of x, y when we will differentiate f, g partially w.r.t. z, for otherwise again p, q will arise as a differentiation of z w.r.t x, y, but p, q are already fixed by (1.3).

Hence, considering x, y, z as independent variables, we will now differentiate f, g partially w.r.t x, y, z. At the first step, differentiate w.r.t x, z to get following pair of two relations respectively, using chain rule:

$$f_x + f_p\phi_x + f_q\psi_x = 0\,, f_z + f_p\phi_z + f_q\psi_z = 0, \qquad (1.6)$$

$$g_x + g_p\phi_x + g_q\psi_x = 0\,, g_z + g_p\phi_z + g_q\psi_z = 0. \qquad (1.7)$$

Notice that partial differentiation of f w.r.t. p, q are denoted by suffix f_p, f_q (not as f_ϕ, f_ψ), but partial differentiation of $p \equiv \phi, q \equiv \psi$ w.r.t x, z are denoted by ϕ_x, ϕ_z (not as p_x, p_z). This is because we are to eliminate two unknown functions ϕ, ψ to get a relation between x, y, z, p, q, f, g, where symbolical meaning of p, q are what was in the system (1.1) as partial derivative of z w.r.t. x, y respectively.

Add 1st relation with ϕ times 2nd for each pair (1.6) & (1.7), to get following two linear algebraic relations for $\psi_x + \phi\psi_z$ and $\phi_x + \phi\phi_z$:

$$f_q(\psi_x + \phi\psi_z) + f_p(\phi_x + \phi\phi_z) + (f_x + \phi f_z) = 0,$$

$$g_q(\psi_x + \phi\psi_z) + g_p(\phi_x + \phi\phi_z) + (g_x + \phi g_z) = 0.$$

By cross-multiplication, we may get expressions for both. However we need one of them, which is LHS of compatibility condition (1.5), the expression is

$$\psi_x + \phi\psi_z = \frac{1}{f_p g_q - f_q g_p}\left[\left(f_x g_p - f_p g_x\right) + \phi\left(f_z g_p - f_p g_z\right)\right]. \qquad (1.8)$$

Note that ϕ in the RHS of (1.8) is not unknown function, because it is actually p.

Now we will find the similar expression for $\phi_y + \psi\phi_z$, which is the RHS of compatibility condition (1.5). To get that expression, we will now differentiate f, g partially w.r.t. y, z, and proceed as before. The detailed steps are skipped here because readers at this point should be able to produce these steps. Finally, we get

$$\phi_y + \psi\phi_z = -\frac{1}{f_p g_q - f_q g_p}\left[\left(f_y g_q - f_q g_y\right) + \psi\left(f_z g_q - f_q g_z\right)\right]. \qquad (1.9)$$

Equating RHS of (1.8) and (1.9), our condition of compatibility in the final form is

$\left(f_x g_p - f_p g_x\right) + \left(f_y g_q - f_q g_y\right) + p\left(f_z g_p - f_p g_z\right) + q\left(f_z g_q - f_q g_z\right) =$
0(1.10)

- Charpit's Method

This method is used to find solution of given non-linear PDE

$$f(x, y, z, p, q) = 0. \qquad (2.1)$$

The main point of Charpit's method is to find a compatible partner PDE

$$g(x, y, z, p, q, a) = 0, \qquad (2.2)$$

where f, g are not functionally dependent, i.e. $\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} \neq 0$, and a is an arbitrary parameter. Clearly compatibility condition (1.10) holds.

Solving (2.1) and (2.2) for p, q, we get $p = p(x, y, z, a), q = q(x, y, z, a)$, and consequently the DE

$$\mathrm{d}z = p(x, y, z, a)\mathrm{d}x + q(x, y, z, a)\mathrm{d}y$$

must have solution $F(x, y, z, a, b) = 0$, which is actually Complete Integral (CI) of given PDE (2.1). Note that the solution of a PDE of n independent variables and one dependent variable, containing exactly n arbitrary constants, is called CI.

Hence, our goal in Charpit's method is to find compatible partner g. This is actually now straightforward, because compatibility condition (1.10) is actually linear 1st order PDE of 5 independent variables x, y, z, p, q and one dependent variable g. Just by regrouping terms of (1.10), we get

$$f_p g_x + f_q g_y + \left(p f_p + q f_q\right) g_z + (-f_x - p f_z) g_p + \left(-f_y - q f_z\right) g_q = 0. \qquad (2.3)$$

This can be solved by Lagrange's method (discussed in Part A). We will find integrals of following equations, known as Charpit's auxiliary equation (AE):

$$\frac{\mathrm{d}x}{f_p} = \frac{\mathrm{d}y}{f_q} = \frac{\mathrm{d}z}{p f_p + q f_q} = \frac{\mathrm{d}p}{-(f_x + p f_z)} = \frac{\mathrm{d}q}{-(f_y + q f_z)}. \qquad (2.4)$$

In practice, we choose suitable pair of fractions from (2.4) involving at least one of p, q, and solving that ODE, we get either $p = p(x, y, z, a)$ or

$q = q(x, y, z, a)$ or some relation involving p, q. In either case we can use given PDE (2.1) to get $p = p(x, y, z, a),\ q = q(x, y, z, a)$.

Last step is to substitute these expressions for p, q into the identity

$$\mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y, \tag{2.5}$$

and solve this DE to get CI of given PDE (2.1) in the form

$$F(x, y, z, a, b) = 0. \tag{2.6}$$

Note that here neither we find general solution (GS) of (2.3) nor we find explicitly the expression for the compatible partner g, as our focus is to find p, q in terms of x, y, z, sufficient to get CI of given PDE.

This outlines Charpit's method.

Problem 1. Find CI of $(p^2 + q^2)y = qz$

Solution: Rewrite given PDE as $f \equiv (p^2 + q^2)y - qz = 0$, and by formula (2.4), Charpit's AE is

$$\frac{\mathrm{d}x}{2py} = \frac{\mathrm{d}y}{2qy-z} = \frac{\mathrm{d}z}{2p^2y+2q^2y-q} = \frac{\mathrm{d}p}{pq} = \frac{\mathrm{d}q}{q^2}$$

From AE, we need one relation between p, q. By inspection, we see that last two fractions yield a simple 1st order ODE $p\mathrm{d}p + q\mathrm{d}q = 0$, solving which we have $p^2 + q^2 = c^2$. Substituting this relation into given PDE, we get

$$q = \frac{c^2y}{z}, p = \frac{c}{z}\sqrt{z^2 - c^2y^2}.$$ [we ignore $\pm$ sign due to arbitrariness of c.]

Then we will solve the DE $p\mathrm{d}x + q\mathrm{d}y = \mathrm{d}z$, which after substituting for p, q from above, becomes after little regrouping

$$\mathrm{d}(z^2 - c^2y^2) = 2c\sqrt{z^2 - c^2y^2}\,\mathrm{d}x.$$

This ODE is easily integrable to obtain

$$z^2 = (cx + d)^2 + c^2y^2.$$

- Special Types of 1st order PDEs

1. Type I: Only p, q present, i.e. given PDE : $f(p, q) = 0$

In this case, it can be easily seen that from AE, we will get either $\mathrm{d}p = 0$ or $\mathrm{d}q = 0$ which means that we can consider either $p = a$ or $q = b$. Substituting either of them in given PDE will definitely give other as constant also. So we will have to solve a simple DE $\mathrm{d}z = a\mathrm{d}x + Q(a)\mathrm{d}y$, for $p = a$. Hence, CI of this special Type PDE is $z = ax + yQ(a) + b$.

Example: $p^2 - q^2 = 4$

Solution: From AE, $\mathrm{d}p = 0 \Longrightarrow p = a, q = \sqrt{a^2 - 4}$. CI: $z = ax + y\sqrt{a^2 - 4} + b$

2. Type V : Reducible to Type I by transformation

Example: $x^2p^2 + y^2q^2 = z^2$

Solution: Rewrite as $(px/z)^2 + (qy/z)^2 = 1$. So if we put $X = \ln x\,, Y = \ln y\,, Z = \ln z$, and calculate $P \equiv \frac{\partial Z}{\partial X} = \frac{xp}{z}\,, Q \equiv \frac{\partial Z}{\partial Y} = \frac{yq}{z}$, so that given PDE now transforms to above form: $P^2 + Q^2 = 1$. So the CI is

$$Z = aX + Y\sqrt{1 - a^2} + b \Longrightarrow \ln z = a \ln x + \sqrt{1 - a^2} \ln y + b$$

3. Type III: Independent variables absent: $f(z, p, q) = 0$

The AE of this Type will always contain simple ODE: $q\mathrm{d}p = p\mathrm{d}q \Longrightarrow p = aq$, substituting into given PDE, $q = Q(a, z)$, so that the DE: $\mathrm{d}z = Q(a, z)(a\mathrm{d}x + \mathrm{d}y)$ is easily integrable.

Example: $p^2z^2 + q^2 = 1$

Solution: From AE, $p = aq \Longrightarrow q = 1/\sqrt{1 + a^2z^2}$, ignoring '−' sign.

Then solve $\sqrt{1 + a^2z^2}\mathrm{d}z = a\mathrm{d}x + \mathrm{d}y$ to get CI:

$$az\sqrt{1 + a^2z^2} + \ln\left(az + \sqrt{1 + a^2z^2}\right) = 2a(ax + y + b)$$

4. Type IV: Separable equation and z absent: $f(x, p) = g(y, q)$

From AE of this Type, we will always get $f_p\mathrm{d}p + f_x\mathrm{d}x = 0\,, g_q\mathrm{d}q + g_y\mathrm{d}y = 0$. From these two, we get $f(x, p) = a, g(y, q) = a$, since actually lhs of 1st & 2nd are actually $\mathrm{d}f, \mathrm{d}g$. Note that we can't take two arbitrary constants, because solving the next DE $\mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y$, 2nd arbitrary constant will appear.

Solving $f(x,p)=a, g(y,q)=a$ for p,q, we must get $p=P(a,x), q=Q(a,y)$, so that the next DE is easily integrable to obtain CI.

Example: $p^2y(1+x^2)=qx^2$

Solution: Rewrite PDE as $f(x,p)\equiv\frac{p^2(1+x^2)}{x^2}=\frac{q}{y}\equiv g(y,q)$, so that from AE, we will get $\frac{p^2(1+x^2)}{x^2}=\frac{q}{y}=a\Longrightarrow q=ay, p=x\sqrt{\frac{a}{1+x^2}}$. Then the next DE is

$\mathrm{d}z=x\sqrt{\frac{a}{1+x^2}}\,\mathrm{d}x+ay\,\mathrm{d}y$, which can be integrated easily to get CI:

$$z=\sqrt{a(1+x^2)}+\frac{y^2}{2}+b.$$

5. Type V: Clairaut's Equation: $z=px+qy+f(p,q)$

For this special from, we always get from AE: $\frac{\mathrm{d}p}{0}=\frac{\mathrm{d}q}{0}\Longrightarrow p=a, q=b$, so that next DE $\mathrm{d}z=a\mathrm{d}x+b\mathrm{d}y$. Thus Clauraut's equation always has CI, obtained by replacing p,q in given PDE simply by a,b in the form: $z=ax+by+f(a,b)$. Note that from DE, we have $z=ax+by+c$, and from given PDE, by substituting for p,q, we have $z=ax+by+f(a,b)$, so that $c=f(a,b)$.

- Higher order linear PDE with constant coefficients

We will consider two independent variables x,y and one dependent variable z, and we will denote here $D\equiv\frac{\partial}{\partial x}, D'\equiv\frac{\partial}{\partial y}$. Note that in the previous topics p,q were used for two partial derivatives $Dz, D'z$. The reason for adopting symbol for the operators here is that it helps expressing formulae in compact forms.

A General linear 2nd order PDE with constant coefficients may be written as

$$[F(D,D')]z\equiv\left[aD^2+bD'^2+2hDD'+2gD+2fD'+c\right]z=f(x,y)\quad(3.1)$$

where a,b,h,g,f,c are constants.

<u>A word of caution about symbols</u>: Observe p,q were merely functions so that p^2 means $(\partial z/\partial x)^2$, whereas D,D' are differential operators, so that D^2 means $\partial^2 z/\partial x^2$. Point is that this topic covers linear 2nd order PDE, so that we will not need to express non-linear terms like $(\partial z/\partial x)^2$ by the operators.