Indian Institute of Technology Kharagpur
End-Semester Examination: Autumn 2022

Date of Examination: 23/11/2022 (FN)
Subject. No: MA51109/MA60049 | Subject Name: Computational Statistics | Duration: 3 Hrs
Department: Mathematics | TOTAL MARKS: 50
Specific Chart, graph paper log book etc. required: None
Special Instruction: None

ANSWER ALL THE QUESTIONS

1. State whether the following statements are *TRUE* or *FALSE*. Justify your answer with a proof or a counter example. No marks will be awarded without justification. [10 marks]

   (a) Let $X \sim N(0,1)$. Then $X^2 \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$.

   (b) Let $X_1, \ldots, X_n$ be iid random variables with mean μ and variance σ^2. Then $var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) \to 0$ as $n \to \infty$.

   (c) Let (X_1, X_2) be jointly normal random variable with correlation coefficient between X_1 and X_2 being zero. Then X_1 and X_2 are independent.

   (d) Let X be a continuous random variable with the probability density function $f(x)$. Then the probability density function of $Z = aX + b$ is given by $\frac{1}{|a|} f\left(\frac{z-b}{a}\right)$.

   (e) The sum of n independent $Bernoulli(p)$ random variables is a Binomial random variable with parameters n and p.

2. Consider a function $f_X(x) : \mathbb{R} \to \mathbb{R}$ defined as

$$f_X(x) = \begin{cases} \frac{c}{x^2} & x = 1, 2, \ldots, \\ 0 & \text{otherwise}, \end{cases}$$

   where $c > 0$ is the positive constant.

   (a) Compute the value of c so that $f_X(x)$ is a probability mass function. [2 marks]

   (b) Does the first and second order moments for this probability mass function exist? [3 marks]

3. Sampling *within* and *over* a unit sphere.

   (a) Give an accept/reject algorithm to generate random sample of a uniform random vector X *within* the unit sphere

$$\{X \in \mathbb{R}^N \mid \|X\|_2 \leqslant 1\}$$

   by bounding it in an N dimensional hypercube. Discuss the efficiency of this accept/reject algorithm as $N \to \infty$. [5 marks]

   (b) Starting with N standard normal random samples, give an algorithm to generate a random sample of a uniformly distributed random vector Y *over* the unit sphere

$$\{X \in \mathbb{R}^N \mid \|X\|_2 = 1\}.$$

   Prove all the necessary results required in this algorithm. [5 marks]

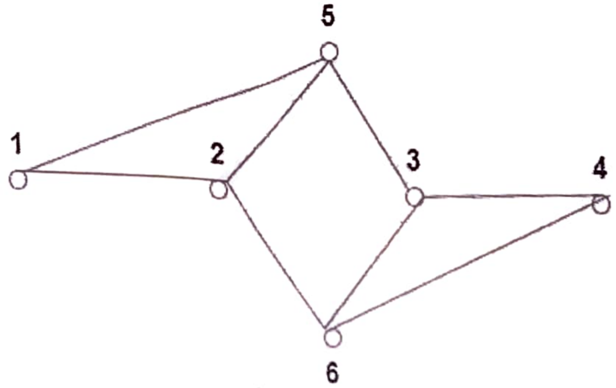
4. Generating a random sample from $Beta(\alpha, \beta)$ density.

(a) Let $X_1 \sim Gamma(\alpha, 1)$ and $X_2 \sim Gamma(\beta, 1)$ be independent random variables. Then prove that $Y = \dfrac{X_1}{X_1 + X_2}$ is $Beta(\alpha, \beta)$. [5 marks]

(b) Let $\alpha = 4$ and $\beta = 5$. Using random samples from $U(0, 1)$, give a procedure to generate a random sample from $Beta(4, 5)$. [5 marks]

(c) Let $\alpha = 0.9$ and $\beta = 0.4$. Using random samples from $U(0, 1)$, give a procedure to generate a random sample from $Beta(0.9, 0.4)$. [5 marks]

5. Consider a random walk on the following graph. Let the random walk is in the node i at the



time instant t. Then it can jump to node j at time instant $t+1$ if i and j are connected with an edge. If node i is connected with ℓ number of nodes, then the transition probability from i to any of the ℓ number of adjacent nodes is $\frac{1}{\ell}$.

(a) Determine the probability transition matrix for this random walk. [3 marks]

(b) Check if $\pi = \begin{bmatrix} \frac{1}{8} & \frac{3}{16} & \frac{3}{16} & \frac{1}{8} & \frac{3}{16} & \frac{3}{16} \end{bmatrix}$ is the stationary distribution of this random walk. [2 marks]

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(c) Give an outline of the approach based on inverse eigenvalue problem to generate a Markov chain on this graph with the stationary distribution as π. [5 marks]

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