Public Choice 2

Example

- Two individuals (1, 2) and three alternatives (x, y, z)
- No individual is ever indifferent between any two alternatives
- $xP_i y \Rightarrow$ i prefers x to y
- Individual i's preference ordering is assumed to be complete and transitive
- Given 3 alternatives, there are only six ways individual 1 can order the alternatives

Example

- He can prefer x to y to z, or he can prefer x to z to y, and so on…
- Same for individual 2
- Hence, there are exactly (6 x 6 =)36 different constellations of individual preferences, or ***preference profiles***, possible in this small society
- Each cell in this table shows a possible pair of rankings of the three alternatives by individuals 1 and 2

Preference Profiles

	Individuals											
Choices	1	2	1	2	1	2	1	2	1	2	1	2
1st	x	x	x	x	x	y	x	y	x	z	x	z
2nd	y	y	y	z	y	x	y	z	y	x	y	y
3rd	z	z	z	y	z	z	z	x	z	y	z	x
1st	x	x	x	x	x	y	x	y	x	z	x	z
2nd	z	y	z	z	z	x	z	z	z	x	z	y
3rd	y	z	y	y	y	z	y	x	y	y	y	x
1st	y	x	y	x	y	y	y	y	y	z	y	z
2nd	x	y	x	z	x	x	x	z	x	x	x	y
3rd	z	z	z	y	z	z	z	x	z	y	z	x
1st	y	x	y	x	y	y	y	y	y	z	y	z
2nd	z	y	z	z	z	z	x	z	z	x	z	y
3rd	x	x	x	y	x	x	z	x	x	y	x	x
1st	z	x	z	x	z	y	z	y	z	z	z	z
2nd	x	y	x	z	x	x	x	z	x	x	x	y
3rd	y	z	y	y	y	z	y	x	y	y	y	x
1st	z	x	z	x	z	y	z	y	z	z	z	z
2nd	y	y	y	z	y	x	y	z	y	x	y	y
3rd	x	z	x	y	x	z	x	x	x	y	x	x

Arrow social welfare function

- Our concern here is whether or not there is a foolproof rule to transform any cell in the table into a social preference relation.

- Such a rule is called an ***Arrow social welfare function***.

- An Arrow social welfare function takes preference profiles and produces social preferences.

- Let R stand for a social preference relation, so xRy means x is socially at least as good as y.

- P is the corresponding strict social preference relation: xPy means x is socially preferred to y; i.e., xRy and ***not*** yRx
- I is the social indifference relation: xIy means x and y are socially indifferent; i.e., xRy **and** yRx

Characteristics

- **Completeness and transitivity**: either *xRy* or *yRx* must hold and *xRy* and *yRz* must imply *xRz*.
  - Majority voting gives non-transitive social rankings.

- **Universality**: An Arrow social welfare function should work no matter what individual preferences happen to be

- **Pareto Consistency**: For any pair of alternatives *x* and *y*, if both individuals prefer *x* to *y*, *x* must be socially preferred to *y*.

- **Non-dictatorship**: if xP_iy implies *xPy* for all *x* and *y*, irrespective of *Pj*, then, *i* is said to be a dictator (his wishes prevail).

Characteristics

- **Independence of irrelevant alternatives**: If people's feelings change about some set of irrelevant alternatives, but do not change about the pair of alternatives x and y, then an Arrow social welfare function must preserve the social ordering of x and y.

- The social preference between x and y must be independent of individual orderings on other pairs of alternatives.

Applying the requirements

- Let's apply the Pareto principal first. It requires that a collective choice rue must respect unanimous opinion – if both 1 and 2 prefer one alternative to another, then that should also be followed by the society

- For example, consider this cell

	Individuals	
Choices	1	2
1st	x	x
2nd	y	z
3rd	z	y

- Pareto requirement says x must be socially preferred to y and x must be socially preferred to z. That is, we must have xPy and xPz.

- Application of Pareto consistency over the entirety of the previous table gives rise to this new table –

Applying the requirements

xPy xPz yPz	xPy xPz	xPz yPz	yPz	xPy	
xPy xPz	xPy xPz zPy	xPz		xPy zPy	zPy
xPz yPz	xPz	xPz yPx yPz	yPx yPz		yPx
yPz		yPx yPz	yPx yPz zPx	zPx	yPx zPx
xPy	xPy zPy		zPx	xPy zPx zPy	zPx zPy
	zPy	yPx	yPx zPx	zPx zPy	yPx zPx zPy

Applying the requirements

- Now let's apply the condition of **Independence of irrelevant alternatives**

- Suppose that when person 1 prefers *x* to *y* to *z* and person 2 prefers *y* to *x* to *z*, an Arrow social welfare function (or, a collective choice rule) declares *x* is socially preferred to *y*, or xPy.

- Then independence requires that xPy hold whenever xP_1y and yP_2x no matter how 1 and 2 rank alternative *z*.

Applying the requirements

- Similarly, if yPx (or xIy) holds when person 1 prefers y to x to z and person 2 prefers x to y to z, then yPx (or xIy) must hold whenever

  $$yP_1x \text{ and } xP_2y$$

- In short, the independence requirement forces an Arrow social welfare function to give rise to social preferences that agree over certain preference profiles

- Independence requires that all the cells in the table where xP_1y and yP_2x must yield identical social rankings of x and y.
- Similarly, all the cells where yP_1x and xP_2y must yield identical social rankings of x and y.

Applying the requirements

- Let's consider the cells again such that we can indicate them in terms of the following preferences –

$$\text{if } xPy\,(xP_1y \text{ and } yP_2x),\text{ cell is marked with} \times$$
$$\text{if } yPx\,(yP_1x \text{ and } xP_2y),\text{ cell is marked with } 0$$

- Similarly we can indicate the social rankings over *x*-*z* and *y*-*z*

Applying the requirements

- The crossed cells all produce the same *x*-*y* social rankings. The circled cells all produce the same *x*-*y* social rankings (which need not be the same as in the crossed cells).

		X	X		X
		X	X		X
0	0			0	
0	0			0	
		X	X		X
0	0			0	

Applying the requirements

- The crossed cells all produce the same *x*-*z* social rankings. The circled cells all produce the same *x*-*z* social rankings (which need not be the same as in the crossed cells).

			X	X	X
			X	X	X
			X	X	X
0	0	0			
0	0	0			
0	0	0			

Applying the requirements

- The crossed cells all produce the same y-z social rankings. The circled cells all produce the same y-z social rankings (which need not be the same as in the crossed cells).

	X			X	X
0		0	0		
	X			X	X
	X			X	X
0		0	0		
0		0	0		

Arrow's Impossibility Theorem

- Does there exist a foolproof rule for discovering, or for defining, social preferences?

- Arrow showed that, if foolproof means consistent with the five requirements above, the answer is No.

- **Statement**: Any Arrow social welfare function which is consistent with the requirements of (1) completeness and transitivity, (2) universality, (3) Pareto consistency, and (5) independence of irrelevant alternatives, makes one person a dictator. Therefore, there is no rule which satisfies all five requirements.

Arrow's Impossibility Theorem

- We start by looking at the preference profile of the first row, second column cell of the first table

- For these preferences Pareto consistency requires xPy and xPz

- There are three and only three complete and transitive social preference orderings which satisfy xPy and xPz.

1. xPy, xPz and yPz
2. xPy, xPz and zPy
3. xPy, xPz and yIz

Case 1: *yPz*

- If *yPz* holds in the first row, second column cell, then independence requires that y be socially preferred to z whenever individual preferences about y and z are the same as they are in that cell.

- Therefore *yPz* holds in all the cells indicated in

	1. yPz			2. yPz	yPz
	yPz			yPz	yPz
	yPz			yPz	yPz

Case 1: yPz

- Now consider the first row, fifth column cell, or cell number 2 in the last table.
- Pareto consistency requires that xPy here, but xPy and yPz implies xPz, by transitivity.
- So in this cell we must also have xPz.
- But if xPz holds in cell number 2, then independence requires that x be socially preferred to z whenever individual preferences about x and z are the same as they are in that cell.
- Therefore, xPz holds in all the cells indicated in the following table

Case 1: *yPz*

			xPz	2. xPz	xPz
			xPz	xPz	3. xPz
			xPz	xPz	xPz

Case 1: *yPz*

- Now we have *xPz* in cell 3.
- We again invoke Pareto consistency and transitivity to conclude that *xPy* must hold in cell 3 as well.
- But this allows us to use independence again to fill in eight more bits of information
- When done, the result is the pattern of social preferences will be like -

Case 1: *yPz*

x y z	x y z	x y z	x y z	x y z	x y z
x z y	x z y	x z y	x z y	x z y	x z y
y x z	y x z	y x z	y x z	y x z	y x z
y z x	y z x	y z x	y z x	y z x	y z x
z x y	z x y	z x y	z x y	z x y	z x y
z y x	z y x	z y x	z y x	z y x	z y x

Case 1: *yPz*

- But the social preferences shown in the last table are identical to person 1's preferences.
- Therefore, in Case 1, 1 is a dictator.
- He gets his way, no matter how 2 feels.
- When you calculate the other 2 cases you will find
- Either 1 or 2 is a dictator