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An inner product is a mapping $\langle ., . \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{K}$ which satisfies the following conditions.

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Then $\langle f + g, h \rangle = \int_a^b \big(f(x) + g(x)\big)\, h(x) dx$

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$$= \langle f, h \rangle + \langle g, h \rangle.$$

- Let $f : [a, b]$ be integrable and $f(x) \geq 0$. Let f is continuous at $x = c$ and $f(c) > 0$ (resp. $f(c) < 0$). Then $\int_a^b f(x)dx > 0$ (resp. $\int_a^b f(x)dx < 0$).

Sol: Case I. Let $a < c < b$.

f is continuous at $x = c$. Take $\epsilon = \frac{f(c)}{2}$ then there exists $\delta > 0$ such that

for each $x \in (c - \delta, c + \delta) \implies |f(x) - f(c)| < \frac{(f(c))}{2}$.

Then $-\frac{(f(c))}{2} < f(x) - f(c) < \frac{(f(c))}{2}$ for all $x \in (c - \delta, c + \delta)$.

Therefore $f(x) > \frac{(f(c))}{2} > 0$ for all $x \in (c - \delta, c + \delta)$.

$\int_a^b f^2(x)dx = \int_a^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^{b} f(x)dx > 0$
as $\int_{c-\delta}^{c+\delta} f^2(x)dx > 0$.

Case II. $c = a$

When $c = a$, we have a neighborhood $(a, a+\delta)$ where f is positive.

$\int_a^b f^2(x)dx = \int_a^{a+\delta} f(x)dx + \int_b^{a+\delta} f(x)dx > 0$

Case-III. $c = b$. Same as Case II.

To show $\langle f, f\rangle = \int_a^b f(x)f(x)dx = 0 \implies f \equiv 0$.

Here f^2 is nonnegative function and continuous. If f is not identically zero, then there exists $c \in [a, b]$ such that $f(c) \neq 0$. Therefore $f^2(c) > 0$. Using above result we have $\int_a^b f(x)f(x)dx > 0$, a contradiction. Hence $f \equiv 0$.

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Hence $\langle f, g \rangle = \int_a^b f(x)g(x)dx$ is not an inner product on $C(\mathbb{R})$.

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We now show that $\langle , \rangle$ is an IP on $\mathbb{V}$.

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Then $x + y = \sum_{i=1}^{k} (a_i + b_i) u_i$. Therefore,

$$\begin{aligned} \langle x + y, z \rangle &= \sum_{i=1}^{k} (a_i + b_i) \overline{c_i} \\ &= \sum_{i=1}^{k} (a_i \overline{c_i} + b_i \overline{c_i}) \\ &= \langle x, z \rangle + \langle y, z \rangle. \end{aligned}$$

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Define $\langle x, y\rangle = \sum\limits_{j=1}^{m} \sum\limits_{i=1}^{k} a_i \overline{b_j} f(u_{\alpha_i}, u_{\beta_j})$.

Check $\langle x, y\rangle = \sum_{j=1}^{m}\sum_{i=1}^{k} a_i\overline{b_j} f(u_{\alpha_i}, u_{\beta_j})$ is an inner product on $\mathbb{V}$.

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$$= \sum_{i=1}^{k} |a_i|^2 \text{ (using the definition of } f\text{)}$$

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$z = c_1 u_{\gamma_1} + \cdots + c_n u_{\gamma_n}$ where $c_l \in \mathbb{K}$ for $l = 1, \ldots, n$.

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- A vectors space together with an inner product is called an inner product space.

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Proof: 1.
$$\begin{aligned} \langle 0, u \rangle &= \langle u - u, u \rangle \\ &= \langle u, u \rangle + \langle -u, u \rangle \\ &= \langle u, u \rangle - \langle u, u \rangle \\ &= 0. \end{aligned}$$

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3. Suppose $u, v, w \in \mathbb{V}$. Then

$$\begin{aligned}\langle u, v + w\rangle &= \overline{\langle v + w, u\rangle} \\ &= \overline{\langle v, u\rangle + \langle w, u\rangle} \\ &= \overline{\langle v, u\rangle} + \overline{\langle w, u\rangle} \\ &= \langle u, v\rangle + \langle u, w\rangle.\end{aligned}$$

4. Suppose $\lambda \in \mathbb{K}$ and $u, v \in \mathbb{V}$. Then

$$\begin{aligned}
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- **[Theorem:]** Let $(\mathbb{V}, \langle ., . \rangle)$ be an inner product space. Let $S = \{\alpha_1, \ldots, \alpha_k\}$ be an orthogonal subset of $\mathbb{V}$ and let $\alpha_i \neq 0$ for $i = 1, \ldots, k$. Then S is linearly independent.

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$\langle c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k, \alpha_i \rangle = \langle 0, \alpha_i \rangle$

$\sum_{j=1}^{k} c_i \langle \alpha_j, \alpha_i \rangle = 0.$

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$c_i = 0$. This is true for $i = 1, \dots, k$.

Hence $S = \{\alpha_1, \dots, \alpha_k\}$ is LI.

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Take $u = (1, 0)$ and $v = (1, 1)$. These two vectors are linearly independent but not orthogonal.

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The next immediate question is that can we construct an orthogonal set from a finite linearly independent set?

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The next immediate question is that can we construct an orthogonal set from a finite linearly independent set?

The answer is yes. Gram Schimdt supplied a process to construct an orthogonal set from a linearly independent finite set.

- Let $(\mathbb{V}, \langle ., . \rangle)$ be an inner product space. Let $u_1, u_2, \ldots, u_k \in \mathbb{V}$ be LI.

How to construct orthogonal vectors $v_1, \ldots, v_k$ using $u_1, u_2, \ldots, u_k$?

Step 1. Take $v_1 = u_1$.

Step 2. We now construct v_2 using v_1 and u_2. Take $v_2 = u_2 + cv_1$ where $c \in \mathbb{K}$. We have to calculate the value of c such that v_2 is orthogonal to v_1. That means

$\langle v_2, v_1 \rangle = \langle u_2, v_1 \rangle + c\langle v_1, v_1 \rangle$

$0 = \langle u_2, v_1 \rangle + c\langle v_1, v_1 \rangle.$

$c = -\frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle}.$

Therefore $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$ is a vector which is perpendicular to v_1. You can easily check that $v_2 \neq 0$, otherwise u_2 is scalar multiple of u_1 which is not possible.

Step 3. We now construct v_3 using v_1, v_2, u_3.

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Take $v_3 = u_3 + c_1 v_2 + c_2 v_1$ is an element in $\mathbb{V}$ where $c_1, c_2 \in \mathbb{K}$. We have to calculate the values of c_1, c_2 such that v_3 is orthogonal to v_1 and v_2. That means

$$\langle v_3, v_1 \rangle = \langle u_3, v_1 \rangle + c_1 \langle v_2, v_1 \rangle + c_2 \langle v_1, v_1 \rangle$$

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Therefore $v_3 = u_3 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1$ is a vector which is perpendicular to v_1 and v_2. You can easily check that $v_3 \neq 0$, otherwise u_3 is a linear combination of u_1 and u_2.

Step 4. Similar way you can calculate v_4 using u_4, v_1, v_2 and v_3.

Step k. $v_k = u_k - \frac{\langle u_k, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \cdots - \frac{\langle u_k, v_{k-1} \rangle}{\langle v_{k-1}, v_{k-1} \rangle} v_{k-1}$. You can easily check that $v_k \neq 0$.

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$v_1 = u_1, v_2 = u_2, \ldots, v_k = u_k$. To calculate v_{k+1} apply above technique on $v_1, \ldots, v_k$ and u_{k+1}.

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Proof: Since $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a linearly independent set, $\alpha_i \neq 0$ for $i = 1, \ldots, n$.

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$\beta_1 = \alpha_1$.

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$\vdots$

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It is clear from the above discussion that $\beta_1, \beta_2, \ldots, \beta_n$ are orthogonal.

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- **[Example:]** Let $(\mathbb{R}^3, \langle ., . \rangle)$ be an inner product space and let $\langle x, y \rangle = \sum_{i=1}^{3} x_i y_i$. Let $u_1 = (0, 1, 2)$, $u_2 = (1, 1, 2)$ and $u_3 = (1, 0, 1)$. Then find orthogonal vectors v_1, v_2, v_3 using u_1, u_2, u_3.

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Sol: $v_1 = u_1 = (0,1,2)$.

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$v_3 = (1,0,1) - (1,0,0) - \frac{2}{5}(0,1,2) = (0, -\frac{2}{5}, \frac{1}{5})$.

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Proof: Using Gram Schimdt process we have an orthogonal set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ such that $\mathsf{ls}(\{\beta_1, \beta_2, \ldots, \beta_n\}) = \mathsf{ls}(\{\alpha_1, \alpha_2, \ldots, \alpha_n\})$.

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- [**Theorem:**] Every non-trivial finite dimensional inner product space has an orthonormal basis.

- **[Theorem:]**Let $(\mathbb{V}, \langle ., . \rangle)$ be an inner product space. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an orthonormal basis. Then for each $x \in \mathbb{V}$ we have

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$$
\begin{aligned}
\langle \alpha x + \beta y, v \rangle &= \langle \alpha x, v \rangle + \langle \beta y, v \rangle \\
&= \alpha \langle x, v \rangle + \beta \langle y, v \rangle \\
&= 0 + 0 = 0 \text{ for all } v \in S.
\end{aligned}
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Hence $S^\perp$ is a subspace of $\mathbb{V}$.

- **[Theorem:]** Let $(\mathbb{V}, \langle ., . \rangle)$ be a **finite dimensional** inner product space. Let $\mathbb{W}$ be a subspace of $\mathbb{V}$. Then $\mathbb{V} = \mathbb{W} \oplus \mathbb{W}^{\perp}$.

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Hence $x \in \operatorname{ls}\{u_{k+1}, \ldots, u_n\}$. Therefore $\mathbb{W}^\perp = \operatorname{ls}\{u_{k+1}, \ldots, u_n\}$. It is clear that $\mathbb{W} \cap \mathbb{W}^\perp = \{0\}$. Hence $\mathbb{V} = \mathbb{W} \oplus \mathbb{W}^\perp$.

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- Let $(\mathbb{V}, \langle ., . \rangle)$ be a finite dimensional IPS. Let $\mathbb{W} \subseteq \mathbb{V}$ be a subspace. For each $x \in \mathbb{V}$ there exists unique $x_1 \in \mathbb{W}$ and $x_2 \in \mathbb{W}^\perp$ such that $x = x_1 + x_2$. The vector x_1 is called the **orthogonal projection** of x into W.

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- Let $(\mathbb{V}, \langle ., . \rangle)$ be a finite dimensional inner product space. Let $\mathbb{W}$ be a subspace of $\mathbb{V}$. Let $\{\alpha_1, \ldots, \alpha_k\}$ be an orthogonal basis of $\mathbb{W}$. Then the orthogonal projection of any vector $x \in \mathbb{V}$ into $\mathbb{W}$ is $\sum\limits_{i=1}^{k} \frac{\langle x, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$.

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A vector space to gather with a norm $||,||$ is called a **normed linear space.**

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   3. $||x+y|| = \sum_{i=1}^{n} |x_i+y_i| \leq \sum_{i=1}^{n}(|x_i|+|y_i|) = \sum_{i=1}^{n} |x_i|+\sum_{i=1}^{n} |y_i| = ||x||+||y||$ for all $x, y \in \mathbb{R}^n$.

- **[Cauchy Schwarz Inequality]** Let $(\mathbb{V}, \langle ., . \rangle)$ be an IPS. Let $x, y \in \mathcal{V}$. Then $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$.

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If $y \neq 0$, take $x - ty$ where $t \in \mathbb{C}$. Then

$0 \leq \langle x - ty, x - ty \rangle$

$= \langle x, x \rangle + \langle x, -ty \rangle + \langle -ty, x \rangle + \langle -ty, -ty \rangle$

$= \langle x, x \rangle - \bar{t}\langle x, y \rangle - t\langle y, x \rangle + |t|^2 \langle y, y \rangle$

Put $t = \frac{\langle x, y \rangle}{\langle y, y \rangle}$.

$$\langle x, x\rangle - \frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}\langle x, y\rangle - \frac{\langle x,y\rangle}{\langle y,y\rangle}\langle y, x\rangle + |t|^2\langle y, y\rangle$$

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The first inequality which we have used in this proof is $0 \geq \langle x-ty, x-ty\rangle$. If $|\langle x,y\rangle|^2 = \langle x,x\rangle\langle y,y\rangle$ hold, then $\langle x-ty, x-ty\rangle = 0$. This says that $x = ty$. Then x and y are linearly dependent.

- Let $(\mathbb{V}, \langle ., . \rangle)$ be an inner product space. Let $x \in \mathbb{V}$. Then we can easily check that $||x|| = (\langle x, x \rangle)^{1/2}$ is a norm on $\mathbb{V}$