From two equations in (5.5), we can eliminate single parameter t, to obtain a relation of the type

$$G(c_1, c_2) = 0. \qquad (5.6)$$

Note that GI (5.3) contains (5.6) for $F \equiv G$, because F is an arbitrary function. Hence, substituting for arbitrary constants c_1, c_2 from equations (5.2b) into (5.6), we obtain our desired integral surface as follows

$$G\big(u(x,y,z), v(x,y,z)\big) = 0. \qquad (5.7)$$

Note that the surface (5.7) contains given curve because of (5.5). Further, this must be integral surface of PDE, as this is derived from GI (5.2a). In fact, it is a particular solution of PDE.

Example: Find the integral surface of the PDE $(x-y)p + (y-x-z)q = z$ through the circle $x^2 + y^2 = 1, z = 1$.

Solution: Lagrange's AE: $\frac{\mathrm{d}x}{x-y} = \frac{\mathrm{d}y}{y-x-z} = \frac{\mathrm{d}z}{z} \Rightarrow$ each ratio= $\frac{\mathrm{d}x+\mathrm{d}y+\mathrm{d}z}{0}$

Hence, we get $\mathrm{d}(x+y+z) = 0 \Rightarrow x+y+z = c_1 = u \Rightarrow x+z = c1 - y$

Next, considering 2nd & 3rd fraction, and using above integral curve, we have

$$\frac{\mathrm{d}y}{y-x-z} = \frac{\mathrm{d}z}{z} \Rightarrow \frac{\mathrm{d}y}{2y-c_1} = \frac{\mathrm{d}z}{z} \Rightarrow \frac{2y-c_1}{z^2} = c_2$$

Substituting for c_1 from 1st integral curve, we get 2nd integral curve

$\frac{y-x-z}{z^2} = c_2$= v

Now, from above discussion, it is clear that for Cauchy problem, we are to find a fixed relation between c_1, c_2 using Cauchy data.

We see that given curve : $x^2 + y^2 = 1$. Putting $z = 1$ in two integral curves,

$$2y = c_1 + c_2 \Rightarrow y = \frac{c1+c2}{2}, y - x = c_2 + 1 \Rightarrow 2x = c_1 - c_2 - 2 \Rightarrow x = \frac{c_1 - c_2 - 2}{2}$$

Substituting for x, y from above into $x^2 + y^2 = 1$, the fixed relation is

$$(c_1 - c_2 - 2)^2 + (c_1 + c_2)^2 = 4 \Rightarrow {c_1}^2 + {c_2}^2 - 2(c_1 + c_2) = 0$$

Finally, we substitute for c_1, c_2 from the integral curves into above relation:

$$z^4(x+y+z)^2 + (y-x-z)^2 - 2z^2[z^2(x+y+z) - (y-x-z)] = 0$$

Remark: In above example, given curve is given in non-parametric form. See the example below, where curve is given in parametric form. This form was taken for theoretical discussion above. Furthermore, above non-parametric equation may be put in parametric form as follows:

$$\Gamma: x(t) = \cos t\,, y(t) = \sin t\,, z(t) = 1, 0 \le t \le 2\pi$$

Thus, from the two integral curves, we must have

$$c_1 = 1 + \sin t + \cos t\,, c_2 = \sin t - \cos t - 1$$

$$\Rightarrow 2\sin t = (c_1 + c_2), 2\cos t = c_1 - c_2 - 2$$

Eliminating t from above equations, we get

$(c_1 + c_2)^2 + (c_1 - c_2 - 2)^2 = 4$, which is the same fixed relation between c_1, c_2, and so substituting for them from integral curves, we get same surface.

Example: Solve the PDE $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ with initial data : given straight line $x_0(s) = 1, y_0(s) = 0, z_0(s) = s$

Solution: AE: $\frac{\mathrm{d}x}{2xy-1} = \frac{\mathrm{d}y}{z-2x^2} = \frac{\mathrm{d}z}{2(x-yz)}$

$\Rightarrow$ each ratio$= \frac{z\mathrm{d}x + \mathrm{d}y + x\mathrm{d}z}{0} = \frac{2x\mathrm{d}x + 2y\mathrm{d}y + \mathrm{d}z}{0}$

Hence, we get $\mathrm{d}(xz+y) = 0\,, \mathrm{d}(x^2+y^2+z) = 0$

$$\Rightarrow xz + y = c_1, x^2 + y^2 + z = c_2$$

Hence, we may take GI in the form

$$x^2 + y^2 + z = \phi(xz+y)$$

We are to find specified form of ϕ such that above integral surface passes through the line $x_0(s) = 1, y_0(s) = 0, z_0(s) = s$. Substituting for x, y, z from line to above surface, we get

$$1 + 0 + s = \phi(1.s + 0) \Rightarrow \phi(s) = 1 + s$$

Thus, desired solution is $x^2 + y^2 + z = 1 + (xz + y)$

Before going to next section, I would like to show an example of finding particular solution for given functional relation between the parameters from CI, and also of finding GI and SI.

Example: Recall the example, discussed in Sec 2 [see 2nd example below (2.6)], where eliminating a, b from $(x-a)^2+(y-b)^2+z^2=25$, we get PDE $z^2(1+p^2+q^2)=25$, so that former is CI of PDE.

Now, write down GI by taking $b=\phi(a)$. Then choosing $\phi(a)=a$, write down particular solution. Finally find SI, if exists.

Solution: Given CI may be expressed as

$$F(x,y,z,a,b) \equiv (x-a)^2+(y-b)^2+z^2-25=0$$

Taking $b=\phi(a)$, ϕ is arbitrary function, we have one parameter family of surface:

$$F\big(x,y,z,a,\phi(a)\big) \equiv (x-a)^2+\big(y-\phi(a)\big)^2+z^2-25=0$$

The GI is the envelope of above surface, which is a-eliminant of above equation and the following equation obtained by partial differentiation of F w.r.t. a

$$F_a \equiv x-a+\big(y-\phi(a)\big)\phi'(a)=0$$

Next choosing $\phi(a)=a$, we get two equations from above two equations:

$$(x-a)^2+(y-a)^2+z^2-25=0, x-a+(y-a)=0 \Longrightarrow 2a=x+y$$

Hence, particular solution of PDE for $\phi(a)=a$: $(x-y)^2+2z^2=50$

Next, to see SI, if exists, we differentiate CI w.r.t. a, b:

$$(x-a)=0, (y-b)=0$$

To eliminate a, b from given CI, we put $a=x, b=y$: $z=\pm 5$

This is required SI. It is easy to verify that this is a solution of PDE.

- Orthogonal Surface to a given System of Surfaces

This is an application of the theory of 1st order linear PDE to geometry. Here we will determine the system of surfaces to a given system of surfaces. Suppose given system is a one-parameter family of surfaces

$$f(x,y,z) = c. \tag{6.1}$$

A system of surfaces will be called orthogonal to given system (6.1), if former cuts each of given surfaces at right angles. Suppose the orthogonal surfaces are given by

$$G(x,y,z) \equiv g(x,y) - z = 0. \tag{6.2}$$

From the geometric interpretation of Lagrange's method for 1st order linear PDE, we know that normal to given surfaces (6.1) at any point M(x,y,z) on the surface is the direction given by $(P,Q,R) = \left(f_x, f_y, f_z\right)$, and the normal at common point M on orthogonal surfaces (6.2) is the direction given by $(p,q,-1)$.

Now since two systems (6.1) and (6.2) are orthogonal, two normals must be perpendicular to each other, which transpires in mathematical language:

$$Pp + Qq = R. \tag{6.3}$$

Note that $P = f_x, Q = f_y, R = f_z$ are known, since f is given function. Now, PDE (6.3) is 1st order linear PDE, which can be solved by Lagrange's method.

Conversely, any solution of linear PDE (6.3) must be orthogonal to every surface characterized by given system (6.1), since relation (6.3) simply means that normal to any solution of (6.3) is perpendicular to the normal to that member of the given system (6.1) which passes through the same point.

Hence, orthogonal surface to given system (6.1) is determined by general linear PDE (6.3), so that the surfaces orthogonal to the system (6.1) are the surfaces generated by the integral curves of characteristic equation of PDE (6.3):

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R} \Longrightarrow \frac{\mathrm{d}x}{f_x} = \frac{\mathrm{d}y}{f_y} = \frac{\mathrm{d}z}{f_z}$$

Example: Find the system of surfaces orthogonal to given system of surfaces given by $z(x+y) = c(3z+1)$, c is a free parameter.

Solution: Write $f(x,y,z) \equiv \frac{z(x+y)}{3z+1} = c$, we compute $f_x = \frac{z}{3z+1}, f_y = \frac{z}{3z+1}$, $f_z = \frac{(x+y)}{(3z+1)^2}$, so that orthogonal surfaces are general solutions of following PDE:

$$f_x p + f_y q = f_z \Longrightarrow \frac{z}{3z+1} p + \frac{z}{3z+1} q = \frac{(x+y)}{(3z+1)^2}$$

AE is $\frac{\mathrm{d}x}{z(3z+1)}=\frac{\mathrm{d}y}{z(3z+1)}=\frac{\mathrm{d}z}{(x+y)}$ ⟹each ratio= $\frac{\mathrm{d}x-\mathrm{d}y}{0}\Longrightarrow u\equiv x-y=c_1$ is one integral curve. Also, writing AE as

$\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}y}{1}=\frac{z(3z+1)\mathrm{d}z}{(x+y)}$ ⟹each ratio= $\frac{x\mathrm{d}x+y\mathrm{d}y-z(3z+1)\mathrm{d}z}{0}$

$\Longrightarrow v\equiv x^2+y^2-2z^3-z^2=c_2$, which is our 2nd integral curve.

Hence, orthogonal surfaces are given by the following system

$x^2+y^2-2z^3-z^2=\phi(x-y)$, ϕ is arbitrary function.

**************End of Part A***************

Part B

Note that in Part A, I show some examples of how to verify for a given solution to be CI of given PDE. In this Part, we will discuss method of finding CI by Charpit's method for a given PDE. Before going to details of this method, I'll quote two results, without proof, as these two results will be used in the discussion of Charpit's method.

Result 1: Lagrange's Theorem for 1st order linear PDE of $n\geq 2$ independent variables x_j and one dependent variable $z(x_1,x_2,\dots,x_n)$:

For convenience, for this result using the symbols $p_j\equiv\partial z/\partial x_j$, we write the PDE as follows

$$P_1p_1+P_2p_2+\cdots+P_np_n=R, \qquad (A.1)$$

where P_js, R are functions of $x_1,x_2,\dots,x_n,z$. Then general integral is given by

$$\phi(u_1,u_2,\dots u_n)=0, \qquad (A.2)$$

where $u_j(x_1,x_2,\dots,x_n,z)=c_j, j=1,2,\dots n$ are n independent solutions of following AE

$$\frac{\mathrm{d}x_1}{P_1}=\frac{\mathrm{d}x_2}{P_2}=\cdots=\frac{\mathrm{d}x_n}{P_n}. \qquad (A.3)$$

Result 2: Pfaffian Differential equation

$$P\mathrm{dx}+\mathrm{Qdy}+\mathrm{Rdz}=0, \qquad (A.4)$$