Computational Statistics

R: random expt.

Ω: sample space.

$\mathcal{F}$: σ-algebra

P: probability measure

$$P: \mathcal{F} \longrightarrow \mathbb{R}$$

$$\underset{\in \mathcal{F}}{A} \longrightarrow P_A$$

Collection of events which is "closed" under complementation and countable union/intersection.

$\rightarrow A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

$\rightarrow A_1, A_2, \ldots \in \mathcal{F}$

$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

i) $P(A) \geq 0 \quad \forall A \in \mathcal{F}$

ii) $P(\Omega) = 1$

iii) If $A_1, A_2, \ldots$ are pairwise disjoint

$A_i \cap A_j = \phi$ for $i \neq j$ and $i, j = 1, 2, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Ex: R: Rolling a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F} = 2^{\Omega}$ = power set of $\Omega = \{\phi, \{1\}, \{2\}, \ldots, \{6\},$
$\{1,2\}, \ldots$
$\{2,4,6\}, \ldots \{1,2,3,4,5,6\}\}$

$P : \mathcal{F} \to \mathbb{R}$

$P\{1\} = \frac{1}{2}, \quad P\{2\} = P\{3\} = P\{4\} = P\{5\} = P\{6\} = \frac{1}{10}$

Ex: R: random expt such that $\Omega = [0,1]$

$\mathcal{F}$ = Borel-σ-algebra generated by intervals of the type $(a,b] \subseteq [0,1]$ $\quad \underbrace{a, b \in [0,1]}$ $\quad A_n = (1-\frac{1}{n}, 1]$

$(a_n, b]$

$\bigcap_{n=1}^{\infty} A_n = \{1\}$

$\underbrace{A^c = [0, 1/3) \cup (2/3, 1]}$ for $\boxed{A = [1/3, 2/3]}$

$a = 1/3, \; b = 2/3$

$P: \mathcal{F} \to \mathbb{R}$

$P(a,b] = b-a$ = length of the interval.

<u>Ex:</u> R: Tossing a coin until the first Heads appear.

$P(H) = p$ $\quad 0 \le p \le 1$

$P(T) = 1-p$

$$\Omega = \{\underset{1/2}{H}, \underbrace{TH, TTH, TTTH, \ldots}_{1/2} \ldots\}$$

Assumption: Tosses are independent.

A_i : Heads in i^{th} toss

$$P(TTH) = P(A_1^c \cap A_2^c \cap A_3) = P(A_1^c)\,P(A_2^c)\,P(A_3) \quad \ldots \text{(independence)}$$

$$= q^2 p$$

Conditional probability.

$A, B \subseteq \Omega$ s.t. $P(B) \neq 0$.

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

① **Product rule:**

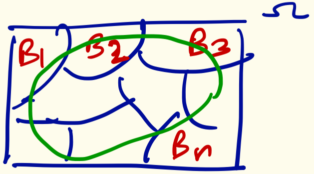
$$P(A\cap B) = P(A|B)\,P(B)$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1 A_2 \cdots A_n)$$

$$= P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 A_2) \cdots P(A_n|A_1 \cdots A_{n-1})$$

② Let $\{B_1, \dots, B_n\}$: disjoint partition of Ω

$B_i \cap B_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{n} B_i = \Omega$



any $A \in \mathcal{F}$, $A = (A\cap B_1) \cup (A\cap B_2) \cdots \cup (A\cap B_n)$

$$P(A) = P(A\cap B_1) + P(A\cap B_2) + \cdots + P(A\cap B_n)$$

$$P(A) = \sum_{i=1}^{n} P(A|B_i)\,P(B_i) \leftarrow \text{Total probability}$$

Baye's rule:

$$P(B_j|A) = \frac{P(A|B_j)\,P(B_j)}{\sum_{i=1}^{n} P(A|B_i)\,P(B_i)}$$

Independence of events.

$A_1, A_2, \cdots$ are said to be independent if for any k and $i_1, i_2, \dots, i_k$

$$P(A_{i_1} A_{i_2} \cdots A_{i_k}) = P(A_{i_1})\,P(A_{i_2}) \cdots P(A_{i_k})$$

Another application of the concept of independence:

Fix an integer $n > 0$. Toss a coin 'n' no. of times

$P(H) = p$, $P(T) = 1-p$

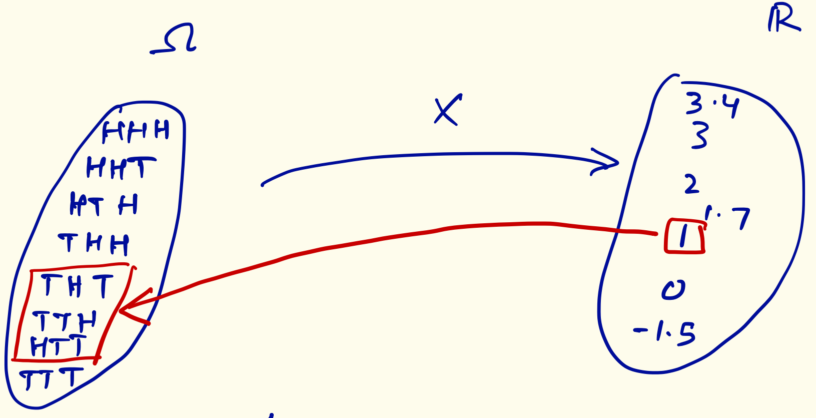
Ω has 2^n elements. $\mathcal{F}$ has $(2)^{2^n}$ elements.

Let $x = x_1 x_2 \dots x_n \in \Omega$

$$P(x) = P(x_1 \dots x_n) = p^{(\text{no. of heads})} \times (1-p)^{(\text{no. of tails})}$$

<u>Random variables:</u> (measurable fn!)

X, a random variable, is a function from Ω to $\mathbb{R}$ / $\mathbb{R}^d$



$$\text{Prob}(X=1) = P(X^{-1}(1))$$