Brief Introduction to Game Theory

Prelude

- “No man is an island” – interdependence

- What is my “guess” about your “choices”?
- Given that, what are my options, responses?
- What would be the outcome of such interaction?
- What if we interact more than once?
- What if my initial “guess” was not fully correct?

- US-China Trade War, Auctions, Voting in Climate Change Discussion, R&D expenditures of firms, pricing of oil by OPEC countries

What is Game Theory?

- Formal way to analyze interaction among a group of rational agents who behave strategically.
- Group, player, interaction, strategic, rational
- Term paper by a group of students
- Focus : Interaction, Rational Agents, Strategy.
- Questions: What (guess, action, outcome), when (simultaneous, sequential, repeated) and how (strategy, nature)?

Rules of the Game

- To understand what is a game, we need to specify four things:

1. Who is playing (agents)?
2. What they are playing with (strategies)?
3. When they are playing?
4. How much they stand to gain/lose (payoffs)?

- Common knowledge

- Forms of representation :

1. Extensive: pictorial depiction of the rules
2. Normal (strategic)

Example 1

- Tennis Game between Nadal and Federar

- Both can smash aces, play long-shots, run towards the net, shoot a volley and so on…

- The most important general principle of such situations - should not do the same thing all the time or systematically – prevent guessing. Else, the opponent will understand your next move and beat you.

- This general idea of "***mixing one's plays***" is well known.

- In actual practice just how does one mix one's plays? What happens when a third possibility (the lob) is introduced?

Example 2

- Grade race. Every one gives more effort with the same result.
- Resultant game **Prisoners' Dilemma**.
- Each professor can make his course look good or attractive by grading it slightly more liberally! The only result is rampant grade inflation. Everyone looks equally good.
- People often think that in every game there must be a winner and a loser.
- In prisoners' dilemma type situations, **both or all players can come out losers**.

Example 3

- There are **win-win games**, too.
- International trade is an example; when each country produces more of what it can do relatively best, all share in the fruits of this international division of labor.
- **But successful bargaining** about the division of the pie is needed if the full potential of trade is to be realized.
- The same applies to many other bargaining situations.

Example 4

- Flat tyre story.

- The story has two important strategic lessons for future party goers. The first is to recognize that the professor may be an intelligent game player. He may suspect some trickery on the part of the students and may use some device to catch them. Given their excuse, the question was the likeliest such device.

- They should have foreseen it and prepared their answer in advance.

- This idea that **one should look ahead to future moves in the game and then reason backward to calculate one's best current action** is a very general principle of strategy.

- Focal point. Convergence of expectations.

Example 4 (Continued)

- By making an advance commitment to the "no excuses" strategy, the professor avoids the temptation to give in to all.
- He must find some way to make a refusal firm and credible.
- The simplest way is to hide behind an administrative procedure or university-wide policy.
- **Commitments, and related strategies, such as threats and promise**.

More examples

- Hawk –chicken.
- Brinkmanship.
- Dating Game

Extensive Form

- **Game Tree**

1. Root
2. Branches
3. Decision Node, Terminal Node (Payoffs)
4. Information set, information partition

- **Consistency Conditions**

1. Single Starting Point
2. No Cycle
3. One Way

- **Predecessor**

1. A node cannot be a predecessor of itself
2. Predecessor's predecessor is also a predecessor
3. Unambiguous sequential ranking
4. Common predecessor

Implications

1. Common predecessor $\Rightarrow$ There cannot exist two roots (α and β are two roots which neither precedes each other , then there must exist one node δ that must precede both α and β)

2. Remember that any node cannot precede itself. Moreover, if α precedes β, and β precedes δ, then α must precede $\delta \Rightarrow$ there cannot by a cycle.

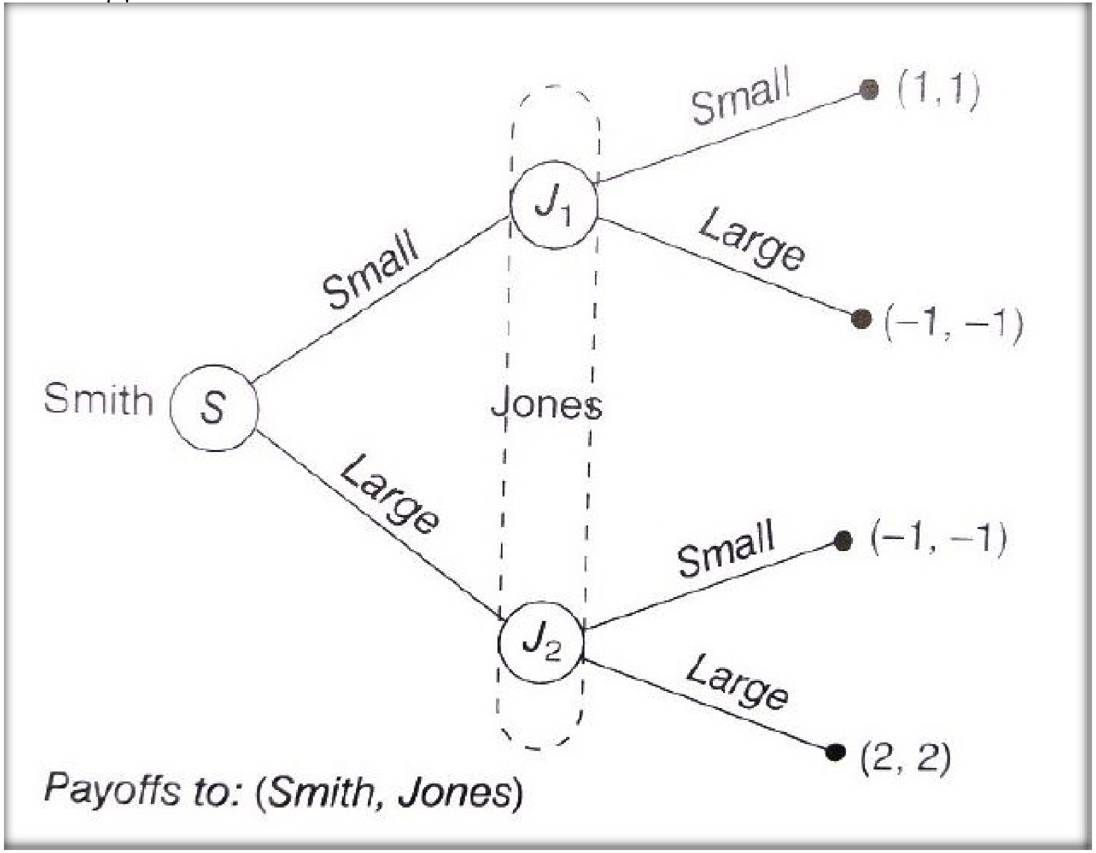
3. Taking all these plus the condition that the nodes can be unambiguously ranked in there occurrence, then there cannot exist more than one branch that may lead to the same node.

4. Hence, from wherever we may start from, we will reach the root and that too by only one way.

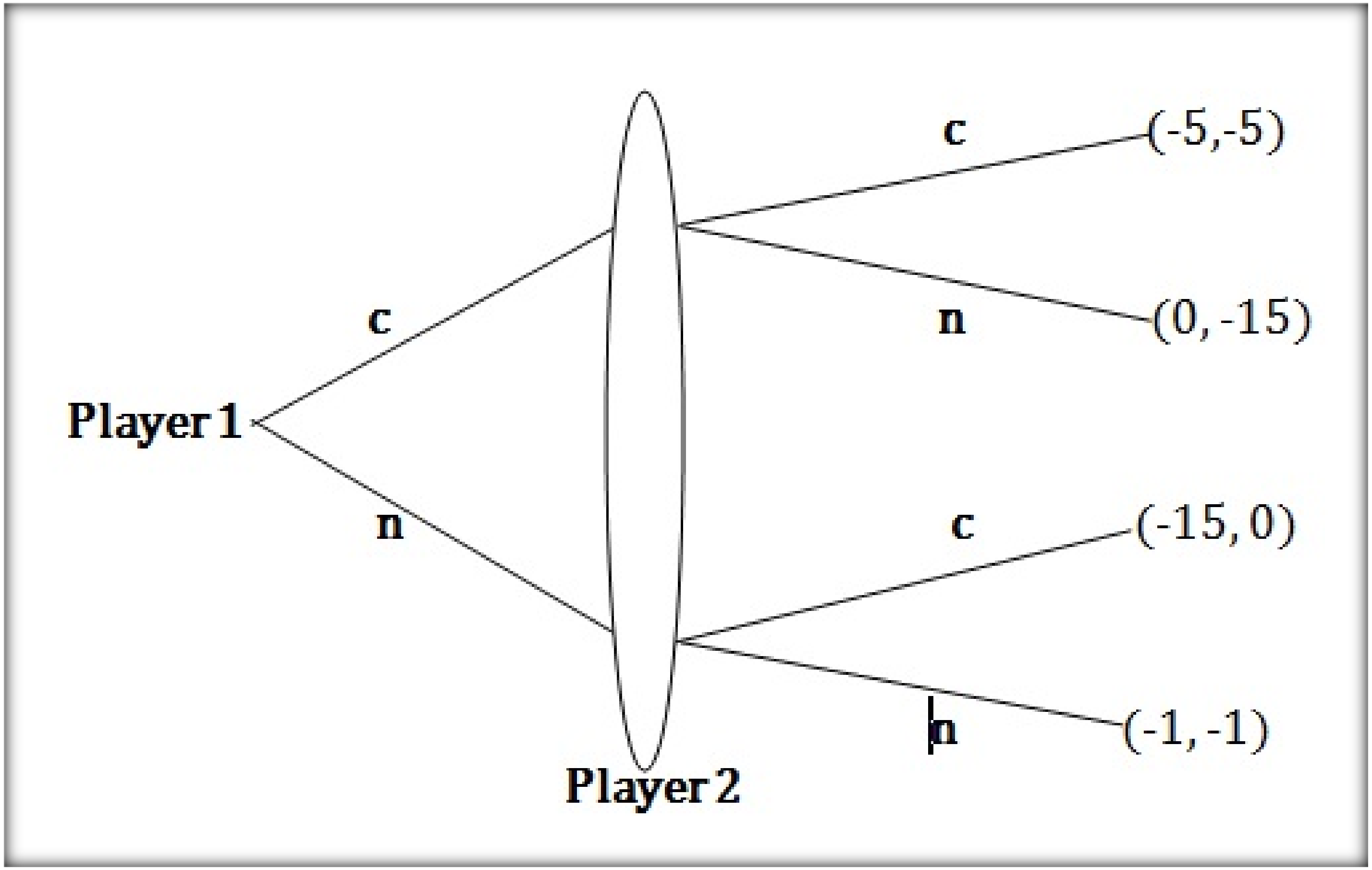
- Chance node: say nature plays a role (good monsoon, bad monsoon)

Information

- **Information set:** Player *i*'s information set at any particular point of the game is the set of different nodes in the game tree that he/she knows might be the actual node but between which he can't distinguish by direct observation
- **Information partition:** collection of information sets available to both players at each stage of the game



Prisoner's Dilemma (Extensive)



Prisoner's Dilemma (Strategic)

P1 \ P2	C	N
C	-5, -5	0, -15
N	-15, 0	-1, -1

Dominance

- Definition: Strategy $S_i^{'}$ **strictly dominates** all other strategies of the i^{th} player if the payoff attached to such strategy is strictly greater than those attached to any other strategy of the i^{th} player's strategy set irrespective of what other players in the game do.

$$\Pi_1(S_1^{'}, S_2^{*}) > \Pi_1(S_1^{''}, S_2^{*})$$
$$\Pi_1(S_1^{'}, S_2^{\#}) > \Pi_1(S_1^{''}, S_2^{\#})$$

- Strategy $S_i^{'}$ **weakly dominates** another strategy $S_i^{''}$ if the payoff attached to such strategy is strictly greater for some strategies of the other player and at least as good as for the rest.

$$\Pi_1(S_1^{'}, S_2) > \Pi_1(S_1^{''}, S_2)$$
$$\Pi_1(S_1^{'}, S_{-2}) \geq \Pi_1(S_1^{''}, S_{-2})$$

Solution

- We do it by **Iterated Elimination of Dominated strategies** (IEDS or IESDS).

$$\Pi_1(S_1^{'}, S_2^{*}) > \Pi_1(S_1^{''}, S_2^{*})$$
$$\Pi_1(S_1^{'}, S_2^{\#}) > \Pi_1(S_1^{''}, S_2^{\#})$$

- Here we have $S_i^{'}$ strictly dominating $S_i^{''}$
- Similarly we can have the case where $S_i^{'}$ weakly dominates $S_i^{''}$
- In both the cases we eliminate $S_i^{''}$ which is the Dominated Strategy.
- Disadvantages: Layers of Rationality, Order of Elimination, Non-existence.

Problem

P2 / P1	H	M	L
H	6, 6	2, 10	2, 8
M	10, 2	5, 5	2, 8
L	8, 2	8, 2	4, 4

P2 / P1	M	L
M	5, 5	2, 8
L	8, 2	4, 4

Nash Equilibrium (NE)

- Let $s_i \in S_i$ for player i whereas s_{-i} is a strategy vector by all players other than player i.

- Best Response: A strategy s_i^* is a best response to a strategy vector s_{-i}^* of the other players if

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*) \ \forall \ s_i$$

- NE: The strategy vector $s^* = \{s_1^*, s_2^*, \ldots, s_N^*\}$ is a NE if

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*) \ \forall \ s_i \ and \ \forall \ i$$

- Requirements: (1) each player plays the best response and (2) such best response strategy is stable such that there is no incentive for departure from such action.

NE (Continued)

- Existence: Relation between IEDS solution and NE
- Uniqueness: Multiple – which one is better?
- Let's solve the PD game! NE and IEDS are same.

P1 \ P2	C	N
C	-5, -5	0, -15
N	-15, 0	-1, -1

Game 2

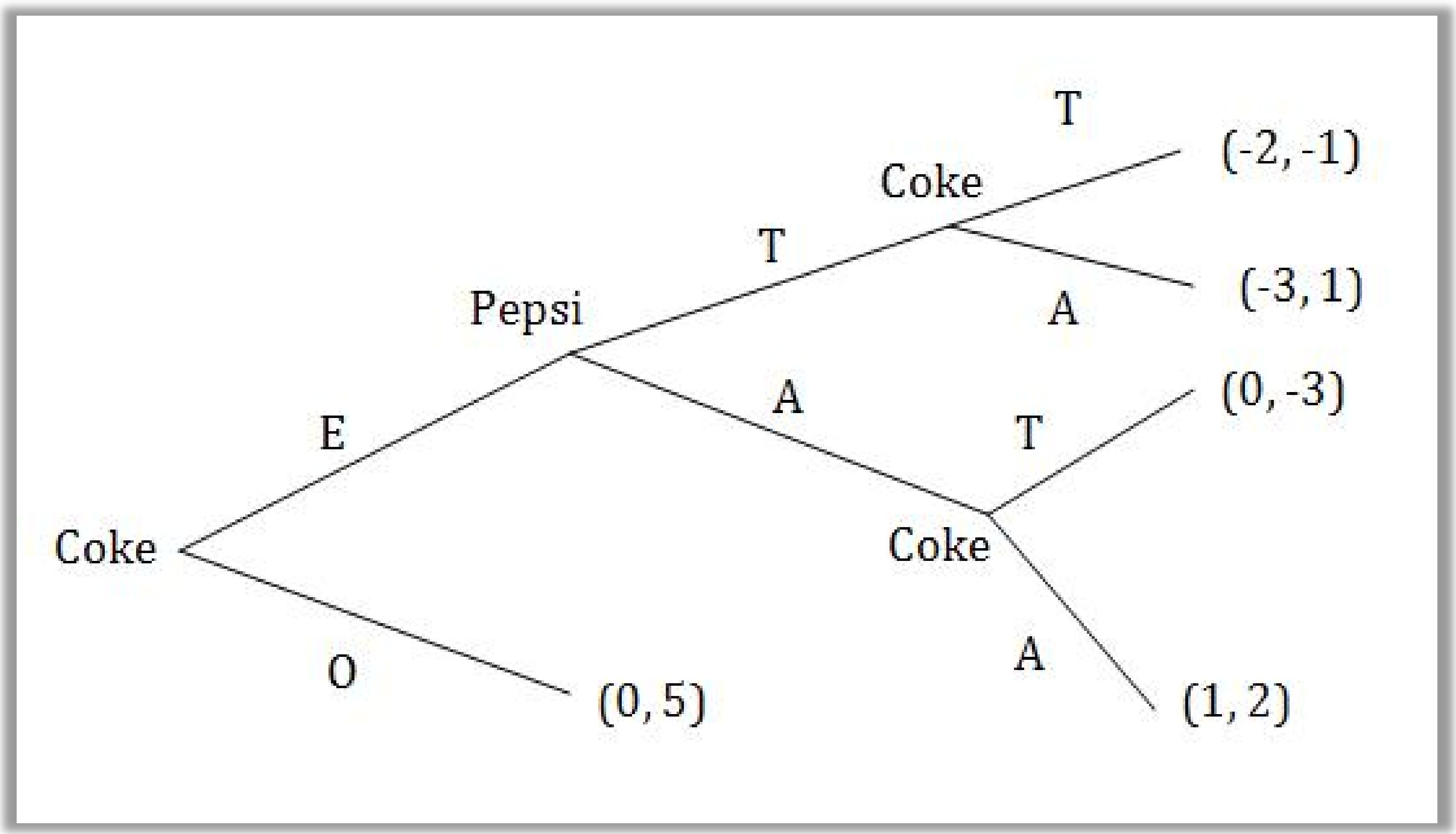
P1 \ P2	L	C	R
T	-13, -8	-1, -4	7, -4
M	-4, -1	4, -1	4, -4
B	1, 2	1, -1	1, -4

Game 2

P1 \ P2	L	C	R
T	-13, -8	-1, -4	7, -4
M	-4, -1	4, -1	4, -4
B	1, 2	1, -1	1, -4

- IEDS Solution: (B, L)
- NE Solutions: {(B, L), (M, C), (T, R)}
- Hence, it is evident that IEDS solution is NE solution (B, L).
- But NE solutions are not always the IEDS solutions (M, C) and (T, R).

The Entry Game



Backward Induction

- Sequential Rationality: take the best decision at the decision node given the future play of the game – hence solve the last stage, then the penultimate stage and finally reach the first stage in the same way…

- Kuhn's Theorem: Every game of perfect information with a finite number of nodes has a solution to BI. If no two payoffs are the same for every player, then there is a unique BI solution.

  Remember, this is what Zermelo (1913) talked about in his Chess Game – hence, also known as Zermelo's Theorem.

- Implication: Same as the IEDS solution in a strategic form game.

Sub-game and SPNE

- Definition: A sub-game of a game is a part of the extensive form, it has nodes and branches AND with three properties –

1. It starts at a single node
2. It contains all the successors to this node
3. If it contains any part of an information set, then it contains all the nodes in that information set.

- Definition: A pair of strategies is a SPNE if the strategies, when confined to any sub-game of the original game, have players playing a NE within that sub-game.

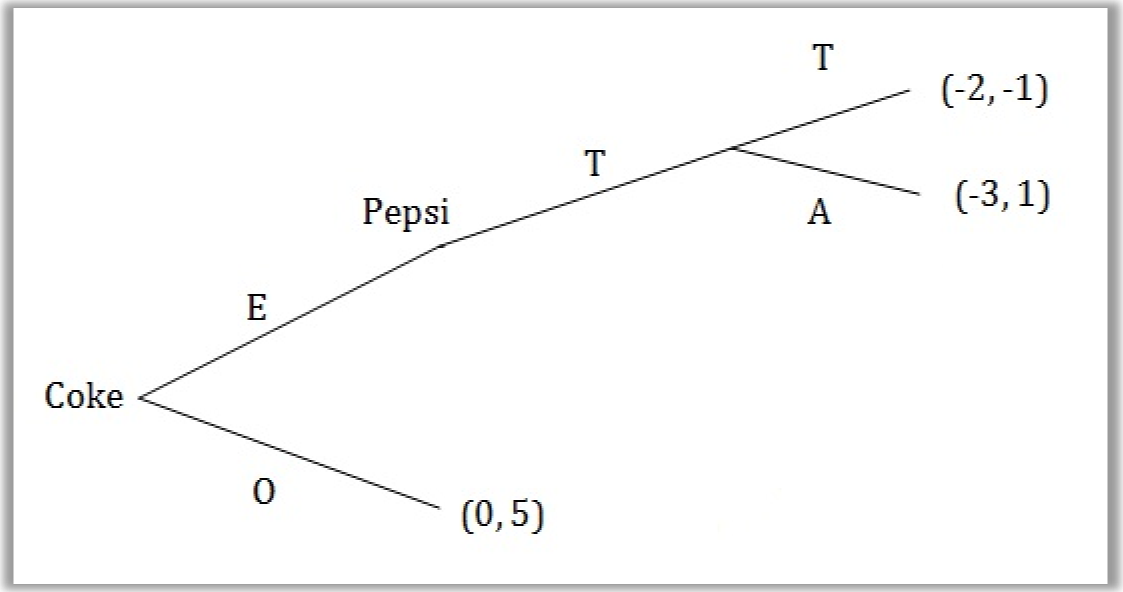
  s_1 and s_2 constitute a SPNE if for every sub-game g, $s_1(g)$ and $s_2(g)$ constitute a NE within the sub-game g.

  Set of strategies for each player such that at every node of the game tree (whether that lies in the equilibrium path or not) the continuation of the same strategy in the sub-game starting at that node is optimal for the player who is taking the decision.

- R. Selten introduced the concept and got the Nobel in 1993!

Commitment

- Suppose Pepsi is committed to play Tough if Coke enters! And Coke thinks this threat to be credible. Then the game boils down to –

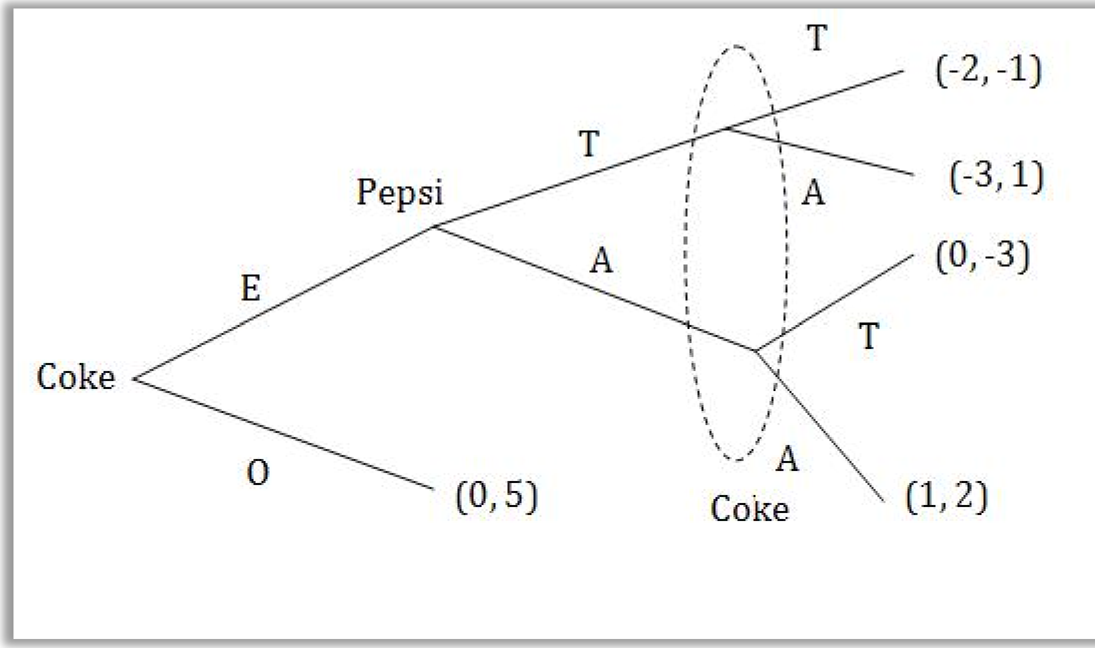


- In this case Coke will not enter at all!!!

- However, result will be different if Coke does not believe Pepsi always playing tough to be a credible threat. It believes that Pepsi can gain more by accommodating!

Credibility

- Let's focus on the issue of credibility a little more. Suppose post-entry choices (A or T) are taken simultaneously.



Pepsi

Coke		T	A
	T	-2, -1	0, -3
	A	-3, 1	1, 2

Credibility (Continued)

		Pepsi	
		T	A
Coke	ET	-2, -1	0, -3
	EA	-3, 1	1, 2
	OT	0, 5	0, 5
	OA	0, 5	0, 5

- The NE turn out to be: (EA, A); (OA, T); (OT; T).

- Reasonable versus Credible: Is it reasonable for Coke to play A against Pepsi's T? Even Pepsi can gain more by playing A instead of T!

- Hence, look into the post-entry game now. (A,T) is not a NE! But (T, T) is a NE and so is (A, A). And both are reasonable.

Credibility (Continued)

- Hence, (EA, A) and (OT, T) are reasonable NE of the entire game.
- Now, given the NE (OT, T) , the post-entry game is never played actually. But, given (T, T) being reasonable, Coke prefers to opt out as it knows that if it enters and Pepsi plays T, then it will also play T and end up loosing, hence, opt out.
- Now, Coke may think playing T for Pepsi may be reasonable but not credible (as Pepsi will gain more by playing A if Coke enters).
- In that case Coke will enter and the game ends at (EA, A).
- And (EA, A) and (OT, T) come out to be SPNE of the entire game.

SPNE

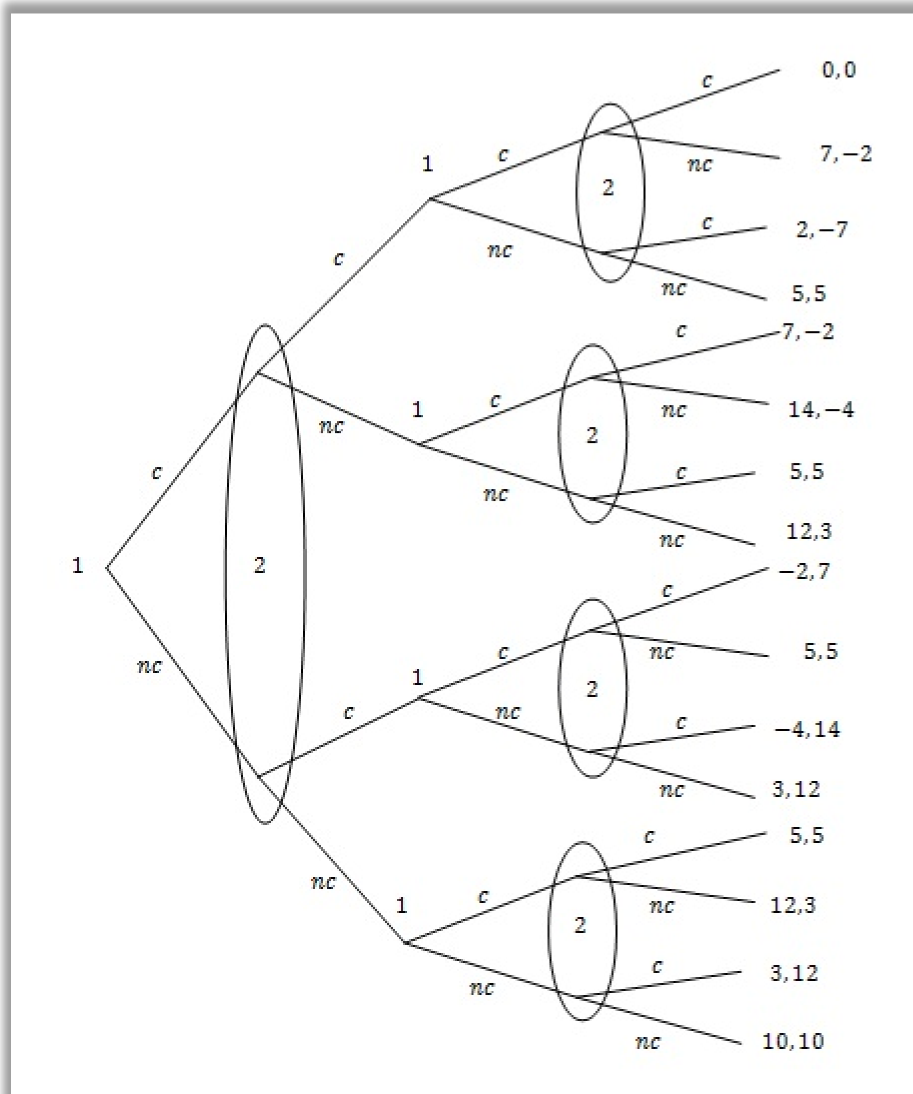
- A strategy set is a SPNE if such strategy set is a Nash equilibrium of every other subgame of the original game.
- That is, the Nash strategy set of a subgame is the SPNE of the entire game.
- That implies that SPNE of a game is a subset of the NE of the game. It may be the proper subset. It may be equal.
- Every finite extensive game has a SPNE.

Repeated Games

- Special category of extensive form games.
- Agents interact more than once.
- Current action shapes future actions and reactions.
- Punishments (rewards) from non-cooperation (cooperation) depends on the credibility of threat.
- Hence, believe only if such outcome is a SPNE.
- Two classifications – finitely repeated and infinitely repeated.
- Stage game: $G = \{S_i, \pi_i; i = 1, 2, \cdots, N\}$ where $\pi_i = \pi_i(S_1, \cdots, S_N)$

Repeated Games (Continued)

- Definition: A repeated game is defined by a stage game G and then umber of times it is repeated T. T can be finite or infinite.



Repeated Games (Continued)

- Consider a finitely repeated game (G, T) with $G = \{S_i, \pi_i; i = 1, 2, \cdots, N\}$
- Suppose the stage game G has exactly one unique NE $(s_1^*, \cdots, s_N^*)$
- This game has a unique SPNE and player I will always play s_i^* in all of the T stages of the game irrespective of what has happened earlier.
- Players are myopic. Think of the last two stages of a finitely repeated game even if T is very large but finite.

Infinitely Repeated Games

- If the stage game payoffs are constant then the total payoff is given by $\frac{\pi}{1-\delta}$
- Let $S = \pi + \delta\pi + \delta^2\pi + \cdots + \delta^t\pi + \cdots\infty$
- $\Rightarrow \delta S = \delta\pi + \delta^2\pi + \cdots + \delta^t\pi + \cdots\infty$
- $\Rightarrow \ S = \frac{\pi}{1-\delta}$
- I confess he doesn't (7, -2), Both confess (0, 0), Both not confess (5, 5)
- Cooperation gives 5 in this and all the subsequent stages $\Rightarrow S = \frac{5}{1-\delta}$
- If I deviate gives me 7 in this period. If $7 > \frac{5}{1-\delta} \Rightarrow 1-\delta > \frac{5}{7} \Rightarrow \delta < \frac{2}{7}$
- Hence, if $\delta > \frac{2}{7}$ then grim trigger will not occur and (NC, NC) can be achieved for all stages and be the SPNE.

Reference

- Prajit Dutta (1999). Strategies and Games. MIT Press. Chapters 2-6; 11; 14-15.