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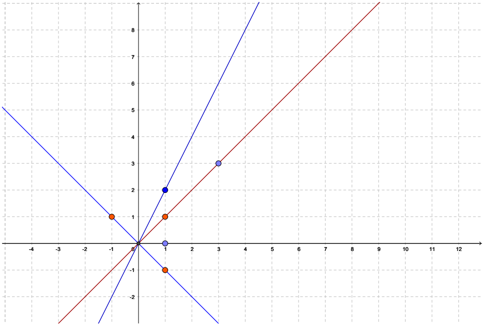

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- [Geometrically] An eigenvector, corresponding to a real nonzero eigenvalue, points in a direction in which it is stretched by the matrix and the eigenvalue is the factor by which it is stretched. If the eigenvalue is negative, the direction is reversed.

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Now we are able to answer our question. If $x^2 + 1$ is the characteristic polynomial of a matrix $A \in \mathbb{M}_2(\mathbb{R})$, then A does not have eigenvalues. Here is that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

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- **[Theorem]** Similar matrices have the same characteristic polynomial. But the converse is not true.

  Proof: Let A and B be two similar matrices.

  Then there exists a nonsingular matrix P such that $P^{-1}AP = B$.
  Then $\det(B - xI)$

  $$= \det(P^{-1}AP - xI)$$

  $$= \det(P^{-1}AP - xP^{-1}P)$$

  $$= \det(P^{-1})\det(A - xI)\det(P) = \det(A - xI).$$

Converse is not true. Consider the following two matrices.

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Then $\det(C) = \begin{vmatrix} A_{n\times n} - xI_n & D_{n\times m} \\ 0_{m\times n} & B_{m\times m} - xI_m \end{vmatrix} = \det(A - xI_n)\det(B - xI_m)$.

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- **[Result]** Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$ and $B \in \mathbb{M}_{n\times m}$, where $m \geq n$.. Then $P_{AB}(x) = x^{m-n} P_{BA}(x)$.

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- **[Result]** Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$ and $B \in \mathbb{M}_{n\times m}$, where $m \geq n$.. Then $P_{AB}(x) = x^{m-n}P_{BA}(x)$.

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- **[Theorem:]** Let $\lambda_1, \ldots, \lambda_p$ be distinct eigenvalues of A. Let $x_1, \ldots, x_p$ be eigenvectors of A corresponding to $\lambda_1, \ldots, \lambda_p$. Then $x_1, x_2 \ldots, x_p$ are LI.

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  $x_{k+1} = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k.$

  $A x_{k+1} = c_1 A x_1 + c_2 A x_2 + \cdots + c_k A x_k.$

  $\lambda_{k+1} x_{k+1} = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_k \lambda_k x_k.$

$$\lambda_{k+1}(c_1x_1 + c_2x_2 + \cdots + c_kx_k) = c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_k\lambda_kx_k.$$

$$(\lambda_{k+1} - \lambda_1)c_1x_1 + \cdots + (\lambda_{k+1} - \lambda_k)c_kx_k = 0$$

$$(\lambda_{k+1} - \lambda_1)c_1 = \cdots = (\lambda_{k+1} - \lambda_k)c_k =$$

$c_1 = \cdots = c_k = 0$ as $\lambda_{k+1} - \lambda_i \neq 0$ for $i = 1, \dots k$.
Hence x_{k+1} is zero a contradiction. Then $k = p$.

- Let $A \in \mathbb{M}_n(\mathbb{F})$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then $\prod_{i=1}^{n} \lambda_i = \det(A)$ and $\sum_{i=1}^{n} \lambda_i = trace(A)$.

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Then we have $\sum_{i=1}^{n} \lambda_i = trace(A)$ and $\prod_{i=1}^{n} \lambda_i = \det(A)$.

Let $A \in \mathbb{M}_n(\mathbb{F})$. The $\det(A)$ and $trace(A)$ are known to you. But you cannot write $\det(A)$ is the product of the eigenvalues of A and $trace(A)$ is the sum of the eigenvalues of A. Because A may not have n number of eigenvalues. For example $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$ in $\mathbb{M}_n(\mathbb{R})$. It has exactly one eigenvalue which is 1. The $\det(A) = 4$ which is not the product of the eigenvalues of A.

If $A \in \mathbb{M}_n(\mathbb{F})$ and the $\det(A)$ and $trace(A)$ are known to you. Then it is always true $\det(A)$ is the product of the eigenvalues of A and $trace(A)$ is the sum of the eigenvalues of A.

- Let $A \in \mathbb{M}_{n\times n}(\mathbb{F})$ and let λ be an eigenvalue of A. Let $E_\lambda(A) = \{x \in \mathbb{F}^n : (A - \lambda I)x = 0\}$ = set of all solutions of the homogeneous equations $(A - \lambda I)x = 0$.

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- The subspace E_λ is a finite dimensional for each eigenvalue λ of A. The dimension of E_λ is called the **geometric multiplicity** of λ with respect to A.

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$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix}$. The rank of $A - 2I$ is 1. Hence the geometric multiplicity of 2 is $3 - 1 = 2$.

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Discussions: Let $B \in \mathbb{M}_n(\mathbb{F})$. Then we can write B in the following way

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We can show that $P^{-1}(\lambda x_j) = \lambda P^{-1} x_j = \lambda e_j$ (Here x_j is the jth column of P and P^{-1} is the inverse of P) $j = 1, \ldots, m$. Then

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So the algebraic multiplicity of λ with respect to A is at least m and the theorem follows.

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- Suppose we have a non-diagonal matrix A and if we are able to show that A is similar to a diagonal matrix, then we can easily find the above information for A.

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  Then you can check that $P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. Hence A is diagonalizable matrix.

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  Therefore $AP_i = d_iP_i$ for $i = 1, \ldots, n$.

Since P is non-singular, we have each P_i is non-zero for each $i = 1, \ldots, n$.

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Hence $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore $P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This implies that $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is not possible.

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Q2. If A is diagonalizable, then how do I calculate such P matrix?

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Consider the following matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ in $\mathbb{M}_n(\mathbb{R})$. Then A has exactly one eigenvalue which is 1 with algebraic multiplicity 1.

So the above argument is not true if we consider some other field $\mathbb{F}$ instead of $\mathbb{C}$.

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$AP = P[\lambda_1 e_1 : \cdots : \lambda_n e_n]$. $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$. Hence A is diagonalizable.

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  So we have at most $n - 1$ eigenvector. A contradiction that a diagonalizable matrix must have n linearly eigenvectors.

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  Contradiction because we assume that the geometric multiplicity of λ_t is strictly less that m_t. Hence the geometric multiplicity of λ_i is m_i for $i = 1, \dots, k$.

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  We have already proved that $E_{\lambda_i}(A) \cap E_{\lambda_j}(A) = \{0\}$ for $1 \leq i, j \leq k$ and $i \neq j$.

  Then $\dim(E_{\lambda_1}(A) + E_{\lambda_2}(A) + E_{\lambda_3}(A) + \cdots + E_{\lambda_k}(A) = \sum_{i=1}^{k} \dim(E_{\lambda_i}(A)) = \sum_{i=1}^{k} m_i = n$.

- **[Theorem:]**Let $A \in \mathbb{M}_n(\mathbb{C})$ and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A with algebraic multiplicities $m_1, \ldots, m_k$ respectively. Then A is diagonalizable if and only if $E_{\lambda_1}(A) \oplus E_{\lambda_2}(A) \oplus E_{\lambda_3}(A) \oplus \cdots \oplus E_{\lambda_k}(A) = \mathbb{C}^n$.

  Proof: We first assume that A is diagonalizable. Then the geometric multiplicity of λ_i is m_i for $i = 1, \ldots, k$.

  We have already proved that $E_{\lambda_i}(A) \cap E_{\lambda_j}(A) = \{0\}$ for $1 \leq i, j \leq k$ and $i \neq j$.

  Then $\dim(E_{\lambda_1}(A) + E_{\lambda_2}(A) + E_{\lambda_3}(A) + \cdots + E_{\lambda_k}(A) = \sum_{i=1}^{k} \dim(E_{\lambda_i}(A)) = \sum_{i=1}^{k} m_i = n$.

  Hence $E_{\lambda_1}(A) \oplus E_{\lambda_2}(A) \oplus E_{\lambda_3}(A) \oplus \cdots \oplus E_{\lambda_k}(A) = \mathbb{C}^n$.

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Let B_i be a basis of $E_{\lambda_i}(A)$ for $i = 1, \ldots, k$.

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It can be proved that $\cup_{i=1}^{k} B_i$ is a basis of $E_{\lambda_1}(A) \oplus E_{\lambda_2}(A) \oplus E_{\lambda_3}(A) \oplus \cdots \oplus E_{\lambda_k}(A) = \mathbb{C}^n$.

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The cardinality of $\cup_{i=1}^k B_i$ is n. Hence we have n linearly independent vectors. Thus A is diagonalizable.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

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We have seen that the algebraic multiplicity equal to the geometric multiplicity for each eigenvalue. Hence A is diagonalizable.