Note: Please check the solution very carefully. If you will find any discrepancy, please let me know. I also request you to please cross check your answers once more using the solution. In this question paper, a list of questions have been made by myself and rest of the questions have been collected from my memory. So I forgot to mention two necessary conditions in two questions. In Q4, I forgot to mention non-trivial word and in Q5, I forgot to mention non-zero linear functionals. So everyone will get full marks in these two questions.

1. Let $\mathbb{M}_n(\mathbb{R})$ be the vectors space of all real matrices of size n over $\mathbb{R}$. Which of the following statement(s) is(are) incorrect?

   (a) $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) : rank(A) \leq n-1\}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.

   (b) $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) : det(A) = 0\}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.

   (c) Let $B \in \mathbb{M}_n(\mathbb{R})$. Then $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) : AB = BA\}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.

   (d) $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) : AA^t = A^tA\}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.

   Ans: (a), (b) and (d).

2. Which of the following statement(s) is(are) correct?

(a) Let $A = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$ and $B = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 1 \leq x, y \leq 2 \right\}$. There are finitely many linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ Such that $T(A) = B$.

(b) Let $A = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$ and $B = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 1 \leq x, y \leq 2 \right\}$. There is no linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ Such that $T(A) = B$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. Then there are two bases C_1 and C_2 in $\mathbb{R}^2$ such that $[T]_{C_1C_1} = A$ and $[T]_{C_2C_2} = B$.

(d) Let $T : \mathbb{M}_n(\mathbb{R}) \to \mathbb{M}_n(\mathbb{R})$ be a linear map defined by $T(A) = A + A^t$ for all $A \in \mathbb{M}_n(\mathbb{R})$. Then $rank(T) = \frac{n(n+1)}{2}$.

Ans: (b) and (d).

3. Let $W = \{AB - BA : A, B \in \mathbb{M}_n(\mathbb{R})\}$. Which of the following statement(s) is(are) correct?

   (a) W is a subspace of $\mathbb{M}_n(\mathbb{R})$.

   (b) $dim(W) = n^2 - 1$

   (c) $W = S$, where S is the set of all matrices A in $\mathbb{M}_n(\mathbb{R})$ such that $trace(A) = 0$.

   (d) W is not a subspace of $\mathbb{M}_n(\mathbb{R})$.

Ans: (a), (b) and (c).

Arguments: Option (a) is trivial.

Option (b). Let $AB - BA$ be an element in W. Since $A = \sum_{i,j} a_{ij} E_{ij}$ and $B = \sum_{i,j} b_{ij} E_{ij}$ where E_{ij} is a matrix whose ijth entry is 1 rest all are zero.

$$E_{ij}E_{kl} - E_{kl}E_{ij} = \begin{cases} 0 & \text{if } j \neq k \text{ and } i \neq l \\ E_{il} & \text{if } j = k \text{ and } i \neq l \\ -E_{kj} & \text{if } j \neq k \text{ and } i = l \\ E_{ii} - E_{jj} & \text{if } j = k \text{ and } i = l \end{cases}$$

Using this fact you can see that $AB - BA$ is linear combination of all E_{ij} with $i \neq j$ together with $E_{ii} - E_{i+1,i+1}$ for $i = 1, \ldots, n-1$. Then $\dim(W) = n^2 - n + (n-1) = n^2 - 1$.

Option (c). It is clear that W is a subspace of S and they have same dimension. Hence $W = S$.

4. Which of the following statement(s) is(are) correct?

   (a) If the vector space $\mathbb{V}$ over $\mathbb{R}$ is isomorphic to the vector space $\mathbb{W}$ over $\mathbb{R}$, then the vector space $\mathbb{V}$ over $\mathbb{Q}$ is isomorphic to the vector space $\mathbb{W}$ over $\mathbb{Q}$.

   (b) If the vector space $\mathbb{V}$ is finite dimensional over $\mathbb{R}$, then the vector space $\mathbb{V}$ over $\mathbb{Q}$ is finite dimensional.

   (c) The (**non-trivial**) vector space $\mathbb{V}$ over $\mathbb{Q}$ is always infinite dimensional no matter whether $\mathbb{V}$ over $\mathbb{R}$ is finite dimensional or infinite dimensional.

   (d) There is a (**non-trivial**) $\mathbb{V}$ such that $\mathbb{V}$ over $\mathbb{Q}$ and $\mathbb{V}$ over $\mathbb{R}$ both are finite dimensional vector spaces.

Answer: If I do not use the word **non-trivial**. Then (a) and (d) are correct options.

Argument: (a). Since $\mathbb{V}(\mathbb{R})$ is isomorphic to $\mathbb{W}(\mathbb{R})$, we have an isomorphism T from $\mathbb{V}$ to $\mathbb{W}$. Since $\mathbb{Q} \subseteq \mathbb{R}$, then T is an isomorphism from $\mathbb{V}(\mathbb{Q})$ to $\mathbb{W}(\mathbb{Q})$.

(d) $\{0\}$ over $\mathbb{Q}$ and $\{0\}$ over $\mathbb{R}$ are both finite dimensional.

If I use the the word **non-trivial**. Then (a) and (c) are correct options.

Argument: (a) same as above.

(b) There are two cases $\mathbb{V}$ over $\mathbb{R}$ is finite dimensional or infinite dimensional. $\mathbb{V}(\mathbb{R})$ is finite dimensional and dimension is n. Then $\mathbb{V}(\mathbb{R})$ is isomorphic to $\mathbb{R}^n(\mathbb{R})$. Using option (a), we have $\mathbb{V}$ over $\mathbb{Q}$ is isomorphic to $\mathbb{R}^n(\mathbb{Q})$, we know that $\mathbb{R}^n(\mathbb{Q})$ is infinite dimensional. Hence $\mathbb{V}(\mathbb{Q})$ is infinite dimensional.

$\mathbb{V}(\mathbb{R})$ is infinite dimensional. Suppose that $\mathbb{V}(\mathbb{Q})$ is finite dimensional, then you can easily conclude that $\mathbb{V}(\mathbb{R})$ is also finite dimensional. A contradiction. Hence $\mathbb{V}(\mathbb{Q})$ is infinite dimensional.

Note: Everyone will get 2 marks for this question.

5. Let $\mathbb{V}$ be a finite dimensional vector space over the field $\mathbb{R}$. Let $f, g \in \mathbb{V}^*$ such that whenever $f(x) \geq 0$, we also have that $g(x) \geq 0$. Which of the following statement(s) is(are) correct?

   (a) $Ker(f) \subsetneq Ker(g)$.

   (b) $Ker(f) = Ker(g)$.

   (c) $f = \alpha g$ for some $\alpha > 0$

   (d) The linear map $T : \mathbb{V} :\to \mathbb{R}^2$ defined by $T(x) = (f(x), g(x))$ for all $x \in \mathbb{V}$, is onto.

Ans: (b) and (c).

Arguments: Since f and g both are non-zero, we have $rank(f) = rank(g) = 1$. Then $nullity(f) = nullity(g) = n - 1$ where $\dim(\mathbb{V}) = n$. Let $\{u_1, \ldots, u_{n-1}\}$ be a basis of $Ker(f)$. We now show that $g(u_i) = 0$ for $i = 1, \ldots, n-1$. As $f(u_i) = 0$, we have $g(u_i) \geq 0$. Suppose $g(u_i) > 0$, then $g(-u_i) < 0$ (as g linear). It is clear that $f(-u_i) = 0$, then $g(-u_i) > 0$ a contradiction. Hence $g(u_i) = 0$ for $i = 1, \ldots, n-1$. This says that $\{u_1, \ldots, u_{n-1}\}$ is a basis of $Ker(g)$. Hence $Ker(f) = Ker(g)$. This says that option (b) is correct.

I have discussed in the tutorial problem that if $Ker(f) = Ker(g)$, then $f = \alpha g$ here $\alpha > 0$ as $f(x) \geq 0$ implies $g(x) \geq 0$, they are linearly dependent. So option (c) is correct.

Since option (b) is correct, so option (a) is not correct.

You can argue that $Ker(T) = Ker(f) = Ker(g)$. If T is onto, then using rank-nullity theorem we have $\dim(\mathbb{V}) = nullity(T) + rank(T) = nullity(f) + 2$. This a contradiction. Hence T is not onto. So option (d) is not correct.

Note: I forgot to give the condition non-zero. Everyone will get 2 marks for this question.

6. Let $\mathbb{V}$ be a finite dimensional vector space over the field $\mathbb{R}$. Let $T : \mathbb{V} \to \mathbb{V}$ be a linear transformation such that $T^2 = T$. Assume that T is not the zero transformation and not the identity transformation. Which of the following statement(s) is(are) correct?

   (a) $Ker(T) \neq \{0\}$

   (b) $\mathbb{V} = Ker(T) \oplus R(T)$

   (c) The transformation $I + T$ is invertible, where I is the identity transformation.

   Ans: (a), (b) and (c).

   Arguments: (a) Suppose $Ker(T) = \{0\}$. Then T is invertible. Therefore $T^{-1}(T^2) = T^{-1}T$ implies $T = I$ which is not possible. Hence $Ker(T) \neq 0$.

   (b) Let $x \in Ker(T) \cap R(T)$. Then $T(x) = 0$ and $T(y) = x$. $x = T(y) = T^2(y) = T(T(y)) = T(x) = 0$. Hence $Ker(T) \cap R(T) = \{0\}$. Using rank nullity theorem and previous argument we have $\mathbb{V} = Ker(T) \oplus R(T)$.

   (c) Let $x \in Ker(T + I)$. Then $T(x) = -x$ this implies $T(x) = T^2(x) = T(T(x)) = T(-x) = -T(x)$. This implies $T(x) = 0$. We know $T(x) = -x$ implies $x = 0$. Hence $I + T$ is one-one and $\mathbb{V}$ is finite dimensional and $I + T$ is a linear operator. Then $i + T$ is bijective. Hnece option (c) is correct.

7. Let $T : \mathbb{P}_2(x,\mathbb{R}) \to \mathbb{P}_3(x,\mathbb{R})$ be a linear transformation defined by

$$T(P(x)) = \int\limits_0^x P(t)dt + P'(x) + p(2)$$

. Which of the following statement(s) is(are) correct?

(a) $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & 2 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ is the matrix representation of T with respect to bases $\{1, x, x^2\}$ of $\mathbb{P}_2(x,\mathbb{R})$ and $\{1, x, x^2, x^3\}$ of $\mathbb{P}_3(x,\mathbb{R})$.

(b) $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$ is the matrix representation of T with respect to bases $\{1, x, x^2\}$ of $\mathbb{P}_2(x,\mathbb{R})$ and $\{1, x, x^2, x^3\}$ of $\mathbb{P}_3(x,\mathbb{R})$.

(c) $Nullity(T) = 1$

(d) $Nullity(T) = 0$

Answer: (a) and (d).

8. Let $\mathbb{V}$ be a finite dimensional vector space and let $T \in \mathbb{L}(\mathbb{V}, \mathbb{V})$ such that $rank(T) \leq rank(T^3)$. Which of the following statement(s) is(are) correct?

   (a) $Ker(T) = R(T)$.

   (b) $Ker(T) \cap R(T) = \{0\}$.

   (c) There exists a nonzero subspace $\mathbb{S}$ subspace of $\mathbb{V}$ such that $Ker(T) \cap R(T) = \mathbb{S}$.

   (d) $Ker(T) \subseteq R(T)$

   Answer: Using the condition $rank(T) \leq rank(T^3)$, we have $R(T) = R(T^3)$ and $Ker(T) = Ker(T^3)$. Let $x \in Ker(T) \cap R(T)$. Then $T(x) = 0$ and $T(y) = x$. This implies $T^3(y) = T^2T(y) = T^2(x) = T(T(x)) = T(0) = 0$. This says that $y \in Ker(T^3) = Ker(T)$. Then $T(y) = 0$. This implies $x = T(y) = 0$. So option (b) is correct.

   Since option (b) is correct, then option (a), (c) and (d) are not correct.

9. Let $\mathbb{W}$ be a finite dimensional vector space over the field $\mathbb{R}$. Let $\mathbb{S}$ be a subspace of $\mathbb{W}$. Then $\mathbb{S} \cap T(\mathbb{S}) \neq \{0\}$ for every isomorphism $T : \mathbb{W} \to \mathbb{W}$ if and only if

   (a) $\mathbb{S} = \mathbb{W}$

   (b) $\dim(\mathbb{S}) < \frac{\dim(\mathbb{W})}{2}$.

   (c) $\dim(\mathbb{S}) = \frac{\dim(\mathbb{W})}{2}$.

   (d) $\dim(\mathbb{S}) > \frac{\dim(\mathbb{W})}{2}$.

   Ans: Only options (d) is correct.

   Arguments: (a) If $\mathbb{W} = \mathbb{S}$, then $\mathbb{S} \cap T(\mathbb{S}) \neq \{0\}$ for every isomorphism T (this is always true). But the converse is not true. That is, $\mathbb{S} \cap T(\mathbb{S}) \neq \{0\}$ for every isomorphism T does not imply $\mathbb{S} = \mathbb{W}$.

   Example: Let $\mathbb{W} = \mathbb{R}^3$ and $\mathbb{S} = \text{LS}(\{(1, 0, 0), (0, 1, 0)\}$. Let T be an isomorphism on $\mathbb{R}^3$. Then $T(\mathbb{S}) = \text{LS}(\{T(1, 0, 0), T(0, 1, 0)\}$. It is clear that (S) and $T(\mathbb{S})$ both the are subspaces of dimension 2. Then $\mathbb{S} + T(\mathbb{S})$ is also a subspace of $\mathbb{V}$.

   If $\mathbb{S} \cap T(\mathbb{S}) = \{0\}$, then the dimension of $\mathbb{S} + T(\mathbb{S})$ is 4 which is not possible. Hence $\mathbb{S} \cap T(\mathbb{S}) \neq \{0\}$ for every isomorphism T but $\mathbb{S} \neq \mathbb{W} = \mathbb{R}^3$.

   We assume that $\mathbb{S} \cap T(\mathbb{S}) \neq \{0\}$. To prove that option (d) is correct.

   Since $\dim(\mathbb{S})$ and $\dim(\mathbb{S})$ are two real numbers, either $\dim(\mathbb{S}) \leq \frac{\dim(\mathbb{W})}{2}$ or $\dim(\mathbb{S}) > \frac{\dim(\mathbb{W})}{2}$.

   We assume that $\dim(\mathbb{S}) \leq \frac{\dim(\mathbb{W})}{2}$. Let $\dim(\mathbb{S}) = k$ and let $\{u_1, \ldots, u_k\}$ be a basis of $\mathbb{S}$. Using extension theorem, we extend $\{u_1, \ldots, u_k\}$ to a basis of $\mathbb{W}$ which is $\{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$.

   Let T be a linear operator such that $T(u_i) = u_{k+i}$, $T(u_{k+i}) = u_i$ for $i = 1, \ldots, k$ and $T(u_i) = u_i$ for $i = 2k + 1, \ldots, n$. You can check that T is an isomorphism. Then $\{T(u_1), \ldots, T(u_k)\}$ is a basis of $T(\mathbb{S})$.

   Let $x \in \mathbb{S} \cap T(\mathbb{S})$. Then $x = c_1u_1 + \cdots + c_ku_k$ and $x = b_1T(u_1) + \cdots + b_k(T(u_k)) = b_1u_{k+1} + \cdots + b_ku_{2k}$. Then $c_1u_1 + \cdots + c_ku_k - b_1u_{k+1} - \cdots - b_ku_{2k} = 0$. This implies $c_1 = c_2 = \cdots = c_k = 0$. Hence $x = 0$. Therefore $\mathbb{S} \cap T(\mathbb{S}) = \{0\}$. So option (b) and (c) are not correct. Therefore $\dim(\mathbb{S}) > \frac{\dim(\mathbb{W})}{2}$. Hence option (d) is correct.

   Converse. Given $\dim(\mathbb{S}) > \frac{\dim(\mathbb{W})}{2}$. To prove that $\mathbb{S} \cap T(\mathbb{S}) \neq \{0\}$ for every isomorphism T.

Suppose that there exists an isomorphism T such that $\mathbb{S} \cap T(\mathbb{S}) = \{0\}$. We know that $\mathbb{S}$ and $T(\mathbb{S})$ are subspaces of $\mathbb{W}$. Then $\mathbb{S} + T(\mathbb{S})$ is also a subspace of $\mathbb{W}$. Then $\dim(\mathbb{S} + T(\mathbb{S})) = 2k$ where k is the dimension of $\mathbb{S}$. Therefore $2k \leq n$ where n is the dimension of $\mathbb{W}$. Hence $k \leq n/2$ a contradiction. Hence $\mathbb{S} \cap T(\mathbb{S}) \neq \{0\}$ for every isomorphism.