Text Book: T. Amaranath, “An Elementary Course in Partial Differential Equations”, 2nd Edition, Narosa, India

References: 1. Ian N Sneddon, “Elements of Partial differentia Equations”, Dover Publication

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Partial Differential Equation

3rd semester core course

Department of Mathematics

IIT Kharagpur

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Division of Lecture: Part A. Part B. Part C

Part A

- What is PDE ? How PDE is formed ? Why PDE is important to study?
- Order and degree of PDE. Definition of Linear & Non-linear PDE
- Classification of 1^{st} order PDE of two independent and one dependent variables as linear/semi linear/quasi linear
- Formulation of PDE by eliminating arbitrary constants/functions
- Classification of integrals as CI, GI, PI, SI
- Lagrange's method of solving 1^{st} order linear PDE. Geometrical interpretation. Method of Characteristics
- Cauchy Problem: Integral surface through given curve
- Orthogonal surface to a given system of surfaces

▪ What is PDE ? How PDE is formed ? Why PDE is important to study?

Symbols: $z = z(x, y), p \equiv \frac{\partial z}{\partial x}, q \equiv \frac{\partial z}{\partial y}$

Suppose ratio of two partial derivatives is proportional to x/y

$\Longrightarrow px - qy = 0$, a PDE

A physical model: Consider z denotes temperature at any point (x, y) on a 2D metal body. When law of physics will be applied, we will have a relation between variables x, y, z and partial derivatives of z of any order, which is a PDE.

<u>Summary</u>

- ❑ When a dependent variable depends on more than one variable, then any relation between them involving derivatives can't be ODE, it will be a PDE.
- ❑ PDE describes physical relation in applied science, hence it is important to study.

▪ Order and degree of PDE. Definition of Linear & Non-linear PDE

General form of PDE for two independent variables and one dependent variable

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \dots\right) = 0$$

The order of highest order partial derivative term of (1.1) is the <u>order of PDE.</u> The exponent of highest order partial order derivative term is called the <u>degree of PDE</u>.

PDE will be <u>linear</u> if ∃ no terms with power higher than one w.r.t. dependent variable z and its partial derivatives. Otherwise PDE will be called <u>nonlinear</u>.

Above definitions remain same for more than two independent variables.

- Classification of 1st order PDE of two independent variables and one dependent variable as linear/quasi-linear/semi-linear/nonlinear

General Form of 1st order 1st degree: $\boldsymbol{Pp + Qq = R}$

- Linear: P, Q, R depend only on x, y
- Semi-linear: P, Q depend only on x, y, but R depend on z
- Quasi-Linear: P, Q, R depend on x, y, z

Examples: $(xy + x^2)p + yq = x^2 \Longrightarrow$ **Linear** (No non-linear term)

$(xy + x^2)p + yq = x^2 + z^2 \Longrightarrow$**Semi-Linear** (Non-linear term presents)

$zp + yq = x^2 + z^2 \Longrightarrow$**Quasi-linear** (Non-linear terms present)

General form of 1st order PDE: $\boldsymbol{f(x, y, z, p, q) = 0}$

- Non-linear: pq-term present

1st order 2nd degree

Example:

$pq + (z + x^2)(p + q) = z^2 \Longrightarrow$**Non-linear** (Non-linear terms present)

▪ Construction of PDE by eliminating arbitrary constants/functions
1st Order

Formation of 1st order PDE by elimination

- Family of surface: $z = f(u)$ or $f(u, v) = 0$, f is arbitrary, u, v are known functions of x, y, z

Differentiate equation of surface once partially w.r.t $x, y \Longrightarrow$ 2 equations

Eliminate f from 3 equations $\Longrightarrow$ 1st order Linear PDE

- Family of surface: $F(x, y, z, a, b) = 0$, a, b arbitrary, F is known function

Differentiate equation of surface once partially w.r.t $x, y \Longrightarrow$ 2 equations

Eliminate f from 3 equations $\Longrightarrow$ 1st order Linear/Semi-linear/Quasi-linear/Non-linear PDE

Given surface is solution of PDE formed, called "Integral Surface" of PDE.

▪ Construction of PDE by eliminating arbitrary constants/functions Higher Order

Formation of 2nd order PDE by elimination

Symbols: $r \equiv \partial^2 z/\partial x^2$, s$\equiv \partial^2 z/\partial x\partial y$, t$\equiv \partial^2 z/\partial y^2$. $f'(u) \equiv \mathrm{d}f/\mathrm{d}u$ etc.

- Family of surface: $z = f(u) + g(v) + w$; f, g are arbitrary functions, u, v, w are known functions

Differentiate twice partially w.r.t. x, y to get p, q, r, s, t (4 equations)

Eliminate f', g', f'', g'' from 5 equations $\Longrightarrow$ 2nd order PDE

Linear if u, v, w depend on x, y only; Non-linear if u, v, w depend on x, y, z.

General form of linear 2nd order PDE:

$$R(x,y)r + S(x,y)s + T(x,y)t + P(x,y)p + Q(x,y)q = W(x,y)$$

Formation of higher order PDE or formation of PDE for more than two independent variables can be done in similar spirit .

▪ Classification of Integrals (Solutions) of a 1st order PDE

General Form: $f(x, y, z, p, q) = 0$

- Complete Integral (CI): $F(x, y, z(x, y), a, b) = 0$, $[a, b$ arbitrary constants, F is known function $]$
- General Integral (GI): $g(u, v) = 0$, g is arbitrary function, u, v known functions of x, y, z.
- Particular Integral (PI): A solution containing no arbitrary function or constants, which can be obtained for particular values of a, b or for some specific forms of g.
- Singular Integral (SI): Envelope of two-parameter family of surface given by CI, if exists, is a solution, which can’t ne obtained from CI or GI.
- Special Solution: [Refer to Text Book] Sometimes there exists solution which doesn’t fall into any of above category.

▪ Methods of finding Integrals of 1st order PDE

Symbols: $F_a \equiv \partial F / \partial a$ etc.

- CI: Using Charpit's Method
- GI: For linear/semi-linear/quasi-linear PDE, using Lagrange's method. For non-linear PDE, GI may be derived from CI by letting $b = \phi(a)$ yielding a one-parameter sub-family $F(x, y, z(x, y), a, \phi(a)) = 0$, then envelope of this subfamily is GI

Process: Eliminate a from given equation and $F_a = 0$.

- PI: Given a condition, PI can be obtained from CI or GI
- SI: Eliminate a, b from CI and from $F_a = 0, F_b = 0$.

▪ Lagrange's Method to find GI (Integral Surface)

General form of 1st linear/semi-linear/quasi-linear PDE

$$P(x,y,z)p+Q(x,y,z)q=R(x,y,z)$$

GI of above PDE is $\phi(u,v)=0$, $u(x,y,z)=c_1$, $v(x,y,z)=c_2$ are two Integral Curves (solutions) of Lagrange's Auxiliary Equation (AE)

$$\frac{\mathrm{d}x}{P}=\frac{\mathrm{d}y}{Q}=\frac{\mathrm{d}z}{R}$$

<u>Geometrical Interpretation</u>: Integral Surfaces of PDE is generated by Integral Curves of AE. Conversely, any Integral Curve of AE generates Integral Surface of PDE.

Explanation: Given PDE geometrically means that two directions (P,Q,R) and $(p,q,-1)$ are perpendicular.

▪ Cauchy Problem: Integral Surface through given curve

General form of 1st order PDE: $f(x.y, z, p, q) = 0$

Suppose $z = \phi(x, y)$ is GI$\Longrightarrow$ Family of Integral Surfaces, ϕ arbitrary.

To find one particular Integral Surface of above family passing through a given curve Γ: $x = x_0(t), y = y_0(t), z = z_0(t), t \in [a, b]$, t is parameter.

Hence, Cauchy Problem is to find PI from GI.

For 1st order linear/semi-linear/quasi-linear PDE $Pp + Qq = R$, GI is obtained by Lagrange's method.

For 1st order non-linear PDE $f(x.y, z, p, q) = 0$, CI $F(x, y, z, a, b) = 0$(F is known) is obtained by Charpit's method. GI can be obtained from CI.

▪ Orthogonal Surface to a given System of Surfaces

Given system: $f(x, y, z) = c \Longrightarrow$ One-parameter family of surface

To find a system of surfaces $z = g(x, y)$, g arbitrary, which cuts each member (for each value of c) of given family at right angle.

$z = g(x, y)$ is GI of 1st order linear/semi-linear/quasi-linear PDE

$$f_x p + f_y q = f_z$$

Explanation: Direction of normal to orthogonal surface is $(p, q, -1)$ and direction of normal to given system of surfaces is $\left(f_x, f_y, f_z\right)$, and above PDE implies two normal are perpendicular to each other.