Time-Series Econometrics

1. Stochastic Process

- Analysis of time-series is based on the modelling of stochastic process.
- A stochastic process is a collection of random variables ordered in time.
- The stochastic process evolves in time according to probabilistic laws.
- An observed **time-series** is considered to be one realization of a **stochastic process.**

Type of Stochastic Process	Definition	Functional Specification
Purely Random (White Noise)	Each element has independent and identical distribution with constant (or zero) mean and constant variance	$Y_t = u_t$ with $E(u_t) = \mu \;\; \forall$ t; $\text{var}(u_t) = \sigma^2 \;\; \forall$ t; $\text{cov}(u_t, u_{t-s}) = 0 \;\; \forall$ s $\neq 0$ For simplicity, one may also assume that: $E(u_t) = 0 \;\; \forall$ t; $\text{var}(u_t) = \sigma^2 \;\; \forall$ t; $\text{cov}(u_t, u_{t-s}) = 0 \;\; \forall$ s $\neq 0$ Example: The random disturbance terms in CLRM, $u_t \sim IIN(0, \sigma^2)$
Random Walk	In each period, the variable takes a random deviation from its previous value, and the deviations are independently and identically distributed in size	$Y_t = Y_{t-1} + u_t$ or $Y_t - Y_{t-1} = u_t$ If u_t is white noise with $E(u_t) = \mu \;\; \forall$ t; $\text{var}(u_t) = \sigma^2 \;\; \forall$ t; $\text{cov}(u_t, u_{t-s}) = 0 \;\; \forall$ s $\neq 0$ => $E(Y_t) = t\mu; \text{var}(Y_t) = t.\sigma^2$
Moving Average (MA) Process	Value of Y at time point t is moving average of the current and past values of the random disturbance term.	Moving Average process of order ***m***: $Y_t = \beta_0 u_t + \beta_1 u_{t-1} + \beta_2 u_{t-2} + \ldots + \beta_m u_{t-m}$ Here, u_t is white noise with $E(u_t) = 0 \;\; \forall$ t; $\text{var}(u_t) = \sigma^2 \;\; \forall$ t; $\text{cov}(u_t, u_{t-s}) = 0 \;\; \forall$ s $\neq 0$ => $E(Y_t) = 0; var(Y_t) = \sigma^2 \left(\sum_{j=0}^{m} \beta_j^2 \right)$
Autoregressive (AR) Process	Value of Y at time point t depends on its previous values and random disturbance at that time point.	Autoregressive process of order ***r***: $Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \ldots + \alpha_r Y_{t-r} + u_t$

ARMA Process	The variable Y has characteristics of both AR and MA	ARMA process of order ***r*** and ***m***, i.e. **ARMA (r, m)**: $Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + ... + \alpha_r Y_{t-r} + u_t +$ $\beta_1 u_{t-1} + \beta_2 u_{t-2} + ... + \beta_m u_{t-m}$ Here, $E(u_t) = 0 \ \forall \text{ t}; \text{var}(u_t) = \sigma^2 \ \forall \text{ t};$ $\text{cov}(u_t, u_{t-s}) = 0 \ \ \forall \text{ s} \neq 0$
ARIMA Process	Differencing non-stationary time-series to make it integrated and modelling the differenced series as ARMA process	ARIMA (r, d, m) means the time-series has to be differenced d times to make it stationary and the stationary time series can be modelled as ARMA(r, m) ARIMA (r, 0, 0) indicates purely AR(r) stationary process ARIMA (0, 0, m) indicates purely MA(m) stationary process ARIMA (0, d, 0) indicates that the time-series is integrated of order d, i.e., I(d)

2. Random Walk Models

Basic Model: $Y_t = \theta_0 + \theta_1 t + \theta_2 Y_{t-1} + \varepsilon_t$

Random Walk	Econometric Specification	
Random walk without drift and trend $\theta_0 = 0; \theta_1 = 0; \theta_2 = 1$	$Y_t = Y_{t-1} + u_t$	This is an AR(1) model If u_t is white noise with $E(u_t) = 0 \ \forall \text{ t}; \text{var}(u_t) = \sigma^2 \ \forall \text{ t};$ $\text{cov}(u_t, u_{t-s}) = 0 \ \ \forall \text{ s} \neq 0$ $E(Y_t) = 0, \text{var}(Y_t) = t\sigma^2$ (Assuming $Y_0 = 0$) Violates the condition of stationarity
Random walk with drift, but no trend $\theta_0 \neq 0; \theta_1 = 0; \theta_2 = 1$	$Y_t = \theta_0 + Y_{t-1} + u_t$	Drift parameter: θ_0 Drift will be upward or downward depending on $\theta_0>0$ or $\theta_0<0$ If u_t is white noise with $E(u_t) = 0 \ \forall \text{ t}; \text{var}(u_t) = \sigma^2 \ \forall \text{ t};$ $\text{cov}(u_t, u_{t-s}) = 0 \ \ \forall \text{ s} \neq 0$ $E(Y_t) = t\theta_0, \text{var}(Y_t) = t\sigma^2$

		(Assuming $Y_0 = 0$) Violates the condition of stationarity
Deterministic Trend $\theta_0 \neq 0; \theta_1 \neq 0; \theta_2 = 0$	$Y_t = \theta_0 + \theta_1 t + u_t$	If u_t is white noise with $E(u_t) = 0 \;\; \forall$ t; $\text{var}(u_t) = \sigma^2 \;\; \forall$ t; $\text{cov}(u_t, u_{t-s}) = 0 \;\; \forall$ s $\neq 0$ $E(Y_t) = \theta_0 + \theta_1 t, \text{var}(Y_t) = \sigma^2$ Violates the condition of stationarity
Random walk with drift and deterministic trend $\theta_0 \neq 0; \theta_1 \neq 0; \theta_2 = 1$	$Y_t = \theta_0 + \theta_1 t + Y_{t-1} + u_t$	Yt is non-stationary
Stochastic process with drift and deterministic trend $\theta_0 \neq 0; \theta_1 \neq 0; \theta_2 < 1$	$Y_t = \theta_0 + \theta_1 t + \theta_2 Y_{t-1} + u_t$	Yt is stationary around a deterministic trend

3. An Example

Consider the model: $Y_t = \rho Y_{t-1} + u_t$

Three possibilities

(1) When $\rho > 1$, the series is non-stationary and explosive. Past shocks have a greater impact than current ones.
(2) When $\rho = 1$, the series is non-stationary shocks persist at full force, and the series is not mean-reverting. This is the random walk model and where the variance increases with *t* and we have the infinite variance problem.
(3) When $\rho < 1$, series is stationary and the effects of shocks die out exponentially according to ρ. The series reverts to its mean.

Typically, we are interested in the last two scenarios, i.e., $\rho = 1$ and $\rho < 1$. The question is whether we have a **unit root** or not (also known as a random walk) i.e., $\rho = 1$?

For the model $Y_t = \rho Y_{t-1} + u_t$ with $\rho < 1$

⇨ $Y_1 = \rho Y_0 + u_1$

⇨ $Y_2 = \rho Y_1 + u_2 = \rho(\rho Y_0 + u_1) + u_2 = \rho^2 Y_0 + \rho u_1 + u_2$

⇨ $Y_3 = \rho Y_2 + u_3 = \rho(\rho^2 Y_0 + \rho u_1 + u_2) + u_3 = \rho^3 Y_0 + \rho^2 u_1 + \rho u_2 + u_3$

This shows that the current error is the sum of all the previous shock weighted by the coefficient (ρ) declining exponentially. How fast the effect of these previous errors will die out depends on the value of ρ

When $\rho < 1$ the time-series is stationary. In this case, the time-series looks jagged and it never wanders too far from the mean. The effect of the errors decay and disappear over time. Impact of recent events are relatively more important than what happened a long time ago.

(1) $Y_t = \alpha + \rho Y_{t-1} + u_t \text{ with} \rho = 1$

(2) $Y_t = \alpha + \rho Y_{t-1} + u_t \text{ with} \rho < 1$

3.1 Model 1

$$Y_t = Y_0 + t\alpha + \sum_{\tau=0}^{t-1} u_{t-\tau}$$

$E(Y_t|Y_0) = Y_0 + t\alpha$ and $\mathrm{var}(Y_t|Y_0) = t\sigma^2$

Both conditional mean and variance depend on time

3.2 Model 2

$$Y_t = Y_0 + \sum_{\tau=0}^{t-1} \rho^\tau (\alpha + u_{t-\tau}) == \sum_{\tau=0}^{\infty} \rho^\tau (\alpha + u_{t-\tau})$$

$E(Y_t) = \alpha \sum_{\tau=0}^{\infty} (\rho^\tau) = \dfrac{\alpha}{1-\rho}$ (Unconditional mean)

$\mathrm{var}(Y_t) = \mathrm{var}\left(\sum_{\tau=0}^{\infty} \rho^\tau (\alpha + u_{t-\tau}) \right) = \sigma^2 \left(\sum_{\tau=0}^{\infty} (\rho^2)^\tau \right) = \dfrac{\sigma^2}{1-\rho^2}$ (Unconditional variance)

Both unconditional mean and variance are constant (independent of time)

3.3 Importance of Stationarity in Time-Series

- It is necessary to test if a time-series is stationary (i.e., if there is no change in property of the series in respect of the mean, variance and autocorrelation structure over time)
- The null hypothesis is generally defined as the presence of a unit root and the alternative hypothesis is either stationarity, trend stationarity or explosive root depending on the test used
- Persistence of shocks for non-stationary series
- Problem of spurious regressions - two variables trending over time can have a high R^2 even if the two are totally unrelated
- Standard assumptions for asymptotic analysis will not be valid for non-stationary variables - testing of hypothesis may not be valid

3.4 Trend Stationary and Difference Stationary

		Decision
General Specification	$Y_t = \theta_0 + \theta_1 t + \theta_2 Y_{t-1} + u_t$	Nature of the series depends on sign and magnitude θ_0, θ_1 and θ_2
Random walk without drift and without trend $\theta_0 = 0; \theta_1 = 0; \theta_2 = 1$	$Y_t = Y_{t-1} + u_t$ Or, $Y_t - Y_{t-1} = u_t$ Or, $\Delta Y_t = u_t$	Difference stationary process (DSP)
Random walk with drift, but no trend $\theta_0 \neq 0; \theta_1 = 0; \theta_2 = 1$	$Y_t = \theta_0 + Y_{t-1} + u_t$ $\Delta Y_t = \theta_0 + u_t$	Y has positive stochastic trend if $\theta_0 > 0$ Y has negative stochastic trend if $\theta_0 < 0$ Difference stationary process (DSP)
Deterministic Trend $\theta_0 \neq 0; \theta_1 \neq 0; \theta_2 = 0$	$Y_t = \theta_0 + \theta_1 t + u_t$ $Y_t - E(Y_t) = u_t$	Trend stationary process (TSP)
Random walk with drift and deterministic trend $\theta_0 \neq 0; \theta_1 \neq 0; \theta_2 = 1$	$Y_t = \theta_0 + \theta_1 t + Y_{t-1} + u_t$ $\Delta Y_t = \theta_0 + \theta_1 t + u_t$	Yt is non-stationary
Stochastic Process with drift and deterministic trend $\theta_0 \neq 0; \theta_1 \neq 0; \theta_2 < 1$	$Y_t = \theta_0 + \theta_1 t + \theta_2 Y_{t-1} + u_t$ $\Delta Y_t = \theta_0 + \theta_1 t + (\theta_2 - 1) Y_{t-1} + u_t$	Yt is stationary around a deterministic trend

4. Testing for Presence of Unit Roots

4.1 Steps for Unit Root Test

1. Define the data as time-series
2. Make log transformation of variables, if necessary. Standard unit root tests assume linearity under both the null and the alternative. Violation of this assumption may cause severe size and power distortions both in finite and large samples
3. Setting lags and differences of the variables
4. Setting the lag length (for ADF and PP tests)
5. Carrying out the test with different types of random walk

4.2 Specification of Different Unit Root Tests

	No Drift; No Trend	Drift; No Trend	Drift and Trend	Hypothesis
Dickey-Fuller (DF) Test	$Y_t = \theta_2 Y_{t-1} + u_t$ $\Delta Y_t = \gamma_2 Y_{t-1} + u_t;$ $\gamma_2 = \theta_2 - 1$	$Y_t = \theta_0 + \theta_2 Y_{t-1} + u_t$ $Y_t - Y_{t-1} = \theta_0 + \theta_2 Y_{t-1} - Y_{t-1} + u_t$ $\Delta Y_t = \theta_0 + \gamma_2 Y_{t-1} + u_t; \gamma_2 = \theta_2 - 1$	$Y_t = \theta_0 + \theta_1 t + \theta_2 Y_{t-1} + u_t$ $\Delta Y_t = \theta_0 + \theta_1 t + \gamma_2 Y_{t-1} + u_t;$ $\gamma_2 = \theta_2 - 1$	$H_0 : \gamma_2 = 0$ $H_1 : \gamma_2 < 0$
Augmented Dickey-Fuller (ADF) Test	$Y_t = \theta_2 Y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta Y_{t-j} + u_t$ $\Delta Y_t = \gamma_2 Y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta Y_{t-j} + u_t;$ $\gamma_2 = \theta_2 - 1$	$Y_t = \theta_0 + \theta_2 Y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta Y_{t-j} + u_t$ $\Delta Y_t = \theta_0 + \gamma_2 Y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta Y_{t-j} + u_t;$ $\gamma_2 = \theta_2 - 1$	$Y_t = \theta_0 + \theta_1 t + \theta_2 Y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta Y_{t-j} + u_t$ $\Delta Y_t = \theta_0 + \theta_1 t + \gamma_2 Y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta Y_{t-j} + u_t;$ $\gamma_2 = \theta_2 - 1$	$H_0 : \gamma_2 = 0$ $H_1 : \gamma_2 < 0$
Phillips-Perron (PP) Test	$Y_t = \theta_2 Y_{t-1} + u_t$ $\Delta Y_t = \theta_2 Y_{t-1} + u_t;$ $\gamma_2 = \theta_2 - 1$	$Y_t = \theta_0 + \theta_2 Y_{t-1} + u_t$ $\Delta Y_t = \theta_0 + \gamma_2 Y_{t-1} + u_t;$ $\gamma_2 = \theta_2 - 1$	$Y_t = \theta_0 + \theta_1 t + \theta_2 Y_{t-1} + u_t$ $\Delta Y_t = \theta_0 + \theta_1 t + \gamma_2 Y_{t-1} + u_t;$ $\gamma_2 = \theta_2 - 1$	$H_0 : \gamma_2 = 0$ $H_1 : \gamma_2 < 0$

4.4 Choice of the Alternative Models

Situation	Function Form	Strategy
When the time series is flat (i.e. doesn't have a trend) and potentially slow-turning around zero	No drift; No Trend	**Non-rejection of the null hypothesis:** Series to be differenced for stationarity **Rejection of the null hypothesis:** Stationary series- need not be differenced
When the time series is flat and potentially slow-turning around a non-zero value	Drift; No Trend	**Non-rejection of the null hypothesis:** Series to be differenced for stationarity **Rejection of the null hypothesis:** Stationary series- need not be differenced
When the time series has a trend in it (either up or down) and is potentially slow-turning around a trend line	Drift; Trend	**Non-rejection of the null hypothesis:** Series to be differenced for stationarity **Rejection of the null hypothesis:** Series is trend stationary; To be analysed by using a time trend instead of differencing the data

Note: If the series is exponentially trending, logarithmic transformation of the series is necessary before differencing it.

Summary of the Steps Involved in the Tests

1. Test for unit roots in the process of the variable with the drift and the time trend.

   If the null hypothesis (H_0: $\gamma_2 = 0$) is not rejected, there are unit roots.

If the null hypothesis rejected, check for the presence of the time trend. If the corresponding null hypothesis is rejected, the process is stationary around a time trend.

   If the coefficient of time variable is significant and presence of unit roots is not rejected, the variable has unit roots with the time trend.

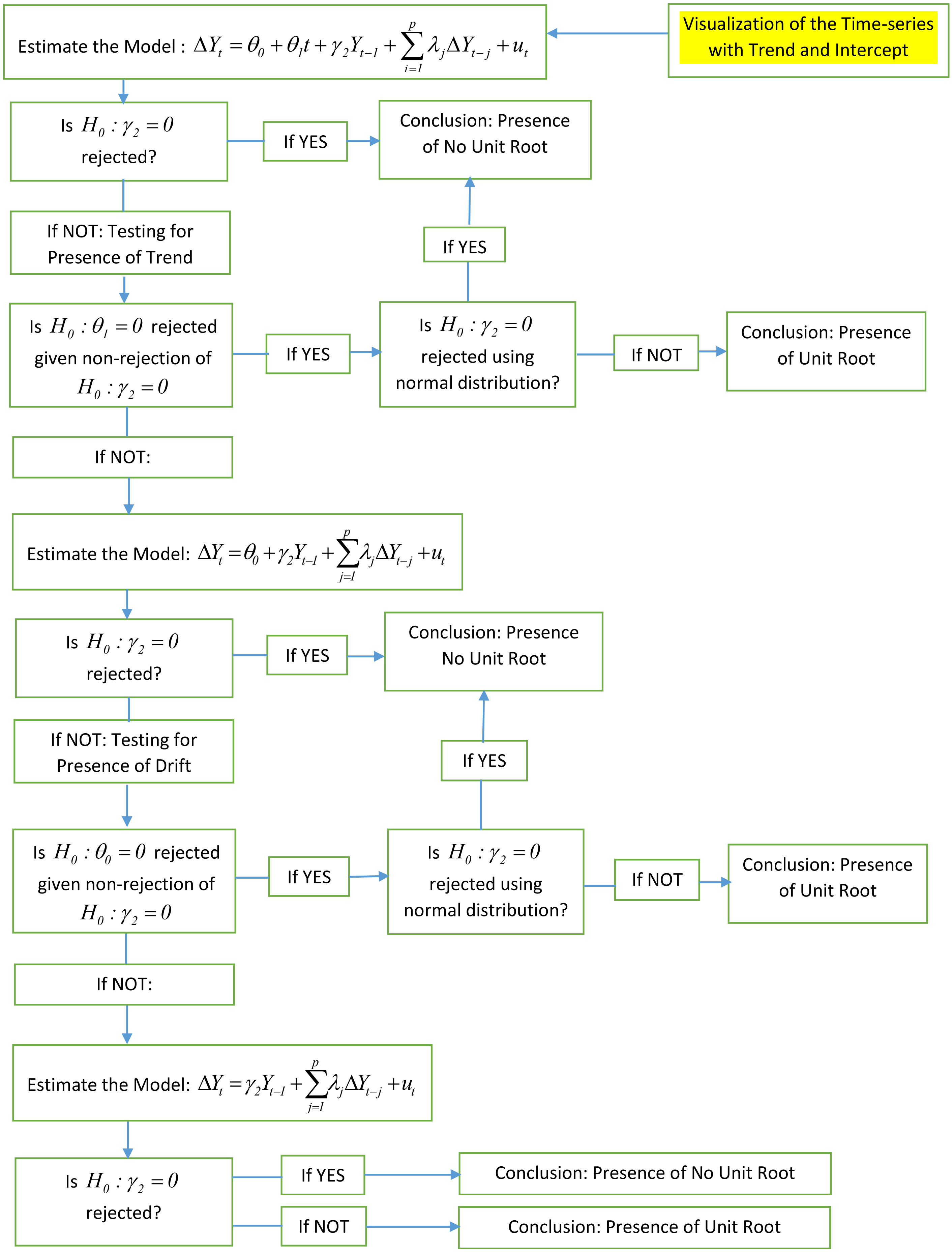
2. If there is no time trend, test for the unit roots with drift.

   Check the process of unit roots. If there are no unit roots, the variable is stationary.

   If a constant term is significant, check the results for unit roots. If the null hypothesis is not rejected, the variable has unit roots with the drift. If the null hypothesis is rejected, the variable has no unit roots.

3. If there is no constant, test for unit roots with no drift and no time trend. If there are no unit roots, the process is stationary.

Selection of Functional Form for Unit Root Test



4.6 Comparative Analysis among the DF, ADF and PP Tests of Unit Root

- Unlike the DF test, the ADF test allows for higher-order autoregressive processes by including the term(s) $\sum_{j=1}^{p} \Delta Y_{t-j}$

- Although the DF and the ADF tests are frequently used in testing for unit roots, there are problems of size distortions and low power. In DF test, the problem of autocorrelation is not corrected for. There is problem of selection of lag length in ADF test. The information criteria such as AIC or BIC often select a low value of the lag length.

- The PP test is based on the similar equation as employed in the DF test (without the lagged differenced terms included in the ADF test).

- The PP test incorporates automatic non-parametric correction procedure for autocorrelated residuals, and usually gives the same conclusions as the ADF tests

- Monte Carlo studies suggest that the PP test has greater power than the ADF test