Computational Statistics

Random number generation.

Method 1: Assume $U[0,1]$ samples are "freely" available.

Let $X \sim f_X(x)$ and we want to generate random samples of X.

Find a transformation $h(u)$ s.t.

$$X = h(U) \quad \text{where} \quad U \sim U(0,1)$$

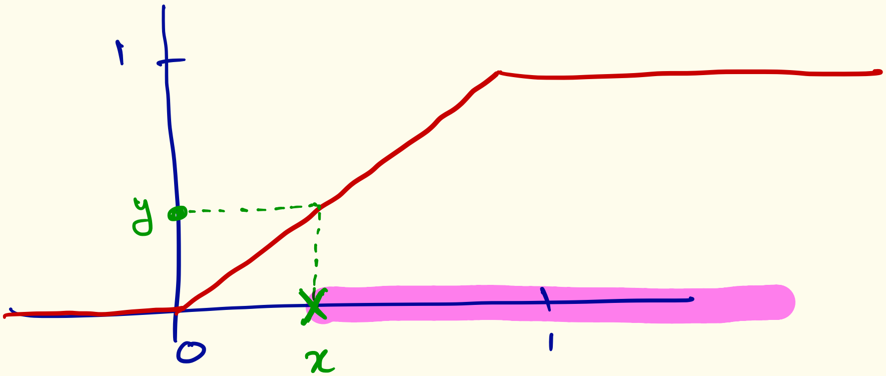
Generate a $U[0,1]$ random sample, let us say u_1

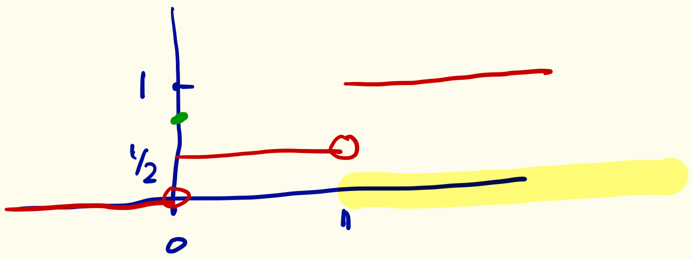
and $x_1 = h(u_1)$ is a sample of $X \sim f_X(x)$

Method 2: Inverse-transform Method

Let X be a random variable with CDF F.

$$F^{-1}(y) = \underset{\min}{\cancel{\inf}} \{x \in \mathbb{R} \mid F(x) \geq y\} \qquad \underline{0 \leq y \leq 1.}$$





$$F^{-1}(3/4) = \min\{x \in \mathbb{R} \mid F(x) \geq 3/4\}$$

$$= 1$$

<u>Result</u>: If $U \sim U[0,1]$, then

$$X = F^{-1}(U)$$

has CDF F.

<u>Pf</u>:

$$\text{Prob}(X \leq x) = \text{Prob}(F^{-1}(U) \leq x)$$

$$= \text{Prob}(U \leq F(x))$$

$$= F(x)$$

$t \in [0,1]$

t $\quad P(U \leq t)$

We should know F / input.

output: random sample from X with CDF F.

step 1. Generate U from $U[0,1]$

step 2: Compute $\underline{X} = F^{-1}(U)$

Method 3: Accept / Reject method

Algo 1: 1) Generate $X \sim U[a,b]$

let $c = \sup\{f(x) : x \in [a,b]\}$

2) Generate $Y \sim U[0,c]$ independently of X.

3) If $Y \leq f(X)$, return $Z = X$

else go to step 1.

Thm: The random variable generated according to Algo 1 has the desired density $f(x)$.

Proof: Define

$\mathcal{A} = \{ (x,y) : 0 \le y \le c \}$, $B = \{ (x,y) : 0 \le y \le f(x) \}$

Claim: (X,Y) is uniform on $\mathcal{A}$.

Let $q(X,Y)$ be the joint density of (X,Y).

$$q(x,y) = \begin{cases} g(x) q(y|x) & \text{if } (x,y) \in \mathcal{A} \\ 0 & \text{o.w.} \end{cases}$$

$q(y|x)$ equals $\frac{1}{c g(x)}$ for $y \in [0, c]$

$$\therefore \boxed{q(x,y) = \frac{1}{c} \text{ for every } (x,y) \in \mathcal{A} \\ = 0 \text{ o.w.}}$$

Let (X^*, Y^*) be the first accepted point in B.

<u>claim:</u> Since (X, Y) is uniform on A

(X^*, Y^*) is uniform on B.

Joint density of (X^*, Y^*) is 1 on B.

$$\text{Marginal of } X^* = \int_0^{f(x)} 1 \, dy = f(x) \qquad \blacksquare$$