Class November 11

(1) $A \in \mathbb{R}^{m \times n}$ $\quad A = U \Sigma V^T$ the SVD. of A

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

$$\|A\|_2 = \sigma_1 = \text{max. mag}(A)$$

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|Ax\|_2$$

Consider $v_1 \in \mathbb{R}^n$ where v_1 is the first col. of V.

$$\underline{A v_1 = \sigma_1 u_1} \leftarrow (*)$$

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \geq \frac{\|Av_1\|_2}{\|v_1\|_2} = \frac{\|\sigma_1 u_1\|_2}{\|v_1\|_2} = \underline{\underline{\sigma_1}}$$

$$\Rightarrow \quad \|A\|_2 \geq \sigma_1$$

Let $x \in \mathbb{R}^n \quad (x \neq 0)$

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

$$\|x\|_2^2 = |c_1|^2 + |c_2|^2 + \cdots + |c_n|^2$$

where $V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

$V \in \mathbb{R}^{n \times n}$

orthogonal.

$$\|x\|_2^2 = x^T x$$

$$= (c_1^T v_1^T + \cdots + c_n v_n^T)(c_1 v_1 + \cdots + c_n v_n)$$

$$= |c_1|^2 + \cdots + |c_n|^2$$

$\left\{ v_i^T v_j = 1 \quad v_i = v_j \right.$

$= 0$

$$Ax = A(c_1 v_1 + \cdots + c_n v_n)$$

$$= c_1 A v_1 + \cdots + c_n A v_n$$

$$= c_1 \sigma_1 u_1 + \cdots + c_r \sigma_r u_r$$

$$\|Ax\|_2^2 = |c_1|^2 \sigma_1^2 + \cdots + |c_r|^2 \sigma_r^2$$

$$\leq \boxed{(|c_1|^2 + \cdots + |c_r|^2)} \sigma_1^2$$

$$\leq \sigma_1^2 \|x\|_2^2$$

$$\Rightarrow \frac{\|Ax\|_2^2}{\|x\|_2^2} \leq \sigma_1^2 \Rightarrow \frac{\|Ax\|_2}{\|x\|_2} \leq \sigma_1 \Rightarrow \underline{\|A\|_2 \leq \sigma_1}$$

$\blacksquare$

$\|A\|_2 = \sigma_1 = \text{max.mag}(A)$

- Let $A \in \mathbb{R}^{n \times n}$ invertible.

$$A = U \Sigma V^T \Rightarrow A^{-1} = V \Sigma^{-1} U^T$$

$$\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_n}$$

$\text{minmag}(A) = \sigma_n$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

$$\Rightarrow 0 < \frac{1}{\sigma_1} \leq \frac{1}{\sigma_2} \leq \cdots \leq \boxed{\frac{1}{\sigma_n}}$$

$$\|A^{-1}\|_2 = \frac{1}{\sigma_n} = \frac{1}{\text{minmag}(A)}$$

$$k_2(A) = \|A\|_2 \|A^{-1}\|_2 = \sigma_1 \cdot \frac{1}{\sigma_n} = \frac{\sigma_1}{\sigma_n} = \frac{\text{maxmag}(A)}{\text{minmag}(A)}$$

• Let $A \in \mathbb{R}^{m \times n}$ $\quad m > n \quad$ and $\quad \operatorname{rank}(A) = n$.

$\sigma_1, \dots, \sigma_n \neq 0$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$

$\sigma_1 \leftarrow$ max mag $\qquad k_2(A) = \dfrac{\sigma_1}{\sigma_n}$

$\sigma_n \leftarrow$ min mag

If A is not full rank,

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \qquad r < \min\{m, n\}$

max.mag $\rightarrow \sigma_1$

min.mag $\rightarrow 0$

$$\Rightarrow k_2(A) = \frac{\text{max.mag}(A)}{\text{min.min}(A)} = \infty$$

★

$A = U \Sigma V^T$

$A \in \mathbb{R}^{m \times n}$

$\begin{bmatrix} \; \end{bmatrix}, \begin{bmatrix} \; \end{bmatrix}$

$$\underbrace{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r}_{> \varepsilon} \geq \underbrace{\sigma_{r+1} \geq \cdots \geq \sigma_n}_{< \varepsilon} \geq 0$$

full column rank matrix

Let $\varepsilon = 10^{-2}$ - be given tolerance.

$$\sigma_r > \varepsilon$$

$$\varepsilon > \sigma_{r+1}$$

numerical rank of $A = r$

(tol. ε) $\leftarrow$ user specification/design parameter.

$A = U \Sigma V^T$ $\qquad$ $\text{rank}(A) = r$

$A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$

$\|A\|_2 = \sigma_1$ ✓

$\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$ ✓

$A = U \Sigma V^T$

$\Rightarrow V A U^T = \Sigma$

$\|V A U^T\|_F^2 = \|A\|_F^2$

$= \|\Sigma\|_F^2$

$= \sigma_1^2 + \cdots + \sigma_r^2$

<u>Observe:</u> $\|A\|_F^2 = \underbrace{\sum_i \sum_j A_{ij}^2} = \text{trace}(A^T A)$

$\|Q_1 A\|_F^2 = \|A\|_F^2$ where $Q_1 \in \mathbb{R}^{m \times m}$ orthogonal

$\|Q_1 A\|_F^2 = \text{trace}((Q_1 A)^T Q_1 A) = \text{trace}(A^T Q_1^T Q_1 A) = \text{trace}(A^T A)$

$= \|A\|_F^2$

$\|Q_1 A Q_2\|_F^2 = \|A\|_F^2$ for Q_1 & Q_2 orthogonal.

Let $A \in \mathbb{R}^{m \times n}$, Let $r = \text{rank}(A) < \underbrace{\min(m,n)} = p$

Rank deficient matrix

$$A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T \quad \text{--- (1)}$$

Let any $\varepsilon > 0$ be given.

<u>claim</u>: There exists a full rank <u>matrix A_ε</u> s.t.

$$\|A - A_\varepsilon\|_2 < \varepsilon.$$

For every rank-deficient matrix A, there is an arbitrarily close full rank matrix.

$$A_\varepsilon = \underbrace{\sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T} + \frac{\varepsilon}{2} \underbrace{u_{r+1} v_{r+1}^T} + \boxed{\frac{\varepsilon}{3}} \underbrace{u_{r+2} v_{r+2}^T} + \cdots$$

$$\cdots + \boxed{\frac{\varepsilon}{3}} \underbrace{u_p v_p^T}$$

$0 < \delta_1 < \varepsilon/2$

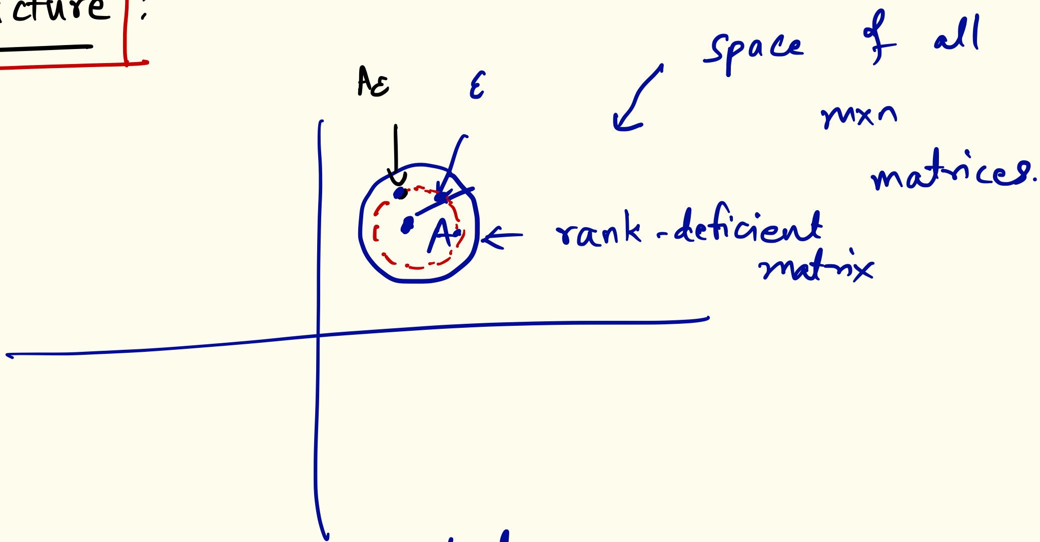
$0 < \delta_{p-(r+1)} < \varepsilon/2$

All singular values $> 0 \Rightarrow A_\varepsilon$: full rank matrix

$$A_\epsilon - A = \boxed{\frac{\epsilon}{2}}\, u_{r+1} v_{r+1}^T + \underbrace{\boxed{\frac{\epsilon}{3}}}_{\delta_1} u_{r+2} v_{r+2}^T + \cdots + \underbrace{\boxed{\frac{\epsilon}{3}}}_{\delta} u_p v_p^T$$

$$\|A - A_\epsilon\|_2 = \|A_\epsilon - A\|_2 = \frac{\epsilon}{2} < \epsilon$$

Hypothetical picture:



Full-rank matrices are abundant.

Low Rank Approximation (LRA) Eckart-Young (1930's)

Let $A \in \mathbb{R}^{m\times n}$ be a given matrix which is full rank. $(m \geq n)$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T \quad : \text{svd of } A.$$

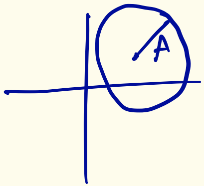
Define $A_k = \sigma_1 u_1 v_1^T + \cdots + \sigma_k u_k v_k^T$ $\quad \sigma_k$: k^{th} singular value.

$k < n$

Then $\underbrace{\| A - A_k \|_2}_{= \sigma} \leq \| A - B \|_2$

for any $B \in \mathbb{R}^{m\times n}$

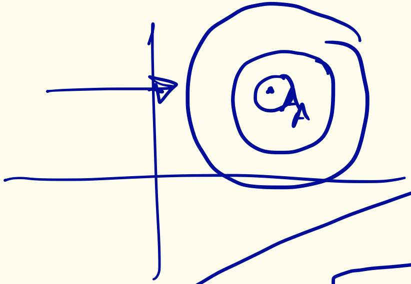
and $\text{rank}(B) \leq k$



$A \in \mathbb{R}^{n \times n}$ and $\text{rank}(A) < n$

A is **NOT** invertible.

$b \notin \text{colspace}(A)$



$\boxed{Ax = b}$ ← what we wanted to solve.

$\boxed{(A + \Delta A)\hat{x} = (b + \Delta b)}$ ← what we really solve.

very large condition number.

$A = \underline{\hspace{3cm}} + \sigma_n u_n v_n^T$

$\sigma_n \to 0$

$(A + \Delta A)$ $\qquad + \tilde{\sigma}_n$

"very small"