

Simplex Method

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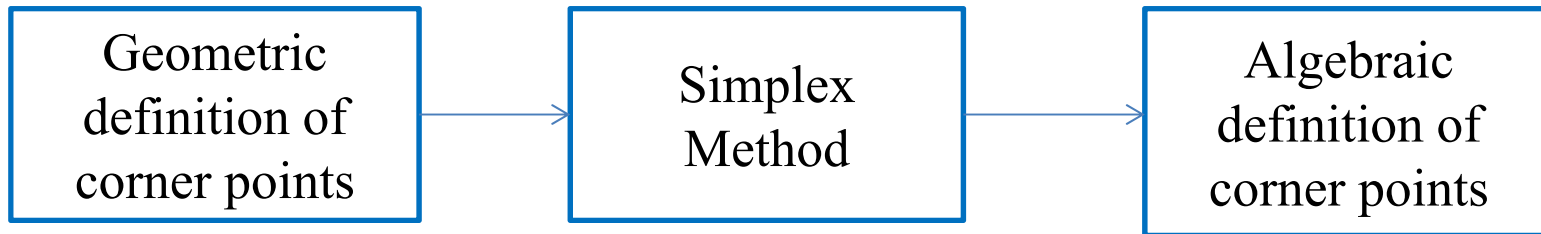
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Introduction: Simplex Method

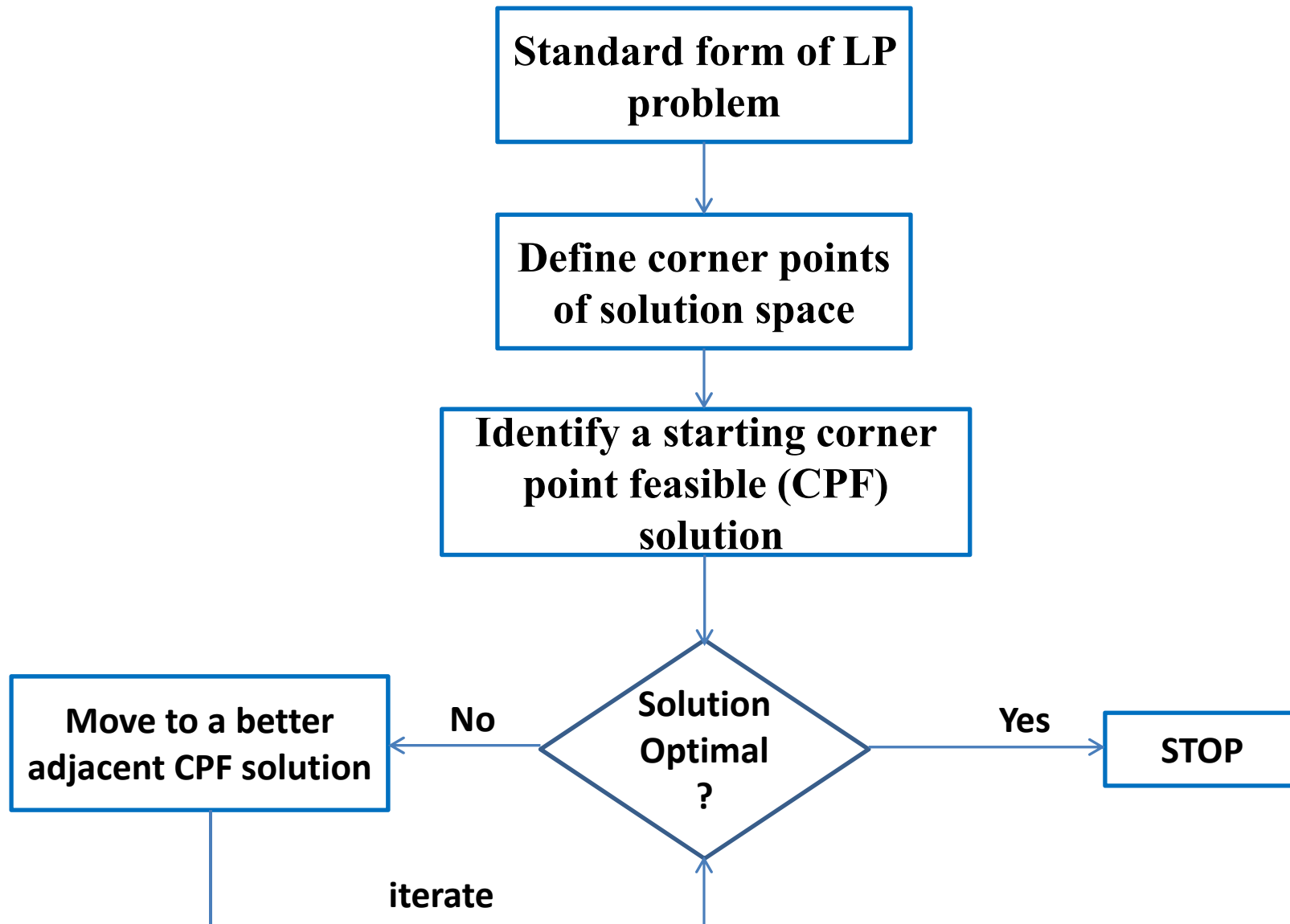
- Graphical method: not Suitable for more than two variables
- Developed by George Dantzig in 1947
- Can be solved any LP model of the following form (called standard form)
 - Maximization objective
 - all functional constraints \leq type,
 - RHS not Negative
 - Nonnegativity constraints on all variables

How Simplex method works?



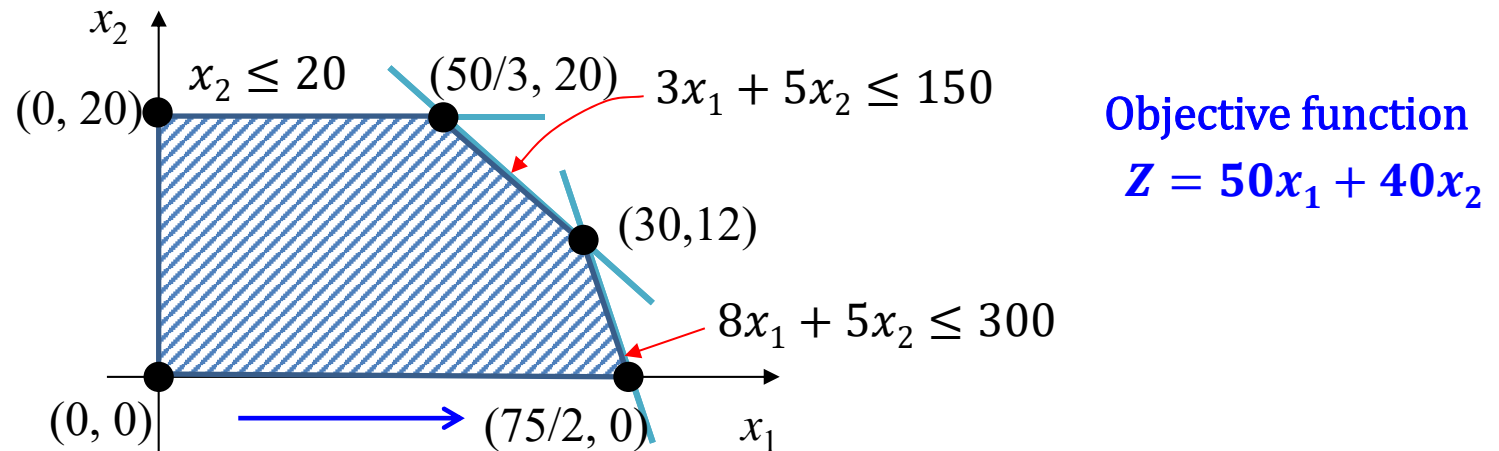
Simplex method translates the geometric procedure to algebraic procedure

Schematic Representation



Simplex Method: Geometrically

Solution space of Tech Edge Problem



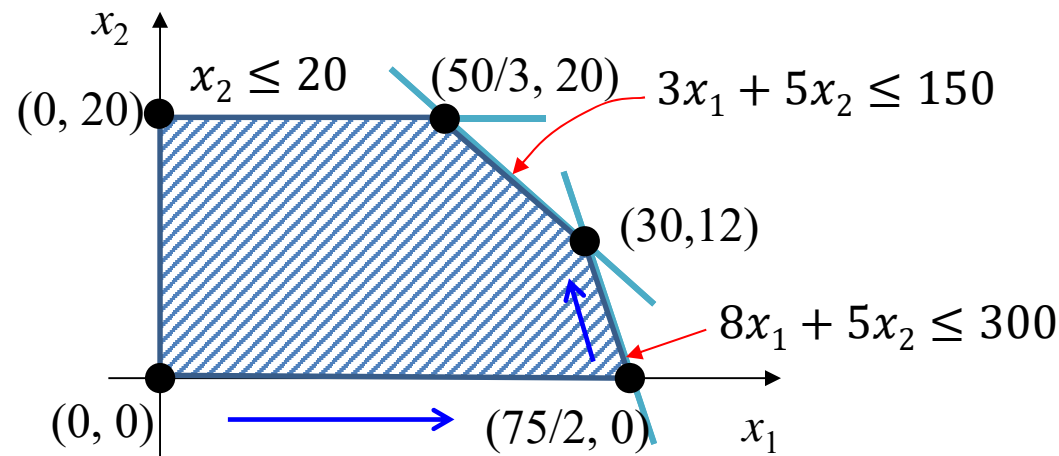
Initialization: Start with CPF Solution $(0, 0)$ (convenient choice)

Optimality test: Is Current CPF solution optimal? No (Better adjacent CPF solution)

Iteration 1: Move to a better adjacent CPF solution. How?

- Since Z improves at a faster rate along x_1 , move along x_1 as far as permitted by the feasible region
- Stop at the intersection of $x_2 = 0$ with $8x_1 + 5x_2 = 300$
- Solve for the intersection. Put $x_2 = 0$: solution $(75/2, 0)$ and $Z = 1875$
- Optimal? Check another adjacent solution

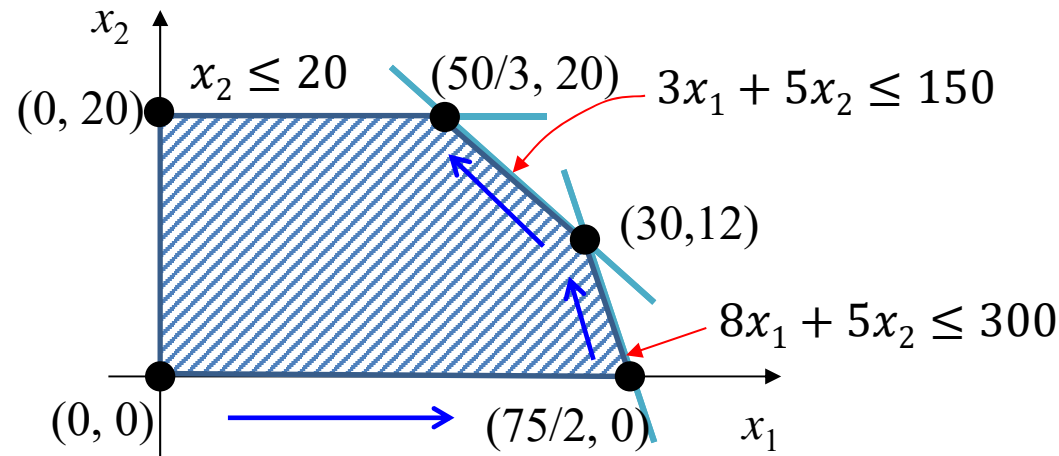
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Iteration 2: Move towards the 3rd CPF solution

- Move along $8x_1 + 5x_2 = 300$ as far as permitted by the feasible region
- Stop at the intersection of $8x_1 + 5x_2 = 300$ with $3x_1 + 5x_2 = 150$
- Solve for the intersection of $3x_1 + 5x_2 = 150$ and $8x_1 + 5x_2 = 300$: Solution $(30, 12)$ and $Z = 1980$
- Optimal ? Check another adjacent point

Contd..



Iteration 3: Move towards the 4th CPF solution

- Move along $3x_1 + 5x_2 = 150$ as far as permitted by the feasible region
- Stop at the intersection with $x_2 = 20$
- Solve for the intersection of $x_2 = 20$ and $3x_1 + 5x_2 = 150$: Solution $(50/3, 20)$ and $Z = 1633.3$
- No better adjacent CPF solutions

Hence, optimal solution $(30, 12)$ with $Z = 1980$

Theorem (without proof): If a CPF solution has no better adjacent CPF solutions, then that CPF solution is the optimal solution.

Simplex: Algebraically

Algebraic procedure is based on solving system of equations

Convert to augmented form

Convert functional inequality constraints to equivalent equality constraints (equations)

- Add slack (for \leq type functional constraint)
- Subtract surplus (for \geq type functional constraint)
- Example

(i) $3x_1 + 5x_2 \leq 150$

$$\Leftrightarrow 3x_1 + 5x_2 + x_3 = 150 \text{ and } x_3 \geq 0,$$

where x_3 is slack variable (amount by which resource availability exceeds its usage)

(ii) $3x_1 + 5x_2 \geq 150$

$$\Leftrightarrow 3x_1 + 5x_2 - x_4 = 150 \text{ and } x_4 \geq 0$$

Where x_4 is surplus variable (e.g. in diet problem surplus of a nutrient in diet plan over its minimum requirement)

(the original form has been augmented by some supplementary variables)

- Augmented form of Tech Edge problem**

$$\begin{array}{ll}
 \text{Maximize} & Z = 50x_1 + 40x_2 \\
 \text{Subject to} & 3x_1 + 5x_2 + x_3 = 150 \\
 & x_2 + x_4 = 20 \\
 & 8x_1 + 5x_2 + x_5 = 300 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{array}$$

Note: Slack variables x_3 , x_4 and x_5 do not enter into objective function

- Interpretation for slack variable**

- (i) If a slack variable = 0 in a solution**

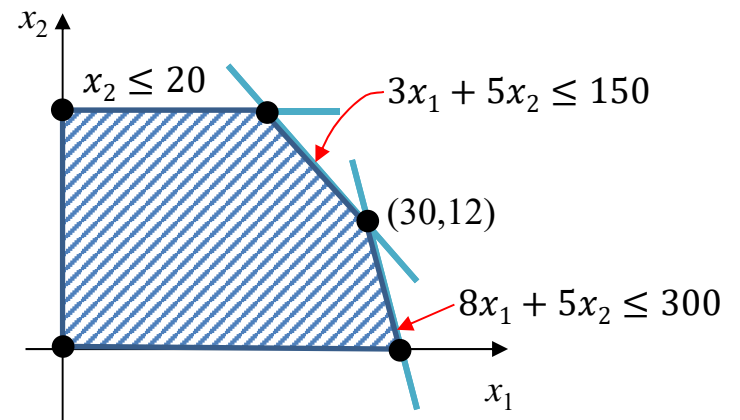
⇒ the solution lies on constraint boundary of the respective functional constraint and the constraint is exactly satisfied, called tight constraint

- (ii) If a slack variable > 0 in a solution**

⇒ the solution lies on the **feasible side** of the respective functional constraint

- (iii) If a slack variable < 0 in a solution**

⇒ the solution lies on the **infeasible side** of the respective functional constraint



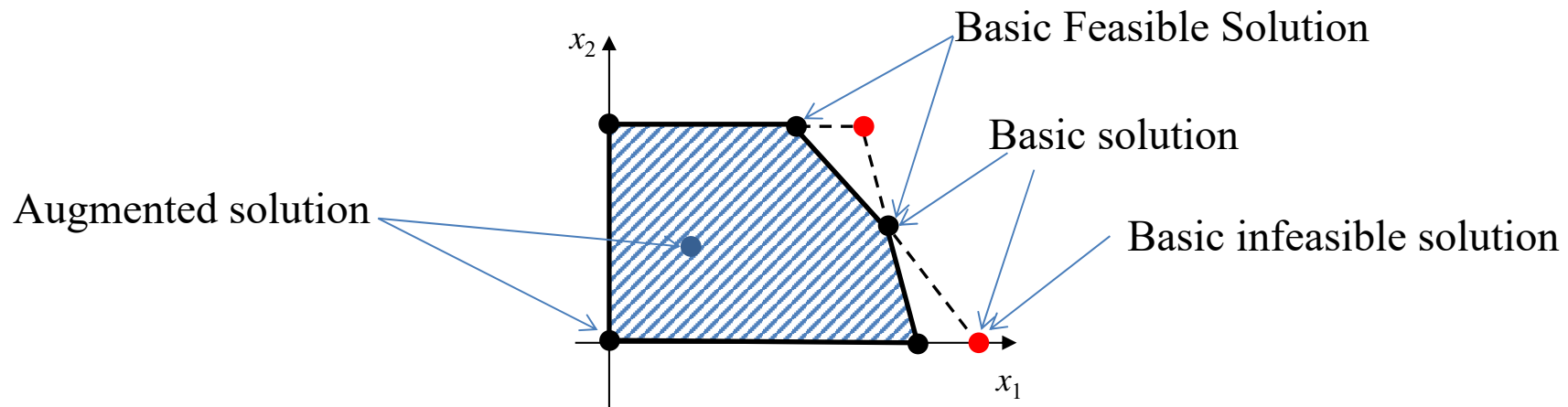
Example: At CPF solution $(30, 12)$, $x_3 = 0$, $x_4 > 0$ and $x_5 = 0$

Important terms for augmented form

➤ **Augmented solution:** solution for (original + slack) variables

e.g. for solution $(0, 0) \rightarrow$ Augmented solution $(0, 0, 150, 20, 300)$

$$(x_1, x_2) \rightarrow (x_1, x_2, x_3, x_4, x_5)$$



➤ **Basic solution:** Augmented corner point solution (could be feasible or infeasible)

- Basic feasible solution (BFS): Augmented CPF solution
- CPF solution $(0, 0)$ is equivalent to BFS $(0, 0, 150, 20, 300)$: difference is due to inclusion of value of slack variables

Contd...

➤ Degrees of freedom of system of equations

= number of variables - number of equations = number of non-basic variables

➤ **Non-basic variables (NBVs):** Set of variables “free” to be set to an arbitrary value (set to zero in Simplex method), and other variables are called **basic variables**

➤ **Set of basic variables (BVs)** is called **Basis**

➤ **Number of basic variables = number of functional constraint**

e.g. In solution (0, 0, 150, 20,300), x_1, x_2 are non-basic variables and x_3, x_4, x_5 are basic variables

➤ To obtain a basic solution, set non-basic variables to zero and solve equations to obtain the value of basic variables.

➤ If a basic solution satisfies non-negativity constraints, the basic solution is a basic feasible solution (BFS)

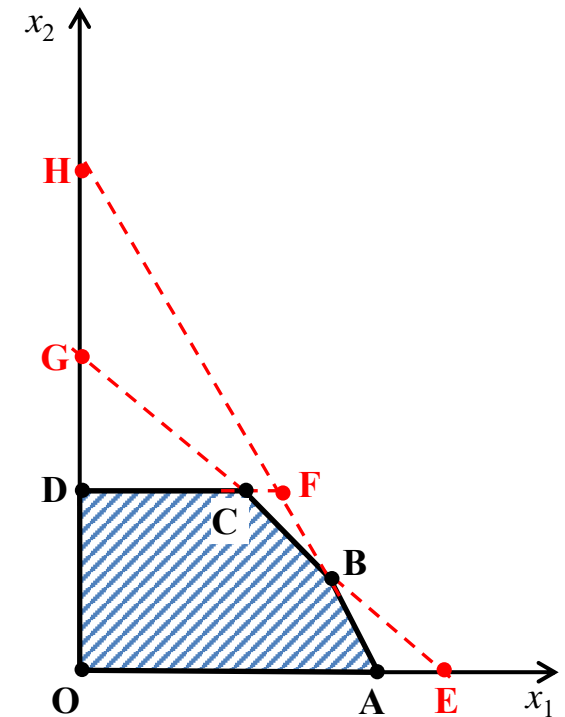
➤ **Maximum number of basic solutions = ${}^{n+m}C_n$**

Example: Basic Solution

➤ Tech edge problem, $n = 2$ and $m = 3 \Rightarrow {}^5C_2 = 10$

$$\begin{aligned} 3x_1 + 5x_2 + x_3 &= 150 \\ x_2 + x_4 &= 20 \\ 8x_1 + 5x_2 + x_5 &= 300 \end{aligned}$$

SN	NBVs (set to zero)	BVs	Solve for basic variables	Associated Corner Point	Feasible?
1	x_1, x_2	x_3, x_4, x_5	150, 20, 30	O	Yes
2	x_1, x_3	x_2, x_4, x_5	30, -10, 150	G	No
3	x_1, x_4	x_2, x_3, x_5	20, 50, 200	D	Yes
4	x_1, x_5	x_2, x_3, x_4	60, -150, -40	H	No
5	x_2, x_3	x_1, x_4, x_5	50, 20, -100	E	No
6	x_2, x_4	x_1, x_3, x_5	Not Possible		
7	x_2, x_5	x_1, x_3, x_4	75/2, 75/2, 20	A	Yes
8	x_3, x_4	x_1, x_2, x_5	50/3, 20, 200/3	C	Yes
9	x_3, x_5	x_1, x_2, x_4	30, 12, 8	B	Yes
10	x_4, x_5	x_1, x_2, x_3	25, 20, -25	F	No



➤ **Two basic feasible solutions are adjacent** if all but one non-basic (basic) variables are the same.

⇒ Moving from one CPF solution to adjacent CPF solution makes one non-basic variable to basic and vice-versa.

Simplex: Algebraically contd...

I. Augmented form

$$\text{Max } Z = 50x_1 + 40x_2 \quad (0)$$

$$\text{S.t. } 3x_1 + 5x_2 + x_3 = 150 \quad (1)$$

$$x_2 + x_4 = 20 \quad (2)$$

$$8x_1 + 5x_2 + x_5 = 300 \quad (3)$$

II. Starting solution

origin (0, 0) – convenient to see Initial BFS, because each equation has only one and different basic variables each with coefficient 1. (**Proper Form**)

So, Initial BFS (0, 0, 150, 20, 300), $Z=0$

III. Iterate

$$Z = 50x_1 + 40x_2$$

Optimal = ? No, why?

coefficient still positive \Rightarrow improvement possible by setting positive values of x_1 and/or x_2

Where to move? (ENTERING VARIABLE)

$$50 > 40 \Rightarrow x_1 \text{ enters the basis}$$

When to stop? (LEAVING VARIABLE)

As much permissible by feasible region

i.e. increase x_1 while keeping the non-basic variable $x_2 = 0$, system of equations reduces to:

$$3x_1 + x_3 = 150$$

$$x_4 = 20$$

$$8x_1 + x_5 = 300$$

Non-negative constraints impose certain restriction on value of x_1

$$x_3 = 150 - 3x_1 \geq 0 \Rightarrow x_1 \leq \frac{150}{3} = 50$$

$$\Rightarrow x_4 = 20 > 0 \Rightarrow \text{No upper bound on } x_1$$

$$x_5 = 300 - 8x_1 \geq 0 \Rightarrow x_1 \leq \frac{300}{8} = 75/2$$

So, take minimum upper bound on x_1

$$x_5 = 0, x_1 = 75/2$$

$\Rightarrow x_1$ can be increased to $75/2$ at which x_5 drops to 0.

so, $x_5 \rightarrow$ **Leaving variable (to become non-basic variable in new BFS)**
 $x_1 \rightarrow$ **Entering variable (to become basic variable in new BFS)**

- Above calculation is referred as the **minimum ratio test** to identify leaving variable.
- Minimum ratio test to determine which basic variable drops to zero first as the entering basic variable is increased

LEAVING VARIABLE RULE

➤ Minimum ratio test

For entering variable x_j , pick the variable as LEAVING corresponding to

$$\min_{for\ all\ i} \left\{ \frac{b_i}{a_{ij}} \right\}, \text{ where } a_{ij} > 0, b_i \geq 0$$

NOTE: b_i can be zero which shows “degeneracy”, to be discussed later

➤ How to determine x_3 and x_4 ?

1. Treat Z as a basic variable and objective function as an equation added to the system of equations.
2. Bring the system of equations in proper form using Gaussian Elimination method i.e., Bring current pattern of coefficients of leaving variable to entering variable by performing elementary row operations.

➤ Elementary Row Operations

- Multiply or divide an equation by a non-zero constant
- Add or subtract a multiple of one equation to another equation.

Initial system of equations

$$Z - 50x_1 - 40x_2 = 0 \quad (0)$$

$$3x_1 + 5x_2 + x_3 = 150 \quad (1)$$

$$x_2 + x_4 = 20 \quad (2)$$

$$8x_1 + 5x_2 + x_5 = 300 \quad (3)$$

- In Equation (3) x_1 should become basic variable by replacing x_5 .
- The pattern of coefficients of x_1 in above equations should be (0,0,0,1), respectively

$$R_{3'} \rightarrow R_3/8 \quad \Rightarrow \quad x_1 + \frac{5}{8}x_2 + \frac{1}{8}x_5 = 75/2 \quad (3')$$

Eliminating x_1 from Equations (0), (1) and (2)

$$R_{0'} \rightarrow R_0 + 50R_{3'} \Rightarrow Z - \frac{70}{8}x_2 + \frac{50}{8}x_5 = 1875 \quad (0')$$

$$R_{1'} \rightarrow R_1 - 3R_{3'} \Rightarrow \frac{25}{8}x_2 + x_3 - \frac{3}{8}x_5 = \frac{75}{2} \quad (1')$$

$$x_2 + x_4 = 20 \quad (2') \text{ [same as Eq. (2)]}$$

- With non-basic variables, $x_2 = x_5 = 0$

New BFS = $(75/2, 0, 75/2, 20, 0)$, $Z = 1875$

Contd..

- Is solution optimal?
- No, why?

$$Z = \frac{70}{8}x_2 - \frac{50}{8}x_5 + 1875$$

Improvement possible by increasing x_2

Entering variable: x_2

Leaving variable ?

Increase x_2 while keeping the current non-basic variable $x_5 = 0$, system of equations reduces to:

$$\text{From (1'), } x_3 = \frac{75}{2} - \frac{25}{8}x_2 \geq 0 \Rightarrow x_2 \leq 12$$

$$\text{From (2'), } x_4 = 20 - x_2 \geq 0 \Rightarrow x_2 \leq 20$$

$$\text{From (3'), } x_1 = \frac{75}{2} - \frac{5}{8}x_2 \geq 0 \Rightarrow x_2 \leq 60$$

so, (1') gives minimum upper bound on x_2

$$x_3 = 0, x_2 = 12$$

$\Rightarrow x_2$ can be increased to 12 at which x_3 drops to 0.

so, $x_3 \rightarrow$ Leaving variable (to become non-basic variable in new BFS)

$x_2 \rightarrow$ Entering variable (to become basic variable in new BFS)

Contd..

- In Equation (1') x_2 should become basic variable by replacing x_3 . Current coefficients of x_3 are (0,1,0,0)
- Perform elementary algebraic operation to make the coefficient of x_2 as (0,1,0,0)

New set of equations

$$Z + \frac{14}{5}x_3 + \frac{26}{5}x_5 = 1980 \quad (0'')$$

$$x_2 + \frac{8}{25}x_3 - \frac{3}{25}x_5 = 12 \quad (1'')$$

$$x_4 - \frac{8}{25}x_3 + \frac{3}{25}x_5 = 8 \quad (2'')$$

$$x_1 - \frac{1}{5}x_3 + \frac{1}{5}x_5 = 30 \quad (3'')$$

New BFS = (30, 12, 0, 8, 0), $Z = 1980$

- Optimal ? Yes
- Why ? $Z = -\frac{14}{5}x_3 - \frac{26}{5}x_5 + 1980$
- No more improvement possible

Contd..

- In summary
- How basic and non-basic sets are changing from one iteration to another.

Iteration	Non-basic	Basic	BFS	Z
Initial	$x_1 = 0, x_2 = 0$	$x_3 = 150, x_4 = 20, x_5 = 300$	$(0, 0, 150, 20, 300)$	0
1	$x_2 = 0, x_5 = 0$	$x_1 = 75/2, x_3 = 75/2, x_4 = 20$	$(75/2, 0, 75/2, 20, 0)$	1875
2	$x_3 = 0, x_5 = 0$	$x_1 = 30, x_2 = 12, x_4 = 8$	$(30, 12, 0, 8, 0)$	1980

