Partial Differential Equation (PDE) Part C

Teacher: A Ganguly

In this last Part of course, we will study three celebrated 2nd order linear PDEs subject to different types of Boundary Conditions (BC) and Initial Conditions (IC):

1. Wave Equation: (1D) $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$, [$x$ is position and t is time]
2. Heat Equation (or Diffusion Equation): (1D) $\frac{\partial\psi}{\partial t} = \alpha\frac{\partial^2\psi}{\partial x^2}$
3. Laplace Equation: $\nabla^2\psi = 0$, [∇^2 is called Laplacian Operator]

   where in 3D Cartesian coordinates, $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

We already know above three PDEs are respectively Hyperbolic, Parabolic and Elliptic [See classification scheme discussed in Part B]. Before going into detailed study of these PDEs, we need some conception about Fourier series.

- Introduction to Fourier Series

Historically it was Great Mathematician Joseph Fourier who in his research paper in 1807 showed that any arbitrary piecewise continuous function $f(x)$ may be represented by an infinite trigonometric series of sine and cosine functions. Fourier's motivation was to express the solution of "Heat Equation" over a metal plate analytically. Hence, the name of the trigonometric series is given as "Fourier series".

The study of the Theory of Fourier Series involves seeking the important question of whether Fourier series converges always to the function $f(x)$ which is being represented by the series, what the principles are to determine Fourier coefficients etc. In this section, I'll not deal with this study, my intention is to quote standard results without proof, as these will be required in later stage when I'll discuss solution of some well-known second order PDEs, e.g. Finding Fourier series solution of Wave Equation needs the following results.

At the outset Fourier series is a way of approximating a periodic function like in Taylor's series expansion for smooth functions. One major concept is that even non-periodic function defined on a finite interval may be represented by Fourier series by making its periodic extensions on full real line, where, of course, our interest will be the value of Fourier series in the finite interval only.

We will consider only real-valued function $f(x)$ of single real variable x.

- Suppose $f(x)$ is a periodic function of period 2π defined on $\mathbb{R}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]. \tag{1.1}$$

Fourier coefficients a_n, b_n can be computed by using following orthogonality property of sine and cosine functions:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \begin{cases} 0\,, m \neq n \\ \pi\,, m = n \end{cases}, \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \begin{cases} 0\,, m \neq n \\ \pi\,, m = n \end{cases} \tag{1.2a}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0, \forall\, m, n\,. \qquad [m, n \text{ are non-negative integers }] \tag{1.2b}$$

Assuming that the infinite series is term by term integrable, we see that if we multiply (1.1) by $\sin(mx)$, $\cos(mx)$ and integrate over the interval $[-\pi, \pi]$, then due to the orthogonal relations (1.2), all but one term in RHS will disappear, thereby getting Fourier coefficients as follows:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\;, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\;,\quad n = 0,1,2,\ldots, \tag{1.3}$$

where for the case of finding a_0, we just integrate (1.1), and use the relation

$$\int_{-\pi}^{\pi} \sin(mx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos(mx)\mathrm{d}x = 0, m, n \neq 0. \tag{1.4}$$

Note that in formula (1.3), I do not separately put a_0, and do not exclude $n = 0$, as in most literature, because for $n = 0$, formula gives correct expression for a_0, and the extra term b_0 is, in fact, identically zero by the formula. Clearly, formula (1.3) yield coefficients if the function $f(x)$ is integrable in the interval. Some points worth mentioning:

- Instead of the interval $[-\pi, \pi]$, we may take any interval of length 2π on $\mathbb{R}$, in formula (1.3), for instance, we may choose $[0,2\pi]$, if periodicity of $f(x)$ is exhibited in $[0,2\pi]$. It is easy to verify that orthogonal relations hold true for any interval of length 2π.
- If instead of $\mathbb{R}$, $f(x)$ is defined only on $[-\pi, \pi]$, that means given $f(x)$ is no longer periodic. In this circumstance, we make a periodic extension of $f(x)$ on $\mathbb{R}$ by $F(x)$, where $F(x) = f(x), x \in [-\pi, \pi]$ and $F(x) = F(x + 2k\pi)$ outside for non-zero integer k, i.e we make a copy of shape of

$f(x)$ in all other interval of length 2π. Note that $F(x)$, thus constructed, is periodic so that above Fourier series representation is possible for it, but our interest lies only in $[-\pi, \pi]$.

- If given $f(x)$ is even or odd function, then Fourier series (1.1) reduces to Fourier-cosine or Fourier-sine series as follows:

$$f(-x) = f(x): b_n = 0 \Longrightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \tag{1.5a}$$

$$f(-x) = -f(x): a_n = 0 \Longrightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin(nx). \tag{1.5b}$$

- If $f(x)$ is defined on $[0, \pi]$, then there are two ways of periodic extension:

  Even extension: Define $f(-x) = f(x), x \in [-\pi, 0]$, and then make identical copies in all intervals of length 2π, by constructing $F(x)$, discussed above for second instance. In this case we have half-range Fourier-cosine series (1.5a), wherein a_ns are given by

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, \mathrm{d}x. \tag{1.6a}$$

  Odd Extension: Define $f(-x) = -f(x), x \in [-\pi, 0]$, and then make identical copies in all intervals of length 2π, by constructing $F(x)$, discussed above for second instance. In this case we have half-range Fourier-sine series (1.5b), wherein b_ns are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, \mathrm{d}x. \tag{1.6b}$$

❖ Suppose $f(x)$ is a periodic function of period $2L$ defined on $\mathbb{R}$

Simply make a change of variables $x \to x' = \pi x / L$ so that we will have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right], \tag{1.7}$$

where the coefficients are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \;, b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \;, \quad n = 0,1,2, \ldots \tag{1.8}$$

Note that all of the arguments for above instances apply for period $2L$, so I'm not repeating those.

Fourier series converges to $f(x)$ at all points in the interval except for discontinuities. At point of discontinuity and at end-points it converges to average of right-hand and left-hand limits of $f(x)$, e.g. At the point $x = -L, L$, value of Fourier series converges to $A = \frac{1}{2}[\lim_{x \to -L+0} f(x) + \lim_{x \to L-0} f(x)]$.

❖ Generalized Fourier Series

We have seen that Fourier Series is an expansion of $f(x)$ in terms of elements of set S of periodic functions $\{1, \sin(nx), \cos(nx)\}_{n=1}^{\infty}$. Now property of the set S is that it is a complete* set of orthogonal functions. There exists many other such set, e.g. $\{P_n(x)\}_{n=0}^{\infty}$ is also a complete set of orthogonal functions, where $P_n(x)$ is Legendre polynomial, defined on $[-1,1]$, and is given by

$$P_n(x) = \sum_{m=0}^{N}(-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m}, N = n/2 \text{ or } (n-1)/2 \tag{1.9a}$$

according as n is even or odd. First three members are $P_0(x) = 1, P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$. Orthogonality relation is

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = \frac{2}{2n+1}\delta_{mn}. \tag{1.9b}$$

Thus a function $f(x)$ can be expressed as an infinite series of Legendre polynomials, known as Fourier-Legendre Series

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\mathrm{d}x. \tag{1.9c}$$

- Fourier series solution of Wave Equation using "Method of Separation"

In Part A, I've derived d'Alembert's solution of 1D Wave Equation using Characteristic directions in the context of infinite string and semi-infinite string. Here, I'll solve 1D Wave Equation for finite string, and use Method of Separation of Variables. The precise problem, to be studied, is given as follows

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}, x \in [0, l], t \geq 0, \quad [\text{c} > 0 \text{ is fixed constant}] \tag{2.1a}$$

where solution $\psi(x, t)$ satisfies following Boundary and Initial conditions:

BC : $\psi(0, t) = \psi(l, t) = 0$, IC: $\psi(x, 0) = f(x), \psi_t(x, 0) = g(x)$ (2.1b)

This type of Problem is known as Initial Boundary Value Problem (IBVP). Note that two independent variables x, t respectively denote position and time. The zero BC on both ends, considered in (2.1b), are called Homogeneous BC. There exists Problem with Non-homogeneous BC, wherein zero value is for one end, and at other end solution has non-zero value [This non-zero term may be a non-zero constant or may be a time-dependent function].

Given BC in Problem (2.1) implies that we are considering a string, tied at both

*For the conception of "Complete Set", see books on Mathematical Analysis ends at $x = 0$ and $x = l$. The transverse vibration of string will start from a disturbance given at equilibrium, and $\psi(x, t)$ denotes the profile of this transverse vibration. IC means that at $t = 0$, the profile ψ of wave and the velocity ψ_t of wave are prescribed by $f(x)$ and $g(x)$. The constant c is the speed of wave generated by vibration.

We will use Method of separation. Its concept is to assume solution in the following separable form:

$$\psi(x, t) = X(x)T(t). \tag{2.2}$$

We are to find X(x) and T(t). Substituting into Wave equation (2.1a),

$$X''T = \frac{1}{c^2}T''X \Longrightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = -k^2. \tag{2.3}$$

where prime denotes ordinary differentiation of factor functions X, T w.r.t. their single arguments. Point is that if the assumed solution would really be a solution, then each ratio in (2.3) must be independent of x, t, since x, t are independent variables. So, we introduce a separation constant $-k^2$, negative sign & square form is just for convenience. Actually, at this point, we do not know whether k is real or complex. However, it can be seen easily that for $k = 0$, we would end up with a zero solution, and hence is rejected:

$$k = 0 \Longrightarrow X(x) = AX + B, \text{BC } A = B = 0.$$

So from now on we will consider $k \neq 0$. Separation (2.3) gives following two ODEs:

$$X'' = -k^2 X,\ T'' = -c^2 k^2 T\ \ (k \neq 0). \tag{2.4}$$

A word of Caution. Do not consider these two as just two simple ODEs for some fixed k, rather consider them as eigenvalue equation for the differential operators□ $\partial^2/\partial x^2$ and $\partial^2/\partial t^2$ with sets of eigenvalues $\{-k^2\}, \{-c^2k^2\}$. That is, I mean to say that, all possible values of k have to be obtained, so that complete solution will be given by the sum of all solutions (eigenvectors) X_k, T_k by the Principle of Linear Superposition. Here the suffix means the eigenvector corresponding to an eigenvalue (a particular value of k). It is easy to write both eigenvectors formally:

$$X_k(x) = C_{1,k}\cos(kx) + C_{2,k}\sin(kx), T_k(t) = D_{1,k}\cos(ckt) + D_{2,k}\sin(ckt), \tag{2.5}$$

where $\{C_{j,k}, D_{j,k}\}$ are four families of arbitrary constants, suffixes just means

⊥Conception of eigenvalue equation for matrix and differential operator are identical.

dependence on k.

Using BC in (2.1b)

$$X_k(0) = X_k(l) = 0 \Longrightarrow C_{1,k} = 0,\ \sin(kl) = 0 \Longrightarrow k = k_n = \frac{n\pi}{l}\ , n = 1,2,3 \ldots \quad (2.6)$$

Equation (2.6) gives all eigenvalues of both equations of (2.4). We see now that the separation constant k is real number, and takes values for all natural numbers.

Substituting for $X(x), T(t)$ into assumed solution (2.2), we get

$$\psi(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[A_n \cos\left(\frac{n\pi c}{l}t\right) + B_n \sin\left(\frac{n\pi c}{l}t\right)\right], \quad (2.7)$$

where for convenience we have identified $A_n = C_{2,k}D_{1,k}|_{k=k_n}, B_n = C_{2,k}D_{2,k}|_{k=k_n}$, so that we are to determine two families of arbitrary constants $\{A_n, B_n\}$.

Using 1st IC in (2.1b),

$$\psi(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right). \quad (2.8)$$

Equation (2.8) is Fourier-sine series [see Sec 1]. Thus we get the constants $\{A_n\}$:

$$A_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x\ , n = 1,2,3, \ldots.. \quad (2.9)$$

From (2.7), differentiating partially w.r.t.

$$\psi_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right)\left[-A_n \sin\left(\frac{n\pi c}{l}t\right) + B_n \cos\left(\frac{n\pi c}{l}t\right)\right]. \quad (2.10)$$

Using 2nd IC in (2.1b),

$$\psi_t(x,0) = g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{l} B_n\right) \sin\left(\frac{n\pi x}{l}\right). \quad (2.11)$$

Equation (2.11) is also Fourier-sine series. So , we get, $\{B_n\}$

$$\frac{n\pi c}{l} B_n = \frac{2}{l}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x\ , n = 1,2,3, \ldots. \quad (2.12a)$$

Hence, final solution of IBVP of (2.1) of 1D Wave Equation for finite string is

$$\psi(x,t) = \frac{2}{l}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[\cos\left(\frac{n\pi c}{l}t\right)\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \quad + \frac{l}{n\pi c} \sin\left(\frac{n\pi c}{l}t\right)\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x\right]. \quad (2.12b)$$

Example: Suppose a string is tightly tied at both ends at $x = 0{,}2$. If the string is released from rest with a initial profile $f(x) = \sin^3(\pi x/2)$, find the solution $\psi(x,t)$ of transverse vibration at any point $x(0 \le x \le l)$ at any instant $t(\ge 0)$. Assume 2 m/s is the speed of wave, and all quantities are in ms unit.

Solution: Transverse vibration of a finite string is described by Wave equation:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{4}\frac{\partial^2\psi}{\partial t^2}, \qquad x \in [0,2], t \ge 0$$

Note that $\psi_t(x,t)$ denotes velocity of wave at any position x. In question, string is released from rest, so that $\psi_t(x,0) = 0$. Also, $\psi(x,t)$ denotes profile of wave at any position x at any instant t, so that according to given data about initial profile, $\psi(x,0) = \sin^3(\pi x/2)$. Thus IC are

$$\psi(x,0) = \sin^3(\pi x/2), \psi_t(x,0) = 0$$

Regarding BC, since string is tied at both ends, we have, as discussed above,

$$\psi(0,t) = \psi(2,t) = 0\,, \forall\ t \ge 0$$

Hence, to solve this IBVP, we will use Method of Separation of Variables. I'll not repeat the steps [see discussion above]. Final solution is

$$\psi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)\cos(n\pi t)\,, A_n = \int_0^2 \sin^3\left(\frac{\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)\mathrm{d}x$$

It is trivial exercise to compute A_n for $n = 1,2,3, \ldots$.

Check that $A_1 = \frac{3}{4}, A_3 = -\frac{1}{4}$ and all other $A_n = 0$. Hence final solution is

$$\psi(x,t) = \frac{3}{4}\sin\left(\frac{\pi x}{2}\right)\cos(\pi t) - \frac{1}{4}\sin\left(\frac{3\pi x}{2}\right)\cos(3\pi t)$$