Solution of Tutorial Problems set-I

Note: All these problems can be solved using the results of Chapter-I.

[**0.0.1**] ***Exercise*** Check that each of the following sets are vector space with respect to usual addition and scalar multiplication.
(i) The set of all real sequences over the field $\mathbb{F} = \mathbb{R}$.
(ii) The set of all bounded real sequences over the filed $\mathbb{R}$.
(iii) The set of all convergent real sequences over the field $\mathbb{R}$.
(iv) $\{(a_n) \mid a_n \in \mathbb{R},\, a_n \to 0\}$ over the field $\mathbb{R}$.
(v) The set of all **eventually** 0 sequences over the field $\mathbb{R}$.We call (x_n) eventually 0 if $\exists k$ s.t. $x_n = 0$ for all $n \geq k$.
(vi) $\mathbb{P}(x)$= $\{p(x) \mid p(x)$ is a real polynomial in $x\}$ over the field $\mathbb{R}$.
(vii) $\mathbb{P}_5(x)$= $\{p(x) \in \mathbb{R}[x] \mid$ degree of $p(x) \leq 5\}$ over the field $\mathbb{R}$.
(viii) $\{A_{n\times n} \mid a_{ij} \in \mathbb{R},\, A$ upper triangular$\}$ over the field $\mathbb{R}$.

[**0.0.2**] ***Exercise*** Consider $\mathbb{P}_n(x)$ and $\mathbb{P}(x)$ over $\mathbb{R}$. Check that each of the following sets is subspace or not.
(i) $\{P(x) \in \mathbb{P}_3(x) \ \mid \ P(x) = ax + b, a, b \in \mathbb{R}\}$.
(ii) $\{P(x) \in \mathbb{P} \ \mid \ P(0) = 0\}$.
(iii) $\{P(x) \in \mathbb{P} \ \mid \ P(0) = 1\}$.
(iv) $\{P(x) \in \mathbb{P} \ \mid \ P(-x) = P(x)\}$.
(v) $\{P(x) \in \mathbb{P} \ \mid \ P(-x) = -P(x)\}$.

[**0.0.3**] ***Exercise*** Fix $A \in \mathcal{M}_n(\mathbb{R})$. Let $\mathbb{U} = \{B \in \mathcal{M}_n(\mathbb{R}) : AB = BA\}$.
a) Show that $\mathbb{U}$ is a subspace of $\mathcal{M}_n(\mathbb{R})$.
b) Let $\mathbb{W} = \{a_0I + a_1A + \cdots a_nA^m | m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, a_i \in \mathbb{R}\}$. Show that $\mathbb{W}$ is a subspace of $\mathbb{U}$.

Sol. a) Let $B, C \in \mathbb{U}$ and $\alpha, \beta \in \mathbb{R}$. To show that $\alpha B + \beta C \in \mathbb{U}$. $A[\alpha B + \beta C] = \alpha AB + \beta AC = \alpha BA + \beta CA = [\alpha B + \beta C]A$. Therefore $\alpha B + \beta C \in \mathbb{U}$. So $\mathbb{U}$ is a subspace.

b) Note that if $r(A)$ is any polynomial, then $Ar(A) = r(A)A$. Hence $r(A) \in \mathbb{U}$. Thus $\mathbb{W} \subseteq \mathbb{U}$.

Let $p(A)$ and $q(A)$ be two polynomials in A. Then $\alpha p(A) + \beta q(A)$ is also polynomial in A. So $\alpha p(A) + \beta q(A) \in \mathbb{W}$. So $\mathbb{W}$ is a subspace.

[**0.0.4**] ***Exercise*** Find basis and dimension for each of the following vector spaces.
(i) $\mathbb{M}_n(\mathbb{C})$ over $\mathbb{R}$.
(ii) $\mathbb{H}_n(\mathbb{C})$, $n \times n$ Hermitian matrices, over $\mathbb{R}$.
(iii) $\mathbb{S}_n(\mathbb{C})$, $n \times n$ Skew-Hermitian matrices, over $\mathbb{R}$.

Sol. (i) $2n^2$.
(ii) n^2.
(iii)n^2.

[**0.0.5**] ***Exercise*** Check whether the following vector space is finite dimensional or infinite dimensional.
(i) The set of all real sequences over the field $\mathbb{F} = \mathbb{R}$.
(ii) The set of all bounded real sequences over the field $\mathbb{R}$.
(iii) The set of all convergent real sequences over the field $\mathbb{R}$.
(iv) $\{(a_n) \mid a_n \in \mathbb{R},\, a_n \to 0\}$ over the field $\mathbb{R}$.
(v) The set of all **eventually** 0 sequences over the field $\mathbb{R}$. We call (x_n) eventually 0 if $\exists k$ s.t. $x_n = 0$ for all $n \geq k$.
(vii) $\mathbb{P}(x)$= $\{p(x) \mid p(x)$ is a real polynomial in $x\}$ over the field $\mathbb{R}$.
(viii) $\mathbb{P}_5(x)$= $\{p(x) \in \mathbb{R}[x] \mid$ degree of $p(x) \leq 5\}$ over the field $\mathbb{R}$.

[**0.0.6**] ***Exercise*** Write 4 proper subspaces of $\mathbb{R}^4$ (a subspace W is called proper if neither W is trivial and nor W equal to whole vector space).

Sol. a) Put $v_1 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^t$. Then $\text{LS}(v_1)$ is a proper subspace.

b) Put $v_2 = \begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}^t$. Then $\text{LS}(v_1, v_2)$ is a proper subspace.

c) Put $v_3 = \begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix}^t$. Then $\text{LS}(v_1, v_2, v_3)$ is a proper subspace.

Therefore $\text{LS}\, v_1, \text{LS}\, v_2, \text{LS}\, v_3, \text{LS}(v_1, v_2), \text{LS}(v_1, v_2, v_3)$ are proper subspace of $\mathbb{R}^3$. There are many proper subspace of $\mathbb{R}^3$.

The following is an extra information:

Claim Any proper subspace of $\mathbb{R}^4$ must be a span of either a) one nonzero vector or b) two linearly independent vectors or c) three linearly independent vectors. Why?

Let $\mathbb{W}$ be a nontrivial subspace. So $\exists$ a nonzero $v_1 \in \mathbb{W}$. If $\mathbb{W} = \text{LS}(v_1)$, it is of type a) and we are done. So let $v_2 \in \mathbb{W} - \text{LS}(v_1)$. By our theorem, v_1, v_2 are linearly independent. If $\mathbb{W} = \text{LS}(v_1, v_2)$, it is of type b) and we are done. So let $v_3 \in \mathbb{W} - \text{LS}(v_1, v_2)$. By our theorem, v_1, v_2, v_3 are linearly independent. If $\mathbb{W} = \text{LS}(v_1, v_2, v_3)$, it is of type c) and we are done. So let $v_4 \in \mathbb{W} - \text{LS}(v_1, v_2, v_3)$. By our theorem, v_1, v_2, v_3, v_4 are linearly independent. Then $\text{LS}(v_1, v_2, v_3, v_4) = \mathbb{R}^4$. Hence W is

[**0.0.7**] ***Exercise*** Show that $u_1, \cdots, u_k \in \mathbb{R}^n$ are linearly independent iff $Au_1, \cdots, Au_k$ are linearly independent for any invertible A_n.

Sol. First we assume that $Au_1, \cdots, Au_k$ are linearly independent. To show $u_1, \cdots, u_k \in \mathbb{R}^n$ are linearly independent.

Suppose $u_1, \cdots, u_n$ are linearly dependent. Then there exits $\alpha_1, \ldots, \alpha_k$ in $\mathbb{F}$ not all zero s.t. $\sum \alpha_i u_i = 0$. Multiplying both side by A. So $A \sum \alpha_i u_i = \sum \alpha_i (Au_i) = 0$. So $Au_1, \ldots, Au_n$ are linearly dependent. A contradiction because $Au_1, \cdots, Au_k$ are linearly independent. Hence $u_1, \cdots, u_k \in \mathbb{R}^n$ are LI.

We now assume that $u_1, \cdots, u_k \in \mathbb{R}^n$ are linearly independent. To show $Au_1, \cdots, Au_k$ are linearly independent.

Suppose $Au_1, \cdots, Au_n$ are linearly dependent. Then $\exists \alpha \neq 0$ s.t. $\sum \alpha_i (Au_i) = 0$. So $0 = \sum \alpha_i (Au_i) = A \sum \alpha_i u_i$. So $A^{-1} 0 = A^{-1} A \sum \alpha_i u_i = \sum \alpha_i u_i$. So $u_1, \ldots, u_n$ are linearly dependent. A contradiction because $u_1, \cdots, u_k \in \mathbb{R}^n$ are linearly independent. Hence $Au_1, \cdots, Au_k$ are LI.

[**0.0.8**] ***Exercise*** Show that $u_1, \cdots, u_k \in \mathbb{V}$ is linearly independent iff $\sum\limits_{i=1}^{k} a_{i1} u_i, \cdots, \sum\limits_{i=1}^{k} a_{ik} u_i$ are linearly independent for any invertible $A_{k \times k}$. Show that $\{u, v\}$ is linearly independent iff $\{u + v, u - v\}$ is linearly independent.

Sol. We first assume that $u_1, \ldots, u_k$ are LI.

Put $w_r = \sum\limits_{i=1}^{k} a_{ir} u_i$.

Then $\begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1k} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} = A^t \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$. To show that $w_1, \ldots, w_k$ are LI.

Suppose $w_1, \ldots, w_k$ are linearly dependent. Then exist $\alpha_1, \ldots, \alpha_k$ not all zero s.t. $\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix} = 0$. So

$$0 = \begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1k} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} = \begin{bmatrix} \beta_1 & \cdots & \beta_k \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix},$$

where $\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} A^t = \begin{bmatrix} \beta_1 & \cdots & \beta_k \end{bmatrix} \neq 0$. Thus $u_1, \ldots, u_k$ are linearly dependent.

We now assume that $w_1, \ldots, w_k$ are LI. To show that $u_1, \ldots, u_k$ are LI.

Suppose $u_1, \ldots, u_k$ are LD. There there exists $c_1, \ldots, c_k \in \mathbb{F}$ not all zero such that $c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = 0$.

Multiplying both side by A^t. $A^t(c_1 u_1 + c_2 u_2 + \cdots + c_k u_k) = 0$

$$\begin{bmatrix}\alpha_1 & \cdots & \alpha_k\end{bmatrix}\begin{bmatrix}a_{11} & \cdots & a_{k1}\\ \vdots & & \vdots\\ a_{1k} & \cdots & a_{kk}\end{bmatrix}\begin{bmatrix}u_1\\ \vdots\\ u_k\end{bmatrix} = 0.$$

$$\begin{bmatrix}\alpha_1 & \cdots & \alpha_k\end{bmatrix}\begin{bmatrix}\sum_{i=1}^{k} a_{i1}u_i\\ \vdots\\ \sum_{i=1}^{k} a_{ik}u_i\end{bmatrix} = 0.$$

This implies that $\sum_{i=1}^{k} a_{i1}u_i, \sum_{i=1}^{k} a_{i2}u_i, \ldots, \sum_{i=1}^{k} a_{ik}u_i$ are LI. A contradiction. Hence $u_1 \ldots, u_k$ are LI.

[**0.0.9**] ***Exercise*** Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let A and B be two non-empty subsets of $\mathbb{V}$. Prove or disprove: $\textsc{ls}(A) \cap \textsc{ls}(B) \neq \{0\} \implies A \cap B \neq \phi$.

Sol. Disprove. Consider $\mathbb{R}^2$ and $A = \{(1,0),(0,1)\}$ and $B = \{(1,2),(2,1)\}$. Notice that $\textsc{ls}(A) = \mathbb{R}^2$ and $\textsc{ls}(B) = \mathbb{R}^2$. Hence $\textsc{ls}(A) \cap \textsc{ls}(B) \neq \{0\}$. But $A \cap B = \phi$

[**0.0.10**] ***Exercise*** Show that a vector space $\mathbb{V}$ over $\mathbb{F}$ has a unique basis if and only if either $\textsc{dim}(\mathbb{V}) = 0$ or $\textsc{dim}(\mathbb{V}) = 1$ and $|\mathbb{F}| = 2$.

Sol. First assume that $\mathbb{V}(\mathbb{F})$ has a unique basis.

There are two cases. Either $\mathbb{V}$ is trivial or non-trivial.

Case I. If $\mathbb{V}$ is trivial, then we are done.

Case II. $\mathbb{V}$ is non-trivial. First we show that $\textsc{dim}\,\mathbb{V} = 1$. Suppose that $\textsc{dim}\,\mathbb{V} \geq 2$. There are two cases.

Case II(a). $\mathbb{V}$ is finite dimensional. Let $\{x_1, \ldots, x_k\}$ be a basis of $\mathbb{V}$ where $k \geq 2$.

Consider $B = \{x_1, x_2, \ldots, x_i, \ldots, x_{j-1}, x_j + x_i, x_{j+1}, \ldots, x_k\}$. To show that B is also a basis of $\mathbb{V}$. Using Problem 0.0.8 in tutorial sheet, you can easily show that B is LI. Since B is LI and $|B| = \textsc{dim}(\mathbb{V})$. Hence B is basis. A contradiction that $\mathbb{V}$ has unique basis. Hence $\textsc{dim}(\mathbb{V}) = 1$.

Case II(b). $\mathbb{V}$ is infinite dimensional. Let $S = \{x_\alpha : \alpha \in I\}$ be a basis of $\mathbb{V}$ where I is an index set. Take $x_\gamma, x_\beta \in S$. Then $x_\gamma + x_\beta \in \mathbb{V}$ as $\mathbb{V}$ is a vector space. But $x_\gamma + x_\beta \notin S$ as S is LI, otherwise there is a vector $x_\gamma + x_\beta$ which is a linear combination of x_γ and x_β.

Consider $B = \Big(\{x_\alpha : \alpha \in I\} - \{x_\gamma\}\Big) \cup \{x_\gamma + x_\beta\}$.

To show that B is a basis of $\mathbb{V}$. Let A be a finite subset of B. There are two cases.

Case I. $A = \{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k}\}$ that means $x_\gamma + x_\beta$ is not in A. Therefore $A \subseteq S$. Since S is LI, then A is LI.

Case II. $A = \{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_\gamma + x_\beta, \ldots, x_{\alpha_k}\}$. Then applying the same techniques as of finite case, we have A is LI. Therefore each finite subset of B is LI. Hence B is LI.

To show $\text{LS}(B) = \mathbb{V}$. It is enough to show $\text{LS}(B) = \text{LS}(S)$. $B - \{x_\gamma + x_\beta\} = S - \{x_\gamma\}$ and you know that $x_\gamma + x_\beta$ is in B. Using this you can easily show that $\text{LS}(B) = \text{LS}(S)$.

Hence B is a basis of $\mathbb{V}$. Therefore $\mathbb{V}$ has two basis a contradiction. Hence $\text{DIM}(V) = 1$.

We now show that $|\mathbb{F}| = 2$. Suppose that $|\mathbb{F}| > 2$. Then $|F|$ has at least one element which is other than additive identity and multiplicative identity. Let $\{x\}$ be a basis of $\mathbb{V}$ and let $\alpha \in \mathbb{F}$ such that α is neither 0 nor 1. Then $\{x\}$ and $\{\alpha x\}$ both are basis of $\mathbb{V}$. A contradiction that $\mathbb{V}$ has unique basis. Hence $|\mathbb{F}| = 2$.

Assume that either $\text{DIM}(\mathbb{V}) = 0$ or $\text{DIM}(\mathbb{V}) = 1$ and $|\mathbb{F}| = 2$. If $\text{DIM}(\mathbb{V}) = 0$, then $\mathbb{V} = \{0\}$. Hence its has unique basis.

If $\text{DIM}(\mathbb{V}) = 1$ and $|\mathbb{F}| = 2$. Suppose that B has two bases they are $\{x\}$ and $\{y\}$. Since $y \in \mathbb{V}$ and $\{x\}$ is basis, then $y = \alpha x$ where $\alpha \in \mathbb{F}$. Since $\mathbb{F}$ has two elements, then $\alpha = 1$ (multiplicative identity). Therefore $y = 1(x) = x$. Hence $\mathbb{V}$ has unique basis.

[**0.0.11**] ***Exercise*** Let $\mathbb{V}$ be an n dimensional vector space over $\mathbb{F}$ and let $\mathbb{F}$ has exactly p elements. Then show that $|\mathbb{V}| = p^n$.

Sol. Let $B = \{u_1, u_2, \ldots, u_n\}$ be a basis of $\mathbb{V}$. Any element in $\mathbb{V}$ can be written as a **unique** linear combination of $u_1, u_2, \ldots, u_n$. That is $c_1u_1 + c_2u_2 + c_3u_3 + \cdots + c_nu_n$ where $c_i \in \mathbb{F}$. For each c_i we have p choices and the choice of c_i does not depend on the choice of c_j for $i \neq j$. Hence $|\mathbb{V}| = p^n$.

[**0.0.12**] ***Exercise*** Check whether vector space $\mathbb{R}$ (set of real numbers) over the field $\mathbb{Q}$ (set rational number) is infinite dimensional or finite dimensional.

Sol. If we are able to show that for each $n \in \mathbb{N}$ there exists a LI subset of $\mathbb{R}(\mathbb{Q})$ containing $n+1$ elements. Then we are done. Let α be a transcendental number. To show that $\{1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^n\}$ is LI.Take $a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0$. We never have non-trivial solution of this equation as α is transcendental. Hence $\{1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^n\}$ is LI. For each n we have a LI set of $n+1$ vectors. Therefore $\mathbb{R}(\mathbb{Q})$ is infinite dimensional.

[**0.0.13**] ***Exercise*** Let $S = \{ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} a \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \}$. Find the values of a for which $\text{LS}(S) \neq \mathbb{R}^3$.

Sol. We find the values of a for which $\text{LS}(S) = \mathbb{R}^3$. Since S contains exactly three and $\text{DIM}(S) = \mathbb{R}^3$, then S is a basis of $\mathbb{R}^3$. This implies S is LI.

Consider $A = \begin{bmatrix} 4 & a & 4 \\ 5 & 2 & 3 \\ 6 & 4 & 2 \end{bmatrix}$. Since S is LI, then A is invertible. Hence $\text{DET}(A) \neq 0$ and $\text{DET}(A) = 8a$. Therefore we have $8a \neq 0$. This implies $a \neq 0$.

We have seen that if $a \in \mathbb{R} - \{0\}$, then $\text{LS}(S) = \mathbb{R}^3$.

Then So $\text{LS}(S) \neq \mathbb{R}^3$ iff $a = 0$.

[**0.0.14**] ***Exercise*** Give 2 bases for the trace 0 real symmetric matrices of size 3×3. Extend these bases to bases of the real symmetric matrices of size 3×3. Extend these bases to bases of the real matrices of size 3×3.

[**0.0.15**] ***Exercise*** Consider $\mathbb{W} = \{v \in \mathbb{R}^6 | v_1 + v_2 + v_3 = 0, v_2 + v_3 + v_4 = 0, v_4 + v_5 + v_6 = 0\}$. Supply a basis for $\mathbb{W}$ and extend it to a basis of $\mathbb{R}^6$.

Sol. For the system: $\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$. Get the RREF: $\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$.

Free variables: v_3, v_5, v_6. Obtain linearly independent solutions by putting a free variable 1 and other free variables 0:

$$\{ \begin{bmatrix} 0 \\ -1 \\ \mathbf{1} \\ 0 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ \mathbf{0} \\ -1 \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ \mathbf{0} \\ -1 \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \}.$$

This is a basis for $\mathbb{W}$. Note that e_1 cannot be a linear combination of these three vectors.!!

To extend add e_1, e_2, e_4. These three with the first basis vector will give e_3. Similarly..... So these 6 will span $\mathbb{R}^6$. They are linearly independent.

[**0.0.16**] ***Exercise*** For what values α are the vectors $(0, 1, \alpha), (\alpha, 1, 0)$ and $(1, \alpha, 1)$ in $\mathbb{R}^3$ linearly independent?

[0.0.17] ***Exercise*** If S and T are two subspaces of a vector spaces having a common complement set W, does it follow that $S = T$?

Sol. Not necessarily. Consider $\mathbb{R}^2(\mathbb{R})$. Take $S = \{(x, 0) : x \in R\}$ and $T = \{(0, y) : y \in R\}$. Consider $W = \text{LS}(\{(1, 1)\})$. It is easy to check W is complement of S and T. But $S \neq T$.

[0.0.18] ***Exercise*** In the vector space $\mathbb{R}^4$, find two different complements of the subspace $S = \{(x_1, x_2, x_3, x_4) : x_3 - x_4 = 0\}$

Sol. We first find a basis of $\mathbb{W}$. Let $(x_1, x_2, x_3, x_4) \in \mathbb{W}$. Then $x_3 - x_4 = 0$. Therefore $(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_3) = x_1(1, 0, 0, 0) + x_2(0, 1, 0, 0) + x_3(0, 0, 1, 1)$ This implies that $\text{LS}(\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\})$. One can easily prove that $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\}$ is LI. Hence $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\}$ is basis of $\mathbb{W}$.

We now extend $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\}$ to a basis of $\mathbb{R}^4$. Take $(0, 0, 0, 1) \in \mathbb{R}^3 - \mathbb{W}$. Using basis extension theorem $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 0, 0, 1)\}$ is LI. It has exactly four elements. Then $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 0, 0, 1)\}$ is a basis of $\mathbb{R}^3$. Therefore $S_1 = \text{LS}(\{(0, 0, 0, 1)\})$ is a complement of $\mathbb{V}$.

Take $(1, 1, 1, 0) \in \mathbb{R}^3 - \mathbb{W}$.

Using basis extension theorem $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1), (1, 1, 1, 0)\}$ is LI. It has exactly four elements. Then $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1), (1, 1, 1, 0)\}$ is a basis of $\mathbb{R}^3$. Therefore $S_2 = \text{LS}(\{(1, 1, 1, 0)\})$ is a complement of $\mathbb{V}$.

[0.0.19] ***Exercise*** Show that a non-trivial subspace S of a finite dimensional vector space $\mathbb{V}$ has two virtually disjoint complements iff $\text{DIM}(S) \geq \frac{\text{DIM}(\mathbb{V})}{2}$.

Sol. We first assume that $\text{DIM}(S) \geq \frac{\text{DIM}(\mathbb{V})}{2}$. To show S has two virtually disjoint complements. Let $\text{DIM}(\mathbb{V}) = n$ and $\text{DIM}(S) = k$. Then $k \geq \frac{n}{2}$. Let $\{x_1, \ldots, x_k\}$ be a basis of S. We extend it to basis of $\mathbb{V}$ that is $\{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$. Take $S_1 = \text{LS}(x_{k+1}, \ldots, x_n\})$. Then S_1 is a complement of S.

Consider $\{x_1, \ldots, x_k, x_{k+1} + x_1, x_{k+2} + x_2, \ldots, x_n + x_{n-k}\},$. This is possible as $k \geq \frac{n}{2}$. To show this set is basis of $\mathbb{V}$. Take $S_2 = \text{LS}(\{x_{k+1} + x_1, x_{k+2} + x_2, \ldots, x_n + x_{n-k}\})$. This is also a complement of S. To show $S_1 \cap S_2 = \{0\}$.

Let $x \in S_1 \cap S_2 = \{0\}$. Then $x \in S_1$ and $x \in S_2$.

$x = c_1 x_{k+1} + c_2 x_{k+2} + \cdots + c_{n-k} x_n$ and

$x = b_1(x_{k+1} + x_1) + b_2(x_{k+2} + x_2) + \cdots + b_{n-k}(x_n + x_{n-k})$

$b_1x_1 + \cdots b_{n-k}x_{n-k} + (b_1 - c_1)x_{k+1} + \cdots + (b_{n-k} - c_{n-k})x_n = 0$

$\{x_1, \ldots, x_{n-k}, x_{k+1}, \ldots, x_n\}$ is LI

Then $b_i = c_i = 0$ for $i = 1, \ldots, n-k$. Hence $x = 0$.

We now assume that S has two virtually disjoint complements.

To show $\text{DIM}\,(S) \geq \frac{\text{DIM}\,(\mathbb{V})}{2}$. Suppose that $\text{DIM}\,(S) < \frac{\text{DIM}\,(\mathbb{V})}{2}$. Let $\text{DIM}\,(\mathbb{V}) = n$ and $\text{DIM}\,(S) = k$. Let S_1 and S_2 be two virtually disjoint complements. Then $\text{DIM}\,(S_1) = n - k = \text{DIM}\,(S_2)$.

$\text{DIM}\,(S_1 + S_2) = \text{DIM}\,(S_1) + \text{DIM}\,(S_2) = 2(n-k) > 2(n - \frac{n}{2}) = n$, a contradiction. Hence $\text{DIM}\,(S) \geq \frac{\text{DIM}\,(\mathbb{V})}{2}$.

[**0.0.20**] ***Exercise*** Find a complement of the subspace $\{(x_1, x_2, \ldots, x_n\} : \sum\limits_{i=1}^{n} x_i = 0\}$ in $\mathbb{R}^n$.

Sol. Same as Exercise 0.0.18