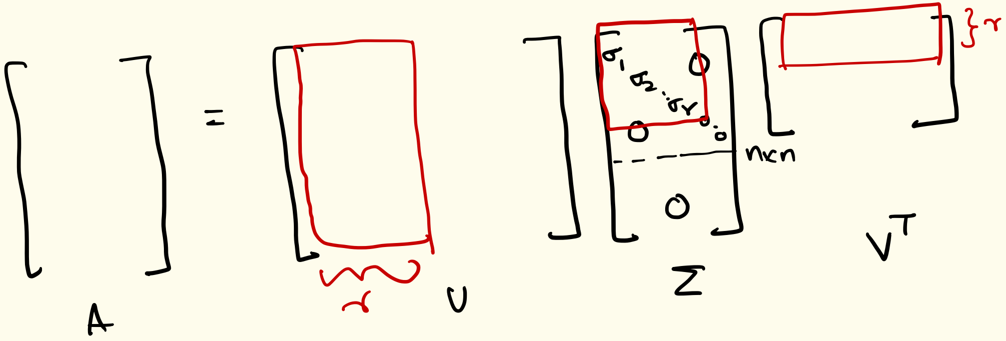
Class November 6

Singular Value Decomposition (SVD)

Let $A \in \mathbb{R}^{m\times n}$ with rank r. Then there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ and a "diagonal" matrix $\Sigma \in \mathbb{R}^{m\times n}$ with nonzero "diagonal" entries $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ such that

$$A = U\Sigma V^T$$

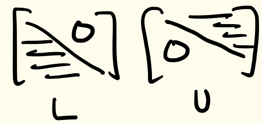


$\sigma_1, \sigma_2, \dots, \sigma_r$: singular values of A.

Decompositions

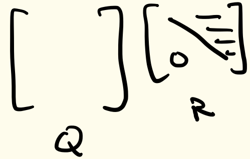
1) LU

A =



2) QR

A =



3) SVD

4) Eigenvalue, eigenvector decomposition.

$$A = U\Sigma V^T$$

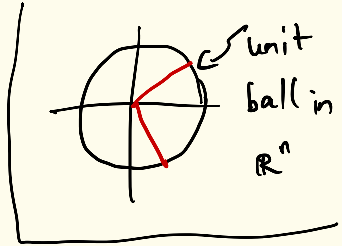
domain

range

$$AV = U\Sigma$$

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix}$$

*



$$v_1, \dots, v_r$$
$$\downarrow \qquad \downarrow$$
$$\text{span}\{u_1, \dots, u_r\} = \mathcal{R}(A) \checkmark$$

$$\dim \mathcal{R}(A) = \text{rank}$$

$$Av_1 = \sigma_1 u_1 \checkmark$$
$$Av_2 = \sigma_2 u_2 \checkmark$$
$$\vdots$$
$$Av_r = \sigma_r u_r \checkmark$$

$$\left.\begin{aligned} Av_{r+1} &= 0 \\ Av_{r+2} &= 0 \\ &\vdots \\ Av_n &= 0 \end{aligned}\right\}$$

$$\mathcal{N}(A) = \{v \in \mathbb{R}^n \mid Av = 0\}$$

$$\text{span}\{v_{r+1}, v_{r+2}, \dots, v_n\} = \mathcal{N}(A) \checkmark$$

$$\dim \mathcal{N}(A) = \text{nullity}$$

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \dots + \sigma_r u_r v_r^T$$

$$= \boxed{\sigma_1 \underbrace{\begin{bmatrix} | \\ | \\ | \end{bmatrix}_{m\times 1}}_{u_1} \underbrace{\begin{bmatrix} \text{———} \end{bmatrix}_{1\times n}}_{v_1^T}} + \sigma_2 \underbrace{\begin{bmatrix} | \\ | \\ | \end{bmatrix}_{m\times 1}}_{u_2} \underbrace{\begin{bmatrix} \text{———} \end{bmatrix}_{1\times n}}_{v_2^T} + \dots$$

$$\dots + \sigma_r \underbrace{\begin{bmatrix} | \\ | \\ | \end{bmatrix}_{m\times 1}}_{u_r} \underbrace{\begin{bmatrix} \text{———} \end{bmatrix}_{1\times n}}_{v_r^T} .$$

A rank r matrix as the sum of r rank-1 matrices.

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

$$A = U \Sigma V^T$$

$$A^T = (U \Sigma V^T)^T = V \Sigma^T U^T$$

orthogonal   "diagonal"-like matrix!   orthogonal

$$A^T = V \Sigma^T U^T$$

$$A^T U = V \Sigma^T$$

$$A^T u_1 = \sigma_1 v_1$$

$$A^T u_2 = \sigma_2 v_2$$

$$\vdots$$

$$A^T u_r = \sigma_r v_r$$

$$A^T u_{r+1} = 0$$

$$\vdots$$

$$A^T u_m = 0$$

$\mathcal{N}(A^T)$

$R(A^T)$

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

Let A be invertible.

$$A = \underset{n\times n}{U}\, \underset{n\times n}{\Sigma}\, \underset{n\times n}{V^T}$$

$$\underset{A}{\begin{bmatrix} & \\ & \end{bmatrix}} = \underset{U}{\begin{bmatrix} & \\ & \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sigma_1 & \cdots & 0 \\ 0 & & \sigma_n \end{bmatrix}} \underset{V^T}{\begin{bmatrix} & \\ & \end{bmatrix}}$$

$$A^{-1} = (U \Sigma V^T)^{-1} = V^{-T} \Sigma^{-1} U^T$$

$$\boxed{A^{-1} = V \Sigma^{-1} U^T}$$

$A \in \mathbb{R}^{m \times n}$ a full column rank matrix.

$m \geq n$

$\Rightarrow \quad A^T A$ is invertible.

$$\boxed{(A^T A)^{-1} A^T} = \text{left inverse / pseudo inverse}$$

SVD of $A = U \Sigma V^T$

$$A^T = V \Sigma^T U^T$$

$$A^T A = V \Sigma^T U^T U \Sigma V^T$$

$$\boxed{A^T A = V (\Sigma^T \Sigma) V^T}$$

$$(A^T A)^{-1} = \left(V (\Sigma^T \Sigma) V^T\right)^{-1} = \boxed{V (\Sigma^T \Sigma)^{-1} V^T}$$

$$(A^T A)^{-1} A^T = V (\Sigma^T \Sigma)^{-1} V^T V \Sigma^T U^T$$

$$\boxed{(A^T A)^{-1} A^T = V \boxed{(\Sigma^T \Sigma)^{-1} \Sigma^T} U^T}$$

diagonal like matrix

$\underline{n \times m}$

$$\left[\begin{array}{ccc|c} \bullet & & 0 & \\ & \ddots & & 0 \\ 0 & & \bullet & \end{array}\right]$$

'n' $\quad n \times n \quad n \times m$

$$\Sigma^T\Sigma = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix}_{n\times n}$$

$$\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \sigma_n \\ & 0 & & \end{bmatrix} = \Sigma$$

$$\Sigma^T\Sigma = \begin{bmatrix} \sigma_1 & & & 0 & 0 \\ & \sigma_2 & \ddots & & \\ 0 & & & \sigma_n & \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{bmatrix}$$

$$(\Sigma^T\Sigma)^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & & 0 \\ & \frac{1}{\sigma_2^2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sigma_n^2} \end{bmatrix}$$

$$(\Sigma^T\Sigma)^{-1}\Sigma^T = \begin{bmatrix} 1/\sigma_1^2 & & & \\ & 1/\sigma_2^2 & & \\ & & \ddots & \\ & & & 1/\sigma_n^2 \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 & 0 \\ & \sigma_2 & & & \\ & & \ddots & & \\ 0 & & & \sigma_n & \end{bmatrix}$$

$$(\Sigma^T\Sigma)^{-1}\Sigma^T = \begin{bmatrix} 1/\sigma_1 & & & & 0 \\ & 1/\sigma_2 & & & \\ & & \ddots & & \\ & & & 1/\sigma_n & \end{bmatrix}_{n\times m}$$

$$A = U\Sigma V^T$$

$$AV = U\Sigma$$

$v_1 \longrightarrow \sigma_1 u_1$ ← max.mag.

$v_2 \longrightarrow \sigma_2 u_2$

$\vdots$

$v_r \longrightarrow \sigma_r u_r$

$\left.\begin{array}{l} v_{r+1} \\ \vdots \\ v_n \end{array}\right\} \longrightarrow 0$ ← minmag

$A \in \mathbb{R}^{2\times 2}$

$$A = U\Sigma V^T$$

$$AV = U\Sigma$$

$v_1 \longrightarrow \sigma_1 u_1$

$v_2 \longrightarrow \sigma_2 u_2$

maxmag$(A) = \|A\|_2$