Contents

0.1 Convex Quadratic Programming Problem

A general convex quadratic programming problem is in the following two forms. Q, is a real symmetric positive definite matrix of order $n \times n$; $c, x \in R^n, b \in R^m$, A is real matrix of order $m \times n$.

$$(QP_1) \qquad \min \frac{1}{2} x^T Q x + c^T x$$
$$subject\ \ to Ax = b$$

$$(QP_2) \qquad \min \frac{1}{2} x^T Q x + c^T x$$
$$subject\ \ to Ax \leq b, x \geq 0$$

Both the quadratic programming problems are convex programming problems since (i) $Q \succ 0 \Rightarrow x^T Q x \ > 0$ and $Ax \leq, = b$ is a linear system.

0.1.1 Solution of (QP_1)

Since (QP_1) is a convex programming problem so KKT conditions are both necessary and sufficient for the existence of solution.

The lagrange function for (QP_1) is

$$L(x,\mu) = f(x) + \mu^T h(x) = \frac{1}{2}x^T Qx + c^T x + \mu^T(Ax - b)$$

KKT optimality conditions are

$$\nabla_x L(x,\mu) = 0$$
$$Ax = b, \mu \in R^m$$

Here $Q \succ 0$ so Q^{-1} exists and $Q = Q^T$ since Q is symmetric.

$$\begin{aligned} &\nabla_x L(x,\mu) = 0 \\ &\equiv Qx + c + A^T\mu = 0 \\ &\equiv x = -Q^{-1}(c + A^T\mu) \end{aligned}$$

What is μ?

$$\begin{aligned} Ax = b \equiv& - AQ^{-1}(c + A^T\mu) = b \\ \equiv& AQ^{-1}A^T\mu = -(b + AQ^{-1}c) \\ \Rightarrow& \mu = -([AQ^{-1}A^T]^{-1}).(b + (AQ^{-1}c)) \end{aligned}$$

For any nonzero vector $x \in R^n$, $x^T AQ^{-1}A^T x = (A^T x)^T Q^{-1}(A^T x) = z^T Q^{-1} z > 0$ since $Q \succ 0$. Hence inverse of $AQ^{-1}A^T$ exists.

Therefore $\mu = -(AQ^{-1}A^T)^{-1}(b + AQ^{-1}c)$. Hence the optimal solution is

$$x = -Q^{-1}[c - A^T(AQ^{-1}A^T)^{-1}(b + AQ^{-1}c))]$$

Example 0.1.1. *(a) Using the above process find the solution of the following quadratic programming problems.*

(b)Write a program for optimal solution formula and verify your answer.

(c)Also, verify your answer in Python in built code.

1. $\min 3x_1^2 + x_2^2 + x_1x_2 + 2x_1 - x_2,\ s.to\ \ x_1 + 2x_2 = 4$
2. $\min 3x_1^2 + x_2^2 + 2x_3^2 + x_1x_2 + x_1x_3 + 2x_1 - x_2 + x_3,\ s.to\ \ x_1 + 2x_2 + x_3 = 4, 3x_1 - x_2 + 3x_3 = 5$

0.1.2 Solution of (QP_2)

$$(QP_2) \qquad \min \frac{1}{2}x^TQx + c^Tx$$
$$subject\ \ to\ \ Ax \leq b, x \geq 0$$

This is also a convex programming problem. Hence KKT optimality conditions are both necessary and sufficient for the existence of solution. Denote

$$L(x, \lambda, \theta) = f(x) + \lambda^T g(x) + \theta(-x) = \frac{1}{2}x^TQx + c^Tx + \lambda^T(Ax - b) + \theta(-x)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}x_ix_j + \sum_{j=1}^{n} c_jx_j + \sum_{i=1}^{m} \lambda_i(\sum_{j=1}^{n}(a_{ij}x_j - b_i)) + \sum_{j=1}^{n} \theta_j(-x_j)$$

KKT optimality conditions are:

- Normal condition:$\nabla_x L(x, \lambda, \theta) = 0.$ That is, $Qx + c + A^T\lambda - \theta = 0.$ which is same as $\sum_{i=1}^{n} q_{ij}x_i + c_j + \sum_{i=1}^{m} a_{ij}\lambda_i - \theta_j = 0$, $j = 1, 2, ..., n$
- Feasibility condition: $Ax \leq b, x \geq 0.$ This is same as $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $x_j \geq 0$, $j = 1, 2, ..., n$. Using slack variables $s_j \geq 0$ these can be written as

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$$

- Complementary slackness condition: $\lambda_i(Ax - b)_i = 0$ and $\theta_jx_j = 0$ (component wise). That is, $\lambda_is_i = 0$ and $\theta_jx_j = 0$
- Dual restriction: $\lambda_i, \theta_j \geq 0$

Summarizing,

$$\nabla_x L(x,\lambda) = 0$$
$$Ax + s = b,$$
$$\lambda_i s_i = 0, \theta_j x_j = 0, \lambda_j \geq 0, x_j \geq 0, \theta_j \geq 0$$

which is same as

$$\sum_{i=1}^{n} q_{ij} x_i + \sum_{i=1}^{m} a_{ij} \lambda_i - \theta_j = 0 = -c_j, j = 1,2,...,n$$
$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i, i = 1,2,...,m$$
$$\lambda_i s_i = 0, \lambda_i, s_i, x_j, \theta_j \geq 0$$

Coefficient of s_i is a unit vector whose i^{th} component is 1 and other components are zero, $i = 1,2,...,m$. Using n artificial variables z_j, $j = 1,2,...,n$ in first equation we can have an identity matrix as basis. So this system can be expressed as a modified linear system as follows.

$$\min \quad z_1 + z_2 + ... + z_n$$
$$subject \ \ to \ \ \sum_{i=1}^{n} q_{ij} x_i + \sum_{i=1}^{m} a_{ij} \lambda_i - \theta_j + z_j = 0 = -c_j, j = 1,2,...,n$$
$$\sum_{j=1}^{n} a_{ij} x_j + s_j = b_i, i = 1,2,...,m$$
$$\lambda_i, s_i, x_j, \theta_j, z_j \geq 0$$

where $\lambda_i s_i = 0$, $\theta_j x_j = 0$ are known as complementary slackness condition. $\lambda_i s_i = 0$ means both can not be nonzero. That is, if λ_i is in the basis then s_i should not be in the basis and vice versa. Both may not be in the basis is also admissible. A variable which is not in the basis takes the value 0 and once this enters into the basis then its value is strictly greater than 0. This rule is applicable to $\theta_j x_j = 0$. This rule is known as restricted basis entry rule. This LPP can be solved using some LPP technique, taking care these restricted basis entry rule in every iteration of simplex table. This method is also known as WOLFES MODIFIED SIMPLEX METHOD. This modified simplex method is discussed in one example in the class. For details please see Quadratic Programming chapter of Engineering Optimization by S.S.Rao.

Exercise: Solve the following quadratic programming problems

1. $\min\ 2x_1^2 + 2x_2^2 + 2x_1x_2 - 4x_1 - 6x_2$ subject to $x_1 + 2x_2 \leq 2$, $x_1, x_2 \geq 0$

2. $\min\ x_1^2 - x_2 - 2x_1$ subject to $2x_1 + 3x_2 \leq 6$, $2x_1 + x_2 \leq 4$, $x_1, x_2 \geq 0$

3. Convert the quadratic programming problem $\min\ x_1^2 - x_2 - 2x_1$ subject to $2x_1 + 3x_2 \leq 6$, $2x_1 + x_2 = 4$, $x_1, x_2 \geq 0$ to a modified linear programming problem using KKT Optimality condition.