Chapter 1

Quadratic Programming Application in Portfolio Analysis:Markowitz Mean Variance Model

- Suppose X amount has to be invested in the assets $A_1, A_2, ..., A_n$ at the time t_0.
- X_i is the amount of investment in A_i
- $P : (A_1, A_2, ..., A_n)$ is known as a portfolio.
- $\sum_{i=1}^{n} X_i = X$.
  X_i may be $+ve$ or $-ve$. If $X_i < 0$ then it is called short selling.
- $w_i = \frac{X_i}{X}$ = Proportion of total investment in i^{th} asset. $\sum_{i=1}^{n} w_i = 1$ or $e^T w = 1$ in vector form.
- r_i = Rate of return of the asset A_i, which is a random variable=$\frac{r_{i,j}-r_{i,j-1}}{r_{i,j-1}}$, r_{ij}=return of i^{th} asset calculated at T_j time $j = 1, 2, ..., n$).
- $\mu_i = E(r_i)$=Mean/Expected/Average return of i^{th} asset for time T=$\frac{1}{n-1}\sum_{j=1}^{n} \frac{r_{i,j}-r_{i,j-1}}{r_{i,j-1}}$
  (For large data set you may use time series to calculate this.)
- R = Rate of return of the portfolio P, which is also a random variable=$\sum_{i=1}^{n} w_i r_i$.

- Expected return of the portfolio $P = \mu_P = E(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \mu_i = w^T \mu$.

1.1 Risk of the portfolio

There are several types of risk functions used in portfolio analysis.

- Variance risk, which is a quadratic function
- Mean absolute deviation, which can be converted to a linear function
- Min-max risk, which can be converted to a linear function
- l_∞ risk, which can be converted to a linear function
- Value at risk, which is a stochastic function
- Conditional value at risk, which is a stochastic function

1.2 Variance risk

Variance of portfolio return is $\sigma^2 = E[(R - E(R))^2]$
Denote the variance of return of i^{th} asset $=\sigma_i^2 = E[(r_i - E(r_i))^2]$.
Covariance of the return of asset i and asset j $=\sigma_{ij} = E[(r_i - E(r_i))(r_j - E(r_j))]$.
Variance of portfolio return is $E[(R - E(R))^2]$, which is known as variance risk.

$$\begin{aligned}
\sigma^2 =& E[(R - E(R))^2] \\
=& E\Big(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \mu_i\Big)^2 \\
=& E\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j (r_i - \mu_i)(r_j - \mu_j)\Big) \\
=& \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j E[(r_i - \mu_i)(r_j - \mu_j)] \\
=& \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = w^T \Omega w,
\end{aligned}$$

where $w = (w_1, w_2, ..., w_n)^T$, $\Omega = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & . & . & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & . & . & \sigma_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ \sigma_{n1} & \sigma_{n2} & . & . & \sigma_n^2 \end{pmatrix}$

1.2.1 Markowitz Model I

$$QP1 \quad \min w^T \Omega w \;\; subject \;\; to \;\; e^T w = 1$$

Example: Q1(a)

Construct Markowitz portfolio optimization models for three assets with following data which represents rate of return of the assets. This should be understood in percentage.

	A	B	C
T_0	10	15	10
T_1	15	20	15
T_2	13	18	11
T_3	15	15	14

T_j denotes time at j^{th} period, A,B,C are assets.

Ans: Denote the return table as

10	15	10
15	20	15
13	18	11
15	15	14
$E(r_1) = 13.25$	$E(r_2) = 17$	$E(r_3) = 12.5$

Denote the deviation matrix as $D = (r_J - E(r_j))$.

D=
$$\begin{array}{|ccc|} \hline -3.25 & -2 & -2.5 \\ 1.75 & 3 & 2.5 \\ -0.25 & 1 & -1.5 \\ 1.75 & -2 & 1.5 \\ \hline \end{array}$$

Then covariance matrix is $\Omega = 1/4D^TD$,

$$\Omega = \begin{array}{|ccc|} \hline 4.1875 & 2 & 3.875 \\ 2 & 4.5 & 2 \\ 3.875 & 2 & 4.25 \\ \hline \end{array}$$

$$QP1: \quad \min \quad w^T \Omega w \ \ s.to \ \ e^T w = 1,$$

which is

$$\min \quad 4.1875w_1^2 + 4.5w_2^2 + 4.25w_3^2 + 4w_1w_2 + 7.75w_1w_3 + 4w_2w_3$$

subject to

$$w_1 + w_2 + w_3 = 1$$

Solution of $QP1$:

$$QP1 \ \ \min w^T \Omega w \ \ subject \ \ to \ \ e^T w = 1$$

$QP1$ is a convex programming problem since the variance matrix $\Omega \succ 0$. Hence this can be solved using KKT optimality conditions. Consider the lagrange function with primal vector w and dual variable μ as

$$L(w, \mu) = w^T \Omega w + \mu(1 - e^T w)$$

KKT optimality conditions for the existence of solution are $\nabla_w L(w, \mu) = 0, e^T w = 1$.
Since $\Omega \succ 0$ so Ω^{-1} exists. Hence
$\nabla_w L(w, \mu) = 0$
$\equiv 2\Omega w - \mu e = 0$
$\equiv 2\Omega^{-1}\Omega w = \Omega^{-1}\mu e$
$\equiv \ w = \frac{\mu}{2}\Omega^{-1}e$
Next, $e^T w = 1 \equiv e^T(\frac{\mu}{2}\Omega^{-1}e) = 1$. Hence $\mu = \frac{2}{e^T\Omega^{-1}e}$, since $\Omega \succ 0$.
Substituting this value in w, we have the optimal portfolio as ANS $w^* = \frac{\Omega^{-1}e}{e^T\Omega^{-1}e}$.
Consider Q1(a). Here optimal portfolio is $w^* = (w_1^*, w_2^*, w_3^*)^T$, where

$$w^* = \frac{\Omega^{-1}e}{e^T\Omega^{-1}e} = \frac{1}{0.32}\begin{bmatrix} 1.5488 & -0.0768 & -1.376 \\ -0.0768 & 0.2848 & -0.064 \\ -1.376 & -0.064 & 1.52 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.45 \\ 0.25 \end{bmatrix}$$

1.2.2 Model II

For given R,

$$(QP2) : \min w^T\Omega w \text{ subject to } \mu^T w = R, e^T w = 1$$

Details discussed in class

This is also a convex programming problem. Hence KKT conditions are both necessary and sufficient for the existence of solution. With dual variables λ_1 and λ_2, consider the lagrange function as $L(w, \lambda_1, \lambda_2) = w^T\Omega w + \lambda_1(R - \mu^T w) + \lambda_2(1 - e^T w)$. Then KKT conditions are $\nabla_w L(w, \lambda_1, \lambda_2) = 0, \mu^T w = R, e^T w = 1$. $\nabla_w L(w, \lambda_1, \lambda_2) = 0$

$\equiv 2\Omega w - \lambda_1\mu - \lambda e = 0$

$\equiv w = \frac{1}{2}\Omega^{-1}(\lambda_1\mu + \lambda_2 e)$

$\mu^T w = R \Rightarrow \lambda_1\mu^T\Omega^{-1}\mu + \lambda_2\mu^T\Omega^{-1}e = 2R$

$e^T w = 1 \Rightarrow \lambda_1 e^T\Omega^{-1}\mu + \lambda_2 e^T\Omega^{-1}e] = 2$

After solving above two equations,

$$\lambda_1 = \frac{det\begin{pmatrix} 2R & \mu^T\Omega^{-1}e \\ 2 & e^T\Omega^{-1}e \end{pmatrix}}{det\begin{pmatrix} \mu^T\Omega^{-1}\mu & \mu^T\Omega^{-1}e \\ e^T\Omega^{-1}\mu & e^T\Omega^{-1}e \end{pmatrix}} \quad \lambda_2 = \frac{det\begin{pmatrix} \mu^T\Omega^{-1}\mu & 2R \\ e^T\Omega^{-1}\mu & 2 \end{pmatrix}}{det\begin{pmatrix} \mu^T\Omega^{-1}\mu & \mu^T\Omega^{-1}e \\ e^T\Omega^{-1}\mu & e^T\Omega^{-1}e \end{pmatrix}}$$

Hence the optimal portfolio is,

$$w^* = \frac{det\begin{pmatrix} 2R & \mu^T\Omega^{-1}e \\ 2 & e^T\Omega^{-1}e \end{pmatrix}}{det\begin{pmatrix} \mu^T\Omega^{-1}\mu & \mu^T\Omega^{-1}e \\ e^T\Omega^{-1}\mu & e^T\Omega^{-1}e \end{pmatrix}}\Omega^{-1}\mu + \frac{det\begin{pmatrix} \mu^T\Omega^{-1}\mu & 2R \\ e^T\Omega^{-1}\mu & 2 \end{pmatrix}}{det\begin{pmatrix} \mu^T\Omega^{-1}\mu & \mu^T\Omega^{-1}e \\ e^T\Omega^{-1}\mu & e^T\Omega^{-1}e \end{pmatrix}}\Omega^{-1}e$$

1.2.3 Model III

$$(QP3) \quad \min w^T \Omega w \text{ subject to } \mu^T w \geq R, e^T w = 1$$

Details discussed in class

$L(w, \lambda_1, \lambda_2) = w^T \Omega w + \lambda_1(\mu^T w - R) + \lambda_2(1 - e^T w)$

KKT Conditions are :

Normalized condition: $\nabla_w L(w, \lambda_1, \lambda_2) = 0 \equiv 2\Omega w + \lambda_1 \mu - \lambda_2 e = 0$

Feasibility conditions: $\mu^T w \geq R, e^T w = 1$

Complementary Slackness condition: $\lambda_1(\mu^T w - R) = 0$

Dual restrictions: $\lambda_1 \geq 0$

This can be converted to following LPP with restricted entry rule.

$\min z_1 + z_2 + ... + z_n + a_1 + a_2$

$2\Omega w + \lambda_1 \mu - \lambda_2 e + z = 0$

$\mu^T w - s + a_1 = R$

$e^T w + a_2 = 1$

$\lambda_1, z_j, s, a_1, a_2 \geq 0$ with respect to the restricted basis entry rule $\lambda_1.s = 0$, $z = (z_1, z_2, ..., z_n)^T$, z_j, a_1, a_2 are artificial variables. s is surplus variable.

This problem can be solve by LPP technique keeping in mind the restricted basis entry rule.

Consider following data of price of one unit of the three assets A, B,C.

	A	B	C
Jan	10	15	10
Feb	15	20	15
Mar	13	18	11
Apr	15	15	14
May	14	16	13

1. With this data set find the optimal portfolio of Markowitz models (Theory discussed in class)

   I: $\min w^T \Omega w \;\; s.to \;\; e^T w = 1$

II :$\min w^T \Omega w \;\; s.to \;\; e^T w = 1, \mu^T w = 0.3$

III: $\min w^T \Omega w \;\; s.to \;\; e^T w = 1, \mu^T w \geq 0.3$

Collect 20 assets data(closing/opening/max/min, any one from BSE or NSE), either month wise or day wise, in excel, for some time period of your choice. Total time period should be large. Asset names should start with any one letter from your first name. Calculate return and variance risk. You may use time series or general mathematical calculations discussed in the class. Save this file as yourrollno. You will use this file for future assignments as well as term project and class test. Throughout this course this data will be used.

- Develop code in Python or R to solve Markowitz model I , find optimal portfolio.
- Develop code in Python or R to solve Markowitz model II , find optimal portfolio.
- Develop code in Python or R to solve Markowitz model III , find optimal portfolio.

Call the saved data for Deviation matrix D–Find Ω matrix— use in-built codes for solving models I,II,III. Save your code as QP1ROLLNO, QP2ROLLNO,QP3ROLLNO. In a folder with name MKROLLNO, save your complete data set and python or R code. Send me for evaluation.

1.3 Portfolio Diagram

(Two asset model)

Consider a portfolio of two assets A_1 A_2. w_1 w_2 are proportions of investment in the assets A_1 and A_2, $w_1 + w_2 = 1$. We may write $w_1 = 1 - \alpha$ and $w_2 = \alpha$, $0 \leq \alpha 1$

Expected rate of return of $A_1 = \mu_1$

Expected rate of return of $A_2 = \mu_2$

Return of the the portfolio $\mu_P = w_1\mu_1 + w_2\mu_2$, $w_1 + w_2 = 1$. $\mu_P = (1 - \alpha)\mu_1 + \alpha\mu_2$

Variance of the portfolio

$$\sigma_P^2(\alpha) = w^T\Omega w = (1-\alpha,\alpha)\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}\begin{pmatrix}1-\alpha\\ \alpha\end{pmatrix}$$
$$=\sigma_1^2(1-\alpha)^2+\sigma_2^2\alpha^2+2\sigma_{12}\alpha(1-\alpha)$$
$$=\sigma_1^2(1-\alpha)^2+\sigma_2^2\alpha^2+2\rho\sigma_1\sigma_2\alpha(1-\alpha)$$

where $\rho=\frac{\sigma_{12}}{\sigma_1\sigma_2}, -1\le\rho\le 1$

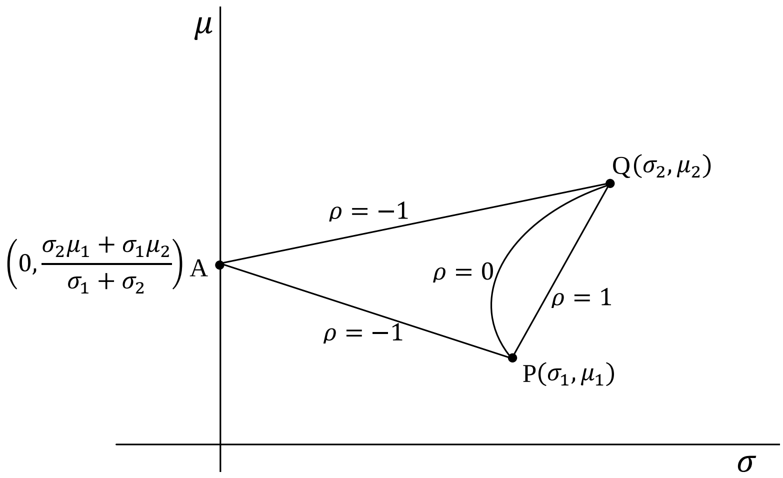

Figure 1.1: Efficient frontier

CASE I: $\rho = 1$.

In this case risk of the portfolio $\sigma_P(\alpha)=(1-\alpha)\sigma_1+\alpha\sigma_2(\alpha)$

and total return of the portfolio $\mu_P(\alpha)=(1-\alpha)\mu_1+\alpha\mu_2$. This is a line segment for $0\le\alpha\le1$, passing through two portfolio points (σ_1,μ_1) and (σ_2,μ_2)

CASE II: $\rho=-1$.

In this case

$$\sigma_P(\alpha) =\mid (1-\alpha)\sigma_1-\alpha\sigma_2(\alpha)\mid$$

So $\sigma_P(\alpha)$ is 0 if $\alpha=\frac{\sigma_1}{\sigma_1+\sigma_2}$. In that case

$$\mu_P=(1-\frac{\sigma_1}{\sigma_1+\sigma_2})\mu_1+\frac{\sigma_1}{\sigma_1+\sigma_2}\mu_2=\frac{\sigma_1\mu_2+\sigma_2\mu_1}{\sigma_1+\sigma_2}$$

$\sigma_P(\alpha) = (1-\alpha)\sigma_1 - \alpha\sigma_2$ if $\alpha \leq \frac{\sigma_1}{\sigma_1+\sigma_2}$ and

$\sigma_P(\alpha) = -[(1-\alpha)\sigma_1 - \alpha\sigma_2(\alpha)]$ if $\alpha \geq \frac{\sigma_1}{\sigma_1+\sigma_2}$.

Hence $\sigma_P(\alpha)$ provides two line segments for $0 \leq \alpha \leq 1$, one passing through (σ_1, μ_1) and $(0, \frac{\sigma_1\mu_2+\sigma_2\mu_1}{\sigma_1+\sigma_2})$; and another passing through (σ_2, μ_2) and $(0, \frac{\sigma_1\mu_2+\sigma_2\mu_1}{\sigma_1+\sigma_2})$.

Case III: $0 < \rho < 1$.

In this case $\sigma^2(\alpha)$ is a convex function of α.

$$\frac{d}{d\alpha}\sigma^2(\alpha) = -2(1-\alpha)\sigma_1^2 + 2\alpha\sigma_2^2 + 2\rho(1-\alpha)\sigma_1\sigma_2 - 2\rho\alpha\sigma_1\sigma_2$$

$$\begin{aligned}\frac{d^2}{d\alpha^2}\sigma^2(\alpha) &= 2\sigma_1^2 + 2\sigma_2^2 + 2\alpha\sigma_2\sigma_2 - 2\rho\sigma_1\sigma_2 \\ &= 2(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma) > 0, since 0 < \rho < 1\end{aligned}$$

Hence $\sigma^2(\alpha)$ is a convex function.

$\frac{d}{d\alpha}\sigma^2(\alpha) = 0$ iff

$$\begin{aligned}&-(1-\alpha)\sigma_1^2 + 2\alpha\sigma_2^2 + 2\rho(1-\alpha)\sigma_1\sigma_2 - 2\rho\alpha\sigma_1\sigma_2 = 0 \\ &\alpha(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\rho - \rho\sigma_1\sigma_2) = \sigma_1^2 + \rho\sigma_1\sigma_2 \\ &\alpha^* = \frac{\sigma_1(1-\rho\sigma_2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\end{aligned}$$

At $\alpha^* = \frac{\sigma_1(1-\rho\sigma_2)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$,

$$\begin{aligned}\mu(\alpha^*) &= Return\ \ of\ \ the\ \ portfolio \\ &= (1-\alpha^*)\mu_1 + \alpha^*\mu_2 \\ &= \left(\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 - \sigma_1^2 + \rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2 - 2\rho\sigma_1\sigma_2}\right)\mu_1 + \left(\frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\right)\mu_2 \\ &= \frac{(\sigma_2^2 - \rho\sigma_1\sigma_2)\mu_1 + (\sigma_1^2 - \rho\sigma_1\sigma_2)\mu_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\end{aligned}$$

Two Fund Theorem

Two efficient funds or portfolios can be established so that any efficient portfolio can be duplicated, in terms of mean and variance, as a combination of these two. All investors seeking efficient portfolios need only to invest in combinations of these two funds. Hence two products can provide a complete investment service.

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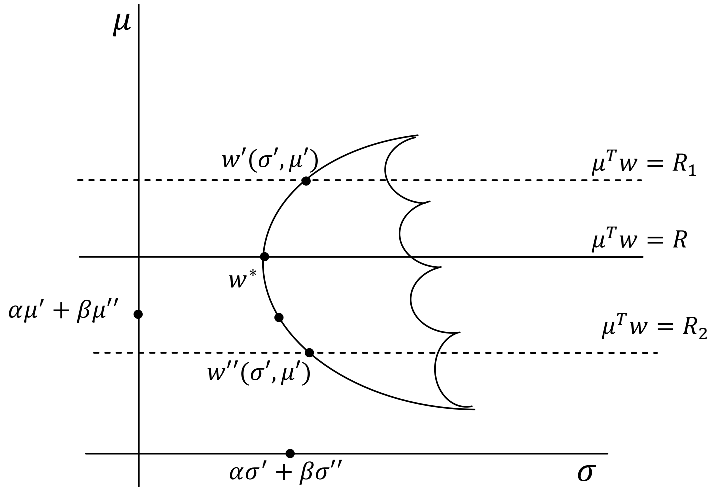

Figure 1.2: Efficient frontier

Proof:Consider a Markowitz MV Model

$$(MV)\min\ w^T\Omega w\ \ subject\ \ to\ \ e^Tw = 1,\ \mu^Tw = R$$

Suppose w' and $w^{''}$ are two efficient portfolios corresponding to the returns R_1 and R_2 respectively. Then $e^Tw' = 1,\ \mu^Tw' = R_1$ and $e^Tw^{''} = 1,\ \mu^Tw^{''} = R_2$

Consider convex combination $w_\lambda = \lambda w' + (1-\lambda)w^{''}$, $0 \le \lambda \le 1$. Then

$$e^Tw_\lambda = e^T(\lambda w' + (1-\lambda)w^{''}) = e^T\lambda w' + (1-\lambda)e^Tw^{''} = \lambda + 1 - \lambda = 1$$

and $\mu^T(\lambda w' + (1-\lambda)w^{''}) = \lambda R_1 + (1-\lambda)R_2$. So w_λ is a feasible portfolio of (MV)

with return R_λ say. Consider the (MV) model with respect to return R_λ:

$$(MV_\lambda) \quad \min \quad w^T \Omega w \quad s.to \ e^T w = 1, \ \ \mu^T w = R_\lambda$$

Next we show that w_λ is an optimal solution of (P_λ) for every λ.
(P_λ) is a convex programming problem for each λ. KKT optimal conditions for (P_λ) are both necessary and sufficient conditions for the existence of solution.
$L(w, \alpha, \beta) = w^T \Omega w + \alpha(1 - e^T w) + \beta(R_\lambda - \mu^T w)$
$\nabla_w L(w, \alpha, \beta) = 2\Omega w_\lambda - \alpha e - \beta\mu$
$= 2\Omega(\lambda w' + (1-\lambda)w'') - \alpha(\lambda e + (1-\lambda)e) - \beta(\lambda\mu + (1-\lambda)\mu)$
$= \lambda(2\Omega w' - \alpha e - \beta\mu) + (1-\lambda)(2\Omega w'' - \alpha e - \beta\mu) = 0$, since w and w' are optimal solution of P, so they satisfy KKT conditions for P.
Hence w_λ is an efficient portfolio whose

$$\begin{aligned} \mu(w_\lambda) &= \lambda R_1 + (1-\lambda)R_2 \\ \sigma^2(w_\lambda) &= \sigma^2(\lambda w' + (1-\lambda)w'') \\ &= \lambda^2 \sigma_w^2 + (1-\lambda)^2 \sigma_w^2 \end{aligned}$$

Example 1.3.1. *Consider your model portfolio (20 asset set). As per your choice fix returns R_1, R_2. Find the corresponding efficient portfolios w' and w''. Construct different portfolios with different values of λ and trace Markowitz efficient frontier.*

1.4 Fractional Programming and Capital Asset Pricing Model

- Let P be a portfolio which has n risky assets $A_1, A_2, \ldots, A_n$ and one risk free asset A_f, with return $\mu_1, \mu_2, ..., \mu_n$ and μ_f respectively.

- $w_j, j = 1, 2, ..., n$ is the proportion of investment in asset A_j and w_f is the proportion of investment in the risk free asset. $\sum_{j=1}^{n} w_j + w_f = 1$.

- Expected return of the portfolio $=\mu_P = \sum_{j=1}^{n} w_j\mu_j + w_f\mu_f$

- Let P_d be the portfolio with risky assets $A_1, A_2, ..., A_n$ only and derived from the original portfolio P. μ_d be the expected rate of return and σ_d^2 be the variance of the derived portfolio respectively.

- Denote $w_r = \sum_{j=1}^{n} w_j$, $w_d = (\frac{w_1}{w_r}, \frac{w_2}{w_r}, ..., \frac{w_n}{w_r})^T = (w_1', w_2', ..., w_n')^T$

Then

$$\begin{aligned} \mu_P &= \sum_{j=1}^{n} w_j\mu_j + w_f\mu_f \\ &= w_r \sum_{j=1}^{n} \frac{w_j}{w_r}\mu_j + w_f\mu_f = w_r\mu_d + w_f\mu_f \\ &= w_r\mu_d + (1-w_r)\mu_f = w_r(\mu_d - \mu_f) + \mu_f \end{aligned}$$

$$\sigma_P^2 = w^T\Omega w + 0 = w_r^2 w_d^T \Omega w_d = w_r^2\sigma_d^2 \quad or \quad w_r = \frac{\sigma_P}{\sigma_d}$$

Hence $\mu_P = \frac{\sigma_P}{\sigma_d}(\mu_d - \mu_f) + \mu_f$ that is, $\frac{\mu_P - \mu_f}{\sigma_P} = \frac{\mu_d - \mu_f}{\sigma_d}$. Since the $w_1, w_2, ..., w_n; w_f$ is unknown so this relation for any portfolio can be expressed as

$$\frac{\mu - \mu_f}{\sigma} = \frac{\mu_d - \mu_f}{\sigma_d}$$

which is a line joining $(0, \mu_f)$ and (σ_d, σ_f). For various combination of $(w_1, w_2, \ldots, w_n, w_f)$,

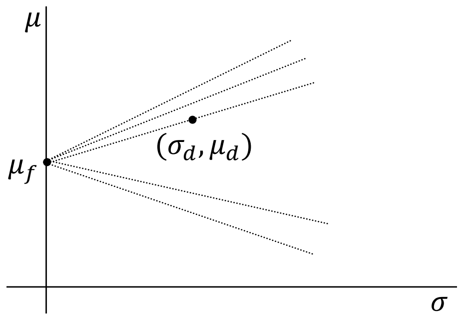

Figure 1.3:

this line takes different forms. Among these lines, the line which is tangent to the upper portion of Markowitz curve is known as CAPITAL MARKET LINE (CML)and the point where CML is tangent to Markowitz curve is known as MARKET PORTFOLIO with risk and return denoted by $M = (\sigma_M, \mu_M)$.

- CML results from the combination of the market portfolio and the risk-free asset.
- Slope of CML ($\frac{\mu_M-\mu_f}{\sigma_M}$)is known as sharp ratio of the market portfolio, which is the maximum slope of the line $\frac{\mu-\mu_f}{\sigma} = \frac{\mu_d-\mu_f}{\sigma_d}$.
- Equation of capital market line for a portfolio P with n - risky assets and one risk free asset is

$$\mu = \frac{\mu_M - \mu_f}{\sigma_M}\sigma + \mu_f$$

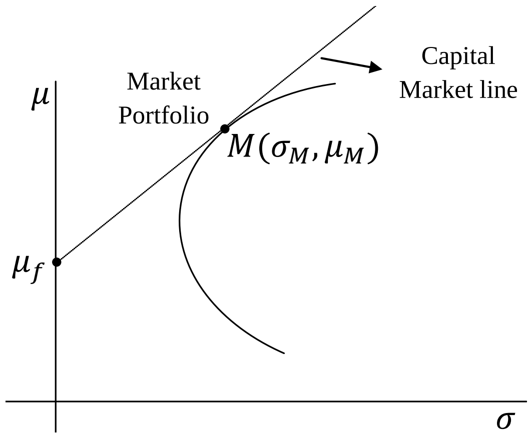

Figure 1.4: Efficient frontier

1.4.1 How to find Market portfolio?

Recall the notations:

- $P = (w_1, w_2, ..., w_n : w_f)$, $w_r = \sum_{j=1}^{n} w_j$.
- Derived portfolio is $P_d = (w_1^{'}, w_2^{'}, ..., w_n^{'})$, where $w_j^{'} = \frac{w_j}{w_r}$ is the investment in risky asset A_j of the derived portfolio.
- Denote $m = (\mu_1, \mu_2, ..., \mu_n)^T$ is the expected rate of return vector for the derived portfolio P_d.
- Expected rate of return of the derived portfolio is $\mu_d = w_d^T m$ and variance risk of the derived portfolio is $\sigma_d^2 = w_d^T \Omega w_d$

Market portfolio is the point on the capital market line when return is maximum, that is, Market portfolio is a point on the efficient frontier where slope of the line $\frac{\mu-\mu_f}{\sigma} = \frac{\mu_d-\mu_f}{\sigma_d}$ is maximum. Hence we need to solve the following fractional optimization problem.

$$\begin{aligned} \max \ & \frac{\mu_d - \mu_f}{\sigma_d} \\ \text{s.t.} \ & e^T w_d = 1 \end{aligned}$$

$$\frac{\mu_d - \mu_f}{\sigma_d} = \frac{m^T w_d - \mu_f}{\sqrt{w_d^T \Omega w_d}}$$

This is a concave programming problem.
$L(w_d, \lambda) = \frac{m^T w_d - \mu_f}{\sqrt{w_d^T \Omega w_d}} + \lambda(1 - e^T w_d)$.
KKT optimality conditions for this optimization problem are:

$$\nabla_w L(w_d, \lambda) = 0, \lambda \in R, e^T w_d = 1$$

$$\begin{aligned} & \nabla_w L(w_d, \lambda) = 0 \\ \equiv \ & \frac{m}{\sqrt{w_d^T \Omega w_d}} - \frac{m^T w - \mu_f}{(w_d^T \Omega w_d)^{3/2}} \Omega w_d = \lambda e \\ \equiv \ & \frac{m}{\sigma_d} - \frac{\mu_d - \mu_f}{\sigma_d^3} \Omega w_d = \lambda e \\ \equiv \ & m\sigma_d^2 - (\mu_d - \mu_f)\Omega w_d = \lambda e \sigma_d^3 \\ \equiv \ & w_d^T(m\sigma_d^2) - (\mu_d - \mu_f) w_d^T \Omega w_d = \lambda \sigma_d^3 w_d^T e = \lambda \sigma_d^3 \\ \equiv \ & \mu_d \sigma_d^2 - \mu_d \sigma_d^2 + \mu_f \sigma_d^2 = \lambda \sigma_d^3 \\ \equiv \ & \lambda = \frac{\mu_f}{\sigma_d} \end{aligned}$$

$$\begin{aligned}
m\sigma_d^2 - (\mu_d - \mu_f)\Omega w_d =& \frac{\mu_f}{\sigma_d} e\sigma_d^3 \\
\sigma_d^2(m - \mu_f e) =& (\mu_d - \mu_f)\Omega w_d \\
\sigma_d^2\Omega^{-1}(m - \mu_f e) =& (\mu_d - \mu_f) w_d \\
\sigma_d^2 e^T\{\Omega^{-1}(m - \mu_f e)\} =& (\mu_d - \mu_f) e^T w_d = (\mu_d - \mu_f) \\
w_d^T \Omega w_d e^T \Omega^{-1}(m - \mu_f e) =& w_d^T m - \mu_f = w_d^T(m - \mu_f e) \\
\Omega w_d e^T \Omega^{-1}(m - \mu_f e) =& (m - \mu_f e)
\end{aligned}$$

$$w_d = \frac{\Omega^{-1}(m - \mu_f e)}{e^T\Omega^{-1}(m - \mu_f e)}$$

$m, \Omega \rightarrow$ Return, Covariance of derived portfolio.

$w_d = (\frac{w_1}{w_r}, \frac{w_2}{w_r}, ..., \frac{w_n}{w_r})^T$, $\sum_{i=1}^n \frac{w_1}{w_r} + w_f = 1$ i.e $w_d^T e = 1 - w_f$ Equation of capital market line for a portfolio P is

$$\begin{aligned}
\mu =& \frac{\mu_m - \mu_f}{\sigma_m}\sigma + \mu_f \\
=& \frac{\sigma}{\sigma_m}(\mu_m - \mu_f) + \mu_f \qquad (1.4.1) \\
=& w_P \mu_m + (1 - w_p)\mu_f \qquad (1.4.2)
\end{aligned}$$

i.e. if the investment is willing to accept risk σ then he has to invest w_p in market portfolio and $(1 - w_p)$ in risk free assets.

Equation of Security Market line:

(Relation between individual asset and market portfolio) Prove that if $M(\sigma_m, \mu_m)$ is the market portfolio and $A(\sigma, \mu)$ is an individual asset then

$$\mu = \beta(\mu_m - \mu_f) + \mu_f,$$

where $\beta = \frac{Cov(A,M)}{\mu_m}$.

Proof:

Consider one risky asset A_i whose expected return is μ_i and S.D. σ_i , M is the market portfolio with expected return μ_m and variance σ_m^2. Suppose the investor comprises of asset A_i with weight w and market portfolio with weight $(1-w)$. Then expected return of the investor portfolio is

$$\begin{aligned} \mu =& w\mu_i + (1-w)\mu_m \\ \sigma^2(w) =& w^2\sigma_i^2 + (1-w)^2\sigma_m^2 + 2\rho w(1-w)\sigma_i\sigma_m \end{aligned} \tag{1.4.3}$$

$\rho =$ correlation between the returns of asset A_i and market portfolio.

If w varies then $\sigma^2(w)$ traces a non linear curve in $\mu - \sigma$ space. At $w = 0, \sigma = \sigma_m, \mu = \mu_m$, which corresponds to the capital market point M. At $w = 1, \sigma = \sigma_i, \mu = \mu_i$, which correspond to the portfolio A

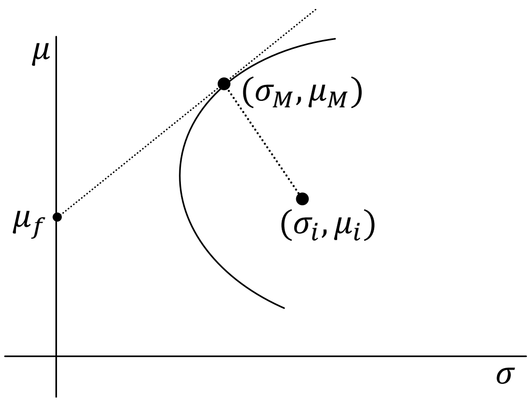

Figure 1.5: Efficient frontier

$$\begin{aligned} \frac{d\mu}{dw} \mid_{w=0} =& \mu_i - \mu_m \\ \frac{d\sigma}{dw} \mid_{w=0} =& \frac{1}{2\sigma \mid_{w=0}} \left(2w\sigma_i^2 - 2(1-w)\sigma_m^2 + 2\rho\sigma_i\sigma_m - 4w\rho\sigma_i\sigma_m\right)_{w=0} \\ =& \frac{-2\sigma_m^2 + 2\rho\sigma_i\sigma_m}{2\sigma(at\ w=0)} = \frac{\rho\sigma_i\sigma_m - \sigma_m^2}{\sigma} \\ =& \frac{\sigma_{im} - \sigma_m^2}{\sigma_m}, \quad \rho = \frac{\sigma_{im}}{\sigma_i\sigma_m} \end{aligned} \tag{1.4.4}$$

$$\frac{d^2\sigma}{dw^2} \mid_{w=0} = \frac{\sigma_i^2}{\sigma_m}(1-\rho^2) \geq 0$$

Hence $\sigma^2(w)$ passes through capital market point M at $w = 0$ and Capital Market Line is tangent to this curve at M.

Slope of the tangent to the curve $\sigma^2(w)$ at $w = 0$ is

$$\frac{d\mu}{d\sigma}\mid_{w=0} = \frac{d\mu}{dw}\frac{dw}{d\sigma}\mid_{w=0} = \left(\frac{\mu_i - \mu_m}{\sigma_{im} - \sigma_m^2}\right)\sigma_m$$

Slope of capital market line = Slope of tangent to $\sigma^2(w)$ at $w = 0$. Hence

$$\frac{\mu_m - \mu_f}{\sigma_m} = \left(\frac{\mu_i - \mu_m}{\sigma_{im} - \sigma_m^2}\right)\sigma_m$$

$$\begin{aligned}
\mu_i - \mu_m =& \frac{\mu_m - \mu_f}{\sigma_m^2}(\sigma_{im} - \sigma_m^2) \\
=& (\mu_m - \mu_f)\left(\frac{\sigma_{im}}{\sigma_i^2} - 1\right) \\
\mu_i =& \mu_m + \left(\frac{\sigma_{im}}{\sigma_i^2} - 1\right)(\mu_m - \mu_f) \\
\mu_i =& \cancel{\mu_m} + \frac{\sigma_{im}}{\sigma_i^2}(\mu_m - \mu_f) - \cancel{\mu_m} + \mu_f \\
\mu_i =& \frac{\sigma_{im}}{\sigma_i^2}(\mu_m - \mu_f) + \mu_f
\end{aligned}$$

Beta ratio$(\beta_i) = \frac{\sigma_{im}}{\sigma_m^2}$=Risk of i^{th} asset in relation to market risk. In general for any individual asset $A = (\sigma, \mu)$, this linear equation can be expressed as

$$\mu = \beta(\mu_m - \mu_f) + \mu_f$$

This equation is known as security market line.