Mixed Strategies

Prelude

- We came across a class of games that the solution methods described there could not solve; in fact, games in that class have ***no Nash equilibrium in pure strategies***.
- To predict outcomes for such games, we need an extension of our concepts of strategies and equilibria.
- This is to be found in the **randomization** of moves.
- Remember the tennis game – we had argued that it is best to ***mix your strategies*** – keep the opponent guessing – never act systematically
- However, randomness doesn't mean choosing each shot half the time or alternating between the two.
- The latter would itself be a systematic action open to exploitation, and a 60–40 or 75–25 random mix may be better than 50–50 depending on the situation.

What is a mixed strategy?

- When players choose to act unsystematically, they pick from among their pure strategies in some random way
- We call a ***random mixture of these two pure strategies*** a mixed strategy
- Such mixed strategies ***cover a whole continuous range***

- At one extreme Strategy A can be chosen with probability 1 (for sure), meaning that Strategy B is never chosen (probability 0); this “mixture” is just the pure strategy A.
- On the other extreme Strategy B can be chosen with probability 1 (for sure), meaning that Strategy A is never chosen (probability 0); this “mixture” is just the pure strategy B.
- In between, the whole set of possibilities of choosing Strategy A with probability π and choosing Strategy B with probability $(1-\pi)$

Expected Payoff

- The payoffs from a mixed strategy are defined as the corresponding ***probability-weighted averages of the payoffs from its constituent pure strategies***.

		NAVRATILOVA	
		DL	CC
EVERT	DL	50, 50	80, 20
	CC	90, 10	20, 80

- The payoff of Evert's mixture (0.75 DL, 0.25 CC) against Navratilova's DL is {(0.75 X 50) + (0.25 X 90)} = (37.5 + 22.5) = 60.

- 60 is Evert's ***expected payoff*** from this particular mixed strategy

Mixed Strategy and NE

- The probability of choosing one or the other pure strategy is a continuous variable that ranges from 0 to 1.
- Each pure strategy is an extreme special case where the probability of choosing that pure strategy equals 1.

- The notion of Nash equilibrium also extends easily to include mixed strategies.
- Nash equilibrium is defined as a list of mixed strategies, one for each player, such that the choice of each is her best choice, in the sense of yielding the highest expected payoff for her, given the mixed strategies of the others.
- Allowing for mixed strategies in a game solves the problem of possible nonexistence of Nash equilibrium
- Nash's celebrated theorem shows that, under very general circumstances a Nash equilibrium in mixed strategies exists

Mixed Strategy NE

- Lets start with the game with no pure strategy NE

		NAVRATILOVA	
		DL	CC
EVERT	DL	50, 50	80, 20
	CC	90, 10	20, 80

- Suppose Evert is not restricted to using only pure strategies and can choose a mixed strategy, perhaps one in which the probability of playing DL on any one occasion is 75%; this makes her probability of playing CC 25%.

- We can calculate Navratilova's expected payoff against this mixture as –

  {(0.75 X 50) + (0.25 X 10)} = (37.5 + 2.5) = 40 if she covers DL, and

  {(0.75 X 20)+ (0.25 X 80)} = (15 + 20) = 35 if she covers CC.

- If Evert chooses this 75–25 mixture, the expected payoffs show that Navratilova can best exploit it by covering DL

Mixed Strategy NE

- When Navratilova chooses DL to best exploit Evert's 75–25 mix, her choice works to Evert's disadvantage because this is a zero-sum game. Evert's expected payoffs are –

$\{(0.75 \text{ X } 50) + (0.25 \text{ X } 90)\} = (37.5 + 22.5) = 60$ if Navratilova covers DL,

$\{(0.75 \text{ X } 80) + (0.25 \text{ X } 20) = (60 + 5) = 65$ if Navratilova covers CC.

- By choosing DL, Navratilova holds Evert down to 60 rather than 65.

- But notice that Evert's payoff with the mixture is still better than the 50 she would get by playing purely DL or the 20 she would get by playing purely CC.

Mixed Strategy NE

- Ideally, Evert would like to find a mix that would be exploitation proof—a mix that would leave Navratilova no obvious choice of pure strategy to use against it

- Evert's ***exploitation-proof mixture*** must have the property that Navratilova gets the same expected payoff against it by covering DL or CC; it must keep Navratilova ***indifferent between her two pure strategies***

- We call this the opponent's ***indifference property***; it is the key to mixed-strategy equilibria in non-zero-sum games

- To find the exploitation-proof mix requires taking a more general approach to describing Evert's mixed strategy so that we can solve algebraically for the appropriate mixture probabilities

Mixed Strategy NE

- We denote the probability of Evert choosing DL by the algebraic symbol p, so the probability of choosing CC is $(1 – p)$.
- We refer to this mixture as Evert's p-mix for short.
- Against the p-mix, Navratilova's expected payoffs are –

$50p + 10(1 - p)$ if she covers DL, and

$20p + 80(1 - p)$ if she covers CC

- For Evert's strategy, her p-mix, to be exploitation proof, these two expected payoffs for Navratilova should be equal.
- That implies $50p + 10(1 - p) = 20p + 80(1 - p)$

  or $30p = 70(1 - p)$; or $100p = 70$; or $p = 0.7$
- Thus, Evert's exploitation-proof mix uses DL with probability 70% and CC with probability 30%

Mixed Strategy NE

- With these mixture probabilities, Navratilova gets the same expected payoff from each of her pure strategies and therefore cannot exploit any one of them to her advantage.

- Evert's expected payoff from this mixed strategy is –

  (50 X 0.7)) + (90 X 0.3) = (35 + 27) = 62 if Navratilova covers DL

  (80 X 0.7) + (20 X 0.3) = (56 + 6) = 62 if Navratilova covers CC

- This expected payoff is better than the 50 that Evert would get if she used the pure strategy DL and better than the 60 from the 75–25 mixture – this mixture is exploitation proof.

- Is it Evert's optimal or equilibrium mixture?

Best Responses and Equilibrium

- To find the ***equilibrium mixtures*** in this game, we return to the method of ***best-response analysis***
- Our first task is to identify Evert's best response to (her best choice of p for) each of Navratilova's possible strategies.
- Since those strategies can also be mixed, they are similarly described by the probability with which she covers DL. Label this q.
- So $(1 - q)$ is the probability that Navratilova covers CC. We refer to Navratilova's mixed strategy as her q-mix
- Now we look for Evert's best choice of p at each of Navratilova's possible choices of q

Best Responses and Equilibrium

- Evert's p-mix gets her the expected payoff –

$$50p + 90(1 - p) \text{ if Navratilova chooses DL, and}$$

$$80p + 20(1 - p) \text{ if Navratilova chooses CC.}$$

- Therefore against Navratilova's q-mix, Evert's expected payoff is

$$[50p + 90(1 - p)]q + [80p + 20(1 - p)](1 - q)$$

- Rearranging the terms, Evert's expected payoff becomes –

$$[20 + 70q] + [60 - 100q]p$$

- Optimisation leads to $q = 0.6$

Best Responses and Equilibrium

- When Navratilova's $q < 0.6$, $[60 - 100q]$ is positive, Evert's expected payoff increases as p increases, and her best choice is $p = 1$, or the pure strategy DL.
- Similarly, when Navratilova's $q > 0.6$, Evert's best choice is $p = 0$, or the pure strategy CC.
- If Navratilova's $q = 0.6$, Evert gets the same expected payoff regardless of p, and any mixture between DL and CC is just as good as any other; any p from 0 to 1 can be a best response.

- If $q < 0.6$, best response is $p = 1$ (pure DL).
- If $q = 0.6$, any p-mix is a best response.
- If $q > 0.6$, best response is $p = 0$ (pure CC).

Best Responses and Equilibrium

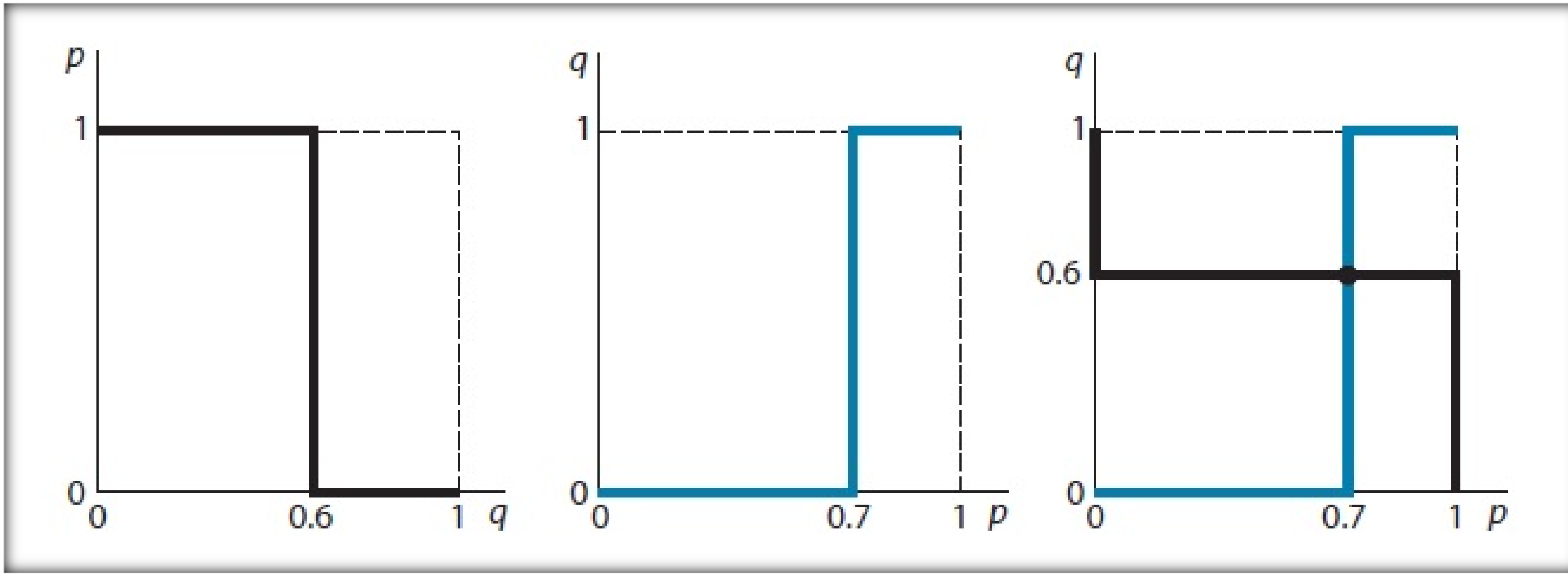
- As a quick confirmation of intuition, observe that when q is low (Navratilova is sufficiently unlikely to cover DL), Evert should choose DL, and when q is high (Navratilova is sufficiently likely to cover DL), Evert should choose CC.

- The exact sense of "sufficiently," and therefore the switching point $q = 0.6$, of course depends on the specific payoffs in the example.

- ***Mixed strategies are just a special kind of continuous strategy, with the probability being the continuous variable***

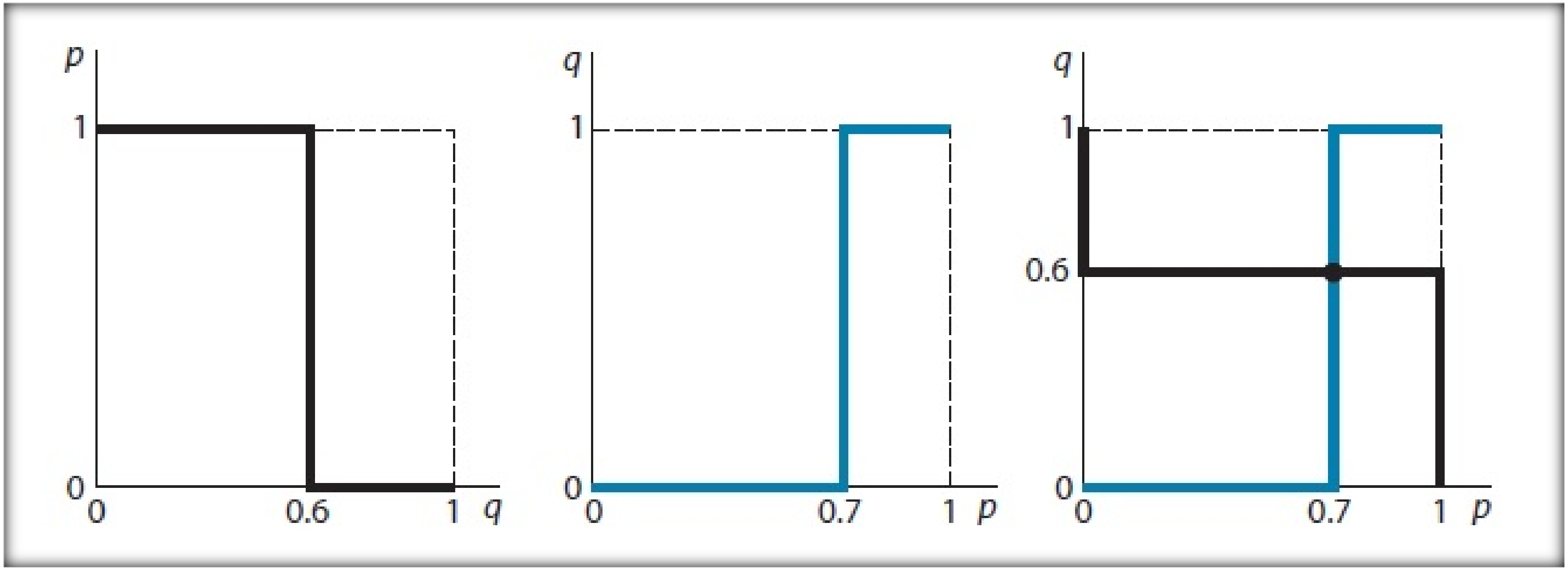
- We have found Evert's best p corresponding to each of Navratilova's choices of q – we have found Evert's best-response rule

Best Response of Evert



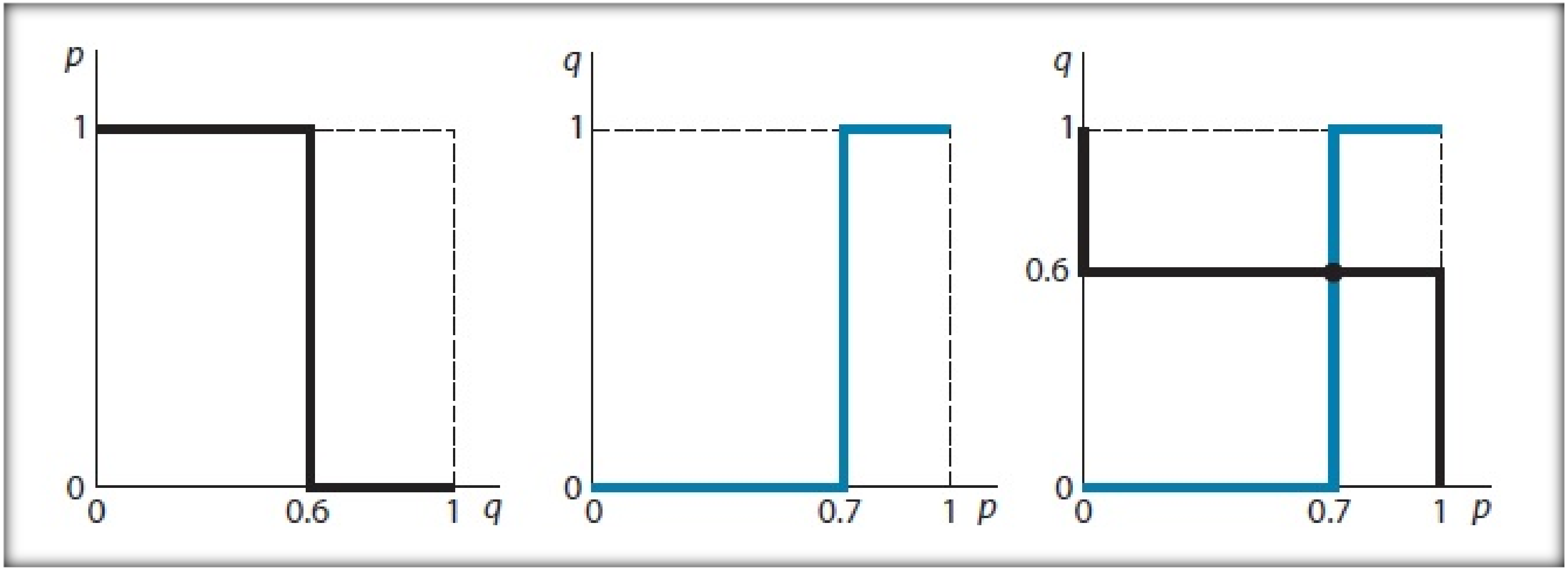
- q on the horizontal axis and p on the vertical axis
- For q less than 0.6, p is at its upper limit of 1; for q greater than 0.6, p is at its lower limit of 0. At q 5 0.6, all values of p between 0 and 1 are equally "best" for Evert
- the best response is the vertical line between 0 and 1

Best Responses of Navratilova



- If $p < 0.7$, best response is $q = 0$ (pure CC).
- If $p = 0.7$, any q-mix is a best response.
- If $p > 0.7$, best response is $q = 1$ (pure DL)

Best Responses and Equilibrium



- Best response curves meet at exactly one point, where p = 0.7 and q = 0.6.
- Here each player's mixture choice is a best response to the other's choice, so the pair constitutes a Nash equilibrium in mixed strategies

Mixed Strategy in Harry meets Sally

- We start with the assurance version of the coordination game (where, LL is preferred by students)

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	1, 1	0, 0
	Local Latte	0, 0	2, 2

- Let p denotes the probability (in Sally's mind) that Harry chooses Starbucks and (1 – p) for LL
- We can describe ***Sally's strategic uncertainty*** as: she thinks that Harry is using a mixed strategy, mixing the two pure strategies, Starbucks and Local Latte, in proportions or probabilities p and (1 - p), respectively

Mixed Strategy in Harry meets Sally

- Given this belief, Sally can calculate her expected pay-offs as – $[(p \times 1) + \{(1 - p) \times 0)\}] = p$ if Sally chooses Starbuck and $\{(p \times 0) + 2(1 - p)\} = 2(1 - p)$ if Sally chooses LL

- Sally is indifferent (or, unsure about her own choice) if – $p=2(1-p)$ or $3p = 2$ or $p = 2/3$

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	1, 1	0, 0
	Local Latte	0, 0	2, 2

Mixed Strategy in Harry meets Sally

- Now, if $p < 2(1 - p)$, implying probability of Harry's going to SB is lower than 2/3, Sally should go to LL

- If $p > 2(1 - p)$, implying probability of Harry's going to SB is higher than 2/3, Sally should go to SB

- If $p = 2/3$, Sally is indifferent (or, unsure about her own choice)

- Similarly, Harry can calculate his expected payoffs given any belief about Sally's q and he is again unsure if $q = 2/3$.

Mixed Strategies of 2 other versions

- In the pure-coordination version, the payoffs from meeting in the two cafés are the same, so the mixed-strategy equilibrium will have $p = ½$ and $q = ½$.

- The Battle-of-sexes version would have mixed-strategy equilibrium as $p = 2/3$ and $q = 1/3$.

Mixing when one has more than 2 Strategies

- Let's add one more strategy to Evert – Lob
- Navratilova mixes DL with probability q and CC with probability $(1\text{-}q)$

		NAVRATILOVA		
		DL	CC	q-mix
EVERT	DL	50, 50	80, 20	$50q + 80(1-q)$, $50q + 20(1-q)$
	CC	90, 10	20, 80	$90q + 20(1-q)$, $10q + 80(1-q)$
	Lob	70, 30	60, 40	$70q + 60(1-q)$, $30q + 40(1-q)$

- Navratilova's p-mix (of Evert) expected payoffs would be –

  $50p_1+10p_2+30(1 - p_1 - p_2)$ if she plays DL and

  $20p_1+80p_2+40(1 - p_1 - p_2)$ if she plays CC

Mixing when one has more than 2 Strategies

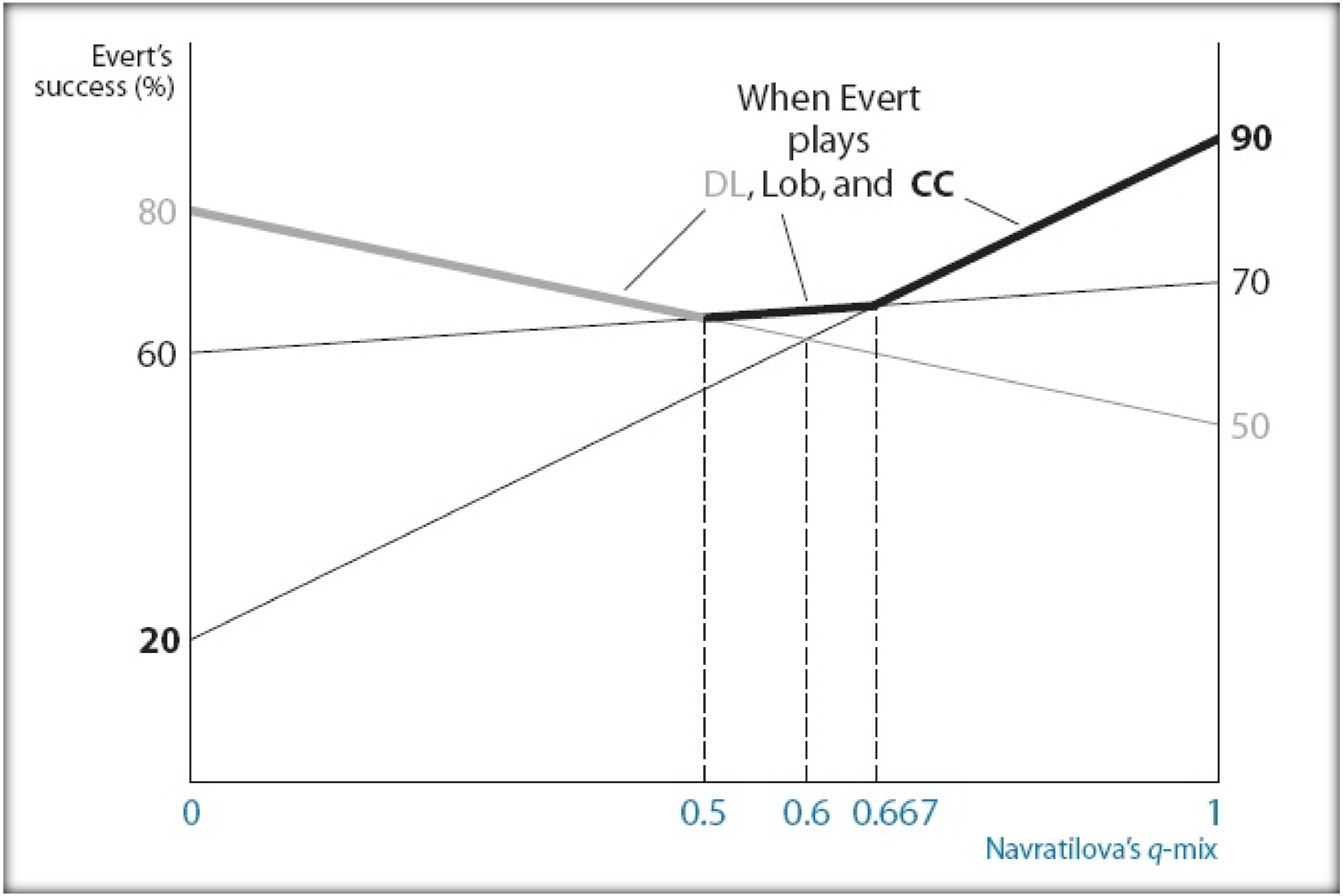
- We will use the logic of best responses to consider Navratilova's optimal choice of q

- For each q, if Navratilova were to choose that q-mix in equilibrium, Evert's best response would be to choose the strategy that gives her (Evert) the highest payoff

- We use the ***upper envelope*** of the response functions as already mentioned the last table –

1. $50q + 80(1-q)$
2. $90q + 20(1-q)$
3. $70q + 60(1-q)$

Mixing when one has more than 2 Strategies



Mixing when one has more than 2 Strategies

- To be more precise about Navratilova's optimal choice of q, we must calculate the coordinates of the kink points in the line showing her worst-case (Evert's best-case) outcomes

- The value of q at the leftmost kink in this line makes Evert indifferent between DL and Lob - That q must equate the two payoffs from DL and Lob when used against the q-mix.

  $50q + 80(1-q) = 70q + 60(1-q)$ implying $q = ½$
- Evert's expected payoff at this point is $\{(50 \text{ X } 0.5) + (80 \text{ X } 0.5)\} = \{(70 \text{ X } 0.5) + (60 \text{ X } 0.5)\} = 65$
- Similarly, the second kink would yield q = 2/3
- Evert's expected payoff would be 66.67

Mixing when one has more than 2 Strategies

- Hence, Navratilova's best option (where Evert's expected payoff is least) would be $q = 0.5$
- Her expected payoff would be 35 (Evert's was 65)

- When Navratilova chooses $q = 0.5$, Evert is indifferent between DL and Lob, and either of these choices gives her a better payoff than does CC.

- Therefore, ***Evert will not use CC at all in equilibrium***.

- CC will be an ***unused strategy in her equilibrium mix***.

Mixing when one has more than 2 Strategies

- Now we can proceed with the equilibrium analysis as if this were a game with just two pure strategies for each player: DL and CC for Navratilova, and DL and Lob for Evert.

- Evert's optimal mixture in this game entails her using DL with probability 0.25 and Lob with probability 0.75.

- Evert's expected payoff from this mixture, taken against Navratilova's DL and CC, respectively, is 65 (as already calculated).

Both with 3 Strategies

- Soccer Penalty Kick Game

		GOALIE		
		Left	Center	Right
KICKER	Left	45, 55	90, 10	90, 10
	Center	85, 15	0, 100	85, 15
	Right	95, 5	95, 5	60, 40

- We can solve exactly like the 2-startegy case, a little more mathematical calculations required
- Kicker mixes with p_R, p_L and $(1 – p_R – p_L)$ then Goalie's expected payoffs –

$$
\begin{array}{ll}
\text{Left:} & 55p_L + 15p_C + 5p_R = 55p_L + 15(1 - p_L - p_R) + 5p_R \\
\text{Center:} & 10p_L + 100p_C + 5p_R = 10p_L + 100(1 - p_L - p_R) + 5p_R \\
\text{Right:} & 10p_L + 15p_C + 40p_R = 10p_L + 15(1 - p_L - p_R) + 40p_R
\end{array}
$$

Both with 3 Strategies

- The opponent's indifference rule says that the kicker should choose p_R and p_L so that all three of these expressions are equal in equilibrium.

- Equating Left and Right equations we have $p_R = (9/7)\ p_L$
- Equate Centre and Right equations plugging this relation
- Get $p_L = 0.355$; $p_R = 0.457$ and $p_C = (1 - p_L - p_R) = 0.188$
- The goalie's payoff from any of his pure strategies against this mixture can then be calculated by using any of the preceding three payoff lines; the result is 24.6

- The goalie's mixture probabilities can be found out to be –

  $q_L = 0.325$, $q_R = 0.561$, and $q_C = 0.113$
- Kicker's payoff = 75.4

Undesirable properties of Mixed Strategy

- Mixed strategy equilibrium yields both players rather ***low expected payoffs***.

- In the H met Sally example, both gets 2/3 (expected payoff) when p (or q) = 2/3 as against the pure strategy payoffs of 1 or 2.

- The reason the two players fare so badly in the mixed-strategy equilibrium is that when they choose their actions independently and randomly, they create a significant probability of going to different places; when that happens, they do not meet, and each gets a payoff of 0.
- The probability of this happening when both are using their equilibrium mixtures is 2 X (2/3) X (1/3) = 4/9 which is quite high

Undesirable properties of Mixed Strategy

- Mixed strategy equilibrium is very ***fragile***

- If either player departs ever so slightly from the exact values $p = 2/3$ or $q = 2/3$, the best choice of the other tips to one pure strategy.

- Once one player chooses a pure strategy, then the other also does better by choosing the same pure strategy, and play moves to one of the two pure-strategy equilibria.

General discussion on Mixed Strategy

- **Weak Sense of Equilibrium**

- The opponent's indifference property implies that in a mixed-strategy equilibrium, each player gets the same expected payoff from each of her two pure strategies, and therefore also gets the same expected payoff from any mixture between them.

- Thus, mixed-strategy equilibria are Nash equilibria only in a weak sense.

- When one player is choosing her equilibrium mix, the other has no positive reason to deviate from her own equilibrium mix.

General discussion on Mixed Strategy

- But she would not do any worse if she chose another mix or even one of her pure strategies.

- Each player is indifferent between her pure strategies, or indeed between any mixture of them, so long as the other player is playing her correct (equilibrium) mix.

General discussion on Mixed Strategy

- **Counterintuitive Changes in Mixture Probabilities in Zero-Sum Games**

- Games with mixed-strategy equilibria may exhibit some features that seem counterintuitive at first glance.

- The most interesting of them is the ***change in the equilibrium mixes that follow a change in the structure of a game's payoffs***.

- Suppose that Navratilova works on improving her skills covering down the line to the point where Evert's success using her DL strategy against Navratilova's covering DL drops to 30% from 50%.

General discussion on Mixed Strategy

- This change in the payoff table does not lead to a game with a pure strategy equilibrium because the players still have opposing interests; Navratilova still wants their choices to coincide, and Evert still wants their choices to differ.

- We still have a game in which mixing will occur.

- But how will the equilibrium mixes in this new game differ from those calculated?

General discussion on Mixed Strategy

- At first glance, many people would argue that Navratilova should cover DL more often now that she has gotten so much better at doing so.

- Thus, the assumption is that her equilibrium q-mix should be more heavily weighted toward DL, and her equilibrium q should be higher than the 0.6 calculated before.

- But the new q is 0.5! The actual equilibrium value for q, 50%, has exactly the opposite relation to the original q of 60% than what many people's intuition predicts

- The intuition misses an important aspect of the theory of strategy: the interaction between the two players – because Navratilova is now so much better at covering DL, Evert uses CC more often in her mix

General discussion on Mixed Strategy

- **Risky and Safe Choices**

- Some strategies are relatively safe; they do not fail disastrously even if anticipated by the opponent but do not do very much better even if unanticipated.

- Other strategies are risky; they do brilliantly if the other side is not prepared for them but fail miserably if the other side is ready

- When the stakes are higher, should you play the risky strategy more or less often than when the stakes are lower?

General discussion on Mixed Strategy

- Mixed strategy equilibrium – you should mix the percentage play and the risky play in exactly the same proportions on a big occasion as you would on a minor occasion.

- Intuition – risky play should be engaged in less often when the occasion is more important.

- Problem: no clear answer!

References

1. Games of Strategy (3rd Edition) by Avinash Dixit, Susan Skeath and David H. Riley Jr.; Viva-Norton [Chapters 7 and 8].