DESIGN AND ANALYSIS OF ALGORITHMS

Lecture 5: HeapSort

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PRIORITY QUEUE

A data structure *implementing a set S* of elements, each associated with a *key*, supporting the following operations:

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- **extract_max(S)** : return element of ***S*** with largest key and remove it from ***S***
- **increase_key(S,x,k)** : increase the value of elements of elements *x*'s key to new value *k*

  (assumed to be as large as current value)

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- An *array*, visualized as a nearly complete *binary tree*
- **Max Heap Property**: The key of a node is $\geq$ than the keys of its children

  $$A[i] \geq A[2i+1] \; , \; A[i] \geq A[2i+1]$$

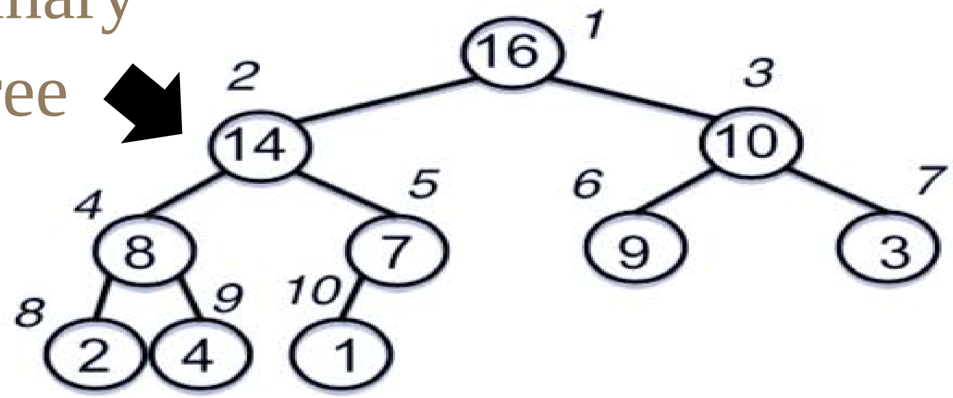
- **Min Heap Property:** the key of a node is $\leq$ than the keys of its children

  $$A[i] \leq A[2i+1] \; , \; A[i] \leq A[2i+1]$$

HEAP AS A TREE

- Root of tree: first elements in the array, corresponding to *i=1*
- parent(i)= i/2: returns index of node's parent
- left(i)=2i   : returns index of node's left child
- Right (i)=2i+1 returns index of node's right child

Binary Tree



Corrosponding array

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

height of a binary heap is **O(log n)**

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other operations : insert, extract_max, heap sort

MAX_HEAPIFY

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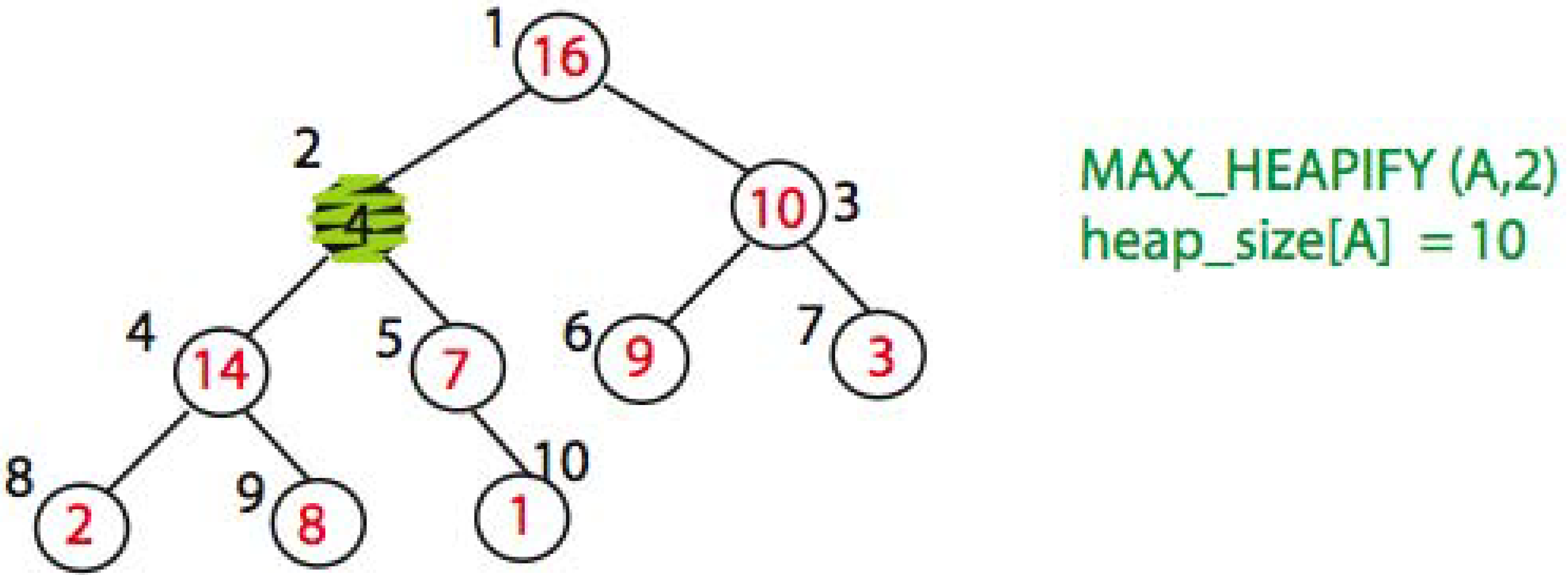
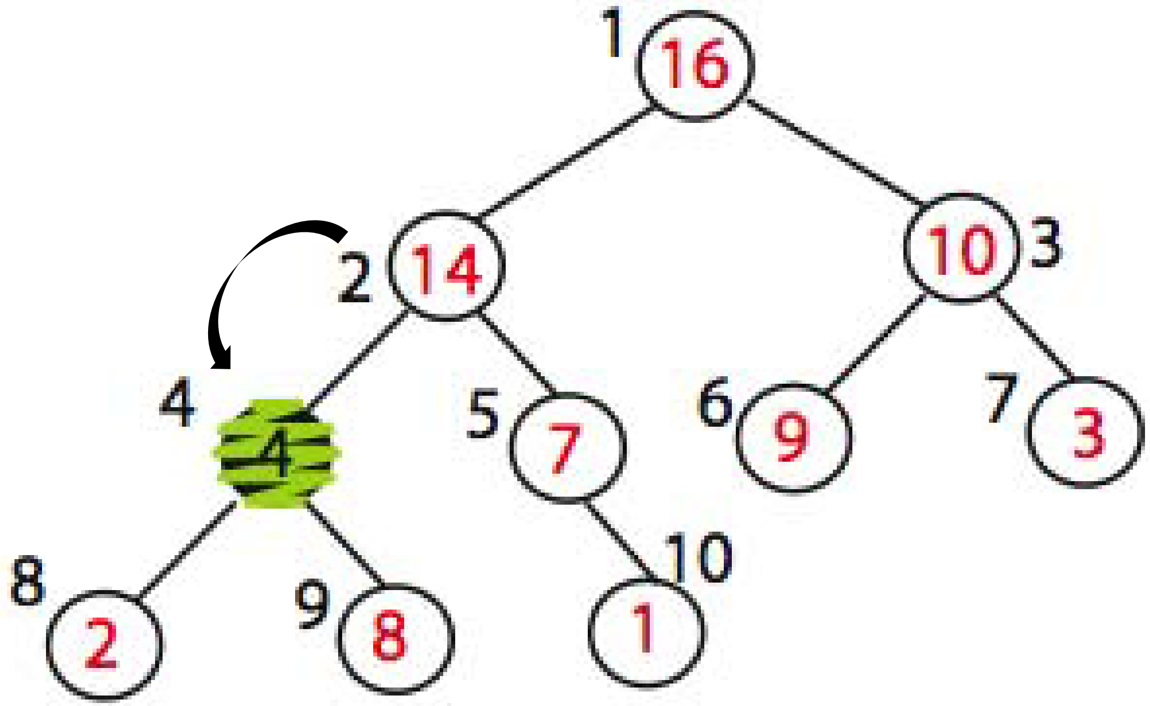
- *If element A[i] violates the max-heap property, correct violation by “trickling” element A[i] down the tree, making the sub tree rooted at index i a max-heap*

Example of Max_heapify



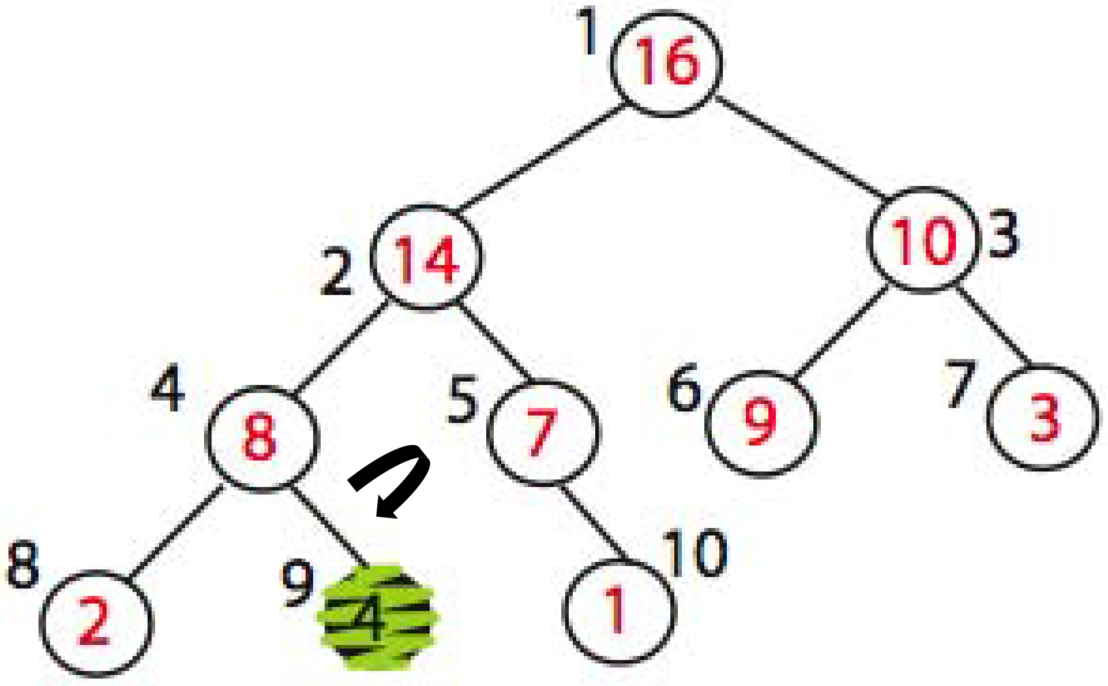
Node 10 is the left child of node 5 but is drawn to the right for convenience

Example of Max_heapify



Exchange A[2] with A[4]
Call MAX_HEAPIFY(A,4)
because max_heap property
is violated

Example of Max_heapify



Exchange A[4] with A[9]
No more calls

Time= ? O(log n)

Max_Heapify

l = left(i)

r = right(i)

if (l <= heap-size(A) and A[l] > A[i])

then largest = l else largest = i

if (r <= heap-size(A) and A[r] > A[largest])

then largest = r

if largest $\neq$ i

then exchange A[i] and A[largest]

Max_Heapify(A, largest)

"pseudo Code"

Build_Max_Heap(A)

Converts A[1…n] to a Max heap

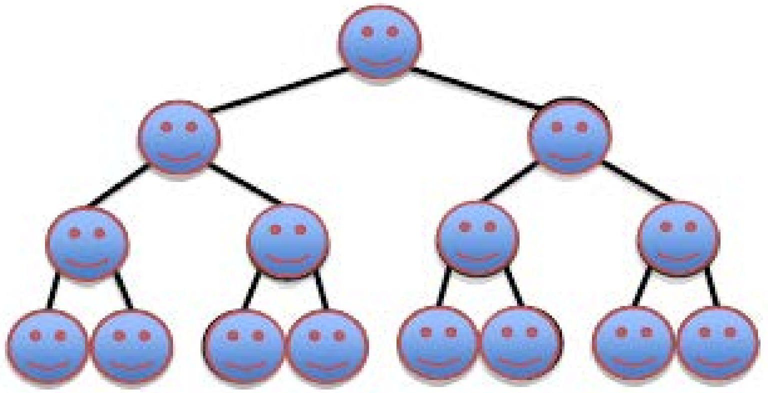
Build_Max_Heap(A):

for i=n/2 downto 1

do

Call “Max_heapify function”

Max_Heapify(A, i)

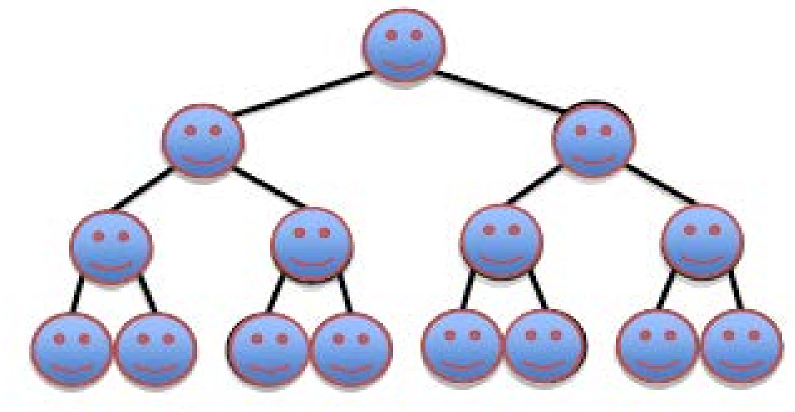


Q. Why start at n/2?

Because elements A[n/2 + 1, … ,n] are all leaves of the tree $2i > n$, for $i > n/2 + 1$

Time=? O(n log n) via simple analysis

Build_Max_Heap(A)



Converts A[1…n] to a Max heap

Build_Max_Heap(A):

for i=n/2 downto 1

do

Max_Heapify(A, i)

Call "Max_heapify function"

Observe however that Max_Heapify takes O(1) for time for nodes that are one level above the leaves, and in general, O(*l*) for the nodes that are *l* levels above the leaves. We have n/4 nodes with level 1, n/8 with level 2, and so on till we have one root node that is ***log n*** levels above the leaves.

Build_Max_Heap(A)

Converts A[1…n] to a Max_heap

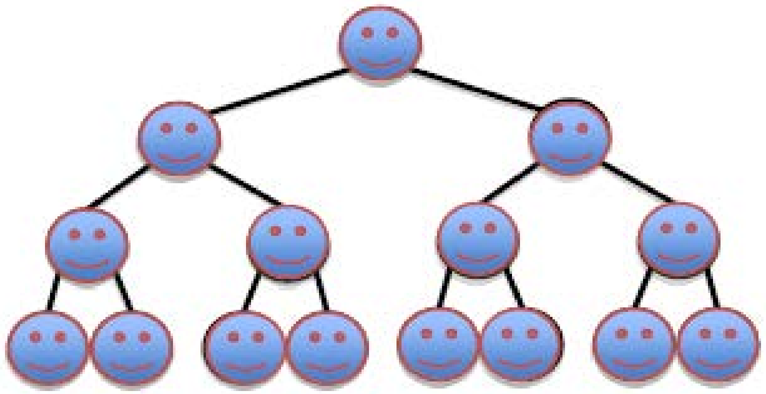
Build_Max_Heap(A):

for i=n/2 down to 1

do

Max_Heapify(A, i)

Call “Max_heapify function”



Total amount of work in the ***for loop*** can be summed as:

n/4 (1 c) + n/8 (2 c) + n/16 (3 c) + … + 1 (lg n c)

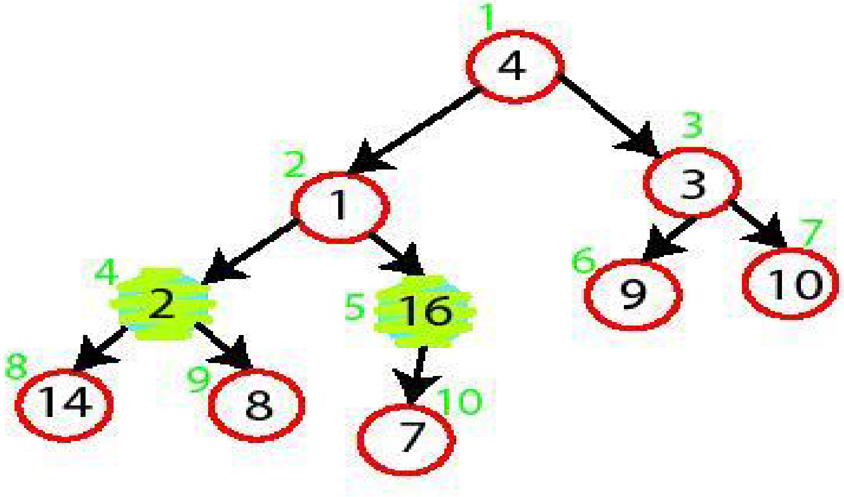
Setting $n/4 = 2^k$ and simplifying we get:

$c\, 2^k(1/2^0 + 2/2^1 + 3/2^2 + \dots (k+1)/2^k)$

The term is brackets is bounded by a constant!

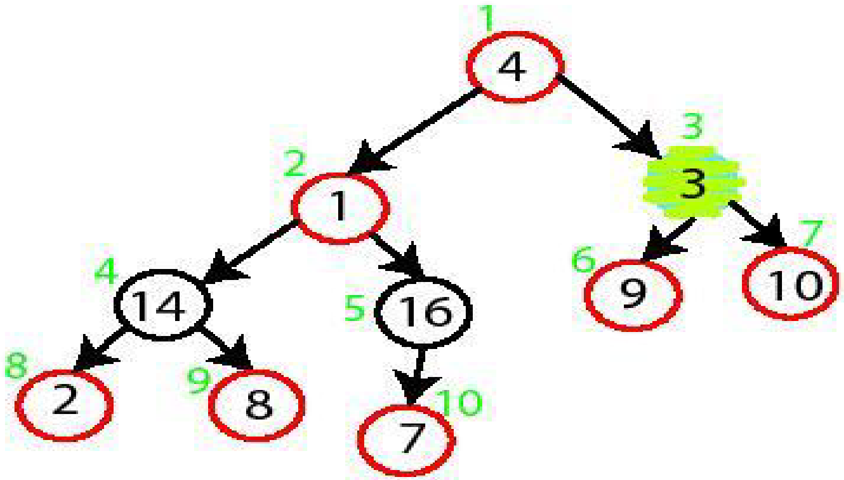
This means that Build_Max_Heap is ***O(n)***

Build_Max_Heap Demo



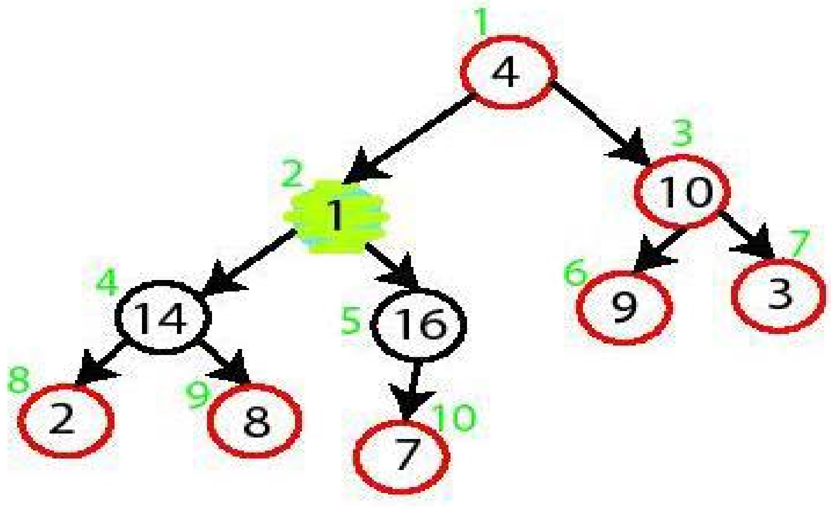
A									
4	1	3	2	16	9	10	14	8	7

MAX-HEAPIFY (A,5)
no change
MAX-HEAPIFY (A,4)
Swap A[4] and A[8]

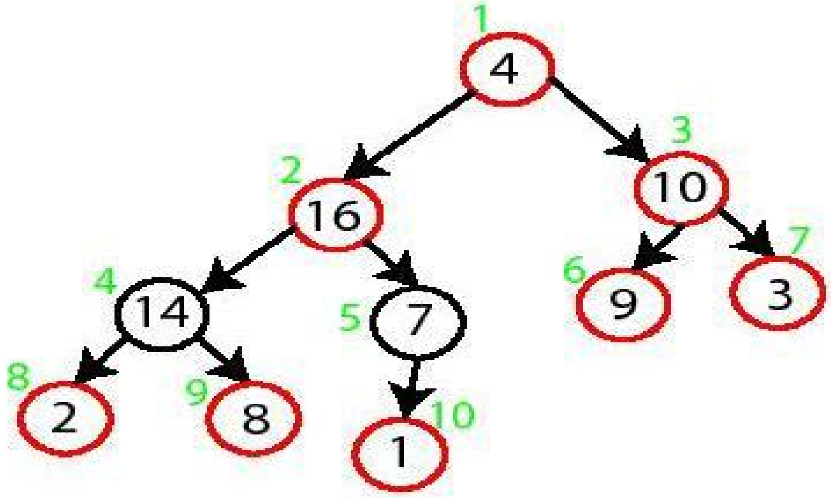


MAX-HEAPIFY (A,3)
Swap A[3] and A[7]

Build_Max_Heap Demo



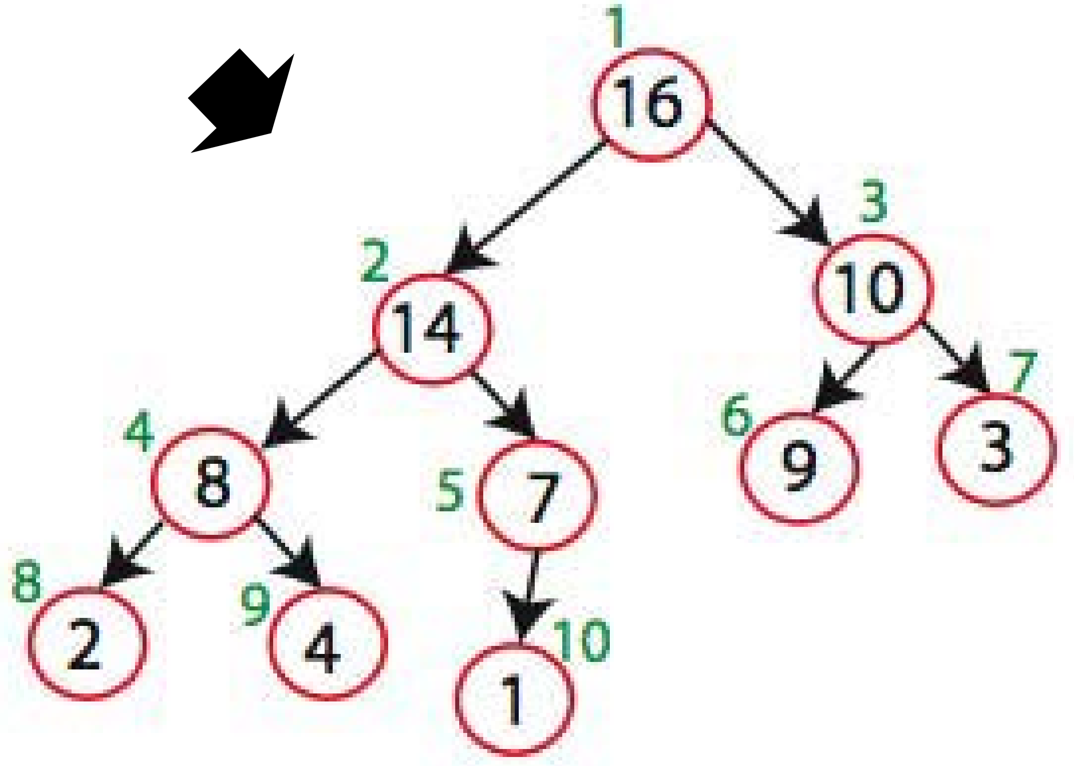
MAX-HEAPIFY (A,2)
Swap A[2] and A[5]
Swap A[5] and A[10]



MAX-HEAPIFY (A,1)
Swap A[1] with A[2]
Swap A[2] with A[4]
Swap A[4] with A[9]

Build_Max_Heap

A	4	1	3	2	16	9	10	14	8	7



Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

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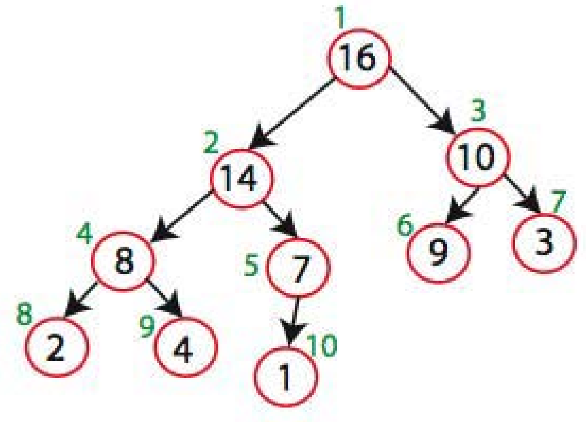
6. Go to Step 2 unless heap is empty.

Example of Heap-Sort

16	14	10	8	7	9	3	2	4	1



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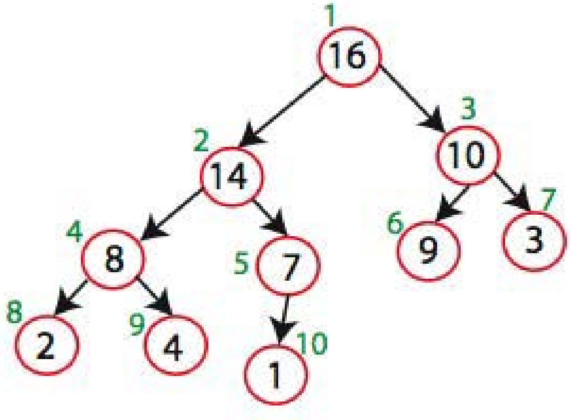
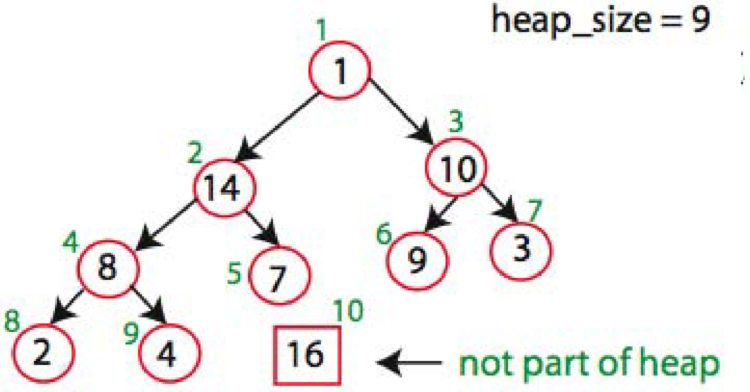


Swap A[10] and A[1]



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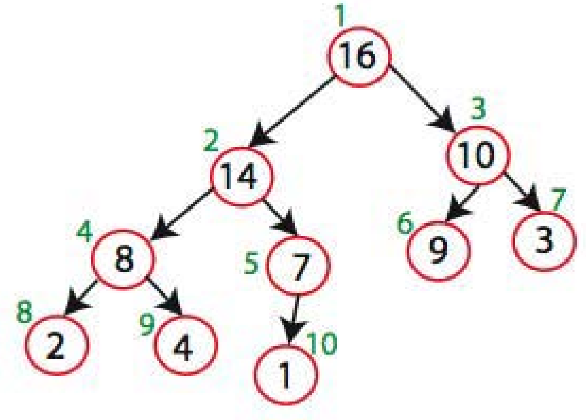

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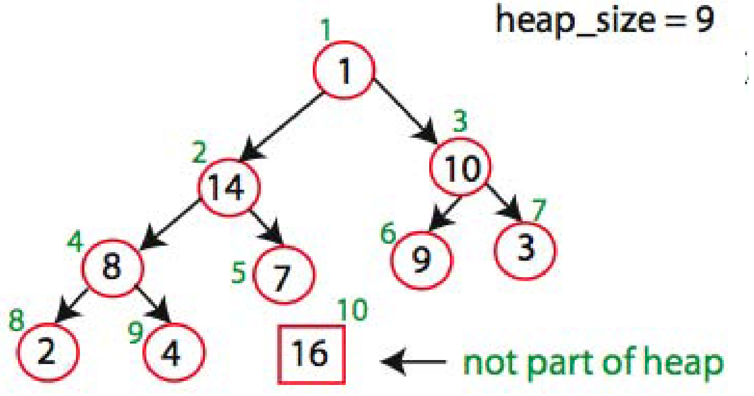
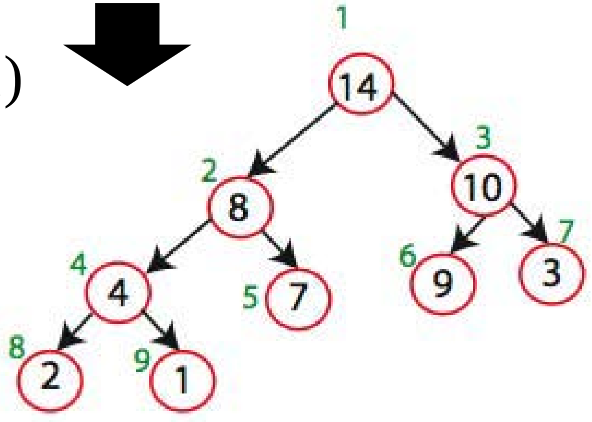


Max_heapify(A,1)



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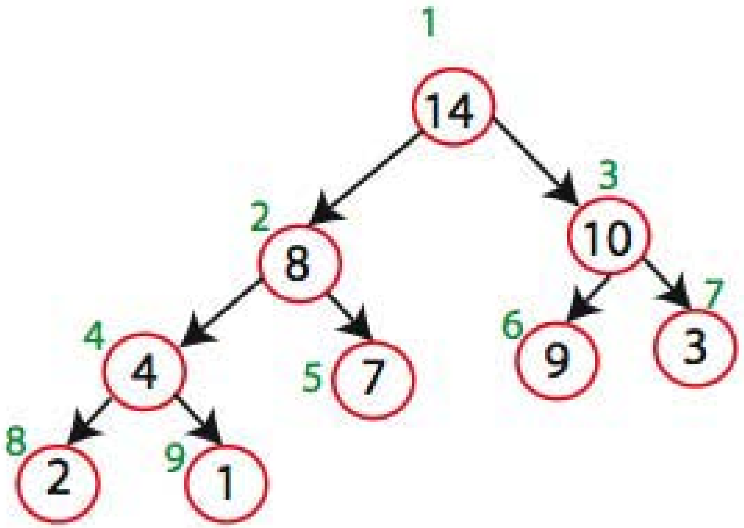

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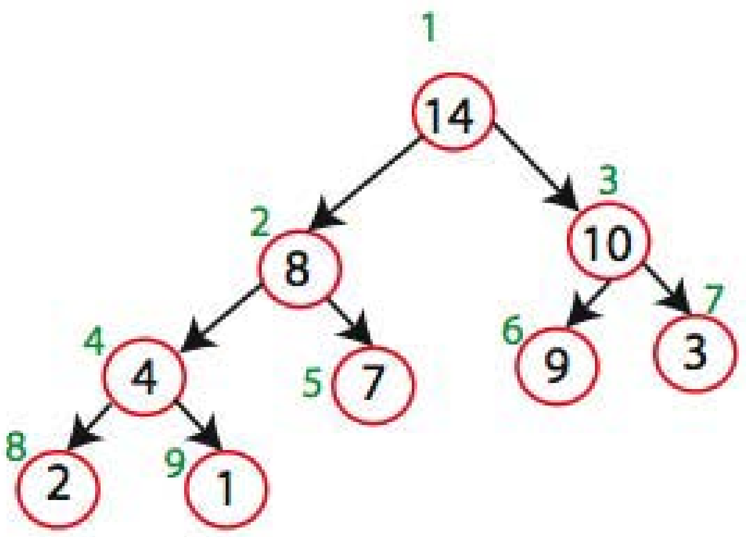
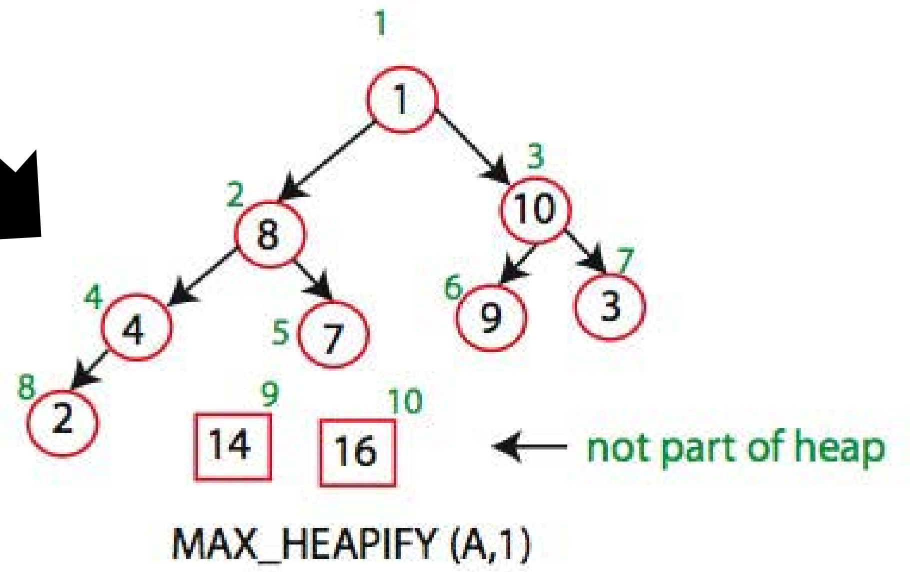


Swap A[9] and A[1]



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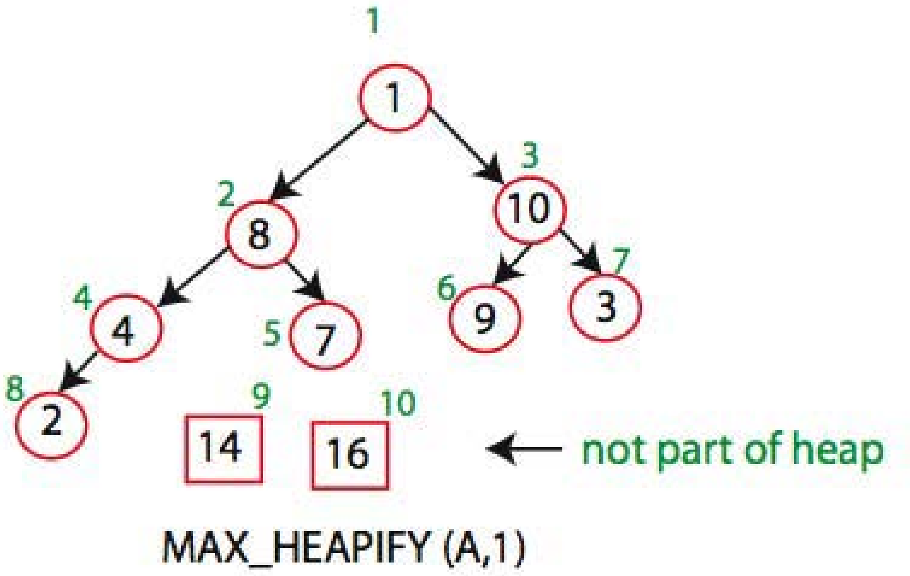

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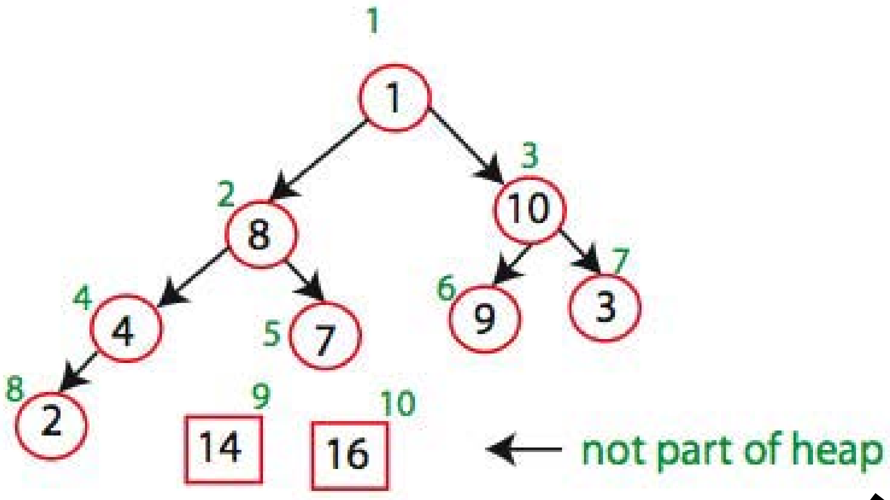
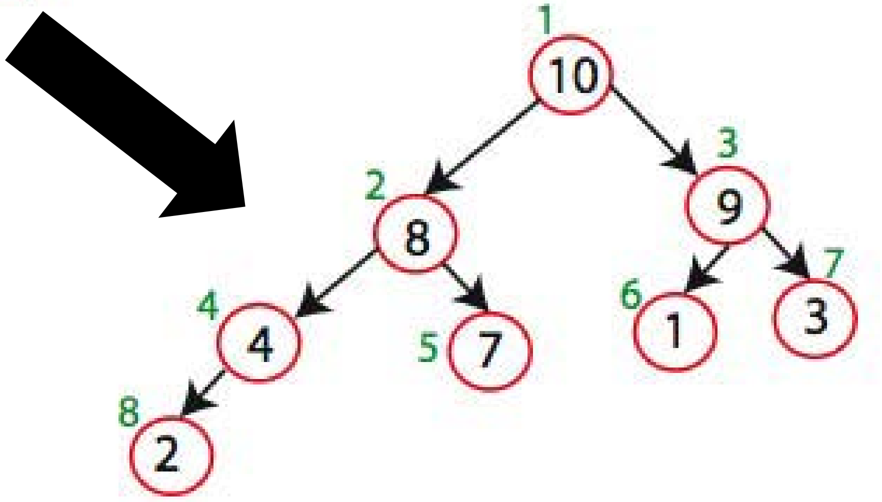


MAX_HEAPIFY (A,1)



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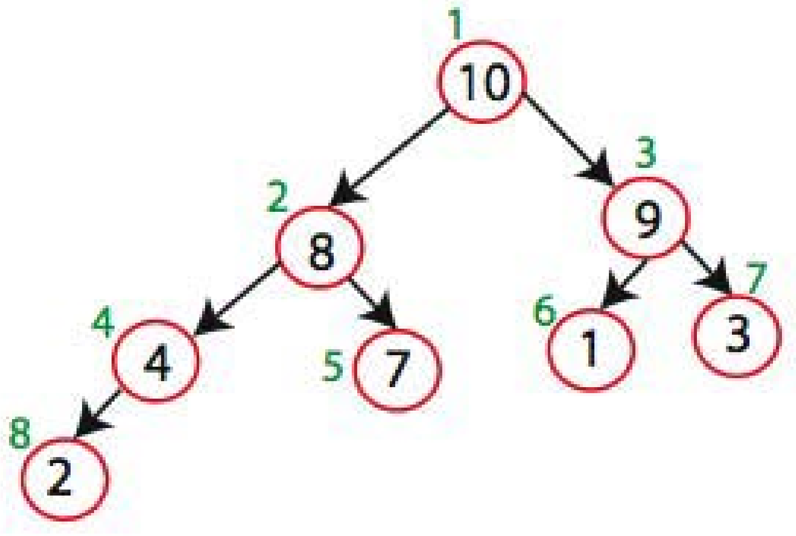

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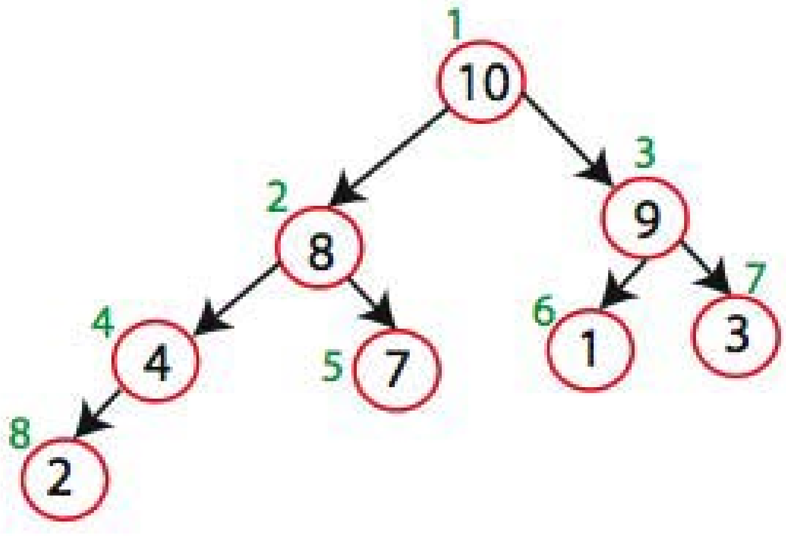
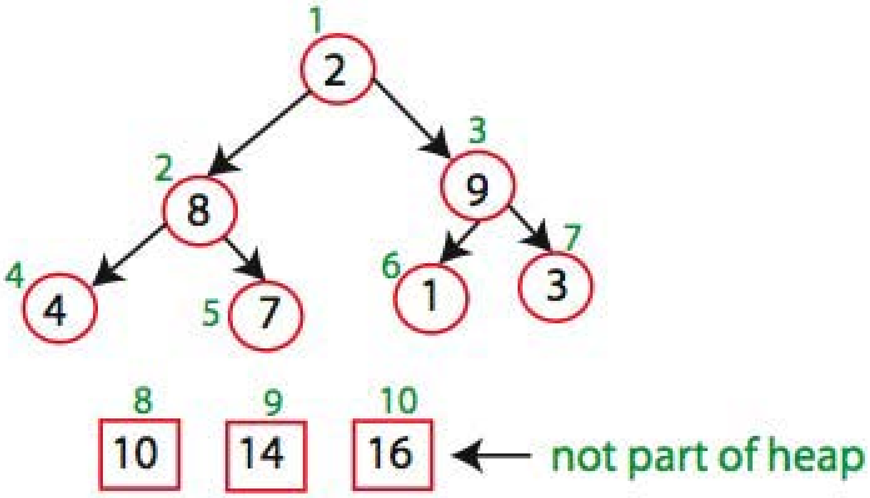


Swap A[8] and A[1]



Heap-Sort

Running time:
After n iterations the Heap is empty every iteration involves a swap and a max_heapify operation; hence it takes O(log n) time

Overall **O(n log n)**