Introduction: Simplex Method

- Graphical method: not Suitable for more than two variables
- Developed by George Dantzig in 1947
- Can be solved any LP model of the following form (called standard form)

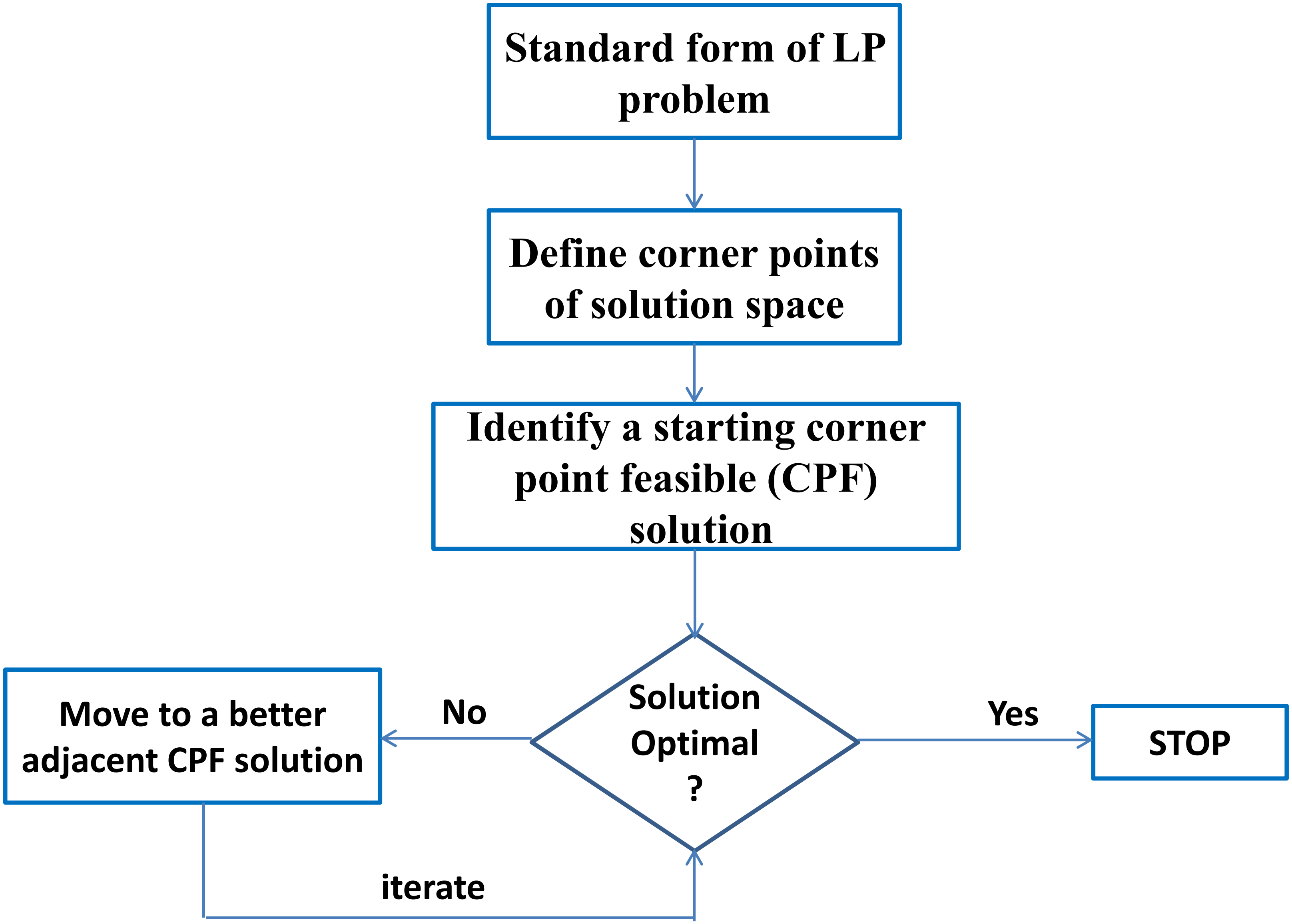
  - Maximization objective
  - all functional constraints $\leq$ type,
  - RHS not Negative
  - Nonnegativity constraints on all variables

How Simplex method works?

Geometric definition of corner points → Simplex Method → Algebraic definition of corner points

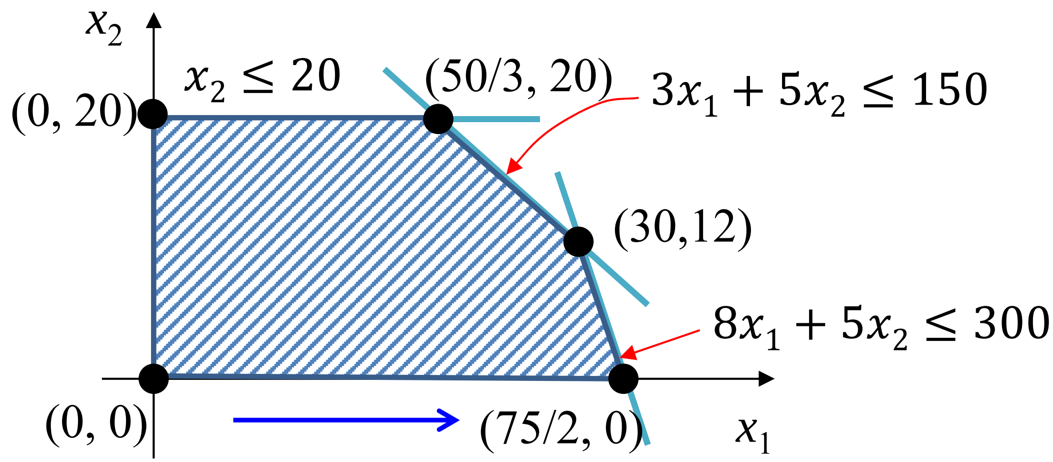
Simplex method translates the geometric procedure to algebraic procedure

Schematic Representation



Simplex Method: Geometrically

Solution space of Tech Edge Problem



Objective function
$\boldsymbol{Z = 50x_1 + 40x_2}$

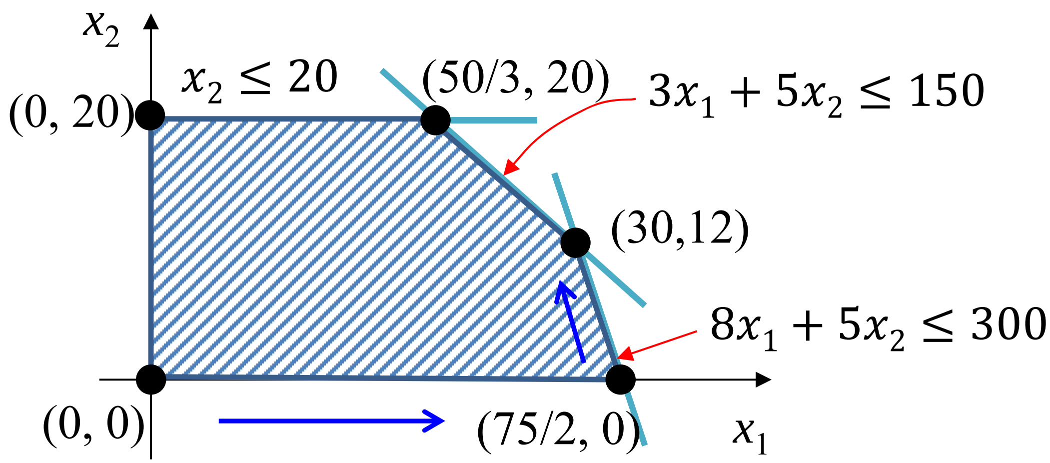
Initialization: Start with CPF Solution (0, 0) (convenient choice)

Optimality test: Is Current CPF solution optimal? No (Better adjacent CPF solution)

Iteration 1: Move to a better adjacent CPF solution. How?

- Since Z improves at a faster rate along x_1, move along x_1 as far as permitted by the feasible region
- Stop at the intersection of $x_2 = 0$ with $8x_1 + 5x_2 = 300$
- Solve for the intersection. Put $x_2 = 0$: solution (75/2, 0) and $Z = 1875$
- Optimal? Check another adjacent solution

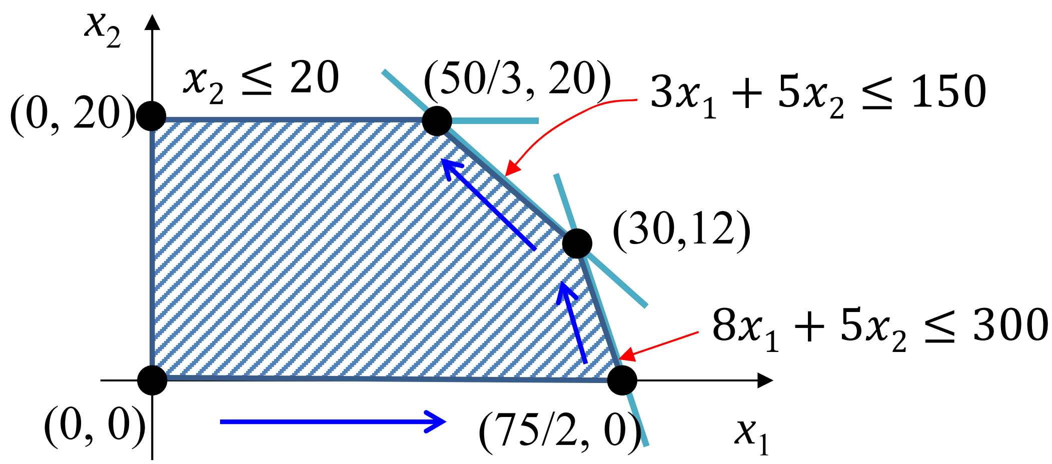
Contd..



Iteration 2: Move towards the 3rd CPF solution

- Move along $8x_1 + 5x_2 = 300$ as far as permitted by the feasible region
- Stop at the intersection of $8x_1 + 5x_2 = 300$ with $3x_1 + 5x_2 = 150$
- Solve for the intersection of $3x_1 + 5x_2 = 150$ and $8x_1 + 5x_2 = 300$: Solution (30, 12) and $Z = 1980$
- Optimal ? Check another adjacent point

Contd..



Iteration 3: Move towards the 4th CPF solution

- Move along $3x_1 + 5x_2 = 150$ as far as permitted by the feasible region
- Stop at the intersection with $x_2 = 20$
- Solve for the intersection of $x_2 = 20$ and $3x_1 + 5x_2 = 150$: Solution (50/3,20) and $Z = 1633.3$
- No better adjacent CPF solutions

Hence, optimal solution (30, 12) with $Z = 1980$

Theorem (without proof): If a CPF solution has no better adjacent CPF solutions, then that CPF solution is the optimal solution.

Simplex: Algebraically

Algebraic procedure is based on solving system of equations

Convert to augmented form

Covert functional inequality constraints to equivalent equality constraints (equations)

- Add slack (for $\leq$ type functional constraint)
- Subtract surplus (for $\geq$ type functional constraint)
- Example

(i) $3x_1 + 5x_2 \leq 150$

$$\Leftrightarrow 3x_1 + 5x_2 + x_3 = 150 \text{ and } x_3 \geq 0,$$

where x_3 is slack variable (amount by which resource availability exceeds its usage)

(ii) $3x_1 + 5x_2 \geq 150$

$$\Leftrightarrow 3x_1 + 5x_2 - x_4 = 150 \text{ and } x_4 \geq 0$$

Where x_4 is surplus variable (e.g. in diet problem surplus of a nutrient in diet plan over its minimum requirement)

(the original form has been augmented by some supplementary variables)

- **Augmented form of Tech Edge problem**

Maximize $Z = 50x_1 + 40x_2$

Subject to
$$3x_1 + 5x_2 + x_3 = 150$$
$$x_2 + x_4 = 20$$
$$8x_1 + 5x_2 + x_5 = 300$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Note: Slack variabled x_3 , x_4 and x_5 do not enter into objective function

- **Interpretation for slack variable**

(i) If a slack variable = 0 in a solution

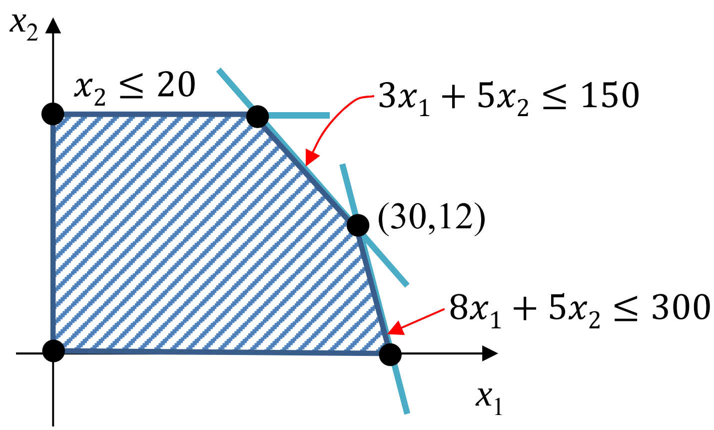
$\Rightarrow$ the solution lies on constraint boundary of the respective functional constraint and the constraint is exactly satisfied, called tight constraint

(ii) If a slack variable > 0 in a solution

$\Rightarrow$ the solution lies on the feasible side of the respective functional constraint

(iii) If a slack variable < 0 in a solution

$\Rightarrow$ the solution lies on the infeasible side of the respective functional constraint



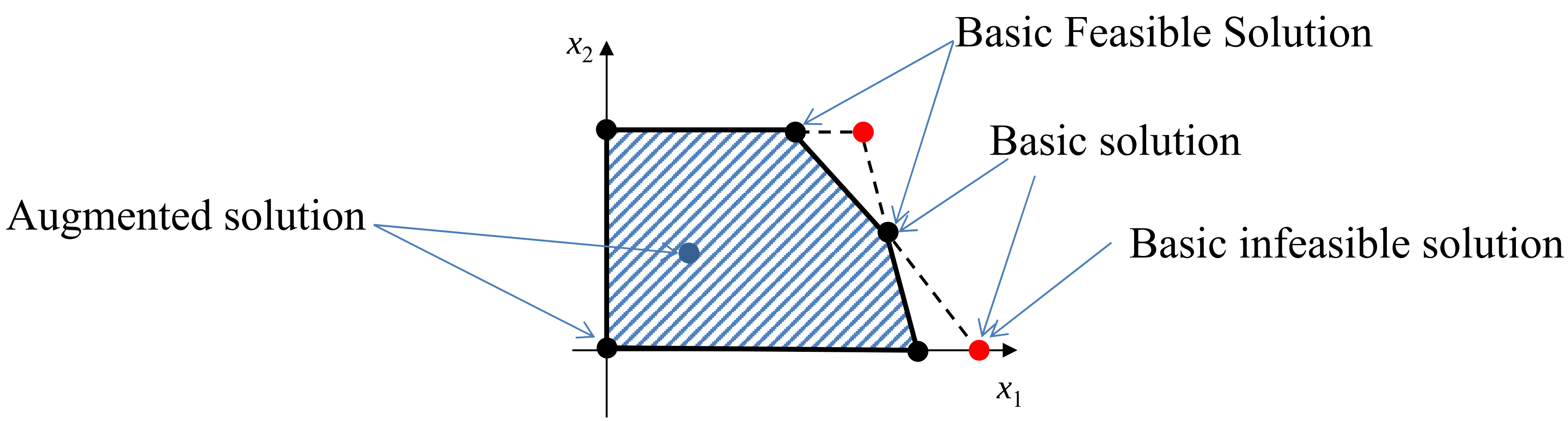
Example: At CPF solution (30,12), $x_3 = 0$, $x_4 > 0$ and $x_5 = 0$

Important terms for augmented form

- **Augmented solution:** solution for (original + slack) variables

e.g. for solution (0, 0) →Augmented solution (0, 0, 150, 20, 300)

$$(x_1, x_2) \rightarrow (x_1, x_2, x_3, x_4, x_5)$$



- **Basic solution:** Augmented corner point solution (could be feasible or infeasible)
  - Basic feasible solution (BFS): Augmented CPF solution
  - CPF solution (0, 0) is equivalent to BFS (0, 0, 150, 20, 300): difference is due to inclusion of value of slack variables

Contd…

- **Degrees of freedom of system of equations**

= number of variables - number of equations = number of non-basic variables

- **Non-basic variables (NBVs):** Set of variables "free" to be set to an arbitrary value (set to zero in Simplex method), and other variables are called **basic variables**
- **Set of basic variables (BVs)** is called **Basis**
- **Number of basic variables = number of functional constraint**

e.g. In solution (0, 0, 150, 20,300), x_1, x_2 are non-basic variables and x_3, x_4, x_5 are basic variables

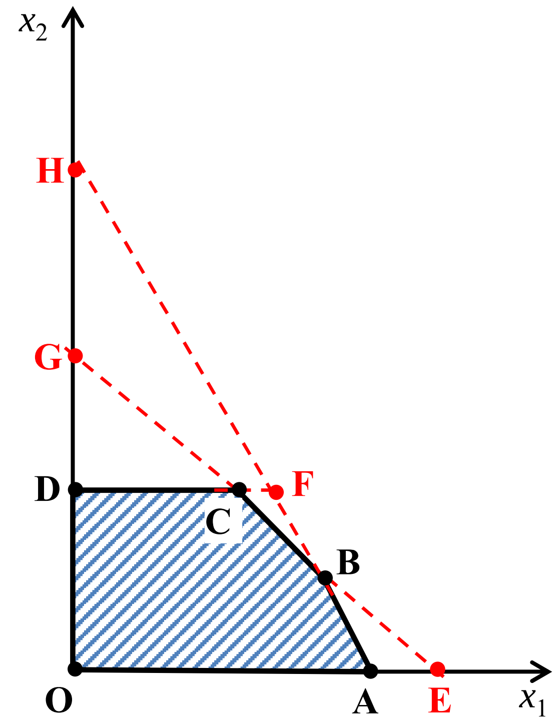
- To obtain a basic solution, set non-basic variables to zero and solve equations to obtain the value of basic variables.
- If a basic solution satisfies non-negativity constraints, the basic solution is a basic feasible solution (BFS)
- Maximum number of basic solutions = ${}^{n+m}C_n$

Example: Basic Solution

$$3x_1 + 5x_2 + x_3 = 150$$
$$x_2 + x_4 = 20$$
$$8x_1 + 5x_2 + x_5 = 300$$

➢ Tech edge problem, $n = 2$ and $m = 3 \Rightarrow {}^5C_2 = 10$

SN	NBVs (set to zero)	BVs	Solve for basic variables	Associated Corner Point	Feasible?
1	x_1, x_2	x_3, x_4, x_5	150, 20, 30	**O**	Yes
2	x_1, x_3	x_2, x_4, x_5	30, -10, 150	**G**	No
3	x_1, x_4	x_2, x_3, x_5	20, 50, 200	**D**	Yes
4	x_1, x_5	x_2, x_3, x_4	60, -150, -40	**H**	No
5	x_2, x_3	x_1, x_4, x_5	50, 20, -100	**E**	No
6	x_2, x_4	x_1, x_3, x_5	*Not Possible*		
7	x_2, x_5	x_1, x_3, x_4	75/2, 75/2, 20	**A**	Yes
8	x_3, x_4	x_1, x_2, x_5	50/3, 20, 200/3	**C**	Yes
9	x_3, x_5	x_1, x_2, x_4	30, 12, 8	**B**	Yes
10	x_4, x_5	x_1, x_2, x_3	25, 20, -25	**F**	No



➢ **Two basic feasible solutions** are **adjacent** if all but one non-basic (basic) variables are the same.

$\Longrightarrow$ Moving from one CPF solution to adjacent CPF solution makes one non-basic variable to basic and vice-versa.

Simplex: Algebraically contd…

I. Augmented form

$$\text{Max } Z = 50x_1 + 40x_2 \quad (0)$$
$$\text{S.t.} \quad 3x_1 + 5x_2 + x_3 = 150 \quad (1)$$
$$x_2 + x_4 = 20 \quad (2)$$
$$8x_1 + 5x_2 + x_5 = 300 \quad (3)$$

II. Starting solution

origin (0, 0) – convenient to see Initial BFS, because each equation has only one and different basic variables each with coefficient 1. (**Proper Form**)

So, Initial BFS (0, 0, 150, 20, 300), Z=0

III. Iterate

$Z = 50x_1 + 40x_2$

Optimal = ? No, why?

coefficient still positive => improvement possible by setting positive values of x_1 and/or x_2

Where to move? (ENTERING VARIABLE)

50 > 40 => x_1 enters the basis

When to stop? (LEAVING VARIABLE)

As much permissible by feasible region

i.e. increase x_1 while keeping the non-basic variable $x_2 = 0$, system of equations reduces to:

$$3x_1 + x_3 = 150$$
$$x_4 = 20$$
$$8x_1 + x_5 = 300$$

Non-negative constraints impose certain restriction on value of x_1

$$x_3 = 150 - 3x_1 \geq 0 \Rightarrow x_1 \leq \frac{150}{3} = 50$$

$\Rightarrow$ $x_4 = 20 > 0 \Rightarrow$ No upper bound on x_1

$$x_5 = 300 - 8x_1 \geq 0 \Rightarrow x_1 \leq \frac{300}{8} = 75/2$$

So, take minimum upper bound on x_1

$$x_5 = 0, x_1 = 75/2$$

$\Rightarrow$ x_1 can be increased to 75/2 at which x_5 drops to 0.

so, x_5 $\rightarrow$ Leaving variable (to become non-basic variable in new BFS)

x_1 $\rightarrow$ Entering variable (to become basic variable in new BFS)

- Above calculation is referred as the **minimum ratio test** to identify leaving variable.
- Minimum ratio test to determine which basic variable drops to zero first as the entering basic variable is increased

LEAVING VARIABLE RULE

- **Minimum ratio test**

For entering variable x_j, pick the variable as LEAVING corresponding to

$$min_{for\ all\ i}\left\{\frac{b_i}{a_{ij}}\right\}, \text{where } a_{ij} > 0, bi \geq 0$$

NOTE: b_i can be zero which shows "degeneracy", to be discussed later

- **How to determine x_3 and x_4 ?**

1. Treat Z as a basic variable and objective function as an equation added to the system of equations.
2. Bring the system of equations in proper form using Gaussian Elimination method

i.e., Bring current pattern of coefficients of leaving variable to entering variable by performing elementary row operations.

- **Elementary Row Operations**
  - Multiply or divide an equation by a non-zero constant
  - Add or subtract a multiple of one equation to another equation.

Initial system of equations

$$Z - 50x_1 - 40x_2 = 0 \qquad (0)$$
$$3x_1 + 5x_2 + x_3 = 150 \qquad (1)$$
$$x_2 + x_4 = 20 \qquad (2)$$
$$8x_1 + 5x_2 + x_5 = 300 \qquad (3)$$

- In Equation (3) x_1 should become basic variable by replacing x_5.
- The pattern of coefficients of x_1 in above equations should be (0,0,0,1), respectively

$$R_{3'} \longrightarrow R_3/8 \qquad \Longrightarrow \qquad x_1 + \frac{5}{8}\,x_2 + \frac{1}{8}x_5 \;=\; 75/2 \qquad (3')$$

Eliminating x_1 from Equations (0), (1) and (2)

$$R_{0'} \longrightarrow R_0 + 50R_{3'} \Longrightarrow \quad Z\; - \frac{70}{8}x_2 + \frac{50}{8}x_5 = 1875 \qquad (0')$$

$$R_{1'} \longrightarrow R_1 - 3R_{3'} \Longrightarrow \quad \frac{25}{8}x_2 + x_3\; - \frac{3}{8}x_5 = \frac{75}{2} \qquad (1')$$

$$x_2 + x_4 = 20 \qquad (2')\ \text{[same as Eq. (2)]}$$

- With non-basic variables, $x_2 = x_5 = 0$

  New BFS = (75/2, 0, 75/2, 20,0), $Z = 1875$

Contd..

- Is solution optimal?
- No, why?

$$Z = \frac{70}{8} x_2 - \frac{50}{8} x_5 + 1875$$

Improvement possible by increasing x_2

Entering variable: x_2

Leaving variable ?

Increase x_2 while keeping the current non-basic variable $x_5 = 0$, system of equations reduces to:

From (1’), $\quad x_3 = \frac{75}{2} - \frac{25}{8} x_2 \geq 0 \Rightarrow x_2 \leq 12$

From (2’), $\quad x_4 = 20 - x_2 \geq 0 \Rightarrow x_2 \leq 20$

From (3’), $\quad x_1 = \frac{75}{2} - \frac{5}{8} x_2 \geq 0 \Rightarrow x_2 \leq 60$

so, (1’) gives minimum upper bound on x_2

$$x_3 = 0, \; x_2 = 12$$

$\Rightarrow x_2$ can be increased to 12 at which x_3 drops to 0.

so, $x_3 \rightarrow$ Leaving variable (to become non-basic variable in new BFS)

$x_2 \rightarrow$ Entering variable (to become basic variable in new BFS)

Contd..

- In Equation (1') x_2 should become basic variable by replacing x_3. Current coefficients of x_3 are (0,1,0,0)
- Perform elementary algebraic operation to make the coefficient of x_2 as (0,1,0,0)

New set of equations

$$Z + \frac{14}{5}x_3 + \frac{26}{5}x_5 = 1980 \qquad (0'')$$

$$x_2 + \frac{8}{25}x_3 - \frac{3}{25}x_5 = 12 \qquad (1'')$$

$$x_4 - \frac{8}{25}x_3 + \frac{3}{25}x_5 = 8 \qquad (2'')$$

$$x_1 - \frac{1}{5}x_3 + \frac{1}{5}x_5 = 30 \qquad (3'')$$

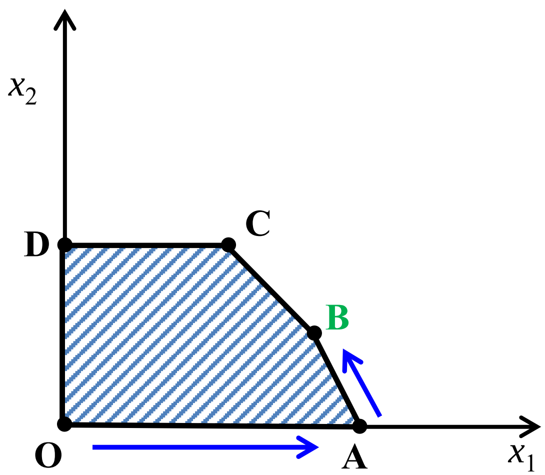
New BFS = (30, 12, 0, 8, 0), $Z = 1980$

- Optimal ? Yes
- Why ? $Z = -\frac{14}{5}x_3 - \frac{26}{5}x_5 + 1980$
- No more improvement possible

Contd..

- In summary
- How basic and non-basic sets are changing from one iteration to another.

Iteration	Non-basic	Basic	BFS	Z
Initial	$x_1 = 0, x_2 = 0$	$x_3 = 150, x_4 = 20, x_5 = 300$	(0, 0, 150, 20, 300)	0
1	$x_2 = 0, x_5 = 0$	$x_1 = 75/2, x_3 = 75/2, x_4 = 20$	(75/2, 0,75/2, 20, 0)	1875
2	$x_3 = 0, x_5 = 0$	$x_1 = 30, x_2 = 12, x_4 = 8$	(30, 12, 0, 8, 0)	1980



SIMPLEX : TABULAR FORM

- The tabular form of the simplex method compactly displays the system of equations yielding the current BF solution.

Main Decisions

- Initial BFS: origin
- Optimality test: current BFS optimal if all profit coefficients ≥ 0
- Entering variable: highest (or most) negative profit coefficient (Pivot column)
- Leaving variable: minimum ratio rule (Pivot row)

Format for Simplex Table

Basis	**List All variables in this row**	**RHS**
List all basic variables in this column	Updated coefficients of all variables in functional constraints	Column for updated value of RHS of all functional constraints
Z	Updated profit coefficients of variables in this row	Updated objective function value

- Tech Edge company example

$$
\begin{aligned}
Z - 50x_1 - 40x_2 &= 0 && (0)\\
3x_1 + 5x_2 + x_3 &= 150 && (1)\\
x_2 + x_4 &= 20 && (2)\\
8x_1 + 5x_2 + x_5 &= 300 && (3)
\end{aligned}
$$

	Basis	x_1	x_2	x_3	x_4	x_5	RHS	Ratio
	x_3	3	5	1	0	0	150	150/3
Iteration 0	x_4	0	1	0	1	0	20	-
	x_5	**8**	5	0	0	1	300	300/8=75/2
	Z	**-50**	-40	0	0	0	0	
	x_3	0	**25/8**	1	0	-3/8	75/2	(75/2)/(25/8) = 12
Iteration 1	x_4	0	1	0	1	0	20	20/1=20
	x_1	1	5/8	0	0	1/8	75/2	(75/2)/(5/8) = 60
	Z	0	**-70/8**	0	0	50/8	1875	
	x_2	0	1	8/25	0	-3/25	**12**	
Iteration 2	x_4	0	0	-8/25	1	3/25	**8**	
	x_1	1	0	-1/5	0	1/5	**30**	
	Z	0	0	14/5	0	26/5	**1980**	

$R_3 \longrightarrow R_3/8$
$R_1 \longrightarrow R_1 - 3R_3$
$R_0 \longrightarrow R_0 + 50R_3$

$R_1 \longrightarrow 8/25R_1$
$R_2 \longrightarrow R_2 - R_1$
$R_3 \longrightarrow R_3 - 5/8R_1$
$R_0 \longrightarrow R_0 + 70/8R_1$

Issues with Simplex Method

1. Tie for
   I. Leaving Variable
   II. Entering Variable
2. Unboundedness
3. Multiple optimal solution

- **Tie for entering variable**

  Example : $z = 50x_1 + 50x_2$

  Optimal solution remains unchanged but number of iteration may be different.

- **Tie for leaving variable**

Degeneracy : At least one of the basic variables has 0 value, called degenerate basic variable, and the corresponding solution is degenerate BFS

Problem : The entering and leaving variable may both have zero values and the system may iterate many times without changing the value of objective function.

- Is it an alarming problem?
- Theoretically, it can be (in pathological cases)
- Not so much in practice

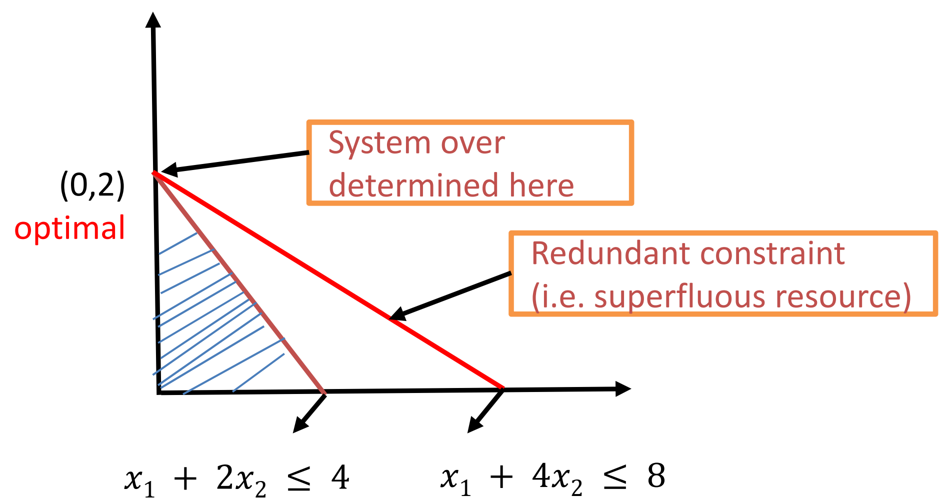
Example:

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 9x_2 \\ \text{subject to} \quad & x_1 + 4x_2 \leq 8 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Perform Simplex iterations to see instance of a basic variable = 0 , => degeneracy

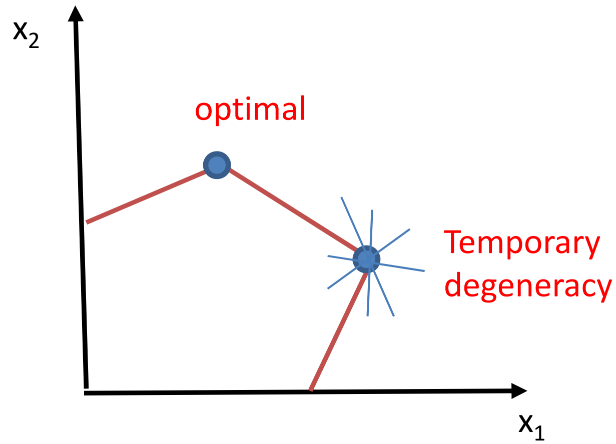
	Basis	x_1	x_2	x_3	x_4	RHS	Ratio	
Iteration 0	x_3	1	$\boxed{4}$	1	0	8	**8/4 = 2**	**Tie**
	x_4	1	2	0	1	4	4/2 = 2	
	Z	-3	**-9**	0	0	0		
Iteration 1	x_2	1/4	1	1/4	0	2	2/(1/4) = 8	
	x_4	**1/2**	0	-1/2	1	0	**0/(1/2) = 0**	
	Z	**-3/4**	0	9/4	0	18		
Iteration 2	x_2	0	1	1/2	-1/2	**2**		
	x_1	1	0	-1	2	**0**		
	Z	0	0	3/2	3/2	**18**		

- Graphically,



- Break tie arbitrarily.
- Practical stand point : The model has at least one redundant constraint.
- Removal of redundant constraint does not change the feasible region.

- Suppose there is a situation as follows



- Do not conclude that system has no optimal solution by iterating in temporary degeneracy.
- In short, break ties arbitrarily.

➢ Unboundedness

- No variable (no limit on entering variable. i.e. entering variable can be increased indefinitely)leaving
- Pivot column coefficients ≤ 0
- Example

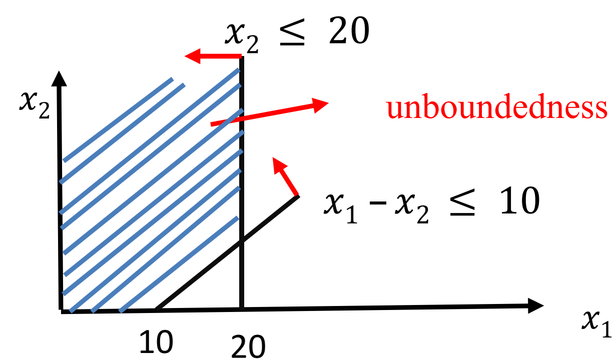
Maximize $Z = x_1 + 2x_2$

Subject to $x_1 - x_2 \leq 10$

$x_1 \leq 20$

$x_1, x_2 \geq 0$

Graphically



In simplex table, the situation will look like as follows:

	Basis	x_1	x_2	x_3	x_4	RHS	Ratio
Iteration 0	x_3	1	-1	1	0	10	--
	x_4	1	0	0	1	20	--
	Z	-1	-2	0	0	0	

A possible entering variable (x_2) with all $a_{ij} \leq 0$

Implication: some relevant constraints missing, and/or parameter estimation incorrect.

➢ Alternative optima

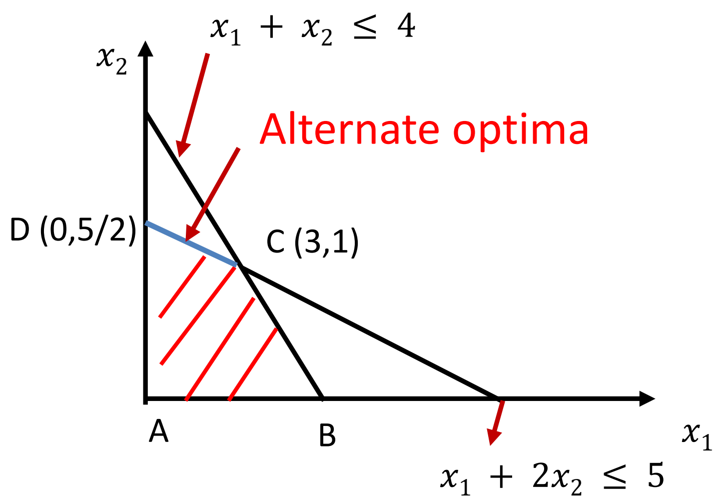
- Objective function line parallel to a constraint.
- Many solution with the same optimal objective function value.
- Example

Maximize $Z = 2x_1 + 4x_2$

subject to $x_1 + 2x_2 \leq 5$

$x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$



- **Simplex iteration-wise**

	Basis	x_1	x_2	x_3	x_4	RHS	Ratio
Iteration 0	x_3	1	**2**	1	0	5	**5/2**
	x_4	1	1	0	1	4	4/1
	Z	-2	**-4**	0	0	0	
Iteration 1	x_2	1/2	1	1/2	0	**5/2**	5
	x_4	1/2	0	-1/2	1	**3/2**	**3**
	Z	**0**	0	2	0	**10**	
Iteration 2	x_2	0	1	1	-1	**1**	
	x_1	1	0	-1	2	**3**	
	Z	**0**	**0**	2	0	**10**	

Current optimal solution = (0, 5/2)

Alternate optimal solution = (3, 1)

- Indication of alternative optimum
  - At optimally, some non-basic variables (in this case x_1) has a zero value for its profit coefficient.
  - Such a non-basic variable can be made to enter the basis without altering optimal objective function value.
  - At the next iteration (taking x_1 as entering variable), the optimal solution is (3,1), $Z = 10$ (point C)

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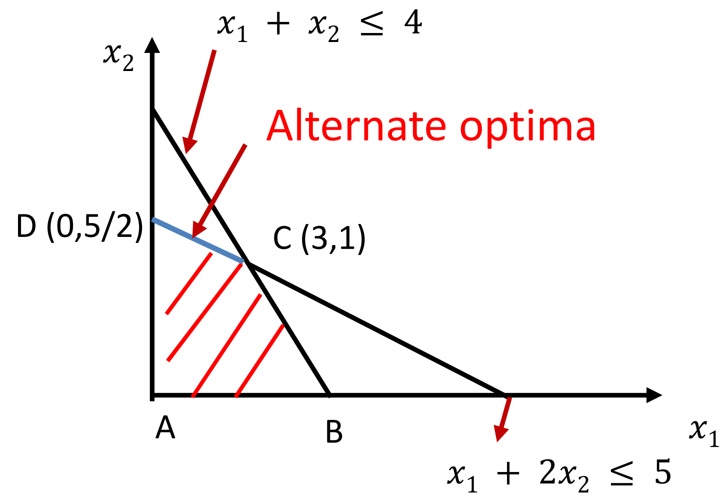

- C and D are counter-points.
- Other points on C,D -> convex combination of points C and D.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} x_1^{\;c} \\ x_2^{\;c} \end{pmatrix} + (1 - \alpha) \begin{pmatrix} x_1^{\;D} \\ x_2^{\;D} \end{pmatrix}, \quad 0 < \alpha < 1$$

- Practically, situations like these can imply dropping a product such as x_1 with no change in objective function.
- **In summary,**

Condition	Indication
Degeneracy	A basic variable takes 0 value => Tie for leaving variable
Unboundedness	$a_{ij} \le 0$ for an entering variable x_j => no leaving variable
Alternative optimal	Profit coefficient (At optimality) of at least one NBV = 0 => multiple optimal solution