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Note: We have not used invertibility of A to show this part. That means if $Au_1, \ldots, Au_k$ are LI where A is any matrix of size n, then $u_1, \ldots, u_k$ are LI.

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Problem 0.0.9: Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let A and B be two non-empty subsets of $\mathbb{V}$. Prove or disprove: $\mathsf{ls}(A) \cap \mathsf{ls}(B) \neq \{0\} \implies A \cap B \neq \phi$.

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Using Problem 0.0.8 in tutorial sheet, you can easily show that B is LI. Since B is LI and $|B| = \dim(\mathbb{V})$. Hence B is basis. A contradiction that $\mathbb{V}$ has unique basis. Hence $\dim \mathbb{V} = 1$.

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Case I. $A = \{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k}\}$ that means $x_\gamma + x_\beta$ is not in A. Therefore $A \subseteq S$. Since S is LI, then A is LI.

Case II. $A = \{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_\gamma + x_\beta, \ldots, x_{\alpha_k}\}$. Then applying the same techniques as of finite case, we have A is LI.

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Hence B is a basis of $\mathbb{V}$. A contradiction that $\mathbb{V}$ has two basis.

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Then $\{x\}$ and $\{\alpha x\}$ both are basis of $\mathbb{V}$. A contradiction that $\mathbb{V}$ has unique basis. Hence $|\mathbb{F}| = 2$.

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For each c_i we have p choices and the choice of c_i does not depend on the choice of c_j for $i \neq j$.

Problem 0.0.11: Let $\mathbb{V}$ be an n dimensional vector space over $\mathbb{F}$ and let $\mathbb{F}$ has exactly p elements. Then show that $|\mathbb{V}| = p^n$.

Answer: Let $B = \{u_1, u_2, \ldots, u_n\}$ be a basis of $\mathbb{V}$. Any element in $\mathbb{V}$ can be written as a **unique** linear combination of $u_1, u_2, \ldots, u_n$.

That is $c_1 u_1 + c_2 u_2 + c_3 u_3 + \cdots + c_n u_n$ where $c_i \in \mathbb{F}$.

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Hence $|\mathbb{V}| = p^n$.

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Let α be a transcendental number. To show that $\{1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^n\}$ is LI.

Take $a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0$.

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Therefore $\mathbb{R}(\mathbb{Q})$ is infinite dimensional.

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Consider $\{x_1, \ldots, x_k, x_{k+1} + x_1, x_{k+2} + x_2, \ldots, x_n + x_{n-k}\}$,. This is possible as $k \geq \frac{n}{2}$.

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To show this set is basis of $\mathbb{V}$.

Take $S_2 = \text{ls}(\{x_{k+1} + x_1, x_{k+2} + x_2, \ldots, x_n + x_{n-k}\})$. This is also a complement of S.

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Then $b_i = c_i = 0$ for $i = 1, \ldots, n-k$. Hence $x = 0$.

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