The Solow Model

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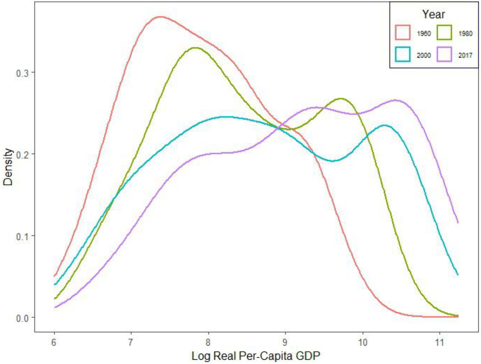
- **Considerable difference in the income level among countries across the world. Why?**

	1960	1980	2000	2017
Mean	7.932	8.465	8.733	9.187
Median	7.881	8.334	8.687	9.324
Mode	6.00	6.43	6.21	6.59
Standard Deviation	0.933	1.072	1.295	1.249
Minimum	6.003	6.428	6.207	6.589
Maximum	9.908	10.323	10.952	11.237
No of Countries	111	111	111	111

Solow Model with Population Growth

Motivation

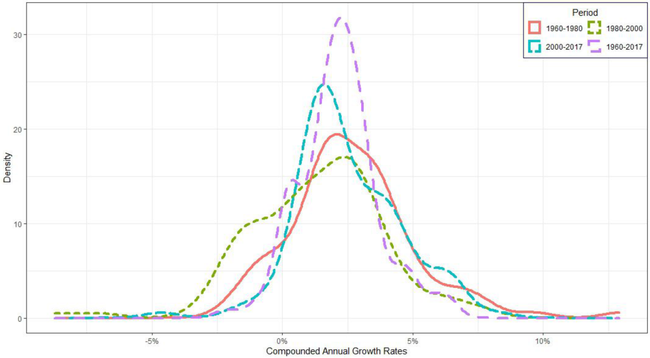


- **Considerable Difference in the growth rate among countries. Why?**

Statistic	1960-1980	1980-2000	2000-2017
Mean	0.0267	0.0134	0.0252
Median	0.0249	0.0149	0.0209
Mode	-0.0077	-0.018	0.0099
Minimum	-0.0228	-0.0873	-0.0466
Maximum	0.1291	0.0754	0.0822
Standard Deviation	0.0242	0.0258	0.0201

Solow Model with Population Growth

Motivation



Solow Model with Population Growth

Motivation

A typical English family with their possessions.

Solow Model with Population Growth

Motivation

A typical Malian family with their possessions.

- **Conditional Convergence:** Growth rate of the rich countries are lower than the poor countries. Why?

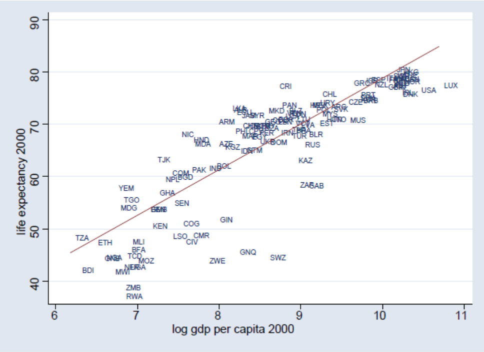
	GDP per capita, 1960	Average annual growth rate, 1960-2017	Years to double
"Rich countries"			
United States	17462.63	0.020	35
Japan	4954.423	0.037	19
France	10020.23	0.024	29
United Kingdom	11909.77	0.021	33
"Poor Countries"			
China	1014.103	0.045	15
India	1040.997	0.032	22
Nigeria	4034.925	0.001	657
Uganda	789.1344	0.014	48
"Growth Miracles"			
Hong Kong	4599.396	0.038	18
Singapore	2644.994	0.057	12
Taiwan	2376.915	0.051	14
South Korea	1113.251	0.061	11
"Growth disasters"			
Venezuela	6749.852	0.002	301
Haiti	1328.218	0.004	173
Madagascar	1438.078	0.001	390
Zimbabwe	1990.924	-0.001	-640

- **Income and Welfare:** Positive relationship between income and consumption

Solow Model with Population Growth

Motivation

- **Income and Welfare:** Positive relationship between income and life expectancy at birth



Solow Model with Population Growth

Kaldor Facts (1963): Long-run

- Percapita income grows over time and its growth rate does not falls
- Physical capital grows over time
- The rate of return to capital, share of capital and labour in national income and capital to output ratio nearly constant
- Growth rate of percapita output differs substantially across countries

- Solow (1956) develops an analytical model to explain *stylized facts* discussed above

- **Time:** Continuos
- **Market:** Perfectly Competitive
- **Technology:** Constant Returns to Scale (CRS) with two factors of Production - capital (K) and labour (N). Output is denoted by Y. The production function follows diminishing marginal productivity with respect to capital and labour. We will confine our analysis to the Cobb-Douglas production function given below.

$$Y(t) = F(K(t), N(t)) = K(t)^{\alpha} N(t)^{1-\alpha}, 0 < \alpha < 1 \tag{1}$$

- Share of capital and labour in national income is constant and denoted by α and $(1-\alpha)$ respectively.

- **Economic Agents:** Firms and Households
- **Firms:** maximize profit by choosing capital and labour. Price of output is normalized to unity. Labour and capital are paid a real wage (w) and real rental (r) $\Rightarrow$ **they are paid in terms of final goods**.
- **Households**: a non-optimizing household saves a constant fraction, $0 < s < 1$ of its total income. Household owns the firm

- **Population:** growing exponentially at an exogenously given rate $0 < n < 1$

$$\frac{\dot{N}(t)}{N(t)} = n$$
$$N(t) = N(0)\, e^{nt}$$

- Capital depreciates at the rate $0 < \delta < 1$
- Percapita output and percapita capital are denoted by y and k and defined as,

$$y = \frac{Y}{N}, k = \frac{K}{N}$$

- **Production function in percapita form:** obtained by dividing both sides of equation (1) by $N(t)$

$$y(t) = f(k(t)) = k(t)^{\alpha}$$

- **Profit Maximization:** CRS technology and perfect competition $\Rightarrow$ factors are paid in terms of their marginal products $\Rightarrow$ product exhausts $\Rightarrow$ zero profit

$$\begin{aligned}
\Pi(t) &= F(K(t), N(t)) - w(t)N(t) - r(t)K(t) \\
\pi(t) &= f(k(t)) - w(t) - r(t)k(t) \\
&= k(t)^{\alpha} - w(t) - r(t)k(t)
\end{aligned}$$

$$\begin{aligned}
k(t) &: r(t) = f'(k(t)) = \alpha k(t)^{\alpha-1} \\
\pi(t) &= 0 \Rightarrow w(t) = f(k(t)) - k(t)f'(k(t)) \\
&= (1-\alpha)k(t)^{\alpha}
\end{aligned}$$

Solow Model with Population Growth

The Fundamental Equation

- Total income of household is spend in consumption and savings. Percapita consumption and percapita savings are denoted by $c(t)$ and $\widetilde{s}(t)$

$$y(t) = c(t) + \widetilde{s}(t)$$

- **Investment:**

$$I(t) = \dot{K}(t) + \delta K(t), \dot{K}(t) = \frac{dK(t)}{dt}$$

$$i(t) = \frac{\dot{K}(t)}{N(t)} + \delta k(t)$$

$$\begin{aligned} \widetilde{s}(t) &= sy(t) = sf(k(t)) \\ &= sk(t)^{\alpha} \end{aligned}$$

- **Goods market equilibrium:** investment equals to savings

$$\frac{\dot{K}(t)}{N(t)} + \delta k(t) = sf(k(t)) \tag{2}$$

- Note, $k(t) = \frac{K(t)}{N(t)}$. Taking log both sides and differentiating with respect to time gives,

$$\begin{aligned} \frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{N}(t)}{N(t)} = \frac{\dot{K}(t)}{K(t)} - n \\ \dot{k}(t) &= \frac{\dot{K}(t)}{N(t)} - nk(t) \end{aligned} \tag{3}$$

- **Substituting equation (2) to equation (3) yields the fundamental equation of Solow model**

$$\begin{aligned} \dot{k}(t) &= sf(k(t)) - (n+\delta)k(t) \\ &= sk(t)^{\alpha} - (n+\delta)k(t) \end{aligned} \tag{4}$$

- We can solve for $k(t) = k(t)^*$ from the fundamental equation and consequently can solve

$$\begin{aligned} y(t)^* &= f\left(k(t)^*\right) = \left(k(t)^*\right)^{\alpha} \\ c(t)^* &= (1-s)\,y(t) = (1-s)f\left(k(t)^*\right) = (1-s)\left(k(t)^*\right)^{\alpha} \\ \widetilde{s}(t)^* &= sy(t) = sf\left(k(t)^*\right) = s\left(k(t)^*\right)^{\alpha} \\ w(t)^* &= f\left(k(t)^*\right) - k(t)^* f'\left(k(t)^*\right) = (1-\alpha)\left(k(t)^*\right)^{\alpha} \\ r(t)^* &= f'\left(k(t)^*\right) = \alpha\left(k(t)^*\right)^{\alpha-1} \end{aligned}$$

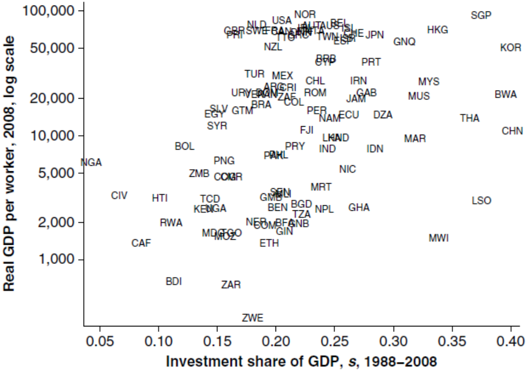
Solow Model with Population Growth
The Long-run

- The long-run is technically known as the Balanced Growth Path (BGP) in the literature where, capital and output grow at identical rate. Note the steady state where, percapita capital stock is neither growing nor decresing is also a BGP where,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{y}(t)}{y(t)} = 0 \Rightarrow \frac{\dot{K}(t)}{K(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{N}(t)}{N(t)} = n$$

- **At steady state:**

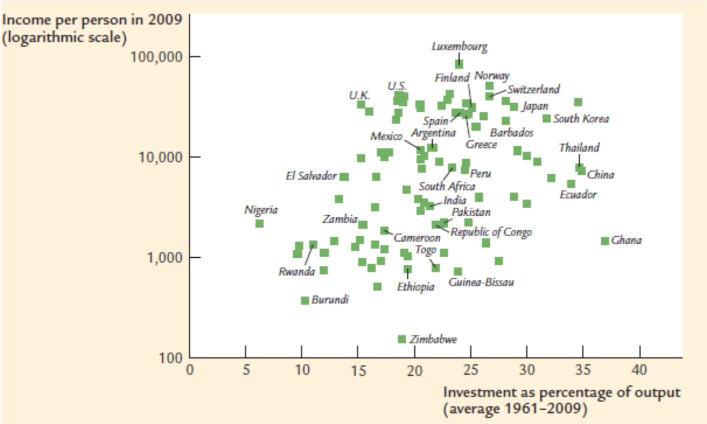
$$
\begin{aligned}
\dot{k}(t) &= sf(k(t)) - (n+\delta)k(t) = 0 \\
\Rightarrow &\quad k_{ss} = \frac{sf(k_{ss})}{n+\delta} = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}} \qquad (5) \\
\Rightarrow &\quad y_{ss} = f(k_{ss}) = k_{ss}^{\alpha} = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} \qquad (6) \\
\Rightarrow &\quad c_{ss} = (1-s)f(k_{ss}) = (1-s)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} \\
\Rightarrow &\quad \widetilde{s}_{ss} = sf(k_{ss}) = s\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} \\
\Rightarrow &\quad w_{ss} = f(k_{ss}) - k_{ss}f'(k_{ss}) = (1-\alpha)(k_{ss})^{\alpha} \\
\Rightarrow &\quad r_{ss} = f'(k_{ss}) = \alpha(k_{ss})^{\alpha-1}
\end{aligned}
$$



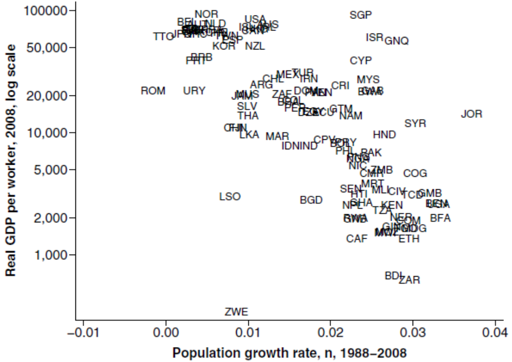
Real GDP per worker, 2008, log scale
100,000
50,000
20,000
10,000
5,000
2,000
1,000
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
Investment share of GDP, s, 1988–2008
SGP
NOR
USA
NLD
JPN
HKG
GNQ
KOR
NZL
PRT
TUR
MEX
CHL
IRN
MYS
ROM
MUS
BWA
BRA
COL
GTM
SYR
DZA
THA
CHN
FJI
MAR
BOL
PRY
IND
IDN
PNG
NGA
ZMB
NIC
MRT
CIV
HTI
TCD
LSO
BEN
BGD
NPL
GHA
TZA
RWA
GIN
MWI
CAF
ETH
BDI
ZAR
ZWE

Solow Model with Population Growth

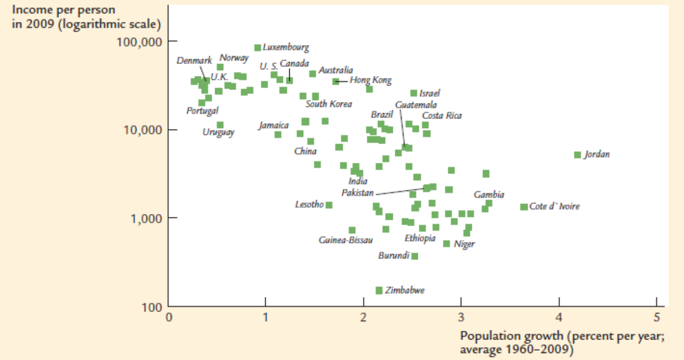
The Long-run



Solow Model with Population Growth
The Long-run

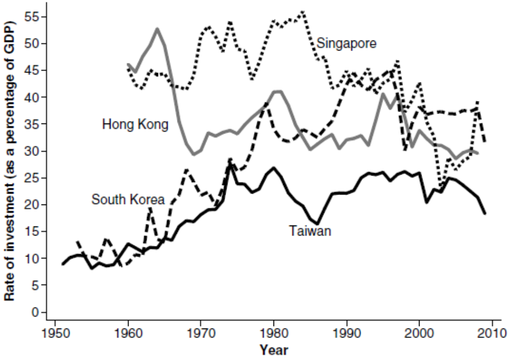


Solow Model with Population Growth
The Long-run



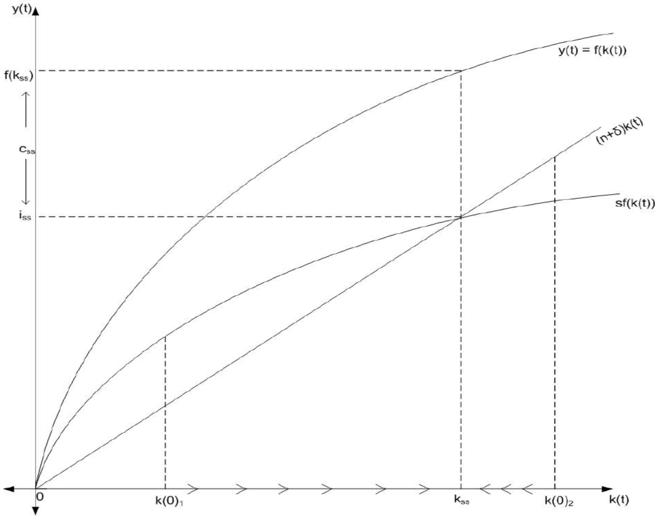
Solow Model with Population Growth

The Long-run



Solow Model with Population Growth

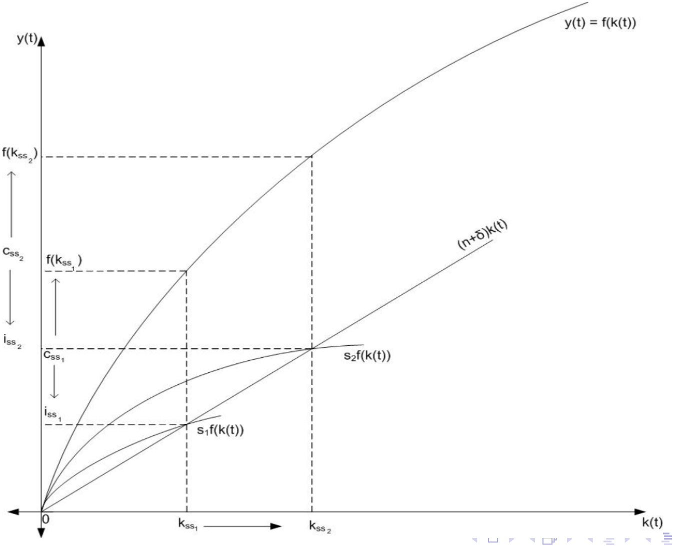
The Phase Diagram



- **Stability:** strict concavity of the production function yields unique and stable steady state.
    - $k(0)_1 < k_{ss} \Rightarrow$ savings is greater than investment $\Rightarrow$ percapita capital stock grows and movings towards steady state
    - $k(0)_2 > k_{ss} \Rightarrow$ savings is less than investment $\Rightarrow$ percapita capital stock shrinks and movings towards steady state

Solow Model with Population Growth

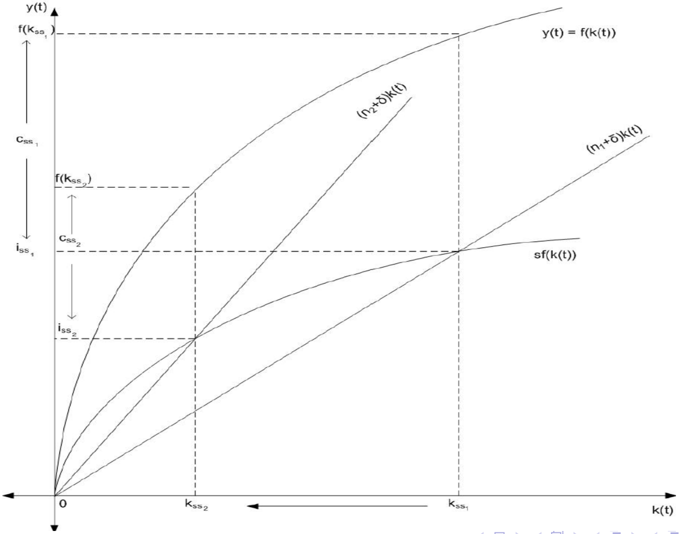
Savings rate rise



- Higher savings rate increases long-run capital stock.
- It in turn increases
  - long-run output, investment (savings) and consumption.
  - factor prices (price of capital and labour) in the long-run.

Solow Model with Population Growth

Population growth rises



- Higher population growth rate reduces long-run capital stock.
- It in turn reduces
  - long-run output, investment (savings) and consumption.
  - factor prices (price of capital and labour) in the long-run.

Solow Model with Population Growth

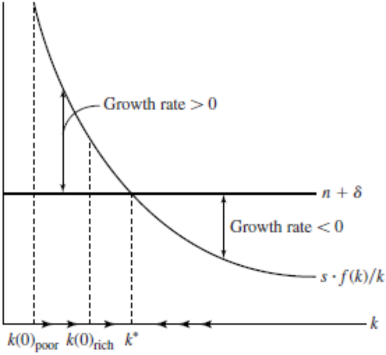
The Short-run Implication

- Equation (4) gives,

$$\begin{aligned}
\gamma_k(t) &= \frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (n+\delta) && (7)\\
&= s(k(t))^{\alpha-1} - (n+\delta) && (8)\\
&= (n+\delta)\left[\left(\frac{k_{ss}}{k(t)}\right)^{1-\alpha} - 1\right]\\
\gamma_y(t) &= \frac{\dot{y}(t)}{y(t)} = \alpha(n+\delta)\left[\left(\frac{y_{ss}}{y(t)}\right)^{\frac{1-\alpha}{\alpha}} - 1\right]
\end{aligned}$$

- Equation (7) shows
  - short-run growth rate depends on the level of capital stock and $(n+\delta)$.
  - short-run growth rate depends of the distance from the steady state

- $s\frac{f(k(t))}{k(t)}$ is measured along the vertical axis

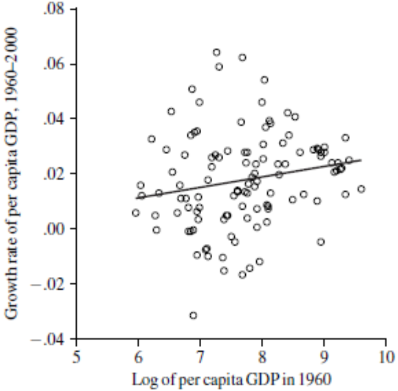


- Solow (1956) predicts that the poor countries should grow faster than the rich countries if they converge to the same steady state $\Rightarrow$ conditional convergence and not an absolute convergence.

Solow Model with Population Growth

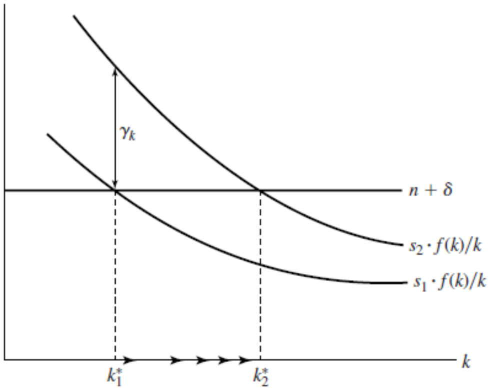
No Absolute Convergence

- **Barro and Sala-i-Martin (2004):** average annual growth rate of 114 countries across the world from 1960 to 2000.

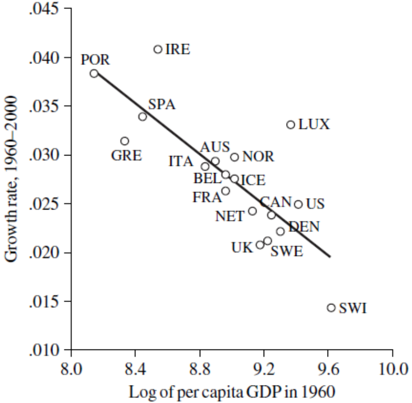


Solow Model with Population Growth

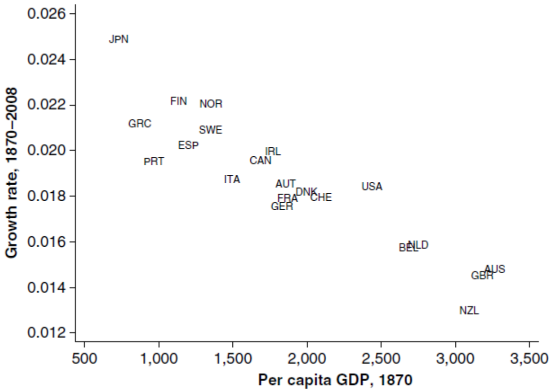
Conditional Convergence



- **Barro and Sala-i-Martin (2004):** average annual growth rate of 18 original OECD countries across (formed in 1961) from 1960 to 2000.



- **Madisson (2010):**



- **Data shows that some countries in the world are rich with higher percapita income and some are poor with lower percapita income.**
  - Equation (6) shows that countries with higher savings rate and/or lower population growth rate and/or lower rate of capital depreciation rate have higher percapita income than others. Therefore, Solow (1956) model gives answer to our first question - why some countries are rich and some are poor in the world.
- Conditional convergence with poor countries growing faster than rich countries can be explained by Solow (1956)
- **Equation (6) shows no growth of percapita output in the long-run. But data shows percapita income in the long-run is not constant. Therefore, a modification of the model is required.**

Solow Model with Population Growth and Technology Growth

Labour Augmenting Technology

- Production function with labour augmenting/Harrod neutral technological progress

$$Y(t) = K(t)^{\alpha} (A(t) N(t))^{1-\alpha} \tag{9}$$

- Technology is growing exponentially at an exogenous rate $0 < g < 1 \Rightarrow \frac{\dot{A}(t)}{A(t)} = g$
- Production function in terms of effective labour force.

$$\begin{aligned} \widehat{y}(t) &= \frac{Y(t)}{A(t) N(t)} = \frac{y(t)}{A(t)} \\ &= f\left(\widehat{k}(t)\right) = \left(\widehat{k}(t)\right)^{\alpha}, \\ \widehat{k}(t) &= \frac{K(t)}{A(t) N(t)} = \frac{k(t)}{A(t)} \end{aligned}$$

- By equating savings with investment we get

$$\frac{\dot{K}(t)}{A(t)\,N(t)} = sf\left(\widehat{k}(t)\right) - \delta\widehat{k}(t) \tag{10}$$

- Note, $\widehat{k}(t) = \frac{K(t)}{A(t)N(t)}$. Taking log both side and differentiating with respect to time gives,

$$\frac{\dot{K}(t)}{A(t)\,N(t)} = \dot{\widehat{k}}(t) + (g+n)\widehat{k}(t) \tag{11}$$

- Substituting equation (11) to (10) gives,

$$\begin{aligned} \dot{\widehat{k}}(t) &= sf\left(\widehat{k}(t)\right) - (n+\delta+g)\,\widehat{k}(t) \\ &= s\left(\widehat{k}(t)\right)^{\alpha} - (n+\delta+g)\,\widehat{k}(t) \end{aligned} \tag{12}$$

- Steady State, $\dot{\widehat{k}}(t) = 0$

$$\widehat{k}_{ss} = \frac{sf\left(\widehat{k}_{ss}\right)}{n+\delta} = \left(\frac{s}{n+\delta+g}\right)^{\frac{1}{1-\alpha}} \tag{13}$$

$$\widehat{y}_{ss} = f\left(\widehat{k}_{ss}\right) = \left(\frac{s}{n+\delta+g}\right)^{\frac{\alpha}{1-\alpha}} \tag{14}$$

- Equation (14) gives,

$$y_{ss}(t) = \left(\frac{s}{n+\delta+g}\right)^{\frac{\alpha}{1-\alpha}} A(t)$$
$$\Rightarrow \frac{\dot{y}_{ss}(t)}{y_{ss}(t)} = \frac{\dot{A}(t)}{A(t)} = g \tag{15}$$

- **Equation (15) shows that percapita income in the long-run is not constant but grows at the rate of technological progress.**

- **The Transitional Dynamics:** Multiply $\widehat{k}(t)^{-\alpha}$ to the both side on equation (12)

$$\dot{\widehat{k}}(t)\,\widehat{k}(t)^{-\alpha} = s - (n+\delta+g)\widehat{k}(t)^{(1-\alpha)} \tag{16}$$

- Define, $x(t) = \widehat{k}(t)^{1-\alpha}$

$$\begin{aligned} \frac{\dot{x}(t)}{x(t)} &= (1-\alpha)\frac{\dot{\widehat{k}}(t)}{\widehat{k}(t)} \\ \dot{x}(t) &= (1-\alpha)\,\dot{\widehat{k}}(t)\widehat{k}(t)^{-\alpha} \end{aligned} \tag{17}$$

- Substituting equation (17) to (16) yields,

$$\dot{x}(t) = (1-\alpha)\,s - (1-\alpha)\,(n+\delta+g)x(t) \tag{18}$$

Solow Model with Population and Technology Growth

The Short-run Implication

- Solution of equation (15): Integrating Factor $= e^{\lambda t}, \lambda = (1-\alpha)(n+\delta+g)$

$$
\begin{aligned}
\frac{d}{dt}\left(e^{\lambda t} x(t)\right) &= (1-\alpha) s e^{\lambda t} \\
\int d\left(e^{\lambda t} x(t)\right) &= \int (1-\alpha) s e^{\lambda t} dt \\
x(t) &= \frac{(1-\alpha) s}{\lambda} + e^{-\lambda t} m \\
x(t) &= \frac{s}{(n+\delta+g)} + \left[x(0) - \frac{s}{(n+\delta+g)}\right] e^{-\lambda t} \quad (19)
\end{aligned}
$$

- $m = x(0) - \frac{s}{(n+\delta+g)}$

Solow Model with Population Growth

The Short-run Implication

- Substituting, $x(t) = \widehat{k}(t)^{1-\alpha}$ and $\widehat{y}(t) = \widehat{k}(t)^{\alpha} = x(t)^{\frac{\alpha}{1-\alpha}}$ in equation (19) gives,

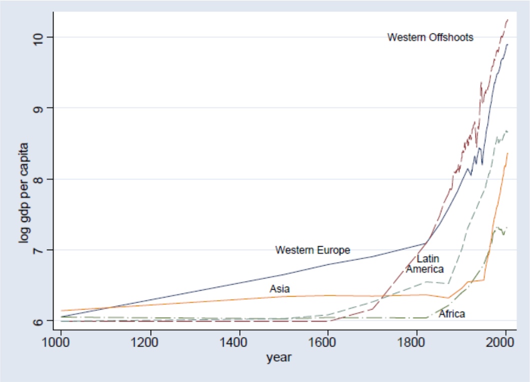
$$
\begin{aligned}
x(t) &= \frac{s}{n+\delta+g} + \left[x(0) - \frac{s}{n+\delta+g}\right] e^{-(1-\alpha)(n+\delta+g)t} \\
\widehat{k}(t) &= \left(\widehat{k}_{ss}^{1-\alpha}(1-e^{-\lambda t}) + \widehat{k}(0)^{1-\alpha} e^{-\lambda t}\right)^{\frac{1}{1-\alpha}} \\
k(t) &= \left(\widehat{k}_{ss}^{1-\alpha}(1-e^{-\lambda t}) + \widehat{k}(0)^{1-\alpha} e^{-\lambda t}\right)^{\frac{1}{1-\alpha}} A(t) \qquad (20) \\
\widehat{y}(t) &= \left(\widehat{y}_{ss}^{\frac{1-\alpha}{\alpha}}(1-e^{-\lambda t}) + \widehat{y}(0)^{\frac{1-\alpha}{\alpha}} e^{-\lambda t}\right)^{\frac{\alpha}{1-\alpha}} \\
y(t) &= \left(\widehat{y}_{ss}^{\frac{1-\alpha}{\alpha}}(1-e^{-\lambda t}) + \widehat{y}(0)^{\frac{1-\alpha}{\alpha}} e^{-\lambda t}\right)^{\frac{\alpha}{1-\alpha}} A(t) \qquad (21)
\end{aligned}
$$

- **Equation (21) shows percapita output and capital stocks grow at the rate of technological progress along the BGP**
- After solving $k(t)$ and $y(t)$ from equations (20) and (21) we can solve,

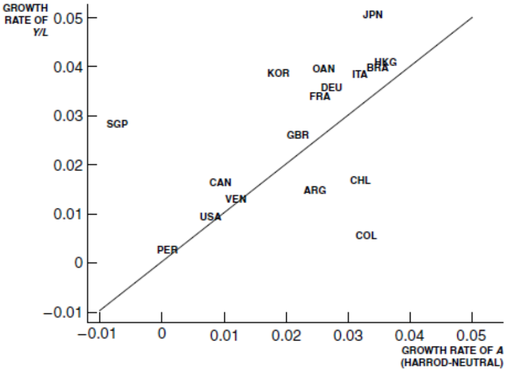
$$
\begin{aligned}
c(t) &= (1-s)\,y(t)\,, \widetilde{s}(t)^{*} = sy(t) \\
w(t) &= f(k(t)) - k(t)^{*} f^{'}(k(t)) = (1-\alpha)(k(t))^{\alpha} \\
r(t) &= f^{'}(k(t)) = \alpha (k(t))^{\alpha-1}
\end{aligned}
$$

Solow Model with Population and Technology Growth

The Industrial Revolution (1760-1840)



Solow Model with Population and Technology Growth



- Solow (1957): equation (9) with $B(t) = A(t)^{1-\alpha}$ gives,

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha \frac{\dot{K}(t)}{K(t)} + (1-\alpha) \frac{\dot{N}(t)}{N(t)} + \frac{\dot{B}(t)}{B(t)}$$

- **Solow Residual:** $\frac{\dot{B}(t)}{B(t)}$
- **Rate of growth of TFP:** $\frac{\dot{A}(t)}{A(t)} = \left(\frac{1}{1-\alpha}\right) \frac{\dot{B}(t)}{B(t)}$

TABLE 2.1 GROWTH ACCOUNTING FOR THE UNITED STATES

	1948–2010	1948–73	1973–95	1995–2000	2000–2010
Output per hour	2.6	3.3	1.5	2.9	2.7
Contributions from:					
Capital per hour worked	1.0	1.0	0.7	1.2	1.2
Information technology	0.2	0.1	0.4	0.9	0.5
Other capital services	0.8	0.9	0.3	0.3	0.7
Labor composition	0.2	0.2	0.2	0.2	0.3
Multifactor productivity	1.4	2.1	0.6	1.5	1.3

- Equation (20) and (21) imply rate of convergence of the percapita capital stock and percapita output towards their steady state at the rate $\lambda = (1-\alpha)(n+\delta+g) > 0$
- Equation (20) and (21) also imply rate of growth of percapita capital stock and percapita output in the long-run is g

- Linearizing equation (12) around the steady state by Taylor series expansion gives,

$$
\begin{aligned}
\frac{\dot{\widehat{k}}(t)}{\widehat{k}(t)} &= s\left(\widehat{k}(t)\right)^{\alpha-1} - (n+\delta+g) \\
&= se^{-(1-\alpha)\log\left(\widehat{k}(t)\right)} - (n+\delta+g) \\
&= -(1-\alpha)\, se^{-(1-\alpha)\log\left(\widehat{k}_{ss}\right)} d\log\left(\widehat{k}(t)\right) \\
&= -(1-\alpha)\, s\widehat{k}_{ss}^{(\alpha-1)} d\log\left(\widehat{k}(t)\right) \\
&= -(1-\alpha)(n+\delta+g)\, d\log\left(\widehat{k}(t)\right) \qquad (22)
\end{aligned}
$$

- Continued from the previous slide,

$$
\begin{aligned}
\frac{d}{dt}\log\left(\widehat{k}\left(t\right)\right) &= -\lambda d\log\left(\widehat{k}\left(t\right)\right) \\
\frac{d}{dt}\log\left(\widehat{y}\left(t\right)\right) &= -\lambda d\log\left(\widehat{y}\left(t\right)\right) && (23) \\
\log\left(\widehat{y}\left(t\right)\right) &= \left(1-e^{-\lambda t}\right)\log\left(\widehat{y}_{ss}\right)+\log\left(\widehat{y}(0)\right)e^{-\lambda t} && (24)
\end{aligned}
$$

- Equation (23) used to estimate the convergence rate, λ by Mankiw, Romer and Weil (MRW) (1992).

- Consider the Production function:

$$Y(t) = K^{\alpha}(t)\,(A(t)N(t))^{1-\alpha} \tag{25}$$

- Divide both sides of equation (25) by $Y(t)^{\alpha}$ and then by $N(t)$ gives,

$$\begin{aligned} y(t) &= A(t)\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} \\ &= A(t)\left(\frac{s}{n+\delta+a}\right)^{\frac{\alpha}{1-\alpha}} \\ \log(y_{ss,i}) &= a + \frac{\alpha}{1-\alpha}\log(s_i) - \frac{\alpha}{1-\alpha}\log(n_i+\delta+g) + \epsilon_i \end{aligned} \tag{26}$$

- $A(t) = Ae^{gt} \Rightarrow \log(A(t)) = \log(A) + gt \Rightarrow \log(A_{ss,i}) = \log(A_i) = a + \epsilon_i$ at the steady state
- a: drift $\Rightarrow$ level of technology common for all countries, ϵ_i: random error $\Rightarrow$ technology level that differs from all the countries
- MRW estimated α from equation (26) using data of different

Solow Model with Population and Technology Growth

Problem: MRW (1992)

- s_i: average annual investment to output ratio from 1965-1980
- $(\delta + g) = 5\%$
- n_i: average annual population growth from 1965-1980

- Data $\alpha = \frac{1}{3}$. Estimated α not matching data. Here, g is the technology growth rate.

Dependent variable: log GDP per working-age person in 1985

Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	5.48	5.36	7.97
	(1.59)	(1.55)	(2.48)
ln(I/GDP)	1.42	1.31	0.50
	(0.14)	(0.17)	(0.43)
$\ln(n+g+\delta)$	−1.97	−2.01	−0.76
	(0.56)	(0.53)	(0.84)
$\overline{R}^2$	0.59	0.59	0.01
s.e.e.	0.69	0.61	0.38
Restricted regression:			
CONSTANT	6.87	7.10	8.62
	(0.12)	(0.15)	(0.53)
$\ln(I/GDP) - \ln(n+g+\delta)$	1.48	1.43	0.56
	(0.12)	(0.14)	(0.36)
$\overline{R}^2$	0.59	0.59	0.06
s.e.e.	0.69	0.61	0.37
Test of restriction:			
p-value	0.38	0.26	0.79
Implied α	0.60	0.59	0.36
	(0.02)	(0.02)	(0.15)

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g+\delta)$ is assumed to be 0.05.

- Introduced human capital in the production function

$$Y(t) = K^{\alpha}(t)\, H(t)^{\beta} \left(A(t)N(t)\right)^{1-\alpha-\beta}, 0 < \alpha < 1, 0 < \beta < 1 \qquad (27)$$

- A fraction s_k and a fraction s_h of total income is spent on accumulating physical capital and human capital respectively. But type of capital depreciate at the δ

$$\begin{aligned} \dot{\widehat{k}}(t) &= s_k f\left(\widehat{k}(t), \widehat{h}(t)\right) - (n+\delta+g)\,\widehat{k}(t) \\ &= s_k \left(\widehat{k}(t)^{\alpha} \left(\widehat{h}(t)\right)^{\beta}\right) - (n+\delta+g)\,\widehat{k}(t) \qquad (28) \end{aligned}$$

$$\begin{aligned} \dot{\widehat{h}}(t) &= s_h f\left(\widehat{k}(t), \widehat{h}(t)\right) - (n+\delta+g)\,\widehat{h}(t) \\ &= s_h \left(\widehat{k}(t)^{\alpha} \left(\widehat{h}(t)\right)^{\beta}\right) - (n+\delta+g)\,\widehat{h}(t) \qquad (29) \end{aligned}$$

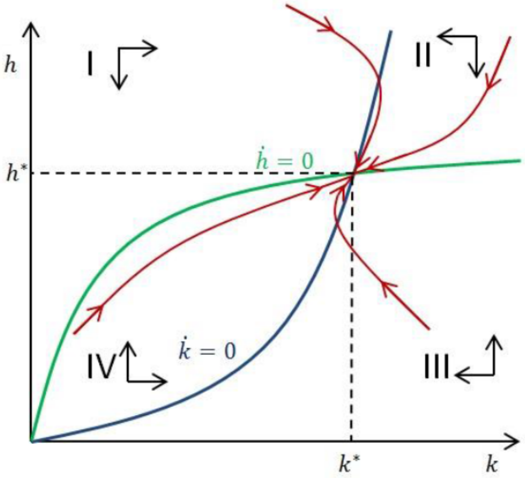
Solow Model with Population and Technology Growth

Problem: MRW (1992)

- **Equation (28):** given $\widehat{k}(t)$ a rise in $\widehat{h}(t)$ increases $\dot{\widehat{k}}(t)$ and vice-versa. Therefore, $\widehat{k}(t)$ rises in the region above the $\dot{\widehat{k}}(t) = 0$ line and falls in the region below the $\dot{\widehat{k}}(t) = 0$ line
- **Equation (29):** given $\widehat{h}(t)$ a rise in $\widehat{k}(t)$ increases $\dot{\widehat{h}}(t)$ and vice-versa. Therefore, $\widehat{h}(t)$ rises in the region to the right side of $\dot{\widehat{h}}(t) = 0$ line and falls in the region to the left side of $\dot{\widehat{h}}(t) = 0$ line
- **Phase diagram shows it is globally stable system**

Solow Model with Population and Technology Growth

Problem: MRW (1992)



- Equations (28) and (29) at steady state: $\dot{\widehat{k}}(t) = \dot{\widehat{h}}(t) = 0$ give,

$$\widehat{k}_{ss} = \left(\frac{s_k^{1-\beta} s_h^{\beta}}{n+\delta+g} \right)^{\frac{1}{1-\alpha-\beta}}, \widehat{h}_{ss} = \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{n+\delta+g} \right)^{\frac{1}{1-\alpha-\beta}}$$

$$\widehat{y}_{ss} = \left(\frac{s_k^{\alpha} s_h^{\beta}}{(n+\delta+g)^{\alpha+\beta}} \right)^{\frac{1}{1-\alpha-\beta}} \tag{30}$$

- Using identical manipulation as done to derive equation (26) from equation (25), we can derive equation (31) using equations (30) and (27)

$$\begin{aligned}\log\left(y_{ss,i}\right) &= a + \frac{\alpha}{1-\alpha-\beta}\log\left(s_{ki}\right) + \frac{\beta}{1-\alpha-\beta}\log\left(s_{hi}\right) \\ &\quad -\frac{\alpha+\beta}{1-\alpha-\beta}\log\left(n_i+\delta+g\right)+\epsilon_i\end{aligned} \quad (31)$$

- MRW estimated α and β from equation (31)

Solow Model with Population and Technology Growth

Problem: MRW (1992)

- MRW got α and β that match with the data. It shows that the augmented Solow model with human capital in the production function matches the *stylized facts.*

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	6.89	7.81	8.63
	(1.17)	(1.19)	(2.19)
ln(I/GDP)	0.69	0.70	0.28
	(0.13)	(0.15)	(0.39)
$\ln(n + g + \delta)$	−1.73	−1.50	−1.07
	(0.41)	(0.40)	(0.75)
ln(SCHOOL)	0.66	0.73	0.76
	(0.07)	(0.10)	(0.29)
$\bar{R}^2$	0.78	0.77	0.24
s.e.e.	0.51	0.45	0.33
Restricted regression:			
CONSTANT	7.86	7.97	8.71
	(0.14)	(0.15)	(0.47)
$\ln(I/GDP) - \ln(n + g + \delta)$	0.73	0.71	0.29
	(0.12)	(0.14)	(0.33)
$\ln(SCHOOL) - \ln(n + g + \delta)$	0.67	0.74	0.76
	(0.07)	(0.09)	(0.28)
$\bar{R}^2$	0.78	0.77	0.28
s.e.e.	0.51	0.45	0.32
Test of restriction:			
p-value	0.41	0.89	0.97
Implied α	0.31	0.29	0.14
	(0.04)	(0.05)	(0.15)

- Is steady state capital stock maximizes the long-run consumption? A social planner's problem is to choose that golden rule level of capital stock that maximizes consumption.

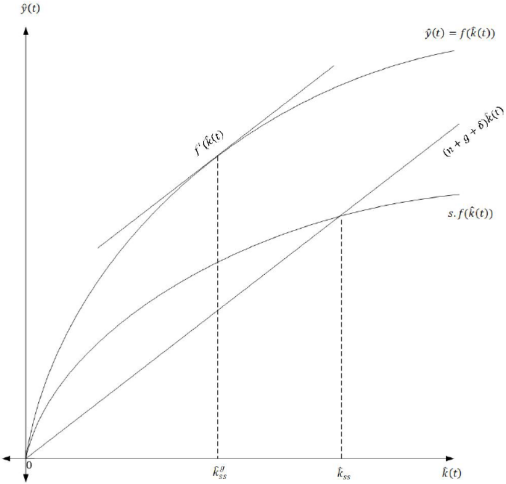
$$
\begin{aligned}
\widehat{c}_{ss} &= (1-s)\, f\left(\widehat{k}_{ss}\right) = f\left(\widehat{k}_{ss}\right) - sf\left(\widehat{k}_{ss}\right) \\
&= f\left(\widehat{k}_{ss}\right) - (n+\delta+g)\widehat{k}_{ss} \\
&= \left(\widehat{k}\,(t)\right)^{\alpha} - (n+\delta+g)\,\widehat{k}\,(t)
\end{aligned} \tag{32}
$$

- Differentiating Equation (32) with respect to $\widehat{k}_{ss}$ gives a golden rule level of capital stock $\left(\widehat{k}_{ss}^{g}\right)$ that maximizes long-run consumption

$$
\begin{aligned}
\widehat{k}_{ss}^{g} &: \quad f'\left(\widehat{k}_{ss}^{g}\right) = (n+\delta+g) \\
\widehat{k}_{ss}^{g} &= \left(\frac{\alpha}{(n+\delta+g)}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Solow Model with Population and Technology Growth

Golden Rule Level of Capital



- We have got,

$$\widehat{k}_{ss} = \left(\frac{s}{n+\delta+g}\right)^{\frac{1}{1-\alpha}}, \widehat{k}^{g}_{ss} = \left(\frac{\alpha}{n+\delta+g}\right)^{\frac{1}{1-\alpha}}$$

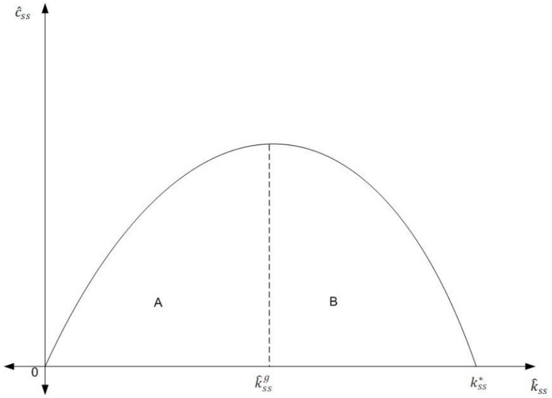
- Steady state capital stock equals to golden rule level of capital stock $\left(\widehat{k}_{ss} = \widehat{k}^{g}_{ss}\right)$ when $s = \alpha$. In this case the allocation of the Solow model is called dynamically efficient. However, the savings rate (s) in Solow model is not optimally chosen. As a result, there is no guarantee to achieve efficient allocation in the Solow model.

- **Over accumulation of capital:** $\widehat{k}_{ss} > \widehat{k}_{ss}^{g} \Rightarrow s > \alpha \Rightarrow$ steady state allocation of the Solow model is dynamically inefficient. Reduction in savings not only increases short-run consumption but it also increases long-run consumption. In fact, any allocation belonging to Region B of the next figure is dynamically inefficient.
- **Under accumulation of capital:** $\widehat{k}_{ss} < \widehat{k}_{ss}^{g} \Rightarrow s < \alpha \Rightarrow$ steady state allocation of the Solow model is neither dynamically efficient nor dynamically inefficient. Increase in savings increases long-run consumption but it reduces short-run consumption.
- **The problem of dynamic inefficiency arises because the savings rate in the Solow model is exogenously given and not optimally chosen.**

Solow Model with Population and Technology Growth

Problem: Dynamic Inefficiency

- Region A: Inconclusive ; Region B: Dynamically Inefficient Region



Thank You