Undergraduate text on

Partial Differential Equations

Lecture Notes for

2^{nd} Year B.Tech. & B.Sc. Students

(MA20103)

By

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Chapter 1
Basic Concepts

1 Definition

Partial Differential Equations (PDEs) occur in different fields of mathematical, engineering, chemical and biological sciences such as sound or wave propagation, diffusion-reaction processes, heat transport, thermodynamics, electromagnetism, elasticity, fluid dynamics, quantum mechanics, biomathematics, finances etc. PDEs act as a bridge to connect the physical (or real) world problems to mathematical framework. This mathematical framework is, sometimes, called as mathematical models. The mathematical models help us perform complicated numerical simulations and predict the behaviors of those physical processes efficinetly and correctly.

PDEs originate when the number of independent variables in a problem is more than one and the depenedent variable depends on all these variables such that we can no longer just consider the ordinary derivatives of the dependent variable but we are required to consider the partial derivatives of the dependent variable with respect to each independent variable, i.e. if $u : \mathbb{R}^n \to \mathbb{R}$ is a function of n-independent variables $x_1, x_2, ..., x_n$ then we will consider an equation involving $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial^2 x_1}, \frac{\partial^2 u}{\partial^2 x_2}, ..., \frac{\partial^2 u}{\partial^2 x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_2}$ etc which is termed as the PDE. In short, *a PDE is a differential equation which contains the unknown function and all its partial derivatives, i.e.*

$$F(t, x_1, x_2, ..., x_n, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial^2 x_1}, ... \frac{\partial^2 u}{\partial x_1 \partial x_2}, ...) = 0. \tag{1.1}$$

1.1 Degree and Order of a PDE

Let $m, n \in \mathbb{N}$. *A PDE is said to be of order m if the order of the highest partial derivative present in that equation is m. A PDE is said to be of degree n if the degree of the highest order partial derivative is n.* For example, in the equation (1.1) the order of the PDE is 2 and degree is 1.

1.2 Example of PDEs

$$\text{Linear (1st order) transport equation: } \frac{\partial u}{\partial t} + D\nabla u = 0 \tag{1.2a}$$

$$\text{Laplace equation (2nd order): } \Delta u = 0 \tag{1.2b}$$

$$\text{Poisson's equation (2nd order): } \Delta u = f \tag{1.2c}$$

$$\text{Diffusion-Reaction/heat equation (2nd order): } \frac{\partial u}{\partial t} - \nabla \cdot D\nabla u = R \tag{1.2d}$$

$$\text{Wave equation (2nd order): } \frac{\partial^2 u}{\partial^2 t} - c^2 \Delta u = 0 \tag{1.2e}$$

$$\text{Helmholtz's equation (2nd order): } \Delta u = -k^2 u \tag{1.2f}$$

$$\text{Schrödinger's equation (2nd order): } i\frac{\partial u}{\partial t} + \Delta u = f \tag{1.2g}$$

$$\text{Unsteady Stokes equation (2nd order): } \frac{\partial u}{\partial t} + \mu \Delta u - \nabla p + f = 0, \tag{1.2h}$$

$$\text{Hamilton-Jacobi equation equation (1st order): } \frac{\partial u}{\partial t} + F(x, u, \nabla u) = 0. \tag{1.2i}$$

2 First order PDEs

From here and on, for any function $z = z(x, y)$ we denote $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

2.1 Classification of first order PDEs

- **Linear PDEs:** A first order pde is said to be linear if it is linear in p, q and z, i.e.

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y), \tag{2.1}$$

  where P, Q, R and S are functions of x and y only. For example, $x^2p + y^2q = xyz + 7$.

- **Semilinear PDEs:** A pde is said to be semilinear if it is linear in p and q, and the coefficients of p and q are the functions of x and y only. In general it is of the form

$$P(x, y)p + Q(x, y)q = R(x, y, z). \tag{2.2}$$

  For example, $x^2p + y^2q = xyz^2$.

- **Quasilinear PDEs:** A pde is said to be quasilinear if it is linear in p and q, and the coefficients of p and q are the functions of x, y and z. It can be given as

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z). \tag{2.3}$$

  For example, $x^2z^2p + y^2z^3q = xyz^2 + 9x^2$.

- **Nonlinear PDEs:** A pde is said to be nonlinear if it does not fall under above three categories. For example, $p^2 + q^2 = pqz$, $pq^2 = 1$ etc.

2.2 Derivation of pdes by elimination of constants

Let us consider the following equation with two arbitrary constants α and β

$$f(x, y, z, \alpha, \beta) = 0, \tag{2.4}$$

where z is function of x and y. We partially differentiate both sides of (2.4) w.r.t. x and y, then

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p = 0, \tag{2.5a}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q = 0. \tag{2.5b}$$

We eliminate α and β from (2.4), (2.5a) and (2.5b). The resulting equation will be required first order pde of the form

$$F(x, y, z, p, q) = 0, \tag{2.6}$$

- **Case 1.** If the number of arbitrary constants is less than the number of independent variables, then the pde obtained by eliminating the constants is not unique.

  Example 2.1. *Obtain the pde from*

  $$z = \alpha x + y \tag{2.7}$$

  Answer. In equation (2.7) we have only one constant. We differentiate (2.7) w.r.t. x and y we get

  $$\frac{\partial z}{\partial x} = \alpha \tag{2.8}$$
  $$\frac{\partial z}{\partial y} = 1. \tag{2.9}$$

  Clearly, the equation (2.9) is the required pde. However, the another one can be obtained by eliminating α from (2.8) and (2.9) and it is given by $z = x\frac{\partial z}{\partial x} + y$.

- **Case 2.** If the number of arbitrary constants is equal to the number of independent variables, then the pde obtained by eliminating the constants is unique.

  Example 2.2. *Obtain the pde from*

  $$az + b = a^2 x + y, \quad a, b \neq 0 \tag{2.10}$$

  Answer. We differentiate (2.10) w.r.t. x and y we get

  $$\frac{\partial z}{\partial x} = a \tag{2.11}$$
  $$\frac{\partial z}{\partial y} = \frac{1}{a}. \tag{2.12}$$

  Up on eliminating the constant, we obtain $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 1$, i.e. $pq = 1$.

- **Case 3.** If the number of arbitrary constants is greater than the number of independent variables, then the pde obtained by eliminating the constants is of order ≥ 1.

  Example 2.3. *Obtain the pde from*

  $$z = ax + by + cxy, \quad a, b, c \neq 0 \tag{2.13}$$

  Answer. We differentiate (2.10) w.r.t. x and y we get

  $$\frac{\partial z}{\partial x} = a + cy, \quad \frac{\partial^2 z}{\partial^2 x} = 0, \tag{2.14}$$
  $$\frac{\partial z}{\partial y} = b + cx, \quad \frac{\partial^2 z}{\partial x \partial y} = c. \tag{2.15}$$

Up on eliminating the constants a and b from (2.14) and (2.15), we obtain

$$a = \frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial x \partial y} y, \quad b = \frac{\partial z}{\partial y} - \frac{\partial^2 z}{\partial x \partial y} x.$$

In (2.13) we use the values of a, b and c to obatin the required pde as

$$z = x\left(\frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial x \partial y} y\right) + y\left(\frac{\partial z}{\partial y} - \frac{\partial^2 z}{\partial x \partial y} x\right) + xy\frac{\partial^2 z}{\partial x \partial y}.$$

3 Derivation of pdes by elimination of arbitrary functions

Let $u, v : \mathbb{R}^3 \to \mathbb{R}$ be real-valued function such that such that $u = f(x, y, z)$, $v = g(x, y, z)$ and

$$\phi(u, v) = 0. \tag{3.1}$$

We differentiate both sides of (3.1) w.r.t. x then

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}} = -\frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}}. \tag{3.2}$$

Similarly, differention w.r.t. y yields

$$\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}} = -\frac{\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}}. \tag{3.3}$$

By (3.2) and (3.3), we have

$$\frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} = \frac{\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}}$$

3.1 Worked Out Examples

Example 3.1. *Obtain the pde from*

$$z = ax + by + a^2 + b^2, \quad a, b, c \neq 0 \tag{3.4}$$

Answer. We differentiate (3.4) w.r.t. x and y we get

$$p = a \quad \text{and} \quad q = b. \tag{3.5}$$

We eliminate the constants a and b from (3.5), then

$$z = px + qy + p^2 + q^2$$

Example 3.2. *Find the pde from*

$$z = (x-a)^2 + (y-b)^2. \tag{3.6}$$

Answer. We differentiate (3.6) w.r.t. x and y, then

$$p = 2(x-a) \quad \text{and} \quad q = 2(y-b). \tag{3.7}$$

We eliminate the constants a and b from (3.7), then

$$p^2 + q^2 = 4z.$$

Example 3.3. *Find the pde of a circle whose center is located in $xy-$plane and radius $r > 0$.*

Answer. Let us consider the center be located at $(h, k, 0)$, then the equation of circle is

$$(x-h)^2 + (y-k)^2 + (z-0)^2 = r^2. \tag{3.8}$$

We differentiate (3.8) w.r.t. x and y, then

$$x - h + zp = 0 \Rightarrow (x-h)^2 = z^2p^2, \tag{3.9}$$
$$y - k + zq = 0 \Rightarrow (y-k)^2 = z^2q^2. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$z^2(p^2 + q^2 + 1) = r^2$$

Example 3.4. *Find the corresponding pde to*

$$\log_e(az-1) = ax + y + b$$

Answer. The given equation is

$$\log_e(az-1) = ax + y + b$$
$$az - 1 = e^{ax+y+b} \tag{3.11}$$

By differentiating (3.11) w.r.t. x and y, then

$$ap = a\frac{\partial z}{\partial x} = ae^{ax+y+b} \Rightarrow p = az - 1, \tag{3.12}$$
$$aq = a\frac{\partial z}{\partial y} = e^{ax+y+b} \Rightarrow aq = az - 1 \Rightarrow a = \frac{1}{z-q} \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$p = \frac{z}{z-q} - 1 \Rightarrow pz - pq + q = 0.$$

Example 3.5. *Find the pde by eliminating the arbitrary function ϕ from*

$$\phi(x+y+z, x^2+y^2-z^2) = 0 \tag{3.14}$$

Answer. The given equation is

$$\phi(u,v) = 0, \tag{3.15}$$

where

$$u = x+y+z, \quad v = x^2+y^2-z^2. \tag{3.16}$$

We will use the formula (3.2) and (3.3). At first we differentiate (3.15) w.r.t. x and then w.r.t. y. Therefore,

$$\begin{aligned}
&\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \\
\Rightarrow &\frac{\partial \phi}{\partial u}(1+p) + 2\frac{\partial \phi}{\partial v}(x-pz) = 0 \\
\Rightarrow &\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}} = -\frac{2(x-pz)}{1+p} \qquad (3.17)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + p\frac{\partial v}{\partial z}\right) = 0 \\
\Rightarrow &\frac{\partial \phi}{\partial u}(1+q) + 2\frac{\partial \phi}{\partial v}(y-qz) = 0 \\
\Rightarrow &\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}} = -\frac{2(y-qz)}{1+q} \qquad (3.18)
\end{aligned}$$

Combining equations (3.17) and (3.18), we obtain

$$(1+q)(x-pz) = (1+p)(y-qz).$$

Example 3.6. *Find the pde by eliminating the arbitrary function f from*

$$z = e^{ax+by}f(ax-by).$$

Answer. The given equation is

$$z = e^{ax+by}f(ax-by). \tag{3.19}$$

By differentiating (3.19) w.r.t. x and y, respectively, we get

$$p = \frac{\partial z}{\partial x} = ae^{ax+by}f'(ax-by) + ae^{ax+by}f(ax-by) \tag{3.20}$$

$$q = \frac{\partial z}{\partial y} = -be^{ax+by}f'(ax-by) + be^{ax+by}f(ax-by) \tag{3.21}$$

We multiply (3.20) by b and (3.21) by a and by adding, we obtain

$$bp + aq = 2abe^{ax+by}f(ax-by) \Rightarrow bp + aq = 2abz \text{ using (3.19).}$$

4 Lagranges equation

Theorem 4.1. *The general solution of the quasilinear equation (Lagrange's equation)*

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z), \tag{4.1}$$

where $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ are continuously differentiable functions of x, y and z (and all not vanishing at the same time) is

$$\phi(u,v) = 0, \tag{4.2}$$

where ϕ is an arbitrary differentiable function w.r.t. u and v and

$$u(x,y,z) = c_1 \quad \text{and } v(x,y,z) = c_2 \tag{4.3}$$

are two independent solutions of the (Pfaffian) system of equations

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}. \tag{4.4}$$

Proof. We skip the proof for the moment. Interested readers can look in to T Amarnath's book, page 13.

□

Working rule. In order to solve $Pp+Qq = R$, we should form auxilary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and find its two integrals $u = a$ and $v = b$, Then $\phi(u,v) = 0$ is the solution of the given equation. Note, however, that the solution can also be written as $u = \psi(v)$ or $v = \xi(u)$.

Geometric interpretation. 1. Consider

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z) \tag{4.5}$$

and

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \tag{4.6}$$

Let

$$z = \phi(x,y) \tag{4.7}$$

represents the solution of equation (4.5). Then (4.7) represents a surface whose normal at any point (x,y,z) has direction ratios $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, 1$, i.e. $p, q, -1$.

Also we know that the simultaneous equation (4.6) represents a family of curves such that the tangent at any point has direction ratios P, Q, R. Rewriting (4.5), we have

$$P(x,y,z)p + Q(x,y,z)q + R(x,y,z)(-1) = 0 \tag{4.8}$$

showing that the normal to surface (4.7) at any point is perpendicular to the number of family of curves (4.6) through that point. Hence the member must touch the surface at that point. Since this holds for each point on (4.7), we conclude that the curves (4.6) lie completely on the surface (4.7) whose differential equation is (4.5).

2.

We know that the curve whose equations are solutions of

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \tag{4.9}$$

are orthogonal to the system of the surfaces whose equation satisfies

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \tag{4.10}$$

From the previous discussion we conclude that the curve of (4.9) lie completely on the surface represented by

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z) \tag{4.11}$$

Hence surfaces represented by (4.10) and (4.11) are orthogonal.

□

The Lagrange's equation and its solution can be divided in to different cases.

Case 1. Suppose that one of the variables is either absent or cancels out from any two fractions of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Example 4.1. *Solve the pde*

$$yzp + xzq = xy.$$

Answer. The given equation is of the form

$$Pp + Qq = R, \tag{4.12}$$

where $P(x,y,z) = yz, Q(x,y,z) = xz$ and $R(x,y,z) = xy$. The Lagrange's auxillary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy} \tag{4.13}$$

Taking first two fractions from (4.13) will yield,

$$x^2 - y^2 = c_1. \tag{4.14}$$

Taking the last two fractions from (4.13) will yield,

$$y^2 - z^2 = c_2. \tag{4.15}$$

From (4.14) and (4.15), the required general integral is

$$\phi(x^2 - y^2, y^2 - z^2) = 0, \ \phi \text{ being an arbitrary function.}$$

Note. For the above problem, we can also write the general solution as $x^2 - y^2 = \phi(y^2 - z^2)$ or $y^2 - z^2 = \phi(x^2 - y^2)$.

Example 4.2. *Solve the pde*

$$zp = -x$$

Answer. The given equation can be put as

$$zp + 0.q = -x, \tag{4.16}$$

The Lagrange's auxillary equations are

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x} \tag{4.17}$$

Taking first two or last two fractions from (4.17) will yield,

$$y = c_1. \tag{4.18}$$

Taking the first and third fractions from (4.17) will yield,

$$2x\,dx + 2z\,dz \Rightarrow x^2 + z^2 = c_2. \tag{4.19}$$

From (4.18) and (4.19), the required general integral is

$$\phi(x^2 + z^2, y) = 0, \ \phi \text{ being an arbitrary function.}$$