(*) Prove that for the model $Y_i = \alpha + \beta x_i + u_i$, $\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2}$ is an unbiased estimator of σ^2

Proof: $Y_i = \alpha + \beta x_i + u_i \Rightarrow \bar{Y} = \alpha + \beta \bar{x} + \bar{u}$

$$\Rightarrow y_i = (\cancel{\alpha} + \beta x_i + u_i - \cancel{\alpha} - \beta \bar{x} - \bar{u}) = \beta x_i + (u_i - \bar{u}) \quad \text{--- (1)}$$

Similarly, $\hat{Y}_i = \hat{\alpha} + \hat{\beta} x_i \Rightarrow \bar{\hat{Y}} = \hat{\alpha} + \hat{\beta} \bar{x}$

$$\Rightarrow \hat{y}_i = (\cancel{\hat{\alpha}} + \hat{\beta} x_i - \cancel{\hat{\alpha}} - \hat{\beta} \bar{x}) = \hat{\beta} x_i \quad \text{--- (2)}$$

Now, $\sum \hat{u}_i^2 = \sum (Y_i - \hat{Y}_i)^2 = \sum \{ (Y_i - \bar{Y}) - (\hat{Y}_i - \bar{\hat{Y}}) \}^2 = \sum (y_i - \hat{y}_i)^2$

$$\Rightarrow \sum \hat{u}_i^2 = \sum \{ \beta x_i + (u_i - \bar{u}) - \hat{\beta} x_i \}^2 = \sum \{ (u_i - \bar{u}) - (\hat{\beta} - \beta) x_i \}^2$$

$$= \underbrace{\sum (u_i - \bar{u})^2}_{A} + \underbrace{(\hat{\beta} - \beta)^2 \sum x_i^2}_{B} - \underbrace{2 (\hat{\beta} - \beta) \sum (u_i - \bar{u}) x_i}_{C}$$

(A): $\sum (u_i - \bar{u})^2 = \sum (u_i^2 + \bar{u}^2 - 2 \bar{u} u_i)$

$$\Rightarrow E\left(\sum (u_i - \bar{u})^2\right) = \sum E(u_i^2) + n E(\bar{u}^2) - 2 E(\bar{u} \sum u_i)$$

$$= n\sigma^2 + n \frac{\sum E(u_i^2)}{n^2} - 2 \frac{E(\sum u_i^2)}{n} \quad (\because E(u_i u_j) = 0)$$

$$= n\sigma^2 + \frac{\cancel{n}\sigma^2}{\cancel{n^2}} - \frac{2 \cancel{n} \sigma^2}{\cancel{n}} = n\sigma^2 + \sigma^2 - 2\sigma^2$$

$$= (n-1)\sigma^2$$

(B): $E\left((\hat{\beta} - \beta)^2 \sum x_i^2\right) = E\left((\sum k_i u_i)^2\right) \sum x_i^2$

$$= \operatorname{var}(\hat{\beta}) \cdot \sum x_i^2 = \left(\sigma^2 / \cancel{\sum x_i^2}\right) \left(\cancel{\sum x_i^2}\right) = \sigma^2$$

(C): $E\left(2 (\hat{\beta} - \beta) (\sum (u_i - \bar{u}) x_i)\right) = 2 E\left((\sum k_i u_i)(\sum (u_i - \bar{u}) x_i)\right)$

$$= 2 \left\{ E\left(\sum k_i x_i u_i^2\right) - \cancel{E\left((\sum k_i u_i)(\bar{u} \sum x_i)\right)} \right\} \quad = 0$$

$$= 2 \frac{\sum x_i^2 E(u_i^2)}{\sum x_i^2} = 2\sigma^2$$

$$\Rightarrow A + B - C = (n-1)\sigma^2 + \sigma^2 - 2\sigma^2$$
$$= (n-2)\sigma^2$$

$$\Rightarrow E(\hat{\sigma}^2) = \frac{E(\sum \hat{u}_i^2)}{n-2} = \frac{\cancel{(n-2)}\sigma^2}{\cancel{(n-2)}} = \sigma^2$$