- [**Exercise 0.0.1**] Can you construct a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that $R(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$?

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Since T is a LT from from a finite dimensional vector space to another space. Using rank nullity theorem we have $2 = nullity(T) + 3$.

Hence $nullity(T) = -1$ which is not possible, Hence there is no such type LT.

- **[Exercise 0.0.2]** Can you construct a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $R(T) = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$?

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Answer: You can easily check that the $rank(T)$ is 2. Using rank nullity theorem , nullity of such transformation if it exists should be 0. This says that it is possible to have such type of LT.

$T(x_1, x_2) = (x_1 + x_2, -x_1, -x_2)$, it is clear that this T is a LT.

$$
\begin{aligned}
R(T) &= \{T(x_1, x_2) : (x_1, x_2) \in \mathbb{R}^2\} \\
&= \{(x_1 + x_2, -x_1, -x_2) : x_1, x_2 \in \mathbb{R}\} \\
&= \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}
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- **[Exercise 0.0.3]** Let $\mathbb{V} = \mathbb{R}^n$ and A be a $n \times n$ matrix. If $Ax = 0$ has a unique solution, then show that $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

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  This implies for each $b \in \mathbb{R}^n$, there exists unique $x \in \mathbb{R}^n$ such that $Ax = b$.

- **[Exercise 0.0.4]** Let $T : \mathbb{V} \to \mathbb{V}$ be a linear map such that $R(T) = Ker(T)$. What can you say about T^2?

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Let $x \in \mathbb{V}$. Then $T^2(x) = T(T(x)) = 0$ as $Ker(T) = R(T)$. Hence T^2 is zero LT.

Converse is not true. That is, there is a LT $T : \mathbb{V} \to \mathbb{V}$ such that $T^2 \equiv 0$ but $R(T) \neq Ker(T)$.

$T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (z, 0, 0)$. Then $T^2 \equiv 0$.

$Ker(T) = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $R(T) = \{(x, 0, 0) : x \in \mathbb{R}$. It is clear that $Ker(T) \neq R(T)$.

- **[Exercise 0.0.5]**Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the field $\mathbb{Q}$. f is a map from $\mathbb{V}$ to $\mathbb{W}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{V}$. Sow that f is a linear transformation.

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Take α is a positive integer, that is $\alpha = m$. Then

$f(mx) = f(x + \cdots + x)$

$= f(x) + \cdots + f(x)$

$= mf(x)$

We now show that $f(-x) = -f(x)$.

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Hence f is a linear transformation.

- **[Exercise 0.0.6]** Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the field $\mathbb{R}$. f is a map from $\mathbb{V}$ to $\mathbb{W}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{V}$. Is f a linear transformation.

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Yes, we can construct a basis B' from B which contains 1 and $\sqrt{2}$. This basis contains exactly one rational number which is 1. If B' contains two rationals 1 and q, then they are linearly dependent which is not possible.

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$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in B' \text{ and irrational} \end{cases}$$

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$0 = \sqrt{2}$ a contradiction. Hence f is non-linear from $\mathbb{R}(\mathbb{R})$ to $\mathbb{R}(\mathbb{R})$ satisfying $f(x+y) = f(x) + f(y)$.

- **[Exercise 0.0.8]** Let f be a linear transformation from $\mathbb{V}$ to $\mathbb{W}$. If S is a subspace of $\mathbb{V}$ then $f(S)$ is a subspace of $\mathbb{W}$. Moreover, if $x_1, \ldots, x_k$ generates S then $f(x_1), \ldots, f(x_k)$ generates $f(S)$.

  Answer: Let $y_1, y_2 \in f(S)$ and $\alpha.\beta \in \mathbb{F}$. Then we have $x_, x_2 \in S$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Take

  $\alpha y_1 + \beta y_2$

  $= \alpha f(x_1) + \beta f(x_2)$

  $= f(\alpha x_1 + \beta x_2)$.

  Since S is a subsapce, we have $\alpha x_1 + \beta x_2 \in S$. Then $f(\alpha x_1 + \beta x_2) \in f(S)$. That is $\alpha y_1 + \beta y_2 \in f(S)$.

Let $y \in f(S)$. Then $f(x) = y$ for some $x \in S$. Since $x_1, \ldots, x_k$ generates S, we have $x = c_1 x_1 + \cdots + c_k x_k$.

$y = f(x) = f(c_1 x_1 + \cdots + c_k x_k) = c_1 f(x_1) + \cdots + c_k f(x_k)$. y is a linear combination of $f(x_1), \cdots, f(x_k)$ and $f(x_1), \cdots, f(x_k)$ are elements of $f(S)$. Hence $f(x_1), \ldots, f(x_k)$ generates $f(S)$.

- **[Exercise 0.0.11]** Let $\mathbb{V}$ be a finite dimensional vector space over the field $\mathbb{F}$. Let $f, g \in \mathbb{V}^*$ be nonzero transformations. Then f and g are **linearly dependent** if and only if $Ker(f) = Ker(g)$.

- **[Exercise 0.0.11]** Let $\mathbb{V}$ be a finite dimensional vector space over the field $\mathbb{F}$. Let $f, g \in \mathbb{V}^*$ be nonzero transformations. Then f and g are **linearly dependent** if and only if $Ker(f) = Ker(g)$.

Answer: We first assume that f and g are LD. Then there exists $\alpha \in \mathbb{F} - \{0\}$ such that $f = \alpha g$. This implies $Ker(f) = Ker(g)$.

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We now assume that $Ker(f) = Ker(g)$. Notice that $rank(f) = rank(g) = 1$. Then $Nullity(f) = Nullity(g) = \dim(\mathbb{V}) - 1$. Let $\dim(\mathbb{V}) = n$.

- **[Exercise 0.0.11]** Let $\mathbb{V}$ be a finite dimensional vector space over the field $\mathbb{F}$. Let $f, g \in \mathbb{V}^*$ be nonzero transformations. Then f and g are **linearly dependent** if and only if $Ker(f) = Ker(g)$.

Answer: We first assume that f and g are LD. Then there exists $\alpha \in \mathbb{F} - \{0\}$ such that $f = \alpha g$. This implies $Ker(f) = Ker(g)$.

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$f(u_i) = g(u_i) = 0$ for $i = 1, \ldots, n-1$ and $f(u_n), g(u_n)$ are non-zero.

Since $f(u_n)$ and $g(u_n)$ are two nonzero elements of $\mathbb{F}$, they are LD and $f(u_n) = \alpha g(u_n)$ for some non-zero $\alpha \in \mathbb{F}$.

Let $x \in \mathbb{V}$. Then $x = c_1 u_1 + \cdots + c_{n-1} u_{n-1} + c_n u_n$.

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Hence f and g are linearly dependent.

- **[Exercise 0.0.13]** Let $\mathbb{V}$ be a finite dimensional vector space over the field $\mathbb{F}$. Let $T \in \mathbb{L}(\mathbb{V}, \mathbb{V})$ such that $rank(T^2) = rank(T)$. Prove that $Ker(T) \cap R(T) = \{0\}$.

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We show that $R(T) = R(T^2)$. Let $y \in R(T^2)$.

Then there exists $x \in \mathbb{V}$ such that $T^2(x) = y$. $T(T(x)) = y$, this implies $y \in R(T)$.

Hence $R(T^2) \subseteq R(T)$. They have same dimension. Therefore $R(T^2) = R(T)$.

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$T^2(y) = T(T(y)) = T(x) = 0$. Hence $y \in Ker(T^2)$. Therefore $y \in Ker(T)$. $T(y) = 0$. Hence $x = 0$.

- **[Exercise 0.0.15]** Let $\mathbb{V}$ be a finite dimensional vector space over the filed $\mathbb{F}$ and let $\mathbb{W}$ be a subspace of $\mathbb{V}$. If f is a linear functional on $\mathbb{W}$. Prove that there is a linear functional g on $\mathbb{V}$ such that $g(v) = f(v)$ for all $v \in \mathbb{W}$.

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  Let $\{v_1, \ldots, v_k\}$ be a basis of $\mathbb{W}$. Then $f(v_i) = \alpha_i$ for $i = 1, \ldots, k$ where $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$.

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We know that there exists a unique linear transformation g such that $g(v_i) = \alpha_i$ for $i = 1, \ldots, k$ and $g(v_i) = \beta_i$ for $i = k+1, \ldots, n$, Where $\beta_i \in \mathbb{F}$ for $i = k+1, \ldots, n$.

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  Let $x \in \mathbb{W}$. Then $x = c_1 v_1 + \cdots + c_k v_k$. Therefore $f(x) = c_1\alpha_1 + \cdots + c_k\alpha_k$ and

  $$g(x) = g(c_1 v_1 + \cdots + c_k v_k) = c_1 g(v_1) + \cdots + c_k g(v_k)$$

  $$g(x) = c_1\alpha_1 + \cdots + c_k\alpha_k = f(x).$$

- **[Exercise 0.0.12]** Let $A \in \mathbb{M}_n(\mathbb{F})$. Let $T : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F})$ defined by $T(B) = AB$ for all $B \in \mathbb{M}_n(\mathbb{F})$. Then prove that there exists a basis Q in $\mathbb{M}_n(\mathbb{F})$ such that $[T]_Q = \begin{bmatrix} A & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & A \end{bmatrix}$

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We use E_{ij} to denote the matrix size n whose ijth entry is 1 and rest of the entries are zero.

Let $Q = \{E_{11}, E_{21}, \ldots, E_{n1}, E_{12}, E_{22}, \ldots, E_{n2}, \ldots, E_{1n}, E_{2n}, \ldots, E_{nn}\}$. Q is a basis of $\mathbb{M}_n(\mathbb{F})$.

Just check that $[T]_Q = \begin{bmatrix} A & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & A \end{bmatrix}$.