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$T : \mathbb{R} \to \mathbb{R}$ be a map defined by $T(x) = x + 1$. Using above theorem you can say that T is not linear.

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  Answer: Yes. Let $(x_1, x_2) \in \mathbb{R}^2$. Then $(x_1, x_2) = x_1 e_1 + x_2 e_2$.

Then
$$\begin{aligned} T(x_1, x_2) &= x_1 T(e_1) + x_2 T(e_2) \\ &= x_1(1,1) + x_2(-1,1) \\ &= (x_1 - x_2, x_1 + x_2) \end{aligned}$$

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  Answer: No it is not possible.

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Proof:

Let $x \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} c_i u_i$.

Define $T(x) = \sum_{i=1}^{n} c_i w_i$. It is clear that T is well defined because $x = \sum_{i=1}^{n} c_i u_i$, this expression unique.

We first show that T is a linear transformation. Take $x, y \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} c_i u_i$ and $y = \sum_{i=1}^{n} d_i u_i$.

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Let $\alpha, \beta \in \mathbb{F}$. $T(\alpha x + \beta y) = T(\sum_{i=1}^{n} (\alpha c_i + \beta d_i) u_i)$.

$$
\begin{aligned}
T(\alpha x + \beta y) &= \sum_{i=1}^{n} (\alpha c_i + \beta d_i) w_i. \\
&= \alpha \sum_{i=1}^{n} c_i w_i + \beta \sum_{i=1}^{n} d_i w_i. \\
&= \alpha T(x) + \beta T(y).
\end{aligned}
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Hence T is linear.

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To show that $U = T$. Let $x \in \mathbb{V}$. Then $x = \sum_{i=1}^{n} a_i u_i$. Using definition of T we have $T(x) = T(\sum_{i=1}^{n} a_i u_i) = \sum_{i=1}^{n} a_i w_i$.

$$U(x) = U(\sum_{i=1}^{n} a_i u_i)$$

$$= \sum_{i=1}^{n} a_i U(u_i) \text{ (applying the definition of linear transformation)}$$

$$= \sum_{i=1}^{n} a_i w_i.$$

Then $U(x) = T(x)$ for all $x \in \mathbb{V}$. Hence $U = T$.

- [Example]

Take the basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$. Take $1, 2, 3 \in \mathbb{R}$. Then using previous theorem we have a unique linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}$ such that $T(e_1) = 1$, $T(e_2) = 2$, $T(e_3) = 3$ and $T(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3$.

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The previous theorem gives a technique to construct a linear transformation from a finite dimensional vector space to another dimensional vector space over the same filed $\mathbb{F}$.

- **[Definition:]** Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation.
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     The subspaces $ker(T)$ is called the **null space** of T and sometimes it is denoted by $N(T)$.

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  2. $R(T) := \{T(x) : x \in \mathbb{V}\}$. you can easily check that $R(T)$ is a subspace of $\mathbb{W}$.

     The subspaces $R(T)$ is called the **range space** of T.

- **[Example:]** $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a map defined by

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$\dim(N(T)) = 1.$

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Let $y = (y_1, y_2) \in R(T)$. Then there exists $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $(y_1, y_2) = T(x_1, x_2, x_3)$.

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The $\dim(R(T))$ is called the **rank** of T and $\dim(N(T))$ is called the **nullity** of T.

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To show $Ker(T) = \{0\}$. Let $x \in ker(T)$. Then $T(x) = 0$ and we know that $T(0) = 0$. Since T is one-one, then $x = 0$. Hence $Ker(T) = \{0\}$.

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Given that $T(u_1), \ldots, T(u_n)$ are LI. To show $u_1, \ldots, u_n$ are LI.

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Given that $T(u_1), \ldots, T(u_n)$ are LI. To show $u_1, \ldots, u_n$ are LI.

$c_1 u_1 + \cdots + c_n u_n = 0_{\mathbb{V}}$.

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$c_1 = c_2 = \ldots = c_n = 0$ as $T(u_1), \dots, T(u_n)$ are LI.

Converse is not true in general. That is if $u_1, \dots, u_n$ are LI, then $T(u_1), \dots, T(u_n)$ may or may not be LI.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ be a LT defined by $T(x_1, x_2) = (x_1 - x_2, x_2 - x_1)$. Take $u_1 = (1, 0)$ and $u_2 = (1, 1)$. Notice that u_1, u_2 are LI. But $T(u_1) = (1, -1)$ and $T(u_2) = (0, 0)$ are LD.

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Let $\{u_1, \ldots, u_k\}$ be a basis of $Ker(T)$. We extend $\{u_1, \ldots, u_k\}$ to a LI set of $m + k$ vectors (this is possible as $\mathbb{V}$ is infinite dimensional) which is $\{u_1, \ldots, u_k, \ldots, u_{k+1}, \ldots, u_{m+k+1}\}$.

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$T_{\mathbb{S}}$ is a LT from $\mathbb{S}$ to $\mathbb{W}$. Then $Ker(T_{\mathbb{S}}) \subseteq Ker(T)$ and $R(T_{\mathbb{S}}) \subseteq R(T)$.

$nullity(T_{\mathbb{S}}) \leq k$ and $rank(T_{\mathbb{S}}) \leq m$. Then $nullity(T_{\mathbb{S}}) + rank(T_{\mathbb{S}}) \leq k + m$ and $\dim(\mathbb{S}) = m + k + 1$. Then rank nullity theorem is not true on $\mathbb{S}$.

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