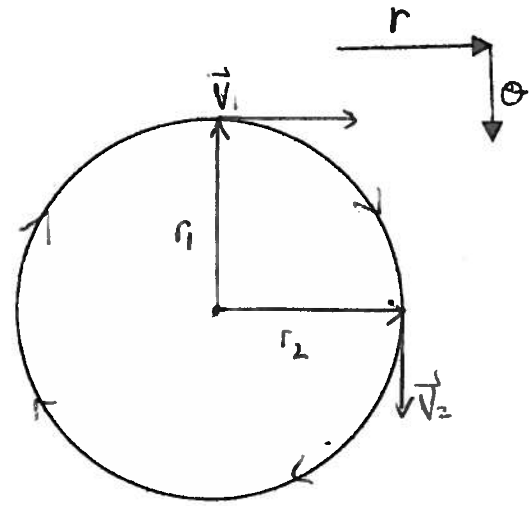
Centripetal Acceleration in Uniform Circular Motion

Consider an object moving in a circle at a constant speed as shown:



- the instantaneous velocity vector, v, is a constant length and always tangent to the path
- since the speed is constant there is no acceleration in the tangential direction
- the acceleration acts only to change the direction of the velocity and is always perpendicular to the velocity vector (radial)

The instantaneous acceleration is the CENTRIPETAL ACCELERATION:

$$\boxed{a_c = \frac{v^2}{r}}$$

a_c – the centripetal acceleration.

v -speed

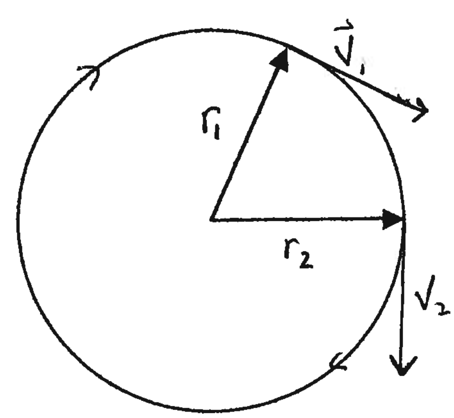
r - radius of the circle

Units check:

DERIVATION OF THE CENTRIPETAL ACCELERATION FORMULA

We start with the average acceleration between two points on the circular path:

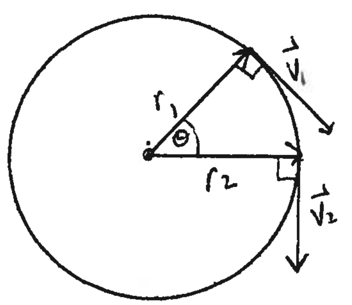
$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{\Delta t}$$

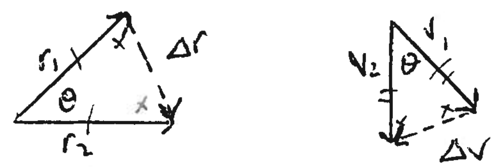


The instantaneous acceleration is found by taking the limit of the average acceleration as the time interval becomes small,

$$a_c = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \qquad \text{......} \quad \text{Eq. 1}$$

To evaluate the limit, we'll allow Δt to become smaller and see how Δr and Δv are related:





The triangles are similar, they are isosceles and they contain the same angle.

We can write the following ratio:

$$\frac{\Delta r}{r} = \frac{\Delta v}{v}$$

or $$\Delta v = v \frac{\Delta r}{r} \qquad \text{.....} \quad \text{Eq. 2}$$

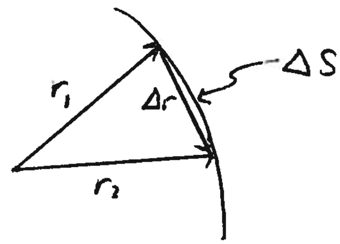
Substituting this relationship in Eq. 1:

$$a_c = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(v \frac{\Delta r}{r}\right) \frac{1}{\Delta t}$$

$$= \frac{v}{r} \lim_{\Delta t \to 0} \left(\frac{\Delta r}{\Delta t}\right) \qquad \text{....} \quad \text{Eq. 3}$$

As the time interval becomes small ($\Delta t \to 0$), the distance Δr approaches the arc length Δs:



But the arc length, Δs, is the distance travelled by the moving object and can also be found by multiplying the speed by the time Δt:

$$\Delta s = v\,\Delta t$$

We can substitute this relationship into the limit as Δt becomes small:

$$\Delta r \approx \Delta s = v\,\Delta t \qquad \text{....Eq. 4}$$

The limit becomes:

$$a_c = \frac{v}{r} \lim_{\Delta t \to 0} \left(\frac{\Delta r}{\Delta t} \right)$$

$$a_c = \frac{v}{r} \lim_{\Delta t \to 0} \left(\frac{v\,\cancel{\Delta t}}{\cancel{\Delta t}} \right)$$

$$a_c = \frac{v}{r} \bullet v$$

$$\boxed{a_c = \frac{v^2}{r}}$$