Contents

Introduction

- † motiva fisicamente lo studio delle soluzioni di Kz
- † parallelo con storia turbo hydro
- † fai un botto di esempi fisici con tante belle foto (parti da mail Onorato e libro zakh per cercarli)

provaprova

Statistics of weakly nonlinear waves

INTRO (what is done in this chapter, write at the end), put citations in here
It is a brief introduction, give reference for deeper understanding, Reference history of the subject (Hassleman, Zakharov, etc)

§1.1 Hamiltonian description of waves in continuous media

We first turn to the construction of a general Hamiltonian method for the description of waves travelling in continuous media. In doing this we greatly borrow from the exceptional treatment in [3].
Questa parte iniziale può essere più o meno ampia in base al tempo a disposizione.
- Hamiltoniana generica per mezzo continuo (da spazio coordinate in una scatola), notazione, commenti sulla stabilità, commenti sul fatto che possa essere già risultato di sviluppo pertubativo (come per le onde oceaniche) e scrittura in spazio-k con 3 onde e 4 onde, simmetrie continue e discrete

to the appendix: - Idea Onorato oscillatore armonico per giustificare nonrisonanza e commento su trasfomrazione canonica per eliminare termini non risonanti (indirizza ad appendice per conto specifico), cita sistemi fisici in cui l'eq è a 4 onde

Our final Hamiltonian is

$$\mathscr{H} = \sum_k \omega_k a_k^* a_k + \frac{1}{2} \sum_{k123} \mathrm{T}_{k123} a_k^* a_1^* a_2 a_3 \delta_{23}^{k1}. \tag{I.1}$$

CITA IL FATTO CHE GLI INDICI DEI k SONO SOPPRESSI

§1.2 Resonant and non-resonant interactions

maybe in the appendix?

§1.3 Perturbation Theory

Often in nonlinear systems no exact solutions (or few of them) are known. We now imagine ourselves in the situation where the interaction term in our Hamiltonian is small enough to allow for the perturbative treatment of the equations of motion, that is the expansion of the solution in orders of some small parameter, and their subsequent calculation order by order. To make the expansion clearer we write an explicit ϵ factor in front of the interaction term † .

Following the derivation of [2] we transform to the action-angle coordinates of the unperturbed quadratic Hamiltonian

$$a_k = \sqrt{I_k} e^{-i\theta_k} \tag{I.2}$$

obtaining

$$\mathcal{H} = \sum_k \omega_k I_k + \frac{\epsilon}{2} \sum_{k123} T_{k123} \sqrt{I_k I_1 I_2 I_3} e^{i(\theta_k+\theta_1-\theta_2-\theta_3)} \delta^{k1}_{23}, \tag{I.3}$$

by assuming that $T \in \mathbb{R}$ (as is the case in a vast class of physical systems) the requirement that $\mathcal{H} \in \mathbb{R}$ implies

$$\mathcal{H} = \sum_k \omega_k I_k + \frac{\epsilon}{2} \sum_{k123} T_{k123} \sqrt{I_k I_1 I_2 I_3} \cos(\Delta\theta^{k1}_{34}) \delta^{k1}_{23}, \tag{I.4}$$

where we defined $\Delta\theta^{k1}_{23} = \theta_k + \theta_1 - \theta_2 - \theta_3$.
We can prove that the change of coordinates (I.2) is canonical by assuming it to be true and recovering a_k and a_k^*'s poisson brackets‡

$$\begin{aligned} \{ia_k^*, a_k\} &= i\left(\frac{\partial a_k^*}{\partial I_k}\frac{\partial a_k}{\partial \theta_k} - \frac{\partial a_k^*}{\partial \theta_k}\frac{\partial a_k}{\partial I_k}\right) && \text{(I.5)}\\ &= -i\left(\frac{i}{2}\frac{1}{\sqrt{I_k}} e^{i\theta_k}\sqrt{I_k}e^{-i\theta_k} + \frac{i}{2}\frac{1}{\sqrt{I_k}} e^{-i\theta_k}\sqrt{I_k}e^{i\theta_k}\right) && \text{(I.6)}\\ &= 1. && \text{(I.7)} \end{aligned}$$

We can thus impose Hamilton equations for the new coordinates (remembering that time dependance of the coordinates is suppressed)

$$\frac{d}{dt} I_k = -\frac{\partial}{\partial\theta_k}\mathcal{H} = 2\epsilon \sum_{123} T_{k123}\sqrt{I_k I_1 I_2 I_3} \sin(\Delta\theta^{k1}_{23})\delta^{k1}_{23} \tag{I.8}$$

$$\frac{d}{dt}\theta_k = \frac{\partial}{\partial I_k}\mathcal{H} = \omega_k + \epsilon \sum_{123} T_{k123}\sqrt{\frac{I_1 I_2 I_3}{I_k}} \cos(\Delta\theta^{k1}_{23})\delta^{k1}_{23}. \tag{I.9}$$

Since we are essentially perturbing an infinite set of harmonical oscillators with a small interaction term, we can euristically assume that the coordinates cannot grow indefinetly to infinity. We shall then be weary of unphysical secular terms artificially introduced by the perturbative expansion. The Poincarè-Lindsted method allows us to remove such terms by a frequency shift

$$\omega_k \to \Omega_k = \omega_k + \epsilon\left(2\sum_p T_{kpkp} I_p - T_{kkkk} I_k\right), \tag{I.10}$$

togheter with a change of the summatory in $\mathcal{H}$ such that $k_2 = k$ & $k_1 = k_3$, $k_3 = k$ & $k_1 = k_2$ and $k_1 = k_2 = k_3 = k$ are excluded from it.
This particular choice is better justified in the APPENDIX WITH LINK, MAKE DUFFING EXAMPLE or in [1].

†The small parameter my be present as a constant in the Hamiltonian (for example the coupling g in the Nonlinear Schrodinger equation) or it may be a placeholder for the smallness of the function T_{k123} in a certain subdomain of k-space (for example the interaction among gravity waves in the small wavenumber limit).

‡Remembering that the true canonical variables are a_k and ia_k^*.

We may now develop perturbation theory, we start by expanding the (unknown) solutions as

$$I_k = I_k^{(0)} + +\epsilon I_k^{(1)} + \epsilon^2 I_k^{(2)} + \mathcal{O}(\epsilon^3) \tag{I.11}$$

$$\theta_k = \theta_k^{(0)} + +\epsilon\theta_k^{(1)} + \epsilon^2\theta_k^{(2)} + \mathcal{O}(\epsilon^3), \tag{I.12}$$

and than substituting them into (I.8) and (I.9).
We now reintroduce explicit time dependance and impose $I_k^{(0)}(0) = \bar{I}_k$ and $I_k^{(1)}(0) = I_k^{(2)}(0) = 0$ to fix initial conditions on the Is and $\theta_k^{(0)}(0) = \bar{\theta}_k$ and $\theta_k^{(1)}(0) = \theta_k^{(2)}(0) = 0$ to fix initial conditions on the θs.

The ϵ^0 order equations are

$$\frac{d}{dt} I_k^{(0)} = 0 \tag{I.13}$$

$$\frac{d}{dt}\theta_k^{(0)} = \Omega_k^{(0)}, \tag{I.14}$$

with solutions

$$I_k^{(0)}(t) = \bar{I}_k \tag{I.15}$$

$$\theta_k^{(0)}(t) = \bar{\theta}_k + \bar{\Omega}_k t, \tag{I.16}$$

where $\Omega_k^{(0)}$ and $\bar{\Omega}_k$ refer to Ω_k with only zeroeth order contribution or initial conditions respectively. ANGOLO HA TERMINE SECOLARE, IN EFFETTI LO METTIAMO IN LEADING ORDER DYNAMIC PROPRIO PER QUESTO MOTIVO FORSE, GIUSTIFICA DOPO DICENDO CHE A NOI IMPORTA CHE AZIONE NON NE ABBIA, SICCOME ANGOLI SONO IN DEI SENI, E FAI NOTARE CHE ALLA FINE IL FREQUENCY SHIFT NON IFLUENZA DINAMICA, COME ATTESO ESSENDO NLO To be precise the ϵ terms in the shifted frequency should be included in the equations for $\theta^{(1)}$ and not $\theta^{(0)}$, we however make this choice to keep the term Ω in a compact form.
This order reproduces the dynamics of an infinite number of integrable systems (for example decoupled harmonic oscillators), with constant actions and angles evolving linearly with time.

At ϵ order the equations of motion are

$$\frac{d}{dt} I_k^{(1)} = 2\sum_{123} T_{k123}\sqrt{I_k^{(0)} I_1^{(0)} I_2^{(0)} I_3^{(0)}}\sin(\Delta\theta_{23}^{k1(0)})\delta_{23}^{k1} \tag{I.17}$$

$$\frac{d}{dt}\theta_k^{(1)} = \sum_{123} T_{k123}\sqrt{\frac{I_1^{(0)} I_2^{(0)} I_3^{(0)}}{I_k^{(0)}}}\cos(\Delta\theta_{23}^{k1(0)})\delta_{23}^{k1}. \tag{I.18}$$

Here the only time dependance lies in $\Delta\theta^{(0)}$ and $I_k^{(1)}(0) = \theta_k^{(1)}(0) = 0$, integrating the equations gives

$$I_k^{(1)}(t) = 2\sum_{123} T_{k123}\sqrt{\bar{I}_k \bar{I}_1 \bar{I}_2 \bar{I}_3}\frac{\delta_{23}^{k1}}{\Delta\bar{\Omega}_{23}^{k1}}\left[\cos(\Delta\bar{\theta}_{23}^{k1}) - \cos(\Delta\bar{\theta}_{23}^{k1} + \Delta\bar{\Omega}_{23}^{k1} t)\right] \tag{I.19}$$

$$\theta_k^{(1)}(t) = \sum_{123} T_{k123}\sqrt{\frac{\bar{I}_1 \bar{I}_2 \bar{I}_3}{\bar{I}_k}}\frac{\delta_{23}^{k1}}{\Delta\bar{\Omega}_{23}^{k1}}\left[\sin(\Delta\bar{\theta}_{23}^{k1} + \Delta\bar{\Omega}_{23}^{k1} t) - \sin(\Delta\bar{\theta}_{23}^{k1})\right]. \tag{I.20}$$

Where $\Delta\bar{\Omega}$ is defined in the same fashion as $\Delta\theta$.

We should be content with this first nontrivial result, but through the sheer power of hindsight † we write also the ϵ^2 order equations only for the action variables (there is no need to actually solve them).

† Developing a statistical theory of the system, the first nontrivial contribution comes from the ϵ^2 order.

Looking at (I.8) we seek to obtain an ϵ^2 equation by substituting I and θ up to their ϵ order terms. By Taylor expanding the square root we obtain four terms of the form

$$\sqrt{(\mathrm{x}+\epsilon\mathrm{y})\tilde{\mathrm{x}}} \underset{\epsilon\to 0}{\sim} \sqrt{\mathrm{x}\tilde{\mathrm{x}}}\left(1+\frac{\epsilon\mathrm{y}}{2\tilde{\mathrm{x}}}\right), \tag{I.21}$$

where, for example, $\mathrm{x}+\epsilon\mathrm{y} = \mathrm{I}_k^{(0)}+\epsilon\mathrm{I}_k^{(1)}$ and $\tilde{\mathrm{x}} = \mathrm{I}_1^{(0)}\mathrm{I}_2^{(0)}\mathrm{I}_3^{(0)}$.
There also appear terms of the form

$$\sin(\mathrm{x}+\epsilon\mathrm{y}) \underset{\epsilon\to 0}{\sim} \sin(\mathrm{x})+\epsilon\mathrm{y}\cos(\mathrm{x}), \tag{I.22}$$

where $\mathrm{x} = \Delta\theta_{23}^{k1(0)}$ and $\mathrm{y} = \Delta\theta_{23}^{k1(1)}$.
By plugging everything into (I.8) we first obtain

$$\frac{d}{dt}\mathrm{I}_k^{(2)} = 2\sum_{123}\mathrm{T}_{k123}\sqrt{\mathrm{I}_k^{(0)}\mathrm{I}_1^{(0)}\mathrm{I}_2^{(0)}\mathrm{I}_3^{(0)}}\left[\frac{1}{2}\left(\frac{\mathrm{I}_k^{(1)}}{\mathrm{I}_k^{(0)}}+\frac{\mathrm{I}_1^{(1)}}{\mathrm{I}_1^{(0)}}+\frac{\mathrm{I}_2^{(1)}}{\mathrm{I}_2^{(0)}}+\frac{\mathrm{I}_3^{(1)}}{\mathrm{I}_3^{(0)}}\right)\sin(\Delta\theta_{23}^{k1(0)})\delta_{23}^{k1}+\Delta\theta_{23}^{k1(1)}\cos(\Delta\theta_{23}^{k1(0)})\delta_{23}^{k1}\right], \tag{I.23}$$

and then by using (I.15), (I.16), (I.19), (I.20) and basic trigonometry we find

$$\begin{aligned}\frac{d}{dt}\mathrm{I}_k^{(2)} = 2\sum_{123456}\mathrm{T}_{k123}\sqrt{\bar{\mathrm{I}}_k\bar{\mathrm{I}}_1\bar{\mathrm{I}}_2\bar{\mathrm{I}}_3\bar{\mathrm{I}}_4\bar{\mathrm{I}}_5\bar{\mathrm{I}}_6}\sum_{i=1}^{4}\frac{\mathrm{T}_{k123}\mathrm{T}_{i456}}{\sqrt{\bar{\mathrm{I}}_i}\Delta\bar{\Omega}_{56}^{i4}} \\ \times\left(\sin(\Delta\bar{\theta}_{23}^{k1}+\Delta\bar{\Omega}_{23}^{k1}t-\sigma_i\Delta\bar{\theta}_{56}^{i4})+\sin(\sigma_i\Delta\bar{\theta}_{56}^{i4}+\sigma_i\Delta\bar{\Omega}_{56}^{i4}t-\Delta\bar{\theta}_{23}^{k1}-\Delta\bar{\Omega}_{23}^{k1}t)\right)\delta_{23}^{k1}\delta_{56}^{i4}.\end{aligned} \tag{I.24}$$

§1.4 Statistical description

Having approximated the solutions to order ϵ we found ourselves with the problem of gathering initial conditions in infinite dimensonal systems † , we shall then renounce the deterministic approach in favour of a probabilistic one.
In general such idea is realized thorugh averaging over infinitely many realizations of the equations of motion with different initial conditions, to then extract average quantities more easily confrontable with experiment. In a nonlinear problem this is again highly non trivial, to simplify the endeavor we assume that a large number of waves is present in the system, in the sense that each mode in Fourier space is highly excited. It is then reasonable to assume the initial phases to be uniformly distributed in the $[0,2\pi]$ segment ‡ . We define the averaging as

$$\langle f(\bar{\theta}_1\ldots\bar{\theta}_N)\rangle_{\bar{\theta}} = \int_0^{2\pi} P(\bar{\theta}_1\ldots\bar{\theta}_N)f(\bar{\theta}_1\ldots\bar{\theta}_N)d\bar{\theta}_1\ldots d\bar{\theta}_N \quad \text{with} \quad P(\bar{\theta}_1\ldots\bar{\theta}_N) = \frac{1}{2\pi^N} \tag{I.25}$$

Looking back at the Hamiltonian (I.1) we see that the phases do not contribute to physical quantities like the energy or the wave number, it is in the action variables that those observables are encoded. We have now a clear plan, to find a kinetic equation, independent of initial conditions, for the action variables.
THe main objective is then

$$\left\langle\frac{d}{dt}\mathrm{I}_k\right\rangle_{\bar{\theta}} = \frac{d}{dt}\langle\mathrm{I}_k\rangle_{\bar{\theta}} = \epsilon\left\langle\frac{d}{dt}\mathrm{I}_k^{(1)}\right\rangle_{\bar{\theta}}+\epsilon^2\left\langle\frac{d}{dt}\mathrm{I}_k^{(2)}\right\rangle_{\bar{\theta}}+ \tag{I.26}$$

-Definizione nk(t), Derivazione alla onorato della 4-WKE, limite al continuo, commenti su assunzioni!, indirizza all'Appendice per metodo Poincarè lindsted.
-Derivazione alla Falkovich della wke, or referral to appendix

† Let us think of the ocean surface for example, measuring its height at a generic instant would be unfeasible.

‡ Unless the original equation of motions are known to have solitonic solutions in a certain regime of k-space, in such case phases could be correlated and not uniformly distributed anymore.

§1.5 Some properties of the kinetic equation

-Proprietà WKE :Irreversibilità, dimostrazione del teorema H, leggi di conservazione e equazioni di continuità

§1.6 Equilibrium stationary state

-Equilibrium solutions and comments

§1.7 Out of equilibrium stationary states

-Out of Equilibrium solutions and comments, argomento fjortoft, argomento zakharov, misto dei due con commenti su forzante e dissipazione ed esempi fisici, ragionamenti dimensionali su forme approssimate di lambda e omega e forme generiche di esponenti, definizioni costanti di KZ e commenti su cnvergenza e trasformata zakh un pò di foto di cascate per sistemi fisici NON MMT

§1.8 Convergence issues (MAYBE)

Beyond the leading order

- † Introduzione al nlo (difficoltà computazionali, wyld diagrammatic technique, zakharov paper, gurarie) e nuovi sviluppi (Rosenhaus etc) suggerimenti per letture di QFT (sceli che notazione usare)
- † come si ottiene la wke da una trattazione con i campi (arriva alle regole di feynman)
- † conti funzione a due punti
- † conti funzione a 4 punti
- † wke nlo
- † commenti e racconti su sviluppi extra (large N and resummation)

$$x = 2y + 3 \tag{II.1}$$

A one dimensional model for Wave Turbulence

† mini intro su storia MMT e outline capitolo

† MMT model con beta generico e sua WKE

† soluzioni KZ con beta generico e controllo sommario convergenza (solo limiti)

† trasformata di Zakharov per controllare davvero convergenza

† commmenti su validita WKE data la presenza di solitoni e quasisolitoni (devi studiarla prima bro)

† MMT nlo, qunatità ancora conservate e discussioni sulla sua convergenza e sulla natura delle correzioni

† tutto quello che riusciamo a trovare come previsione teorica

Numerical study of the Wave Kinetic Equation

- † intro e obbiettivi
- † WavKinS struttura base e funzionamento (che algoritmi usa)
- † simulazioni leading order e commenti sulle stesse
- † simulazioni nlo e commenti, confronti con aspettative teoriche

Bibliography

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{3} G. F. VE. Zakharov and V. LVov, Kolmogorov Spectra of Turbulence I, Springer-Verlag, 1992.