

Contents

I	Statistics of weakly nonlinear waves	4
§1.1	Hamiltonian description of waves in continuous media	4
§1.2	Resonant and non-resonant interactions	4
§1.3	Perturbation Theory	4
§1.4	Random Phase Approximation	7
§1.5	Statistics over the actions and the thermodynamic limit	9
§1.6	Some properties of the kinetic equation	11
§1.7	Equilibrium stationary state	11
§1.8	Out of equilibrium stationary states	11
§1.9	Convergence issues (MAYBE)	11
II	Beyond the leading order	12
III	A one dimensional model for Wave Turbulence	13
IV	Numerical study of the Wave Kinetic Equation	14

Introduction

- motiva fisicamente lo studio delle soluzioni di Kz
- parallelo con storia turbo hydro
- fai un botto di esempi fisici con tante belle foto (parti da mail Onorato e libro zakh per cercarli)

provaprova

Statistics of weakly nonlinear waves

INTRO (what is done in this chapter, write at the end), put citations in here

It is a brief introduction, give reference for deeper understanding, Reference history of the subject (Hassleman, Zakharov, etc)

§1.1 Hamiltonian description of waves in continuous media

We first turn to the construction of a general Hamiltonian method for the description of waves travelling in continuous media. In doing this we greatly borrow from the exceptional treatment in [3].

Questa parte iniziale può essere più o meno ampia in base al tempo a disposizione.

- Hamiltoniana generica per mezzo continuo (da spazio coordinate in una scatola), notazione, commenti sulla stabilità, commenti sul fatto che possa essere già risultato di sviluppo perturbativo (come per le onde oceaniche) e scrittura in spazio-k con 3 onde e 4 onde, simmetrie continue e discrete

to the appendix: - Idea Onorato oscillatore armonico per giustificare nonrisonanza e commento su trasformazione canonica per eliminare termini non risonanti (indirizza ad appendice per conto specifico), cita sistemi fisici in cui l'eq è a 4 onde

Our final Hamiltonian is

$$\mathcal{H} = \sum_k \omega_k a_k^* a_k + \frac{1}{2} \sum_{k_{123}} T_{k_{123}} a_k^* a_1^* a_2 a_3 \delta_{23}^{k_1}. \quad (\text{I.1})$$

CITA IL FATTO CHE GLI INDICI DEI k SONO SOPPRESSI

§1.2 Resonant and non-resonant interactions

maybe in the appendix?

§1.3 Perturbation Theory

Often in nonlinear systems no exact solutions (or few of them) are known. We now imagine ourselves in the situation where the interaction term in our Hamiltonian is small enough to allow for the perturbative treatment of the equations of motion, that is the expansion of the solution in orders of some small parameter, and their subsequent calculation order by order. From a physical viewpoint the smallness of the interaction term corresponds to a separation of the fast time scale on which the

linear term operates from the slow time scale of the nonlinear one.

To make the expansion clearer we write an explicit ϵ factor in front of the interaction term [†].

Following the derivation of [2] we transform to the action-angle coordinates of the unperturbed quadratic Hamiltonian

$$a_k = \sqrt{I_k} e^{-i\theta_k} \quad (\text{I.2})$$

obtaining

$$\mathcal{H} = \sum_k \omega_k I_k + \frac{\epsilon}{2} \sum_{k123} T_{k123} \sqrt{I_k I_1 I_2 I_3} e^{i(\theta_k + \theta_1 - \theta_2 - \theta_3)} \delta_{23}^{k1}, \quad (\text{I.3})$$

by assuming that $T \in \mathbb{R}$ (as is the case in a vast class of physical systems) the requirement that $\mathcal{H} \in \mathbb{R}$ implies

$$\mathcal{H} = \sum_k \omega_k I_k + \frac{\epsilon}{2} \sum_{k123} T_{k123} \sqrt{I_k I_1 I_2 I_3} \cos(\Delta\theta_{34}^{k1}) \delta_{23}^{k1}, \quad (\text{I.4})$$

where we defined $\Delta\theta_{23}^{k1} = \theta_k + \theta_1 - \theta_2 - \theta_3$.

We can prove that the change of coordinates (I.2) is canonical by assuming it to be true and recovering a_k and a_k^* 's poisson brackets[‡]

$$\{ia_k^*, a_k\} = i \left(\frac{\partial a_k^*}{\partial I_k} \frac{\partial a_k}{\partial \theta_k} - \frac{\partial a_k^*}{\partial \theta_k} \frac{\partial a_k}{\partial I_k} \right) \quad (\text{I.5})$$

$$= -i \left(\frac{i}{2} \frac{1}{\sqrt{I_k}} e^{i\theta_k} \sqrt{I_k} e^{-i\theta_k} + \frac{i}{2} \frac{1}{\sqrt{I_k}} e^{-i\theta_k} \sqrt{I_k} e^{i\theta_k} \right) \quad (\text{I.6})$$

$$= 1. \quad (\text{I.7})$$

We can thus impose Hamilton equations for the new coordinates (remembering that time dependence of the coordinates is suppressed)

$$\frac{d}{dt} I_k = -\frac{\partial}{\partial \theta_k} \mathcal{H} = 2\epsilon \sum_{123} T_{k123} \sqrt{I_k I_1 I_2 I_3} \sin(\Delta\theta_{23}^{k1}) \delta_{23}^{k1} \quad (\text{I.8})$$

$$\frac{d}{dt} \theta_k = \frac{\partial}{\partial I_k} \mathcal{H} = \omega_k + \epsilon \sum_{123} T_{k123} \sqrt{\frac{I_1 I_2 I_3}{I_k}} \cos(\Delta\theta_{23}^{k1}) \delta_{23}^{k1}. \quad (\text{I.9})$$

Since we are essentially perturbing an infinite set of harmonical oscillators with a small interaction term, we can euristically assume that the coordinates cannot grow indefinitely to infinity. We shall then be weary of unphysical secular terms artificially introduced by the perturbative expansion. The Poincarè-Lindsted method allows us to remove such terms by a frequency shift

$$\omega_k \rightarrow \Omega_k = \omega_k + \epsilon \left(2 \sum_p T_{kp kp} I_p - T_{kk kk} I_k \right), \quad (\text{I.10})$$

togheter with a change of the summatory in \mathcal{H} such that the trivial interactions[§] $k_2 = k$ & $k_1 = k_3$, $k_3 = k$ & $k_1 = k_2$ and $k_1 = k_2 = k_3 = k$ are excluded from it.

[†]The small parameter may be present as a constant in the Hamiltonian (for example the coupling g in the Nonlinear Schrodinger equation) or it may be a placeholder for the smallness of the function T_{k123} in a certain subdomain of k -space (for example the interaction among gravity waves in the small wavenumber limit).

[‡]Remembering that the true canonical variables are a_k and ia_k^* .

[§]We call them trivial as they do not correspond to a net exchange of energy/action among different Fourier modes

This particular choice is better justified in the [APPENDIX WITH LINK, MAKE DUFFING EXAMPLE](#) or in [1].

We may now develop perturbation theory, we start by expanding the (unknown) solutions as

$$I_k = I_k^{(0)} + \epsilon I_k^{(1)} + \epsilon^2 I_k^{(2)} + \mathcal{O}(\epsilon^3) \quad (\text{I.11})$$

$$\theta_k = \theta_k^{(0)} + \epsilon \theta_k^{(1)} + \epsilon^2 \theta_k^{(2)} + \mathcal{O}(\epsilon^3), \quad (\text{I.12})$$

and then substituting them into (I.8) and (I.9).

We now reintroduce explicit time dependance and impose $I_k^{(0)}(0) = \bar{I}_k$ and $I_k^{(1)}(0) = I_k^{(2)}(0) = 0$ to fix initial conditions on the I s and $\theta_k^{(0)}(0) = \bar{\theta}_k$ and $\theta_k^{(1)}(0) = \theta_k^{(2)}(0) = 0$ to fix initial conditions on the θ s.

The ϵ^0 order equations are

$$\frac{d}{dt} I_k^{(0)} = 0 \quad (\text{I.13})$$

$$\frac{d}{dt} \theta_k^{(0)} = \Omega_k^{(0)}, \quad (\text{I.14})$$

with solutions

$$I_k^{(0)}(t) = \bar{I}_k \quad (\text{I.15})$$

$$\theta_k^{(0)}(t) = \bar{\theta}_k + \bar{\Omega}_k t, \quad (\text{I.16})$$

where $\Omega_k^{(0)}$ and $\bar{\Omega}_k$ refer to Ω_k with only zeroeth order contribution or initial conditions respectively. Notice that the ϵ terms in the shifted frequency should be included in the equations for $\theta^{(1)}$ and not $\theta^{(0)}$, we however make this choice to keep all terms linear with time together (and thus leading in the expansion).

This order reproduces the dynamics of an infinite dimensional integrable system (for example infinitely many decoupled harmonic oscillators), with constant actions and angles evolving linearly with time.

At ϵ order the equations of motion are

$$\frac{d}{dt} I_k^{(1)} = 2 \sum_{123} T_{k123} \sqrt{I_k^{(0)} I_1^{(0)} I_2^{(0)} I_3^{(0)}} \sin(\Delta \theta_{23}^{k1(0)}) \delta_{23}^{k1} \quad (\text{I.17})$$

$$\frac{d}{dt} \theta_k^{(1)} = \sum_{123} T_{k123} \sqrt{\frac{I_1^{(0)} I_2^{(0)} I_3^{(0)}}{I_k^{(0)}}} \cos(\Delta \theta_{23}^{k1(0)}) \delta_{23}^{k1}. \quad (\text{I.18})$$

Here the only time dependance lies in $\Delta \theta^{(0)}$ and $I_k^{(1)}(0) = \theta_k^{(1)}(0) = 0$, integrating the equations gives

$$I_k^{(1)}(t) = 2 \sum_{123} T_{k123} \sqrt{\bar{I}_k \bar{I}_1 \bar{I}_2 \bar{I}_3} \frac{\delta_{23}^{k1}}{\Delta \bar{\Omega}_{23}^{k1}} \left[\cos(\Delta \bar{\theta}_{23}^{k1}) - \cos(\Delta \bar{\theta}_{23}^{k1} + \Delta \bar{\Omega}_{23}^{k1} t) \right] \quad (\text{I.19})$$

$$\theta_k^{(1)}(t) = \sum_{123} T_{k123} \sqrt{\frac{\bar{I}_1 \bar{I}_2 \bar{I}_3}{\bar{I}_k}} \frac{\delta_{23}^{k1}}{\Delta \bar{\Omega}_{23}^{k1}} \left[\sin(\Delta \bar{\theta}_{23}^{k1} + \Delta \bar{\Omega}_{23}^{k1} t) - \sin(\Delta \bar{\theta}_{23}^{k1}) \right]. \quad (\text{I.20})$$

Where $\Delta \bar{\Omega}$ is defined in the same fashion as $\Delta \theta$.

We should be content with this first nontrivial result, but through the sheer power of hindsight[†] we write also the ϵ^2 order equations only for the action variables (there is no need to actually solve them).

Looking at (I.8) we seek to obtain an ϵ^2 equation by substituting I and θ up to their ϵ order terms. By Taylor expanding the square root we obtain four terms of the form

$$\sqrt{(x + \epsilon y)\tilde{x}} \underset{\epsilon \rightarrow 0}{\sim} \sqrt{x\tilde{x}} \left(1 + \frac{\epsilon y}{2\tilde{x}} \right), \quad (\text{I.21})$$

where, for example, $x + \epsilon y = I_k^{(0)} + \epsilon I_k^{(1)}$ and $\tilde{x} = I_1^{(0)} I_2^{(0)} I_3^{(0)}$. There also appear terms of the form

$$\sin(x + \epsilon y) \underset{\epsilon \rightarrow 0}{\sim} \sin(x) + \epsilon y \cos(x), \quad (\text{I.22})$$

where $x = \Delta\theta_{23}^{k1(0)}$ and $y = \Delta\theta_{23}^{k1(1)}$.

By plugging everything into (I.8) we first obtain

$$\frac{d}{dt} I_k^{(2)} = 2 \sum_{123} T_{k123} \sqrt{I_k^{(0)} I_1^{(0)} I_2^{(0)} I_3^{(0)}} \left[\frac{1}{2} \left(\frac{I_k^{(1)}}{I_k^{(0)}} + \frac{I_1^{(1)}}{I_1^{(0)}} + \frac{I_2^{(1)}}{I_2^{(0)}} + \frac{I_3^{(1)}}{I_3^{(0)}} \right) \sin(\Delta\theta_{23}^{k1(0)}) \delta_{23}^{k1} + \Delta\theta_{23}^{k1(1)} \cos(\Delta\theta_{23}^{k1(0)}) \delta_{23}^{k1} \right], \quad (\text{I.23})$$

and then by using (I.15), (I.16), (I.19), (I.20) and basic trigonometry we find

$$\begin{aligned} \frac{d}{dt} I_k^{(2)} = 2 \sum_{123456} T_{k123} \sqrt{\bar{I}_k \bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4 \bar{I}_5 \bar{I}_6} \sum_{i=0}^3 \frac{T_{k123} T_{i456}}{\sqrt{\bar{I}_i} \Delta \bar{\Omega}_{56}^{i4}} \\ \times \left(\sin(\Delta \bar{\theta}_{23}^{k1} + \Delta \bar{\Omega}_{23}^{k1} t - \sigma_i \Delta \bar{\theta}_{56}^{i4}) + \sin(\sigma_i \Delta \bar{\theta}_{56}^{i4} + \sigma_i \Delta \bar{\Omega}_{56}^{i4} t - \Delta \bar{\theta}_{23}^{k1} - \Delta \bar{\Omega}_{23}^{k1} t) \right) \delta_{23}^{k1} \delta_{56}^{i4}, \end{aligned} \quad (\text{I.24})$$

where σ_i is equal to +1 if $i = 0, 1$ and alternatively is -1. When $i = 0$ it represents functional dependence on k .

§1.4 Random Phase Approximation

Having approximated the solutions to order ϵ we found ourselves with the problem of gathering initial conditions in infinite dimensional systems[‡], we shall then renounce the deterministic approach in favour of a probabilistic one.

In general such idea is realized thorough averaging over infinitely many realizations of the equations of motion with different initial conditions, to then extract average quantities more easily confrontable with experiment. In a nonlinear problem this is again highly non trivial, to simplify the endeavor we assume that a large number of waves is present in the system, in the sense that each mode in Fourier space is highly excited. It is then reasonable to assume that, after a time evolution proportional to the minimum value of $\frac{1}{\omega_k}$ in the range of physical interest, the phases θ would be uniformly distributed in the $[0, 2\pi]$ segment[§]. This means that whatever our initial conditions, given that

[†]Developing a statistical theory of the system, the first nontrivial contribution comes from the ϵ^2 order.

[‡]Let us think of the ocean surface for example, measuring its height at a generic instant would be unfeasible.

[§]This is known as the random phase approximation. One shall be careful as if the original equations of motion are known to have solitonic solutions in a certain regime of k -space, in such case phases could be correlated and the assumption would not hold.

$\bar{I}_k \neq 0$ almost everywhere and the nonlinear contribution being slower than the linear one, we may actually assume some new initial conditions on θ s drawn from the following distribution

$$\langle f(\bar{\theta}_1 \dots \bar{\theta}_N) \rangle_{\bar{\theta}} = \int_0^{2\pi} P(\bar{\theta}_1 \dots \bar{\theta}_N) f(\bar{\theta}_1 \dots \bar{\theta}_N) d\bar{\theta}_1 \dots d\bar{\theta}_N \quad \text{with} \quad P(\bar{\theta}_1 \dots \bar{\theta}_N) = \frac{1}{2\pi^N} \quad (\text{I.25})$$

Looking back at the Hamiltonian (I.1) we see that the phases do not contribute to physical quantities like the energy or the wave number, it is in the action variables that those observables are encoded. We have now a clear plan, to find a kinetic equation, independent of initial conditions, for the action variables.

The main objective is then

$$\left\langle \frac{d}{dt} I_k \right\rangle_{\bar{\theta}} = \frac{d}{dt} \langle I_k \rangle_{\bar{\theta}} = \left\langle \frac{d}{dt} I_k^{(0)} \right\rangle_{\bar{\theta}} + \epsilon \left\langle \frac{d}{dt} I_k^{(1)} \right\rangle_{\bar{\theta}} + \epsilon^2 \left\langle \frac{d}{dt} I_k^{(2)} \right\rangle_{\bar{\theta}} \quad (\text{I.26})$$

To zeroeth order, being constant, is null. We average over the ϵ order equation (I.17) (with subbed zeroeth order solutions)

$$\left\langle \frac{d}{dt} I_k^{(1)} \right\rangle_{\bar{\theta}} = 2 \sum_{123} T_{k123} \sqrt{\bar{I}_k \bar{I}_1 \bar{I}_2 \bar{I}_3} \left\langle \sin(\Delta \bar{\theta}_{23}^{k1} + \Delta \bar{\Omega}_{23}^{k1} t) \right\rangle_{\bar{\theta}} \delta_{23}^{k1}. \quad (\text{I.27})$$

Making the probability distribution explicit and isolating the term depending on phases we obtain

$$\left\langle \sin(\Delta \bar{\theta}_{23}^{k1} + \Delta \bar{\Omega}_{23}^{k1} t) \right\rangle_{\bar{\theta}} = \left\langle 2 \text{Im} \left(e^{i \Delta \bar{\theta}_{23}^{k1}} e^{i \Delta \bar{\Omega}_{23}^{k1} t} \right) \right\rangle_{\bar{\theta}} = \frac{1}{2\pi^4} \int_0^{2\pi} e^{i \bar{\theta}_k} e^{i \bar{\theta}_1} e^{i \bar{\theta}_2} e^{i \bar{\theta}_3} d\bar{\theta}_k d\bar{\theta}_1 d\bar{\theta}_2 d\bar{\theta}_3 = 0 \quad (\text{I.28})$$

To first order we have a trivial kinetic equation, and must then go to ϵ^2 order to find nontrivial results, luckily we have already written I 's Hamilton equations to second order.

The $\bar{\theta}$ dependent part of equation (I.24) may be rewritten as

$$e^{+i\sigma_i \Delta \bar{\theta}_{56}^{i4} - i \Delta \bar{\theta}_{23}^{k1}} \left[e^{-i \Delta \bar{\Omega}_{23}^{k1}} \left(e^{i\sigma_i \Delta \bar{\Omega}_{56}^{i4} t} - 1 \right) \right] \quad (\text{I.29})$$

We shall focus on the $i=0$ term and extend the results to the other ones. Isolating the exponential with $\bar{\theta}$ in (I.29) and averaging we get

$$\left\langle e^{i(\bar{\theta}_4 + \bar{\theta}_2 + \bar{\theta}_3 - \bar{\theta}_5 - \bar{\theta}_6 - \bar{\theta}_1)} \right\rangle_{\bar{\theta}}. \quad (\text{I.30})$$

This term is different from 0 only if the total exponent is null. It acts as a Kroenecker's delta on the 3 out of the 6 sums, imposing either $k_4 = k_1$ & $k_2 = k_5$ & $k_3 = k_6$ or $k_4 = k_1$ & $k_2 = k_6$ & $k_3 = k_5^\dagger$. The full averaged $i=0$ term, with (I.30) enforced, is

$$4 \sum_{123} T_{k123} T_{k132} \bar{I}_1 \bar{I}_2 \bar{I}_3 \frac{\sin(\Delta \bar{\Omega}_{23}^{k1} t)}{\Delta \bar{\Omega}_{23}^{k1}} \delta_{23}^{k1}, \quad (\text{I.31})$$

where the property $T_{k123} = T_{k132}$ was used[†].

Looking at the cases $i=1,2,3$ the only differences are:

- $i=1 \longrightarrow$ the same as $i=0$ except for the exchange $\bar{I}_1 \longleftrightarrow \bar{I}_k$;

[†]The combinations with $k_2 = k_4$ or $k_2 = k_3$ were excluded from the sum with the shift (I.10)

[‡]In the case of complex T the property would hold as well with a complex conjugate on one side.

- $i = 2 \rightarrow$ the same as $i = 0$ except for an overall minus sign;
- $i = 3 \rightarrow$ the same as $i = 2$ except for $\bar{I}_2 \longleftrightarrow \bar{I}_3$.

The final result is

$$\frac{d}{dt} \langle I_k^{(2)} \rangle = 4 \sum_{123} T_{k123}^2 \bar{I}_k \bar{I}_1 \bar{I}_2 \bar{I}_3 \left(\frac{1}{\bar{I}_k} + \frac{1}{\bar{I}_1} - \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} \right) \frac{\sin(\Delta \bar{\Omega}_{23}^{k1} t)}{\Delta \bar{\Omega}_{23}^{k1}} \delta_{23}^{k1}, \quad (\text{I.32})$$

where it is not anymore necessary to account in the sum for trivial interactions, as for $k_2 = k$ & $k_1 = k_3$, $k_3 = k$ & $k_1 = k_2$ and $k_1 = k_2 = k_3 = k$ the r.h.s is null.

We may ignore to this order the frequency shift as Taylor expanding the sine function shows its contribution to be of order ϵ^3 .

By using one of the possible definitions of the Dirac's delta function

$$\lim_{a \rightarrow \infty} \frac{\sin(ax)}{\pi x} = \delta(x), \quad (\text{I.33})$$

into (I.32) together with (I.26) and the assumption that enough time has passed, finally

$$\frac{d}{dt} \langle I_k \rangle = 4\pi\epsilon^2 \sum_{123} T_{k123}^2 \bar{I}_k \bar{I}_1 \bar{I}_2 \bar{I}_3 \left(\frac{1}{\bar{I}_k} + \frac{1}{\bar{I}_1} - \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} \right) \delta(\Delta \omega_{23}^{k1}) \delta_{23}^{k1}. \quad (\text{I.34})$$

There are some important remarks on this last equation.

The presence of the Dirac's delta defines a resonance manifold[†] of the Fourier modes k_1, k_2 and k_3 interacting with k , showing that only resonant term contribute to the net interaction between different modes[‡].

Based on difference preferences one could define a nonlinear time $\tau = t\epsilon^2$, as in [2], and absorb the ϵ^2 term into the time derivative or just include it again into T , we opt for the latter and will not write it explicitly in the future.

The equation can be readily extended to the case $T \in \mathcal{C}$ through the substitution $T^2 \rightarrow |T|^2$.

We solved the problem of initial conditions on the phases but not yet on the action variables and we are still dealing with infinite sums, since we defined our system in a finite coordinate space. The next section builds on this.

§1.5 Statistics over the actions and the thermodynamic limit

We now take the thermodynamic (continuum) limit ($L \rightarrow \infty$) to turn our set of infinitely many coupled equation into an integral one, easier to approach analytically. In Fourier space the limit takes the form

$$\vec{\Delta k} = \frac{2\pi}{L} \rightarrow 0 \quad \Lambda^* \rightarrow \mathcal{R}^d \quad (\text{I.35})$$

and we define

$$\tilde{I}_k = \frac{I_k}{(\Delta k)^d} \rightarrow \tilde{I}(k) \quad \sum_k (\Delta k)^d \rightarrow \int d^d k \quad \frac{(\delta_{23}^{k1})^d}{(\Delta k)^d} \rightarrow \delta^d(k + k_1 - k_2 - k_3). \quad (\text{I.36})$$

[†]even if at this discrete stage $\Delta\omega$ is not a function over \mathcal{R} yet, and should be argued to be densely valued around 0 in the continuum limit

[‡]This can be seen as an a posteriori justification for the elimination of nonresonant three wave interaction terms.

The action variables and interaction coefficients become functions of the now continuous coordinates of Fourier space, the sums turn into integrals and the Kroenecker's deltas become a Dirac's delta. Equation (I.34), suppressing again all dimensional indexes, becomes

$$\frac{d}{dt} \langle \tilde{I}(k) \rangle = 4\pi \int dk_1 dk_2 dk_3 T^2(k, k_1, k_2, k_3) \tilde{I}(k) \tilde{I}(k_1) \tilde{I}(k_2) \tilde{I}(k_3) \left(\frac{1}{\tilde{I}(k)} + \frac{1}{\tilde{I}(k_1)} - \frac{1}{\tilde{I}(k_2)} - \frac{1}{\tilde{I}(k_3)} \right) \delta(\Delta\omega_{23}^{k1}) \delta(k + k_1 - k_2 - k_3). \quad (I.37)$$

In performing the limit we must be careful to require that

$$\lim_{\epsilon, \Delta k \rightarrow 0} \frac{(\Delta k)^d}{\epsilon} = 0, \quad (I.38)$$

as otherwise we could not use definition (I.33) to extract a delta from (I.32).

In the following we will recover the previous discrete notation for compactness, while still working in the continuous limit.

Our equation is still deterministic regarding the action function. We shall assume a stochastic distribution on the initial data defining the mean value[†] of I_k respect both distribution at a generic time

$$n(k, t) = n_k = \langle I_k \rangle_{\tilde{I}, \bar{\theta}}, \quad (I.39)$$

and requiring them to be uncorrelated at $t = 0$

$$\langle \tilde{I}_i \tilde{I}_j \tilde{I}_k \rangle_{\tilde{I}} = \langle \tilde{I}_i \rangle_{\tilde{I}} \langle \tilde{I}_j \rangle_{\tilde{I}} \langle \tilde{I}_k \rangle_{\tilde{I}} = \bar{n}_i \bar{n}_j \bar{n}_k \quad \text{for } i \neq j \neq k, \quad (I.40)$$

basically a gaussian distribution on initial conditions **JUSTIFY THIS INTUITIVELY**. Based on dimensional analysis we call n_k the wave action density of the system

Averaging over (I.37) we obtain

$$\frac{d}{dt} n_k = 4\pi \int dk_1 dk_2 dk_3 T_{k123}^2 \bar{n}_k \bar{n}_1 \bar{n}_2 \bar{n}_3 \left(\frac{1}{\bar{n}_k} + \frac{1}{\bar{n}_1} - \frac{1}{\bar{n}_2} - \frac{1}{\bar{n}_3} \right) \delta(\Delta\omega_{23}^{k1}) \delta_{23}^{k1}. \quad (I.41)$$

We find ourselves with an equation describing the time evolution of the average action value per Fourier mode, given an initial distribution. Unfortunately this equation holds now for a small time after $t = 0$, since it does not account for successive interactions between different waves. We need some form of closure. **DOES IT MAKE SENSE?, THINK AGAIN**

We make a last crucial assumption, time evolution does not spoil at successive instants the random phase and amplitude assumptions on initial conditions. Intuitively we can think of the separation of time scales on which the linear and nonlinear term act as allowing for the linear term to keep introducing chaos into the system.

This assumption let us substitute $\bar{n}_k \rightarrow n(k, t) = n_k$, giving us a self-consistent integro-differential equation for the evolution of n_k , the celebrated Wave Kinetic Equation

$$\frac{d}{dt} n_k = 4\pi \int dk_1 dk_2 dk_3 T_{k123}^2 n_k n_1 n_2 n_3 \left(\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} \right) \delta(\Delta\omega_{23}^{k1}) \delta_{23}^{k1}. \quad (I.42)$$

Given that we are assuming gaussian statistics, but at the same time letting the mean values vary with time through interactions among modes it is customary to call the approximation quasi-gaussian.

[†]Working with a_k variables n_k would be the second moment of the distribution, not the first.

§1.6 Some properties of the kinetic equation

-Proprietà WKE :Irreversibilità, dimostrazione del teorema H, leggi di conservazione e equazioni di continuità

§1.7 Equilibrium stationary state

-Equilibrium solutions and comments

§1.8 Out of equilibrium stationary states

-Out of Equilibrium solutions and comments, argomento fjortoft, argomento zakharov, misto dei due con commenti su forzante e dissipazione ed esempi fisici, ragionamenti dimensionali su forme approssimate di λ e ω e forme generiche di esponenti, definizioni costanti di KZ e commenti su convergenza e trasformata zakh un pò di foto di cascate per sistemi fisici NON MMT

§1.9 Convergence issues (MAYBE)

Beyond the leading order

- Introduzione al nlo (difficoltà computazionali, wyld diagrammatic technique, zakharov paper, gurarie) e nuovi sviluppi (Rosenhaus etc) suggerimenti per letture di QFT (sceli che notazione usare)
- come si ottiene la wke da una trattazione con i campi (arriva alle regole di feynman)
- conti funzione a due punti
- conti funzione a 4 punti
- wke nlo
- commenti e racconti su sviluppi extra (large N and resummation)

$$x = 2y + 3 \tag{II.1}$$

A one dimensional model for Wave Turbulence

- mini intro su storia MMT e outline capitolo
- MMT model con beta generico e sua WKE
- soluzioni KZ con beta generico e controllo sommario convergenza (solo limiti)
- trasformata di Zakharov per controllare davvero convergenza
- commenti su validità WKE data la presenza di solitoni e quasisolitoni (devi studiarla prima bro)
- MMT nlo, quantità ancora conservate e discussioni sulla sua convergenza e sulla natura delle correzioni
- tutto quello che riusciamo a trovare come previsione teorica

Numerical study of the Wave Kinetic Equation

- intro e obiettivi
- WavKinS struttura base e funzionamento (che algoritmi usa)
- simulazioni leading order e commenti sulle stesse
- simulazioni nlo e commenti, confronti con aspettative teoriche

Bibliography

- [1] S. NAZARENKO, *Wave Turbulence*, Springer Berlin Heidelberg, 2011.
- [2] M. ONORATO AND G. DEMATTEIS, *A straightforward derivation of the four-wave kinetic equation in action-angle variables*, Journal of Physics Communications, 4 (2020), p. 095016.
- [3] G. F. VE. ZAKHAROV AND V. LVOV, *Kolmogorov Spectra of Turbulence I*, Springer-Verlag, 1992.