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Introduction

- cita peierls 1922
- usa articolo nazarenko(desktop) e intro storica suo libro per storia tramite articoli
- Reference history of the subject (Hassleman, Zakharov, etc)
- motiva fisicamente lo studio delle soluzioni di Kz
- parallelo con storia turbo hydro
- fai un botto di esempi fisici con tante belle foto (parti da mail Onorato e libro zakh per cercarli)

Statistics of weakly nonlinear waves

This chapter is devoted to the derivation and analysis of the wave kinetic equation and its stationary solutions, assuming generic systems with nonlinearity of the four-wave kind, essentialy meaning that not only energy cascades may originate, but also wave action ones. Our account mainly draws from [3] and [4]. A more rigorous and modern derivation may be found in [2].
First we introduce a general model for nonlinear waves, we later expand it through standard perturbation theory and assume homogeneus statistics over the phases and gaussian statistics on the free theory action variables. By requiring that said assumptions are not spoiled by time evolution we find a late time asymptotic closure, resulting in the aforementioned wave kinetic equation. The conserved quantities of this new statistical description are discussed, and a notion of entropy is introduced.
Stationary solutions of the equation are discussed, both in the case of a closed system and in the case of an exchange of energy and wave action with the environment, trough forcing and dissipation terms opportunely separated in k-space.

§1.1 Hamiltonian description of waves in continuous media

We first turn to the construction of a general Hamiltonian method for the description of waves travelling in continuous media. In doing this we greatly borrow from the exceptional treatment in [4].
Questa parte iniziale può essere più o meno ampia in base al tempo a disposizione.
- Hamiltoniana generica per mezzo continuo (da spazio coordinate in una scatola), notazione, commenti sulla stabilità, commenti sul fatto che possa essere già risultato di sviluppo pertubativo (come per le onde oceaniche) e scrittura in spazio-k con 3 onde e 4 onde, simmetrie continue e discrete , espansione in small amplitudes nazarenko con forma generica tensoriale, write conditions under which the linear part can be written as omega a astar (COPIA SPUDORATAMENTE NAZARENKO per intro a sistema dinamico ed esempi fisici x 3 e 4 onde con foto dei sistemi di interesse sia 3 che 4 onde!!! AMPLIANDO, interaction representation)

to the appendix: - Idea Onorato oscillatore armonico per giustificare nonrisonanza e commento su trasfomrazione canonica per eliminare termini non risonanti (indirizza ad appendice per conto specifico), cita sistemi fisici in cui l'eq è a 4 onde parti da spazio fourier e analizza bene dimensionalmente la hamiltoniana e le sue componenti

COME CI ARRIVIAMO QUI? SPIEGA BENE Our final Hamiltonian is

$$\mathrm{H} = \sum_k \omega_k a_k^* a_k + \frac{1}{2} \sum_{k123} \mathrm{T}_{k123} a_k^* a_1^* a_2 a_3 \delta_{23}^{k1}. \tag{I.1}$$

CITA IL FATTO CHE GLI INDICI DEI k SONO SOPPRESSI

§1.2 Resonant and non-resonant interactions

maybe in the appendix?

§1.3 Perturbation Theory

Often in nonlinear systems no exact solutions (or few of them) are known. We now imagine ourselves in the situation where the interaction term in our Hamiltonian is small enough to allow for the perturbative treatment of the equations of motion, that is the expansion of the solution in orders of some small parameter, and their subsequent calculation order by order. From a physical viewpoint the smalness of the interaction term corresponds to a separation of the fast time scale on which the linear term operates from the slow time scale of the nonlinear one.
To make the expansion clearer we write an explicit ϵ factor in front of the interaction term † .

Following the derivation of [3] we transform to the action-angle coordinates of the unperturbed quadratic Hamiltonian

$$a_k = \sqrt{\mathrm{I}_k} e^{-i\theta_k} \tag{I.2}$$

obtaining

$$\mathrm{H} = \sum_k \omega_k \mathrm{I}_k + \frac{\epsilon}{2} \sum_{k123} \mathrm{T}_{k123} \sqrt{\mathrm{I}_k \mathrm{I}_1 \mathrm{I}_2 \mathrm{I}_3}\, e^{i(\theta_k+\theta_1-\theta_2-\theta_3)} \delta_{23}^{k1}, \tag{I.3}$$

by assuming that $\mathrm{T} \in \mathbb{R}$ (as is the case in a vast class of physical systems), $\mathrm{H} \in \mathbb{R}$ implies

$$\mathrm{H} = \sum_k \omega_k \mathrm{I}_k + \frac{\epsilon}{2} \sum_{k123} \mathrm{T}_{k123} \sqrt{\mathrm{I}_k \mathrm{I}_1 \mathrm{I}_2 \mathrm{I}_3} \cos(\Delta\theta_{34}^{k1}) \delta_{23}^{k1}, \tag{I.4}$$

where we defined $\Delta\theta_{23}^{k1} = \theta_k + \theta_1 - \theta_2 - \theta_3$.
We can prove that the change of coordinates (I.2) is canonical by assuming it to be true and recovering a_k and a_k^*'s poisson brackets‡

$$\begin{aligned} \{i a_k^*, a_k\} &= i \left(\frac{\partial a_k^*}{\partial \mathrm{I}_k} \frac{\partial a_k}{\partial \theta_k} - \frac{\partial a_k^*}{\partial \theta_k} \frac{\partial a_k}{\partial \mathrm{I}_k} \right) && \text{(I.5)} \\ &= -i \left(\frac{i}{2} \frac{1}{\sqrt{\mathrm{I}_k}} e^{i\theta_k} \sqrt{\mathrm{I}_k} e^{-i\theta_k} + \frac{i}{2} \frac{1}{\sqrt{\mathrm{I}_k}} e^{-i\theta_k} \sqrt{\mathrm{I}_k} e^{i\theta_k} \right) && \text{(I.6)} \\ &= 1. && \text{(I.7)} \end{aligned}$$

We can thus impose Hamilton equations for the new coordinates (remembering that time dependance of the coordinates is suppressed)

†The small parameter my be present as a constant in the Hamiltonian (for example the coupling g in the Nonlinear Schrodinger equation) or it may be a placeholder for the smallness of the function T_{k123} in a certain subdomain of k-space (for example the interaction among gravity waves in the small wavenumber limit).

‡Remembering that the true canonical variables are a_k and $i a_k^*$.

$$\frac{d}{dt}\mathrm{I}_k = -\frac{\partial}{\partial\theta_k}\mathrm{H} = 2\epsilon\sum_{123}\mathrm{T}_{k123}\sqrt{\mathrm{I}_k\mathrm{I}_1\mathrm{I}_2\mathrm{I}_3}\sin(\Delta\theta^{k1}_{23})\delta^{k1}_{23} \tag{I.8}$$

$$\frac{d}{dt}\theta_k = \quad \frac{\partial}{\partial\mathrm{I}_k}\mathrm{H} = \omega_k + \epsilon\sum_{123}\mathrm{T}_{k123}\sqrt{\frac{\mathrm{I}_1\mathrm{I}_2\mathrm{I}_3}{\mathrm{I}_k}}\cos(\Delta\theta^{k1}_{23})\delta^{k1}_{23}. \tag{I.9}$$

Since we are essentially perturbing an infinite set of harmonical oscillators with a small interaction term, we can euristically assume that the coordinates cannot grow indefinetly to infinity. We shall then be weary of unphysical secular terms artificially introduced by the perturbative expansion. The Poincarè-Lindsted method allows us to remove such terms by a frequency shift

$$\omega_k \to \Omega_k = \omega_k + \epsilon\left(2\sum_p \mathrm{T}_{kpkp}\mathrm{I}_p - \mathrm{T}_{kkkk}I_k\right), \tag{I.10}$$

togheter with a change of the summatory in H such that the trivial interactions† $k_2 = k$ & $k_1 = k_3$, $k_3 = k$ & $k_1 = k_2$ and $k_1 = k_2 = k_3 = k$ are excluded from it.
The shift (I.10) intuitively corresponds to the self-interaction of waves, not resulting in a net exchange of energy among modes, and thus directly changing the characteristic frequency of the free equation.
This particular choice is better justified in the APPENDIX WITH LINK, MAKE DUFFING EXAMPLE or in [2].

We may now develop perturbation theory, we start by expanding the (unknown) solutions as

$$\mathrm{I}_k = \mathrm{I}^{(0)}_k + +\epsilon\mathrm{I}^{(1)}_k + \epsilon^2\mathrm{I}^{(2)}_k + \mathcal{O}(\epsilon^3) \tag{I.11}$$

$$\theta_k = \theta^{(0)}_k + +\epsilon\theta^{(1)}_k + \epsilon^2\theta^{(2)}_k + \mathcal{O}(\epsilon^3), \tag{I.12}$$

and than substituting them into (I.8) and (I.9).
We now reintroduce explicit time dependance and impose $\mathrm{I}^{(0)}_k(0) = \bar{\mathrm{I}}_k$ and $\mathrm{I}^{(1)}_k(0) = \mathrm{I}^{(2)}_k(0) = 0$ to fix initial conditions on the Is and $\theta^{(0)}_k(0) = \bar{\theta}_k$ and $\theta^{(1)}_k(0) = \theta^{(2)}_k(0) = 0$ to fix initial conditions on the θs.

The ϵ^0 order equations are

$$\frac{d}{dt}\mathrm{I}^{(0)}_k = 0 \tag{I.13}$$

$$\frac{d}{dt}\theta^{(0)}_k = \Omega^{(0)}_k, \tag{I.14}$$

with solutions

$$\mathrm{I}^{(0)}_k(t) = \bar{\mathrm{I}}_k \tag{I.15}$$

$$\theta^{(0)}_k(t) = \bar{\theta}_k + \bar{\Omega}_k t, \tag{I.16}$$

where $\Omega^{(0)}_k$ and $\bar{\Omega}_k$ refer to Ω_k with only zeroeth order contribution or initial conditions respectively. Notice that the ϵ terms in the shifted frequency should be included in the equations for $\theta^{(1)}$ and not $\theta^{(0)}$, we however make this choice to keep all terms linear with time togheter (and thus leading in the expansion). This order reproduces the dynamics of an infinite dimensional integrable system (for example infinitely

† we call them trivial as they do not correspond to a net exchange of energy/action among different Fourier modes

many decoupled harmonic oscillators), with constant actions and angles evolving linearly with time.

At ϵ order the equations of motion are

$$\frac{d}{dt}\mathrm{I}_k^{(1)} = 2\sum_{123}\mathrm{T}_{k123}\sqrt{\mathrm{I}_k^{(0)}\mathrm{I}_1^{(0)}\mathrm{I}_2^{(0)}\mathrm{I}_3^{(0)}}\sin(\Delta\theta_{23}^{k1(0)})\delta_{23}^{k1} \tag{I.17}$$

$$\frac{d}{dt}\theta_k^{(1)} = \sum_{123}\mathrm{T}_{k123}\sqrt{\frac{\mathrm{I}_1^{(0)}\mathrm{I}_2^{(0)}\mathrm{I}_3^{(0)}}{\mathrm{I}_k^{(0)}}}\cos(\Delta\theta_{23}^{k1(0)})\delta_{23}^{k1}. \tag{I.18}$$

Here the only time dependance lies in $\Delta\theta^{(0)}$ and $\mathrm{I}_k^{(1)}(0) = \theta_k^{(1)}(0) = 0$, integrating the equations gives

$$\mathrm{I}_k^{(1)}(t) = 2\sum_{123}\mathrm{T}_{k123}\sqrt{\bar{\mathrm{I}}_k\bar{\mathrm{I}}_1\bar{\mathrm{I}}_2\bar{\mathrm{I}}_3}\,\frac{\delta_{23}^{k1}}{\Delta\bar{\Omega}_{23}^{k1}}\left[\cos(\Delta\bar{\theta}_{23}^{k1}) - \cos(\Delta\bar{\theta}_{23}^{k1} + \Delta\bar{\Omega}_{23}^{k1}t)\right] \tag{I.19}$$

$$\theta_k^{(1)}(t) = \sum_{123}\mathrm{T}_{k123}\sqrt{\frac{\bar{\mathrm{I}}_1\bar{\mathrm{I}}_2\bar{\mathrm{I}}_3}{\bar{\mathrm{I}}_k}}\,\frac{\delta_{23}^{k1}}{\Delta\bar{\Omega}_{23}^{k1}}\left[\sin(\Delta\bar{\theta}_{23}^{k1} + \Delta\bar{\Omega}_{23}^{k1}t) - \sin(\Delta\bar{\theta}_{23}^{k1})\right]. \tag{I.20}$$

Where $\Delta\bar{\Omega}$ is defined in the same fashion as $\Delta\theta$.

We should be content with this first nontrivial result, but through the sheer power of hindsight[†] we write also the ϵ^2 order equations only for the action variables (there is no need to actually solve them).
Looking at (I.8) we seek to obtain an ϵ^2 equation by substituting I and θ up to their ϵ order terms. By Taylor expanding the square root we obtain four terms of the form

$$\sqrt{(\mathrm{x}+\epsilon\mathrm{y})\tilde{\mathrm{x}}} \underset{\epsilon\to 0}{\sim} \sqrt{\mathrm{x}\tilde{\mathrm{x}}}\left(1+\frac{\epsilon\mathrm{y}}{2\tilde{\mathrm{x}}}\right), \tag{I.21}$$

where, for example, $\mathrm{x}+\epsilon\mathrm{y} = \mathrm{I}_k^{(0)} + \epsilon\mathrm{I}_k^{(1)}$ and $\tilde{\mathrm{x}} = \mathrm{I}_1^{(0)}\mathrm{I}_2^{(0)}\mathrm{I}_3^{(0)}$.
There also appear terms of the form

$$\sin(\mathrm{x}+\epsilon\mathrm{y}) \underset{\epsilon\to 0}{\sim} \sin(\mathrm{x}) + \epsilon\mathrm{y}\cos(\mathrm{x}), \tag{I.22}$$

where $\mathrm{x} = \Delta\theta_{23}^{k1(0)}$ and $\mathrm{y} = \Delta\theta_{23}^{k1(1)}$.
By plugging everything into (I.8) we first obtain

$$\frac{d}{dt}\mathrm{I}_k^{(2)} = 2\sum_{123}\mathrm{T}_{k123}\sqrt{\mathrm{I}_k^{(0)}\mathrm{I}_1^{(0)}\mathrm{I}_2^{(0)}\mathrm{I}_3^{(0)}}\left[\frac{1}{2}\left(\frac{\mathrm{I}_k^{(1)}}{\mathrm{I}_k^{(0)}}+\frac{\mathrm{I}_1^{(1)}}{\mathrm{I}_1^{(0)}}+\frac{\mathrm{I}_2^{(1)}}{\mathrm{I}_2^{(0)}}+\frac{\mathrm{I}_3^{(1)}}{\mathrm{I}_3^{(0)}}\right)\sin(\Delta\theta_{23}^{k1(0)})\delta_{23}^{k1} + \Delta\theta_{23}^{k1(1)}\cos(\Delta\theta_{23}^{k1(0)})\delta_{23}^{k1}\right], \tag{I.23}$$

and then by using (I.15), (I.16), (I.19), (I.20) and basic trigonometry we find

$$\begin{aligned}\frac{d}{dt}\mathrm{I}_k^{(2)} = 2\sum_{123456}\mathrm{T}_{k123}\sqrt{\bar{\mathrm{I}}_k\bar{\mathrm{I}}_1\bar{\mathrm{I}}_2\bar{\mathrm{I}}_3\bar{\mathrm{I}}_4\bar{\mathrm{I}}_5\bar{\mathrm{I}}_6}\sum_{i=0}^{3}\frac{\mathrm{T}_{k123}\mathrm{T}_{i456}}{\sqrt{\bar{\mathrm{I}}_i}\Delta\bar{\Omega}_{56}^{i4}} \\ \times\left(\sin(\Delta\bar{\theta}_{23}^{k1} + \Delta\bar{\Omega}_{23}^{k1}t - \sigma_i\Delta\bar{\theta}_{56}^{i4}) + \sin(\sigma_i\Delta\bar{\theta}_{56}^{i4} + \sigma_i\Delta\bar{\Omega}_{56}^{i4}t - \Delta\bar{\theta}_{23}^{k1} - \Delta\bar{\Omega}_{23}^{k1}t)\right)\delta_{23}^{k1}\delta_{56}^{i4},\end{aligned} \tag{I.24}$$

where σ_i is equal to $+1$ if $i = 0, 1$ and alternatively is -1. When $i = 0$ it represents functional dependance on k.

[†]Developing a statistical theory of the system, the first nontrivial contribution comes from the ϵ^2 order.

§1.4 Random Phase Approximation

Having approximated the solutions to order ϵ we found ourselves with the problem of gathering initial conditions in infinite dimensonal systems [†] , we shall then renounce the deterministic approach in favour of a probabilistic one.
In general such idea is realized thorugh averaging over infinitely many realizations of the equations of motion with different initial conditions, to then extract average quantities more easily confrontable with experiment. In a nonlinear problem this is again highly non trivial, to simplify the endeavor we assume that a large number of waves is present in the system, in the sense that each mode in Fourier space is highly excited. It is then reasonable to assume that, after a time evolution proportional to the minimum value of $\frac{1}{\omega_k}$ in the range of physical interest, the phases θ would be uniformly distributed in the $[0,2\pi]$ segment [‡] . This means that whatever our initial conditions, given that $\bar{\mathrm{I}}_k \neq 0$ almost everywhere and the nonlinear contribution being slower than the linear one, we may actually assume some new initial conditions on θs drawn from the following distribution

$$\left\langle f(\bar{\theta}_1\ldots\bar{\theta}_N)\right\rangle_{\bar{\theta}} = \int_0^{2\pi} P(\bar{\theta}_1\ldots\bar{\theta}_N) f(\bar{\theta}_1\ldots\bar{\theta}_N) d\bar{\theta}_1\ldots d\bar{\theta}_N \quad \text{with} \quad P(\bar{\theta}_1\ldots\bar{\theta}_N) = \frac{1}{2\pi^N} \tag{I.25}$$

Looking back at the Hamiltonian (I.1) we see that the phases do not contribute to physical quantities like the energy or the wave number, it is in the action variables that those observables are encoded. We have now a clear plan, to find a kinetic equation, independent of initial conditions, for the action variables.
The main objective is then

$$\left\langle \frac{d}{dt}\mathrm{I}_k\right\rangle_{\bar{\theta}} = \frac{d}{dt}\left\langle \mathrm{I}_k\right\rangle_{\bar{\theta}} = \left\langle \frac{d}{dt}\mathrm{I}_k^{(0)}\right\rangle_{\bar{\theta}} + \epsilon\left\langle \frac{d}{dt}\mathrm{I}_k^{(1)}\right\rangle_{\bar{\theta}} + \epsilon^2\left\langle \frac{d}{dt}\mathrm{I}_k^{(2)}\right\rangle_{\bar{\theta}} \tag{I.26}$$

To zeroeth order, being constant, is null. We average over the ϵ order equation (I.17) (with subbed zeroeth order solutions)

$$\left\langle \frac{d}{dt}\mathrm{I}_k^{(1)}\right\rangle_{\bar{\theta}} = 2\sum_{123} \mathrm{T}_{k123}\sqrt{\bar{\mathrm{I}}_k\bar{\mathrm{I}}_1\bar{\mathrm{I}}_2\bar{\mathrm{I}}_3}\left\langle \sin(\Delta\bar{\theta}^{k1}_{23} + \Delta\bar{\Omega}^{k1}_{23}\, t)\right\rangle_{\bar{\theta}} \delta^{k1}_{23}. \tag{I.27}$$

Making the probability distribution explicit and isolating the term depending on phases we obtain

$$\left\langle \sin(\Delta\bar{\theta}^{k1}_{23} + \Delta\bar{\Omega}^{k1}_{23}\, t)\right\rangle_{\bar{\theta}} = \left\langle 2\,\mathrm{Im}\left(e^{i\Delta\bar{\theta}^{k1}_{23}} e^{\Delta\bar{\Omega}^{k1}_{23} t)}\right)\right\rangle_{\bar{\theta}} = \frac{1}{2\pi^4}\int_0^{2\pi} e^{i\bar{\theta}_k} e^{i\bar{\theta}_1} e^{i\bar{\theta}_2} e^{i\bar{\theta}_3} d\bar{\theta}_k d\bar{\theta}_1 d\bar{\theta}_2 d\bar{\theta}_3 = 0 \tag{I.28}$$

To first order we have a trivial kinetic equation, and must then go to ϵ^2 order to find nontrivial results, luckily we have already written I's Hamilton equations to second order.

The $\bar{\theta}$ dependent part of equation (I.24) may be rewritten as

$$e^{+i\sigma_i\Delta\bar{\theta}^{i4}_{56} - i\Delta\bar{\theta}^{k1}_{23}}\left[e^{-i\Delta\bar{\Omega}^{k1}_{23}}\left(e^{i\sigma_i\Delta\bar{\Omega}^{i4}_{56} t} - 1\right)\right] \tag{I.29}$$

We shall focus on the $i = 0$ term and extend the results to the other ones. Isolating the exponential with $\bar{\theta}$ in (I.29) and averaging we get

$$\left\langle e^{i(\bar{\theta}_4+\bar{\theta}_2+\bar{\theta}_3-\bar{\theta}_5-\bar{\theta}_6-\bar{\theta}_1)}\right\rangle_{\bar{\theta}}. \tag{I.30}$$

[†] Let us think of the ocean surface for example, measuring its height at a generic instant would be unfeasible.

[‡] This is known as the random phase approximation. One shall be careful as if the original equations of motion are known to have solitonic solutions in a certain regime of k-space, in such case phases could be correlated and the assumption would not hold.

This term is different from 0 only if the total exponent is null. It acts as a Kroenecker's delta on the 3 out of the 6 sums, imposing either $k_4 = k_1$ & $k_2 = k_5$ & $k_3 = k_6$ or $k_4 = k_1$ & $k_2 = k_6$ & $k_3 = k_5$†.
The full averaged $i = 0$ term, with (I.30) enforced, is

$$4\sum_{123} \mathrm{T}_{k123}\mathrm{T}_{k123}\bar{\mathrm{I}}_1\bar{\mathrm{I}}_2\bar{\mathrm{I}}_3 \frac{\sin(\Delta\bar{\Omega}^{k1}_{23}\, t)}{\Delta\bar{\Omega}^{k1}_{23}}\delta^{k1}_{23}, \tag{I.31}$$

where the property $\mathrm{T}_{k123} = \mathrm{T}_{k132}$ was used‡.
Looking at the cases $i = 1, 2, 3$ the only differences are:

- $i = 1 \longrightarrow$ the same as $i = 0$ except for the exchange $\bar{\mathrm{I}}_1 \longleftrightarrow \bar{\mathrm{I}}_k$;
- $i = 2 \longrightarrow$ the same as $i = 0$ except for an overall minus sign;
- $i = 3 \longrightarrow$ the same as $i = 2$ except for $\bar{\mathrm{I}}_2 \longleftrightarrow \bar{\mathrm{I}}_3$.

The final result is

$$\frac{d}{dt}\left\langle \mathrm{I}^{(2)}_k \right\rangle = 4\sum_{123} \mathrm{T}^2_{k123}\bar{\mathrm{I}}_k\bar{\mathrm{I}}_1\bar{\mathrm{I}}_2\bar{\mathrm{I}}_3 \left(\frac{1}{\bar{\mathrm{I}}_k} + \frac{1}{\bar{\mathrm{I}}_1} - \frac{1}{\bar{\mathrm{I}}_2} - \frac{1}{\bar{\mathrm{I}}_3}\right) \frac{\sin(\Delta\bar{\Omega}^{k1}_{23}\, t)}{\Delta\bar{\Omega}^{k1}_{23}}\delta^{k1}_{23}, \tag{I.32}$$

where it is not anymore necessary to account in the sum for trivial interactions, as for
$k_2 = k$ & $k_1 = k_3$, $k_3 = k$ & $k_1 = k_2$ and $k_1 = k_2 = k_3 = k$ the r.h.s is null.
We may ignore to this order the frequency shift as Taylor expanding the sine function shows its contribution to be of order ϵ^3.
By using one of the possible definitions of the Dirac's delta function

$$\lim_{a\to\infty} \frac{\sin(a\mathrm{x})}{\pi \mathrm{x}} = \delta(\mathrm{x}), \tag{I.33}$$

into (I.32) togheter with (I.26) and the assumption that enough time has passed , finally

$$\frac{d}{dt}\left\langle \mathrm{I}_k \right\rangle = 4\pi\epsilon^2\sum_{123} \mathrm{T}^2_{k123}\bar{\mathrm{I}}_k\bar{\mathrm{I}}_1\bar{\mathrm{I}}_2\bar{\mathrm{I}}_3 \left(\frac{1}{\bar{\mathrm{I}}_k} + \frac{1}{\bar{\mathrm{I}}_1} - \frac{1}{\bar{\mathrm{I}}_2} - \frac{1}{\bar{\mathrm{I}}_3}\right) \delta(\Delta\omega^{k1}_{23})\delta^{k1}_{23}. \tag{I.34}$$

There are some important remarks on this last equation.
The presence of the Dirac's delta defines a resonance manifold§ of the Fourier modes k_1, k_2 and k_3 interacting with k, showing that only resonant term contribute to the net interaction between different modes¶.
Based on difference preferences one could define a nonlinear time $\tau = t\epsilon^2$, as in [3], and absorb the ϵ^2 term into the time derivative or just include it again into T, we opt for the latter and will not write it explicitely in the future.
The equation can be readily extended to the case $\mathrm{T} \in \mathbb{C}$ through the substitution $\mathrm{T}^2 \to |\mathrm{T}|^2$.
We solved the problem of initial conditions on the phases but not yet on the action variables and we are still dealing with infinite sums, since we defined our system in a finite coordinate space. The next section builds on this.

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†The combinations with $k_2 = k_4$ or $k_2 = k_3$ were excluded from the sum with the shift (I.10)

‡In the case of complex T the property would hold as well with a complex conjugate on one side.

§even if at this discrete stage $\Delta\omega$ is not a function over $\mathbb{R}$ yet, and should be argued to be densely valued around 0 in the continuum limit

¶This can be seen as an a posteriori justification for the elimination of nonresonant three wave interaction terms.

§1.5 Statistics over the actions and the thermodynamic limit

We now take the thermodynamic (continuum) limit ($L \to \infty$) to turn our set of infinitely many coupled equation into an integral one, easier to approach analytically. In Fourier space the limit takes the form

$$\vec{\Delta k} = \frac{2\pi}{L} \to 0 \qquad \Lambda^* \to \mathbb{R}^d \tag{I.35}$$

and we define

$$\tilde{\mathrm{I}}_k = \frac{\mathrm{I}_k}{(\Delta k)^d} \to \tilde{\mathrm{I}}(k) \qquad \sum_k (\Delta k)^d \to \int d^d k \qquad \frac{(\delta^{k1}_{23})^d}{(\Delta k)^d} \to \delta^d(k + k_1 - k_2 - k_3). \tag{I.36}$$

The action variables and interaction coefficents become functions of the now continuous coordinates of Fourier space, the sums turn into integrals and the Kroenecker's deltas become a Dirac's delta. Equation (I.34), suppressing again all dimensional indexes, becomes

$$\frac{\partial}{\partial t}\langle \tilde{\mathrm{I}}(k)\rangle = 4\pi \int dk_1 dk_2 dk_3$$
$$\mathrm{T}^2(k, k_1, k_2, k_3)\bar{\mathrm{I}}(k)\bar{\mathrm{I}}(k_1)\bar{\mathrm{I}}(k_2)\bar{\mathrm{I}}(k_3)\left(\frac{1}{\bar{\mathrm{I}}(k)} + \frac{1}{\bar{\mathrm{I}}(k_1)} - \frac{1}{\bar{\mathrm{I}}(k_2)} - \frac{1}{\bar{\mathrm{I}}(k_3)}\right)\delta(\Delta\omega^{k1}_{23})\delta(k + k_1 - k_2 - k_3). \tag{I.37}$$

In performing the limit we must be careful to require that

$$\lim_{\epsilon, \Delta k \to 0} \frac{(\Delta k)^d}{\epsilon} = 0, \tag{I.38}$$

as otherwise we could not use definition (I.33) to extract a delta from (I.32).
In the following we will recover the previous discrete notation for compactness, while still working in the continous limit.

Our equation is still deterministic regarding the action function. We shall assume a stochastic distribution on the inital data defining the mean value† of I_k respect both distribution at a generic time

$$n(k, t) = n_k = \langle \mathrm{I}_k \rangle_{\bar{\mathrm{I}}, \bar{\theta}}, \tag{I.39}$$

and requiring them to be uncorrelated at $t = 0$

$$\langle \bar{\mathrm{I}}_i \bar{\mathrm{I}}_j \bar{\mathrm{I}}_k \rangle_{\bar{\mathrm{I}}} = \langle \bar{\mathrm{I}}_i \rangle_{\bar{\mathrm{I}}} \langle \bar{\mathrm{I}}_j \rangle_{\bar{\mathrm{I}}} \langle \bar{\mathrm{I}}_k \rangle_{\bar{\mathrm{I}}} = \bar{n}_i \bar{n}_j \bar{n}_k \qquad \text{for} \qquad i \neq j \neq k, \tag{I.40}$$

basically a gaussian distribution on initial conditions JUSTIFY THIS INTUITIVELY. As long that this distribution remains close to being gaussian the whole statistical dynamics is described by the variance n_k. Based on dimensional analysis we call n_k the wave action density of the system‡.
Averaging over (I.37) we obtain

$$\frac{\partial}{\partial t} n_k = 4\pi \int dk_1 dk_2 dk_3 \mathrm{T}^2_{k123} \bar{n}_k \bar{n}_1 \bar{n}_2 \bar{n}_3 \left(\frac{1}{\bar{n}_k} + \frac{1}{\bar{n}_1} - \frac{1}{\bar{n}_2} - \frac{1}{\bar{n}_3}\right)\delta(\Delta\omega^{k1}_{23})\delta^{k1}_{23}. \tag{I.41}$$

We find ourselves with an equation describing the time evolution of the average action value per Fourier mode, given an initial distribution.
The equation still depends on initial conditions and does not allow to look for stationary states, given the

† Working with a_k variables n_k would be the second moment of the distribution, not the first.
‡ $\mathrm{N} = \int n_k dk$ has the units of an action.

chaotic nature of nonlinear systems it is also very sensitive on said initial conditions. What we are looking for is a kinetic equation. what about successive interactions?? We make a last crucial assumption, time evolution does not spoil at successive instants the random phase and amplitude assumptions on inital conditions. Intuitively we can think of the separation of time scales on which the linear and nonlinear term act as allowing for the linear term to keep introducing chaos into the system.
This assumption let us subsitute $\bar{n}_k \to n(k,t) = n_k$, giving us a self-consistent integro-differential equation for the evolution of n_k, the celebrated Wave Kinetic Equation (WKE)

$$\frac{\partial}{\partial t} n_k = 4\pi \int dk_1 dk_2 dk_3 \mathrm{T}^2_{k123} n_k n_1 n_2 n_3 \left(\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} \right) \delta(\Delta\omega^{k1}_{23}) \delta^{k1}_{23}. \tag{I.42}$$

Given that we are assuming gaussian statistics, but at the same time letting the mean values of the infinite distributions at every point k vary with time depending on each other values, it is customary to call the approximation quasi-gaussian. The l.h.s. is often called the collisional integral, again for the reason that it quantifies the interactions among different modes, and represented through the symbol $\mathtt{I}\{n_k\}$.
This same equation obtained through explicit perturbative expansion around the free theory may be found looking directly at the time derivative of $n_k(t)$. Given the nonlinearity of the equations of motion, $\frac{\partial}{\partial t} n_k(t)$ will depend on the fourth moment, whose time derivative will depend on the sixth one and so on. This produces an hierarchy of equations in which the closure problem, that we addressed through $\bar{n}_k \to n(k,t) = n_k$, is more evident. Under similar line of tought as in the previous section, assuming an n-points correlator to be gaussian leads to a closed equation, the higher the n, the higher the order of the kinetic equation obtained. For the first nontrivial one said assumption should be on the 6-points function.

§1.6 Main properties of the kinetic equation

The Microscopic Hamiltonian (I.1) conserves energy, momentum and wave number (or wave action) as a consequence of its invariance under time translation, space translation and phase shifting† respectively. A first question would be which of said conserved quantities are inherited by the WKE.
Let us start with energy, we separe the linear and nonlinear terms in (I.1)

$$\mathrm{H} = \mathrm{H}_2 + \mathrm{H}_{\mathrm{int}}. \tag{I.43}$$

The derivation of the kinetic equation was based off the assumption that $\mathrm{H}_{\mathrm{int}} \ll \mathrm{H}_2$ and the obtained equation clearly shows that at ϵ^2 order the only role of $\mathrm{H}_{\mathrm{int}}$ is to redistibute energy amidst modes. This is enough to hypothesize that the conserved quantity may be the average linear energy

$$\mathrm{E} = \langle \mathrm{H}_2 \rangle_{\bar{\mathrm{I}},\bar{\theta}} = \int \omega_k n_k dk. \tag{I.44}$$

Its conservation may be easily checked by utilizing (I.42)

$$\frac{d}{dt}\mathrm{E} = \int \omega_k \frac{\partial}{\partial t} n_k dk = \int \omega_k \mathtt{I}\{n_k\}\, dk = \tag{I.45}$$

$$4\pi \int dk dk_1 dk_2 dk_3 \mathrm{T}^2_{k123} n_k n_1 n_2 n_3 \omega_k \left(\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} \right) \delta(\Delta\omega^{k1}_{23}) \delta^{k1}_{23} = \tag{I.46}$$

$$\pi \int dk dk_1 dk_2 dk_3 \mathrm{T}^2_{k123} n_k n_1 n_2 n_3 (\omega_k + \omega_1 - \omega_2 - \omega_3) \left(\frac{\omega_k}{n_k} + \frac{\omega_1}{n_1} - \frac{\omega_k}{n_2} - \frac{\omega_k}{n_3} \right) \delta(\Delta\omega^{k1}_{23}) \delta^{k1}_{23} = 0. \tag{I.47}$$

† A three wave interaction would not conserve the wave action

The simmetry properties of T and the rest of the integrand were exploited to properly rename the dummy integrated variables. The delta function over frequencies allows to determine that the integral is null. It is crucial to remember that the usage of the delta function is allowed only if the integral converges, such condition must be checked case by case and thus conservation of energy is not to be taken for granted†. Defining the energy density as $\mathcal{E}_k = \omega_k n_k$, we cast (I.45) as a continuity equation

$$\frac{\partial}{\partial t}\mathcal{E}_k + \vec{\nabla}\cdot\vec{\mathrm{P}} = 0 \tag{I.48}$$

$$\vec{\nabla}\cdot\vec{\mathrm{P}} = -\omega_k \mathrm{I}\{n_k\}, \tag{I.49}$$

where P is the energy flux and the vector signs were made explicit to avoid ambiguity with future notation.

We now move on to wave action conservation, given the original simmetry it is reasonable to assume that the new conserved quantity is

$$\mathrm{N} = \int n_k dk. \tag{I.50}$$

By substituting into (I.42) all four wave number variable are now integrated over. From the simmetries of T, namely $(k, k_1 \to k_2, k_3)$, and the fact that the remaing part of the integrand is antisymmetryc for that same exchange, we conclude that the integral is null. Thus (I.50) is a conserved quantity. In the same fashion as (I.48) we may produce the continuity equation for the wave action density

$$\frac{\partial}{\partial t}n_k + \vec{\nabla}\cdot\vec{\mathrm{Q}} = 0 \tag{I.51}$$

$$\vec{\nabla}\cdot\vec{\mathrm{Q}} = -\mathrm{I}\{n_k\}, \tag{I.52}$$

where Q is the wave action flux.
From a similar argument as energy but using the deltas over wave numbers it may be shown that additional conserved quantities are the components of momentum

$$\Pi = \int k n_k dk, \tag{I.53}$$

of lesser importance as they do not lead to cascade solutions.IS THIS TRUE?
Having built a statistical theory, it is customary to ask whether the time evolution is irreversible. This is commonly done by producing a definition of entropy for which an H-theorem holds, thus showing that the statistical treatment of the microscopic theory, invariant under time reversal, introduces a time arrow in the dynamics.
Given the quasi-gaussian approximation we may consider the full system as infinitely many closed systems, one for each k, slowly exchanging energy. Appealing again to the perturbative nature of the interaction term we may formulate a kind of ergodic assumption: very loosely speaking they interact, given that they exchange energy, but interact so little that we may assume each of them to be a closed system with its own Boltzmann statistics.‡
Considering that the bigger n_k is, the more probable it is to observe waves with wave vector k in the full system, we may well use it instead of the number of microstates in the Boltzmann entropy. Thus the entropy of each of them, measuring temperature in Joules to avoid the introduction of another constant and omitting the insertion of another one to make n_k adimensional, is $\mathrm{S}_k = \ln(n_k)$. Coupling togheter all different wave vectors we obtain

$$\mathrm{S} = \int dk \ln(n_k). \tag{I.54}$$

†See [4] for a more detailed analysis.

‡We are sorry for butchering statistical mechanics in this way, we do so hoping to give an intuitive origin for the formula for entropy

We shall now check that a Boltzmann H-theorem holds for such definition. Let us look at the time dependance of entropy, by using the kinetic equation we find

$$\frac{d}{dt}\mathrm{S} = \int \frac{dk}{n_k}\frac{\partial}{\partial t}n_k = \int dk \frac{\mathrm{I}\{n_k\}}{n_k} = \int dk dk_1 dk_2 dk_3 \mathrm{D}_{k123} \tag{I.55}$$

$$= 4\pi \int dk dk_1 dk_2 dk_3 |\mathrm{T}_{k123}|^2 n_k n_1 n_2 n_3 \left(\frac{1}{n_k^2} + \frac{1}{n_k n_1} - \frac{1}{n_k n_2} - \frac{1}{n_k n_3}\right)\delta(\Delta\omega_{23}^{k1})\delta_{23}^{k1}, \tag{I.56}$$

by renaming the integrated dummy variables we may reformulate the integrand as
$\frac{1}{4}\left(\mathrm{D}_{k123} + \mathrm{D}_{1k23} + \mathrm{D}_{23k1} + \mathrm{D}_{321k}\right)$, recognizing the explicited square of a sum of fractions we find

$$\frac{d}{dt}\mathrm{S} = 4\pi \int dk dk_1 dk_2 dk_3 |\mathrm{T}_{k123}|^2 n_k n_1 n_2 n_3 \left(\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right)^2 \delta(\Delta\omega_{23}^{k1})\delta_{23}^{k1} \geq 0, \tag{I.57}$$

thus proving that entropy as defined cannot decrease with time.

§1.7 Equilibrium stationary state

Having now a working definition of entropy, we may look for stationary solution, i.e. solutions maximising the entropy. Through the insertions of Lagrange multipliers we may choose and fix the constant of motion values.
We are looking for an n_k such that

$$\frac{\delta}{\delta n_k}\left(\mathrm{S} - \frac{\mathrm{E}}{T} + \frac{u\cdot\Pi}{T} + \frac{\mu\mathrm{N}}{T}\right). \tag{I.58}$$

Where we could call T the temperature, μ the chemical potential and u the impulse.
Using the previously given definitions of E and N

$$\frac{\delta}{\delta n_k}\int dk' \left[\ln(n_{k'}) - \frac{\omega_{k'}}{T}n_{k'} + \frac{u\cdot k'}{T}n_{k'} + \frac{\mu}{T}n_{k'}\right] = \frac{1}{n_k} - \frac{\omega_k}{T} + \mu + u\cdot k = 0, \tag{I.59}$$

where the obvious solution is

$$n_k = \frac{T}{\omega_k - u\cdot k - \mu}, \tag{I.60}$$

equivalent to the Gibbs Ensamble. It is easy to see that (I.60) is a solution of the wave kinetic equation, inserting it into $\left(\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right)$ in (I.42) yealds

$$\frac{1}{T}\left(\omega_k + \omega_1 - \omega_2 - \omega_3 + u\cdot k + u\cdot k_1 - u\cdot k_2 - u\cdot k_3\right), \tag{I.61}$$

that can be readily see to be null thanks to the deltas over frequencies and wave vectors.
From now on we shall choose $u = 0$ as we are not interested in states with overall momentum different than zero. This solutions is called the Rayleigh-Jeans distribution, from the analogy with statistical mechanics.
By choosing $\mu = 0$, meaning that the total number of waves in the system is not fixed to a chosen value, we see that $n_k = \frac{T}{\omega_k}$ implies $\mathcal{E}_k = T$, showing equipartition of energy for such a state.
To have the energy of this state finite an high-k cutoff or a low-k one are required, meaning that under or over a certain lenght there is no oscillation. This however makes perfectly sense in a classical theory, where the granularity of any medium is ignored in order to describe it through differential equations or there is always a finite size to the medium. In this case those cutoffs only reinstate natural properties of the system.
Closed systems with a resonant three wave interaction do not conserve wave action, and thus relax to $n_k = \frac{T}{\omega_k}$. Four wave closed systems relax instead to the more general solution.

§1.8 Out of equilibrium stationary states

Let us now imagine that our system is not closed, energy and action may be introduced and dissipated. Examples of forcing may be wind blowing on the surface of the sea, a ship moving trough a body of water, radio waves scattering off plasma or experimentally pumping atoms into a condensate. Dissipation may be due to viscosity in a fluid, interactions with a tank or a basin, Landau damping, superfluid particles in a magnetic trap reaching enough energy to escape it, generation of quasiparticles or interaction with inhomogenities. CHECK CORRECTNESS OF PHYSICAL EXAMPLES IN PAPERS
now fjiortoft argument and then zakh formalization.
In many physical settings those mechanisms operate only in a limited region of wave vector space (k-space), for example a ship would mainly generate waves of wavelenght comparable to its own lenght, Landau damping peaks for certain rather low wavelenghts (energetic waves) and interaction with a tank would dissipate water eaves of lenght comparable to the tank dimension. In such case at least part of k-space would still locally behave almost as a closed system, with the difference that energy and wave action flows in an out slowly due to the nonlinear coupling of modes. A region of this type in k-space is usually called the inertial range, and it is limited by neighbouring pumping and dissipations regions.
It is natural to ask if there is a preferred flow of conserved quantities in the inertial range and if there exist nontrivial stationary out of equilibrium states in such open systems.

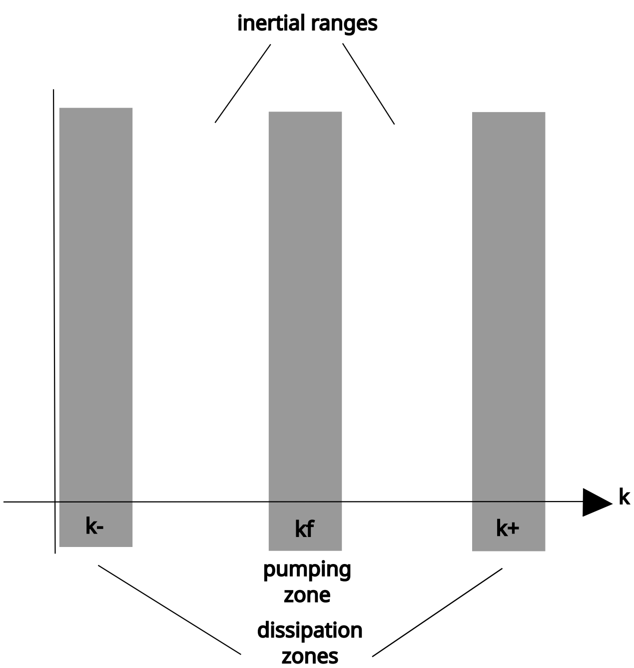

Figure 1: Pictorial representation of Fjortoft argument

To begin answering the first question we may start with a generic argument due to Fjortoft. We assume the system to be isotropic and concerns ouserlves only with the wave vector absolute value[†]. We imagine a system with a localized forcing around k_f in k-space, inserting per unit time energy density $\mathcal{E}_f$ and wave action density n_f. Then are present two dissipating regions at k_- and k_+ extracting from the system per unit time relatively $\mathcal{E}_-$, $\mathcal{E}_+$, n_- and n_+. We assume $k_- \ll k_f \ll k_+$ so that two inertial ranges are present.

Assuming that the dissipating mechanisms are able to fully absorb what is injected in the forcing region, we may require that energy and wave action are conserved:

$$\omega_{k_f} n_f = \omega_{k_-} n_- + \omega_{k_+} n_+ \tag{I.62}$$

$$n_f = n_- + n_+, \tag{I.63}$$

where we used $\mathcal{E}_k = \omega_k n_k$. A crucial assumption for this argument, true for many physical systems, is that ω_k is a monotonical function of its arguments and it grows as k grows.
Let us now pretend that almost all the wave action density is dissipated at k_+, i.e. $n_f \simeq n_+$. Using the definition of $\mathcal{E}$

$$n_f = \frac{\mathcal{E}_f}{\omega_{k_f}} \simeq \frac{\mathcal{E}_+}{\omega_{k_+}}, \tag{I.64}$$

[†]If until now we meant the whole vector with k, now it represents only its modulus.

leading to

$$\frac{\mathcal{E}_+}{\mathcal{E}_f} \simeq \frac{\omega_{k_+}}{\omega_{k_f}}. \tag{I.65}$$

Given that k_f and k_+ where assumed to be well separated and monotonicity of ω_k implies $\omega_{k_+} \gg \omega_{k_f}$, we find

$$\mathcal{E}_+ \gg \mathcal{E}_f, \tag{I.66}$$

implying that more energy is dissipated than what was introduced, clearly paradoxical. This for now only shows that the full wave action flux, if different than zero, cannot fully flow from low to high k. It does suggests however that the majority of wave action will be transfered towords lower k that where it was introduced, we will later prove this in a more rigorous way.
Turning to the energy flux let us now pretend that energy density is instead almost fully dissipated at k_-, meaning that $\mathcal{E}_f \simeq \mathcal{E}_-$, we follow the same logic as before to find

$$\mathcal{E}_f = \omega_{k_f} n_f \simeq \omega_{k_-} n_- \quad \longrightarrow \quad \frac{n_f}{n_-} \simeq \frac{\omega_{k_-}}{\omega_{k_f}}. \tag{I.67}$$

Again $\omega_{k_-} \ll \omega_{k_f}$ thus implying the paradoxical conclusion that $n_- \gg n_f$, more wave action is dissipated than introduced. As before this only shows that the energy flux cannot fully go towords lower k, and it suggests that said flux will mainly move towords high k. We will later prove this as well.
This argument hints at the physical picture that we will later draw through more careful mathematical analysis. In a generic system with nonlinear interaction among k-modes and monotonical dispersion relation, if some mechanism allow for injection of conserved quantities in a limited region and other mechanisms extract those same quantities both at much higher and much lower wave number, the nonlinear interaction may allow for out of equilibrium stationary states where energy flow towords high k and wave action towords low k. Even at its early stage we can already appreciate how powerful this result may be, given its generality and the wide array of settings that can manifest such phenomenology. But let us not get ahead of ouselves, we have for now only a blurred picture of the situation.

First we start by the most general way of introducing forcing and dissipation in the kinetic equation. We add a term Γ_k, resuming within itself the dynamics of wave generation and damping, multiplied to n_k, as it intuively makes sense that the higher the number of waves of a certain lenght, the more effective the generation or destruction of them. We thus write

$$\frac{\partial}{\partial t} n_k = \mathtt{I}\{n_k\} + \Gamma_k n_k, \tag{I.68}$$

where again $\mathtt{I}\{n_k\}$ stands for the collisional integral and the time dependance is implicit. It is easy to convince ourselves that where $\Gamma_k > 0$ it represents the growth rate of waves and where $\Gamma_k < 0$ it represents the dampening rate. A torough examinations of the conditions that must be present on Γ_k to have a consistent theory is presentend in [4], we here avoid it as to keep this introduction straightforward.

Guided by the intuition gathered through Fjortoft's argument, we look for stationary solutions of the kinetic equation with active fluxes of energy or wave action. As we expect the fluxes to bring conserved quantities across an eventual inertial range, we study the equation without the Γ_k term. Given our choice of locality on Γ_k we make now a reasonable universality assumption, n_k in the inertial range does not depend on the explicit form of the pumping and dissipating term.
We look for a solution with a power law spectrum, to this end from now on we assume the system to be isotropic and concern ourselves only with dependance on the wave number modulus k. Our ansatz is

$$n_k = A k^{-\nu}. \tag{I.69}$$

where A is an arbitrary constant. This ansatz was first introduced by Zakharov in 1965 [5], inspired by similar behavior in hydrodynamics discovered by Kolmogorov on on dimensional grounds in 1941 [1].
A proper analysis would consist in substituting (I.69) into (I.42) and perform the so-called Zakharov transform, to put the integrand in a form where the values of ν for which $\mathtt{I}\{n_k\} = 0$ are obvious. For an analysis of such method in the general case we remind again the reader to [4].
IF THERE IS TIME AT THE END INSERT HERE ZAKHAROV TRANSFORM EXPLANATION
We choose instead the route of dymensional analysis. After inserting our ansatz in the kinetic equation, we perform the change of variables $k_i = q_i k$ to leave a dimensionless integral. To proceed we have to restrict the possible forms of the interaction term $\mathrm{T}_{k123} = \mathrm{T}(k, k_1, k_2, k_3)$, we require it to be an homogeneus function of degree β. At the same time we assume the dispersion relation to be of the type $\omega_k = |k|^{\alpha\dagger}$. We obtain

$$\mathtt{I}\{n_k\} = 4\pi A^3 k^{3d--3\nu+2\beta-d-\alpha} \int dq_1 dq_2 dq_3 q_1^{d-1} q_2^{d-1} q_3^{d-1}$$
$$|\mathrm{T}(1, q_1, q_2, q_3)|^2 q_1^{-\nu} q_2^{-\nu} q_3^{-\nu} \left(1 + q_1^\nu - q_2^\nu - q_3^\nu\right) \delta(1 + |q_1|^\alpha - |q_2|^\alpha - |q_3|^\alpha) \delta(1 + q_1 - q_2 - q_3), \quad \text{(I.70)}$$

where, thanks to the isotropy assumption, we moved to generalized spherical coordinates in d dimensions and ignored any constant resulting from angular integration. As for the delta functions, their known property under integration was used:

$$\delta(\lambda x) = \frac{1}{\lambda}\delta(x).$$

We will refer in the following to the adimensional integral as $\tilde{\mathtt{I}}(\nu)$. Let us first turn to the energy flux, with a slight redefinition of the energy density, to account for spherical coordinates, the continuity equation (I.48) turns to

$$\tilde{\mathcal{E}}_k = \mathrm{S} k^{d-1} k^\alpha n_k \quad \text{(I.71)}$$
$$\frac{\partial}{\partial t}\tilde{\mathcal{E}}_k + \frac{\partial}{\partial k}\mathrm{P}_k = 0, \quad \text{(I.72)}$$

where S is the surface of a unit radius sphere in d dimensions and P_k is the radial component in spherical coordinates of the flux vector, again depending only on the wave vector modulus.
By using (I.42) we find

$$\frac{\partial}{\partial t}\tilde{\mathcal{E}}_k = \mathrm{S} k^{d-1+\alpha} \mathtt{I}\{n_k\} = \mathrm{S} A^3 k^{3d-1-3\nu+2\beta} \tilde{\mathtt{I}}(\nu), \quad \text{(I.73)}$$

that togheter with (I.72) and the fundamental theorem of calculus gives

$$\mathrm{P}_k = -4\pi \mathrm{S} A^3 \tilde{\mathtt{I}}(\nu) \int_0^k dx x^{3d-1-3\nu+2\beta} = -4\pi \mathrm{S} A^3 \tilde{\mathtt{I}}(\nu) \frac{k^{3d+-3\nu+2\beta}}{3d - 3\nu + 2\beta}. \quad \text{(I.74)}$$

A perfectly analogous process lead to a similar expression for the wave action flux Q_k, with a difference due to the absence of the dispersion relation in the wave action density definition‡ $\tilde{n_k} = \mathrm{S} k^{d-1} n_k$. The result is

$$\mathrm{Q}_k = -4\pi \mathrm{S} A^3 \tilde{\mathtt{I}}(\nu) \frac{k^{3d-3\nu+2\beta-\alpha}}{3d - 3\nu + 2\beta - \alpha}. \quad \text{(I.75)}$$

We already know that the Rayleigh-Jeans state (I.60) is a stationary solution, hence $\tilde{\mathtt{I}}(\alpha) = 0$. This implies that in the case of a closed system there are no fluxes of conserved quantities traversing the system, as to be expected from thermalization.

† For surface gravity waves $\alpha = \frac{1}{2}$ and for the nonlinear Schrodinger equation $\alpha = 2$
‡ Again slightly modified to account for the isotropy assumption

Returning to the open system previously discussed we see that nontrivial stationary solutions[†] must have constant fluxes. If they had null fluxes there would be no transfer of energy and wave action across the inertial range, incompatible with the presence of a forcing region and a dissipating one. If instead the fluxes were to be dependant on k in the inertial range, there would be no balance between the quantities flowing into a certain mode k and those leaving it, leading to $\frac{\partial}{\partial t} n_k \neq 0$.
Looking at (I.74) and (I.75) we see that the only way a possible stationary solution could yeald finite flux is to have an exponent ν which is a zero of the denominator, we thus obtain the 2 scalings

$$\nu_P = \frac{2}{3}\beta + d, \tag{I.76}$$
$$\nu_Q = \frac{2}{3}\beta + d - \frac{1}{3}\alpha. \tag{I.77}$$

Setting a general constant value for the fluxes, applying l'Hopital rule and assuming that what we found are really the exponent of stationary solutions, i.e $\tilde{\mathrm{I}}(\nu_P) = \tilde{\mathrm{I}}(\nu_Q) = 0$, we find

$$\lim_{\nu \leftarrow \nu_P} \mathrm{P}_k = \mathrm{P}_0 = \frac{4}{3}\pi \mathrm{S} A^3 \tilde{\mathrm{I}}'(\nu_P), \tag{I.78}$$
$$\lim_{\nu \leftarrow \nu_Q} \mathrm{Q}_k = \mathrm{Q}_0 = \frac{4}{3}\pi \mathrm{S} A^3 \tilde{\mathrm{I}}'(\nu_Q), \tag{I.79}$$

showing that we should also check in the specific cases that the first derivative of the dimensionless collision integral with respect to the power law solution's exponent must be finite. The value of said derivative determines the direction of the flow of the relative conserved quantity.
Expressing the generic prefactor A as a function of the flux in each case and renaming it properly we finally obtain the famous Kolmogorv-Zakharov solutions

$$n_k^P = C_{KZ}^P \mathrm{P}_0^{\frac{1}{3}} k^{\frac{2}{3}\beta + d}, \tag{I.80}$$
$$n_k^Q = C_{KZ}^Q \mathrm{Q}_0^{\frac{1}{3}} k^{\frac{2}{3}\beta + d - \frac{1}{3}\alpha}. \tag{I.81}$$

Where the C_{KZ} are the Kolmogorov-Zakharov constants defined as $C_{KZ}^P = (\frac{3}{4\pi \tilde{\mathrm{I}}'(\frac{2}{3}\beta + d)})$ and $C_{KZ}^Q = (\frac{3}{4\pi \tilde{\mathrm{I}}'(\frac{2}{3}\beta + d - \frac{1}{3}\alpha)})$.
The value of P_0 and Q_0 will instead depend on the specific mechanisms at work in the pumping an damping regions. Their value may be extimated from the specific form of Γ_k.
For example for the energy flux one has to look at the continuity equation outside of the inertial range, easily obtainable from (I.68)

$$\frac{\partial}{\partial t}\tilde{\mathcal{E}}_k + \frac{\partial}{\partial k}\mathrm{P}_k = \Gamma_k \tilde{\mathcal{E}}_k. \tag{I.82}$$

When in a stationary solutions the time derivative part is null and by finite integration from a generic k_0 to infinity we obtain

$$\mathrm{P}_\infty - \mathrm{P}_{k_0} = \int_{k_0}^{\infty} dk \Gamma_k \tilde{\mathcal{E}}_k. \tag{I.83}$$

By recognizing that by placing k_0 in the inertial range $\mathrm{P}_{k_0} = P_0$ and in any reasonable physical setting $P_\infty = 0$ DISCUSS CONVERGENCE, Complete argument for direction of the cascade, MAYBE DISCUSS LIMITS, PUT PLOTS FOR EXPERIMENTAL CONFIRMATION (STEAL FROM NEWELL), AND THEN GO BACK TO THE START TO CONCLUDE THE CHAPTER.

[†] If they exist

Beyond the leading order

- Introduzione al nlo (difficoltà computazionali, wyld diagrammatic technique, zakharov paper, gurarie) e nuovi sviluppi (Rosenhaus etc) suggerimenti per letture di QFT (sceli che notazione usare)
- come si ottiene la wke da una trattazione con i campi (arriva alle regole di feynman)
- conti funzione a due punti
- conti funzione a 4 punti
- wke nlo
- commenti e racconti su sviluppi extra (large N and resummation)

$$x = 2y + 3 \tag{II.1}$$

A one dimensional model for Wave Turbulence

- mini intro su storia MMT e outline capitolo
- MMT model con beta generico e sua WKE
- soluzioni KZ con beta generico e controllo sommario convergenza (solo limiti)
- trasformata di Zakharov (se già introdotta in chap 1 solo usarla) e controllo convergenza LO integrando
- commmenti su validita WKE data la presenza di solitoni e quasisolitoni (devi studiarla prima bro)
- MMT nlo, qunatità ancora conservate e discussioni sulla sua convergenza e sulla natura delle correzioni
- tutto quello che riusciamo a trovare come previsione teorica

Numerical study of the Wave Kinetic Equation

- intro e obbiettivi
- WavKinS struttura base e funzionamento (che algoritmi usa)
- simulazioni leading order e commenti sulle stesse
- simulazioni nlo e commenti, confronti con aspettative teoriche

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